Portfolio optimization under solvency constraints: a dynamical approach

Sujith Asanga[1], Alexandru Asimit[2], Alexandru Badescu[3,*], and Steven Haberman[4]

Abstract

We develop portfolio optimization problems for a non-life insurance company seeking to find the minimum capital required, which simultaneously satisfies solvency and portfolio performance constraints. Motivated by standard insurance regulations, we consider solvency capital requirements based on three criteria: Ruin Probability, Conditional Value-at-Risk and Expected Policyholder Deficit ratio. We propose a novel semiparametric formulation for each problem and explore the advantages of implementing this methodology over other potential approaches. When liabilities follow a Lognormal distribution, we provide sufficient conditions for convexity for each problem. Using different expected Return on Capital target levels, we construct efficient frontiers when portfolio assets are modelled with a special class of multivariate GARCH models. We found that the correlation between asset returns plays an important role in the behaviour of the optimal capital required and the portfolio structure. The stability and out-of-sample performance of our optimal solutions are empirically tested with respect to both the solvency requirement and portfolio performance, through a double rolling window estimation exercise.

Keywords: Portfolio optimization; Capital requirements; Solvency constraint; Multivariate GARCH; Double rolling window.

JEL Classification: C58, G11, G22.

[1]Department of Mathematics and Statistics, University of Calgary, Calgary, Alberta T2N 1N4, Canada. `sujith.asanga@gmail.com`.

[2]Cass Business School, City University, London EC1Y 8TZ, United Kingdom. `E-mail: asimit@city.ac.uk`.

[3]Corresponding author*: *Department of Mathematics and Statistics, University of Calgary, Calgary, Alberta T2N 1N4, Canada.* `E-mail: abadescu@math.ucalgary.ca`. Phone: 1-403-2203963.

[4]Cass Business School, City University, London EC1Y 8TZ, United Kingdom. `E-mail: S.Haberman@city.ac.uk`.

1 Introduction

Insurance regulation has played an important role in securing policyholders and investors against various types of risk. One of its primary objectives is the establishment of an initial capital amount required to be held by insurance companies, in order to offer protection in the case of unexpected events. There has been an extensive literature on capital adequacy and its relationship to risk measures. For example, the Value-at-Risk (VaR), one of the most popular tools used in financial risk management, constitutes the basis for the Solvency II regulatory standards which applies to insurance companies in European Union (EU) (e.g. Sandström, 2006). In order to overcome some of the VaR pitfalls (e.g. VaR is not sub-additive), Artzner *et al.* (1999) introduced the notion of coherent risk measures. A discussion about their applications to capital requirements in insurance is provided in Artzner (1999). Amongst the coherent risk measures, an important special case is represented by the Expected Shortfall (ES), which plays a crucial role for the development of the Swiss Solvency Test (SST) (FOPI, 2004). The class of coherent measures has been further extended to convex measures by Föllmer and Schied (2002). For an overview of theoretical properties of various well-known risk measures used as solvency capital requirements, we refer to Dhaene *et al.* (2006). For a more recent survey on applications of risk measures in portfolio management we refer to Krokhmal *et al.* (2011).

The standard approach used in connecting minimum capital standards to risk measures relies on the investment of solvency capital into a single "eligible" security, often taken as a risk-free asset. However, if the regulator allows the financial institution (e.g. insurance company in our case) to use a portfolio of such "eligible" assets, investing only into the risk-free asset may not be efficient. For example, Balbas (2008) showed that the investment of the capital requirement into a risk-free asset is not optimal in several important cases, and he provided an example based on a Conditional Value-at-Risk ($CVaR$) (a risk measure introduced by Rockafellar and Uryasev, 2000) and Black-Scholes assumptions. Artzner *et al.* (2009) provided a brief discussion on the efficient use of capital and risk measures in the case of multiple traded assets, while Farkas *et al.* (2012) gave a comprehensive theoretical background on the same issue. However, none of the above studies provide empirical examples on how minimum capital and its optimal allocation are obtained. Moreover, despite their popularity, these optimization problems are typically treated separately in the actuarial literature. The use of both initial capital and portfolio weights as decision variables for optimization problems has only been recently proposed. For example, Mankai and Bruneau

(2012) introduced a joint optimization problem by maximizing the expected return on risk-adjusted capital subject to a $CVaR$ constraint, while Asimit *et al.* (2012) developed a minimum capital requirement problem based on a Ruin Probability (RP) constraint. However, both studies assumed a static setting and did not investigate the behaviour of the optimal solutions and portfolio performance over time.

In this paper we introduce three joint optimization problems for a non-life insurance company in a dynamic framework. Each problem is constructed by minimizing the initial capital subject to two types of constraints. The first category is represented by solvency requirements according to a particular insurance regulation, while the second constraint, which is the same for all problems, is given through a portfolio performance measure.[5] Since shareholders usually require a gain on their investment, we use the expected Return on Capital (ROC) as our portfolio performance measure. Other choices for measuring performance are suggested in Cherny and Madan (2009), among others.

Motivated by the Solvency II and SST directives, the first two solvency criteria are based on a target value for the RP and a negative $CVaR$ of the insurer's net loss, respectively, both computed over a predefined period of time (e.g. one year horizon). Since the RP constraint is equivalent to a negative VaR, the two criteria considered agree with the mathematical definition of a solvency requirement given by Djehiche and Hörfelt (2005) for a general risk measure. The third solvency criterion uses an upper bound for the Expected Policyholder Deficit (EPD), which was introduced by Butsic (1994) as a new measure of insolvency risk. The EPD criteria has played an important role in establishing the US Risk-based Capital (RBC) regulatory system (e.g. see NAIC, 2009). Analyses and comparisons of the three capital standards have been previously considered in the literature. For example, Holzmüller (2009) and Cummins and Phillips (2009) provided detailed assessments of the RBC, SST and Solvency II directives. Barth (2000) compared the RP and EPD approaches and found that the latter increases the insolvency risk for larger insurers. Eling *et al.* (2009) investigated the RP, EPD and ES in a mean-variance setup using data from a German non-life insurance company.

Our objective is to provide a detailed analysis of the optimal capital required and its portfolio allocation for all three solvency criteria. In constructing the insurer net loss, we model only two sources of risk, namely the market

[5] A conceptually similar problem has been very recently proposed in the financial literature by Santos *et al.* (2012). They develop an optimization problem which minimizes the capital required subject to a Basel II criteria (i.e. a target number of VaR violations within a year) and a lower bound for the expected portfolio return. However, their problem is constructed as a single optimization problem (i.e. the only decision variable is the portfolio weight), since the capital requirement is given explicitly by the maximum between current-day VaR and the average one-day-ahead VaR over the last 60 business days.

(assets) and insurance (liabilities) risks. The dynamical structure is introduced by assuming that the portfolio's assets follow a Multivariate Generalized Autoregressive Conditional Heteroskedastic (MV-GARCH) model. There is a considerable number of MV-GARCH specifications proposed in the financial econometrics literature (e.g. conditional covariance matrix, factor, and conditional covariance and correlation models) and for recent surveys we refer to Bauwens *et al.* (2006) and Silvennoinen and Terasvirta (2009). In this study, we focus on the class of Dynamic Conditional Correlation (DCC) models introduced by Engle (2002). There are at least three important reasons for this choice. Firstly, these models are not heavily parametrized and therefore, are appropriate for large scale estimation and risk management problems (e.g. see Engle and Sheppard, 2008). Secondly, their forecasting performance is not significantly outperformed by richer competitors (e.g. see Laurent *et al.*, 2012). Thirdly, we wish to analyze the effect of a time-varying correlation matrix between portfolio's assets on the optimal solutions, by comparing it with the constant and zero correlation cases. The insurance liability is modelled with a univariate random variable.

One of the major issues with implementing the proposed problems is related to their convexity. Since the expected ROC constraint is linear in both capital and weights, the focus remains on the convexity of the solvency constraints. The standard approach for dealing with $CVaR$ optimization is based on a Monte-Carlo type of approximation, and this leads to a linear programming (LP) reformulation for the initial problem (e.g. see Rockafellar and Uryasev, 2000, 2002, and Krokhmal *et al.*, 2002, among others). Tian *et al.* (2010) used the similar prescription for solving asset-liability mean-variance portfolio optimization problems under $CVaR$ constraints. Alexander *et al.* (2006) pointed out that the LP reformulation becomes less efficient when the number of Monte-Carlo paths becomes large. For the RP problem, closed-form expression and/or convex reformulation are rarely available. There are two streams of literature dealing with probability (chance) constraints. The first direction consists of using Monte-Carlo type estimators based on indicator functions and performing further appropriate approximations (e.g. see Boyd and Vandenbergue, 2004, for convex approximations by eliminating the indicator function, Nemirovski and Shapiro, 2006, for Bernstein scheme convex approximation, Luedtke and Ahmed, 2008, for non-convex mixed-integer programming (MIP) reformulation, among others). The second direction formulates and solves the chance constraints as VaR-constrained optimization (e.g. see Larsen *et al.*, 2002, for algorithms based on iterative $CVaR$ optimizations, Gaivoronski and Pflug, 2004, for scenario-based methods and Wozabal *et al.*, 2008, for a difference of convex functions reformulation).

In order to avoid the above convexity issues, we propose a semiparametric approach for reformulating the solvency constraints using the empirical distribution based on asset returns scenarios generated according to the MV-GARCH models and the given parametric specification of the liability distribution. For the *RP*-constrained optimization, this methodology can be viewed as a generalization of the semiparametric algorithm proposed in Asimit *et al.* (2012). When liabilities are Lognormal distributed, we derive sufficient convexity conditions for all three solvency constraints. Our numerical examples are constructed based on two 3-asset portfolios. The first portfolio is formed with one "risk-free" asset (US T-Bills) and two risky assets (NASDAQ and NYSE), while the second consists of the S&P 500 index and two exchange traded funds which track the investment results of US Treasury and Corporate Bond indices. The parameters are estimated from daily returns using the two-stage estimation methodology introduced by Engle and Sheppard (2001) for three covariance specification: DCC-GARCH, CCC-GARCH (constant correlation) and UNI-GARCH (no-correlation). The liability parameters are estimated based on monthly aggregate claim amounts from property insurance provided by a European Union-based insurance company. Using different level of shareholders' expected *ROC*, we construct efficient frontiers for a one-month horizon. All three solvency constraints indicate a similar type of behaviour in the sense that the DCC-GARCH is the most conservative model in terms of capital requirements. We run a double rolling window estimation exercise (re-estimate asset and liability parameters over a given period) to compare the out-of-sample performance of our models. The results indicate that the DCC specification outperforms the CCC and the no-correlation ones, in terms of both the solvency constraint and return on capital performance, for portfolios with strongly correlated assets. The univariate GARCH structure is slightly preferred for lower correlated asset portfolios. The time-varying correlation also plays an important role in the portfolio structure. We further found that the portfolio weights are generally stable over the rolling window period, while the optimal total assets exhibit significant variation.

The rest of the paper is organized as follows. In the next section, we introduce the optimization problems based on the solvency and expected *ROC* constraints and illustrate the semiparametric approach for solving them. Models for both assets and liabilities and discussions on the convexity of the proposed methods are provided in Section 3. An extensive empirical analysis is performed in Section 4. We conclude the paper in Section 5.

2 Preliminaries and solvency constrained optimization

We consider a discrete-time framework with the set of trading dates indexed by $\mathcal{T} = \{0, 1, \ldots, T\}$. The market consists of a portfolio of n assets with the gross return[6] process over the period $[t, t+1]$ defined by $\mathbf{R}_{t+1} = (R_{1,t+1}, \ldots, R_{n,t+1})^T$. We denote by $\mathcal{F}_t$, the historical information on the asset return evolution up to time t, so that $\mathcal{F}_t = \sigma(\mathbf{R}_1, \ldots, \mathbf{R}_t)$. For convenience, we use the following notations for conditional probabilities, expectations and variances: $\Pr(\cdot|\mathcal{F}_t) = \Pr_t(\cdot)$, $E[\cdot|\mathcal{F}_t] = E_t[\cdot]$ and $Var[\cdot|\mathcal{F}_t] = Var_t[\cdot]$. Moreover, we use majuscules for random variables (except for cases when un upper script associated to a random variable may be interpreted as a realization of that random variable) and non-capital letters for deterministic quantities.

We introduce three optimization problems based on different solvency criteria for a non-life insurance company within a one-period setting, $[t, t+\tau]$, where τ is the solvency horizon satisfying $\tau \leq T - t$. First, we denote by p_t the aggregate premium available for investment at time t. In addition, we assume that shareholders provide a regulatory initial capital of size c_t. Without loss of generality, no other premiums are collected and no capital is issued or retired between t and $t+\tau$, and therefore, the total invested amount is $p_t + c_t$. Let $\mathbf{x}_t = (x_{1,t}, \ldots, x_{n,t})^T$ be the portfolio weights whose components satisfy the standard budget constraint, $\sum_{i=1}^{n} x_{i,t} = 1$ and the no short sales constraint, $x_{i,t} \geq 0$, $i = 1, \ldots, n$. Since our problem is designed as a single-period optimization, no rebalancing is allowed during the solvency period.

To fully describe the setup, we let the insurer's liability be modelled by a univariate random variable $Y_{t+\tau}$. This represents the aggregate claim amount over the solvency horizon which is assumed to be paid at time $t+\tau$. At this point, no particular assumptions regarding the conditional distributions of $\mathbf{R}_{t+\tau}$ and $Y_{t+\tau}$ are made, and no premium calculation principle is assigned for p_t. We define the insurer's net loss as the difference between the liability and portfolio value over the solvency horizon:

$$L_{t,t+\tau} = Y_{t+\tau} - (p_t + c_t)\mathbf{R}_{t+\tau}^T \mathbf{x}_t.$$

Since both the capital requirement and portfolio allocations are decision variables in our joint optimization problems, we can view the net loss r.v. as a function of these quantities (i.e. $L_{t,t+\tau} := L(c_t, \mathbf{x}_t)$). For convenience, we assume there are no other sources of risk other than the ones modelled through Y and $\mathbf{R}$, and there are no transaction or

[6] The gross return process is defined here as the ratio between the terminal and initial asset prices, and thus is non-negative.

other friction costs.

Each optimization problem proposed in the following subsections is characterized by minimizing the capital requirement c_t, subject to two key constraints. The first constraint is a solvency capital requirement imposed by the insurer's regulator, and is based on one of the following criteria: RP, $CVaR$ and EPD. Next, we define the gross ROC over the investment period:

$$ROC_{t,t+\tau} = -\frac{L_{t,t+\tau}}{c_t} \; . \tag{2.1}$$

Since shareholders typically require a rate of return on the provided capital, the second constraint is introduced as a portfolio performance measure based on a target level for shareholders' expected return. For each type of solvency, we provide a novel semiparametric approach, which allows us to reformulate the constraints and further implement the optimization without costly computational effort.

2.1 Optimization with RP constraint

The use of ruin probability constraints is motivated by the Solvency II Regime, which applies to any EU based insurance company, and consists of identifying the capital required to maintain a target level for the ruin probability over a specified period of time. Thus, we define the RP-constrained problem as follows:

$$\begin{aligned} &\min_{c_t,\mathbf{x}_t} c_t \\ \text{s.t.}\quad & E_t\big[\mathbb{1}_{\{L_{t,t+\tau}>0\}}\big] \leq 1-\alpha, \\ & E_t\big[ROC_{t,t+\tau}\big] \geq ROC^{\alpha}, \\ & \mathbf{1}^T\mathbf{x}_t = 1, \;\; \mathbf{x}_t \geq \mathbf{0}, \;\; c_t \geq 0. \end{aligned} \tag{2.2}$$

Here, α represents the specified solvency level, $\mathbb{1}_{\{\cdot\}}$ is the indicator function and ROC^{α} is the lower bound for the shareholders' expected return on capital, which also depends on α. We notice that the solvency probability constraint in (2.2) can be reformulated as a Value-at-Risk constraint, where the VaR of a loss random variable Z at a confidence level α is defined by

$$VaR^{\alpha}(Z) := \inf\{z \in \Re : \Pr(Z \leq z) \geq \alpha\}.$$

Indeed, it immediately follows that:

$$E_t\big[\mathbb{1}_{\{L_{t,t+\tau}\ >0\}}\big] \leq 1-\alpha \Leftrightarrow \Pr_t\big(L_{t,t+\tau} > 0\big) \leq 1-\alpha \Leftrightarrow VaR_t^{\alpha}(L_{t,t+\tau}) \leq 0,$$

where VaR_t^{α} is the value at risk conditional on the historical asset return evolution up to time t. The main difficulty in dealing with this type of problem is the convexity of the chance constraint. Closed-form expressions for the ruin probability only exist in very few special cases. For example, if we assume that $L_{t,t+\tau}$ has a multivariate Gaussian distribution, then (2.2) can be rewritten as a Second Order Cone optimization, which can be efficiently implemented with appropriate solvers. Asimit *et al.* (2012) found a closed form expression for such a problem in the absence of the short-sales and *ROC* constraints. However, when $Y_{t+\tau}$ and $R_{t+\tau}$ do not belong to the same family of distributions, we may not be able to even identify the distribution of $\mathrm{L}_{t,t+\tau}$.

A standard approach in the chance constrained programming literature is to use a fully nonparametric method for approximating the conditional expectation in (2.2). This can be done by using Monte-Carlo simulations for both assets and liabilities. The solvency condition can be thus reformulated as:

$$\frac{1}{m}\sum_{j=1}^{m}\mathbb{1}_{\{Y_{t+\tau}(j)-(p_t+c_t)\mathbf{R}_{t+\tau}^{T}(j)\mathbf{x}_t>0\}} \leq 1-\alpha. \tag{2.3}$$

Here, m is the number of Monte-Carlo simulations and $Y_{t+\tau}(j)$ and $\mathbf{R}_{t+\tau}(j)$ represent the j^{th} generated path for liabilities and assets conditional on $\mathcal{F}_t$. Due to the presence of the indicator function, the optimization problem is still non-convex. As was already mentioned in the introduction, several approaches such as convex approximations or non-convex Mixed Integer Programming (MIP) representations have been recently proposed in the literature to handle the non-parametric constraint. In general, their implementation becomes less efficient when m is large, which is generally required for a better accuracy of the Monte-Carlo estimator. Another alternative is to construct an equivalent condition to (2.3) by finding an appropriate confidence level which requires a reasonable small value for the number of Monte-Carlo paths; however, this depends on the data used and requires a calibration procedure. In order to avoid such issues, we use a conditional version of the semiparametric approach proposed in Asimit *et al.* (2012). This methodology is based on a pre-specified parametric conditional liability distribution and scenario-based asset returns. Using the notation, $E\big[\cdot|\mathcal{F}_t \bigcup\{\mathbf{R}_{t+\tau}=\mathbf{R}_{t+\tau}(j)\}\big] = E_t^{(j)}\big[\cdot\big]$, and using the double expectation rule, we reformulate the initial problem:

$$\begin{aligned} &\min_{c_t, \mathbf{x}_t} c_t \\ \text{s.t.} \quad &\tfrac{1}{m} \sum_{j=1}^{m} E_t^{(j)} \left[\mathbb{1}_{\{Y_{t+\tau} - (p_t + c_t)\mathbf{R}_{t+\tau}^T(j)\mathbf{x}_t > 0\}} \right] \leq 1 - \alpha, \\ &E_t \left[ROC_{t,t+\tau} \right] \geq ROC^{\alpha}, \\ &\mathbf{1}^T \mathbf{x}_t = 1, \;\; \mathbf{x}_t \geq \mathbf{0}, \;\; c_t \geq 0. \end{aligned} \tag{2.4}$$

The expectation in the solvency constraint is taken with respect to the r.v. Y. A sufficient condition for the convexity of (2.4) is that $E_t^{(j)} \left[\mathbb{1}_{\{Y_{t+\tau} - (p_t + c_t)\mathbf{R}_{t+\tau}^T(j)\mathbf{x}_t > 0\}} \right]$ is convex in $(c_t,\ \mathbf{x}_t)$, for any $j = 1, \ldots, m$. This is equivalent to having a conditionally convex survival function for the liability $Y_{t+\tau}$. Most of the survival functions used for modelling claim data posses this property (some not on their entire domain) and all our empirical results in Section 4 will be based on such a distribution.

2.2 Optimization with $CVaR$ constraint

The $CVaR$ was introduced by Rockafellar and Uryasev (2000) as an alternative coherent risk measure to VaR, which quantifies the loss severity in the case of default. For general random variables, the $CVaR$ is defined as a weighted average of the corresponding VaR and conditional expected losses which strictly exceed VaR. When losses have a continuous distribution function, $CVaR$ coincides with ES (e.g. see Acerbi and Tasche, 2002, and Hürliman, 2003), which constitute the basis for quantifying the target capital according to the Swiss Solvency Test (EIOPA, 2011), that applies to all Swiss based insurance companies.

Following a similar approach as in the RP-constrained case, we define the following optimization problem with a $CVaR$ solvency constraint:

$$\begin{aligned} &\min_{c_t, \mathbf{x}_t} c_t \\ \text{s.t.} \quad &CVaR_t^{\beta} \left(L_{t,t+\tau} \right) \leq 0, \\ &E_t \left[ROC_{t,t+\tau} \right] \geq ROC^{\beta}, \\ &\mathbf{1}^T \mathbf{x}_t = 1, \;\; \mathbf{x}_t \geq \mathbf{0}, \;\; c_t \geq 0. \end{aligned} \tag{2.5}$$

Here, β is the confidence level for $CVaR$ and ROC^{β} is the associated lower bound for our performance measure.

$CVaR$ is a more conservative measure of risk than VaR given the same confidence level. In the empirical analysis from Section 4, we shall relate the confidence levels for each of the risk measures by, $\beta = 1 - 2(1-\alpha)$, such that VaR^{α} is the median of the worst $1-\beta$ events. This is also satisfied by the values used in the Solvency II and SST directives ($\alpha = 99.5\%$ and $\beta = 99\%$).

There are various ways of formulating $CVaR$ in the literature. The most appropriate representation for our context is the one provided by Pflug (2000) and Rockafellar and Uryasev (2000), who define $CVaR$ as the solution of an optimization problem. For a general loss r.v. Z we have:

$$CVaR^{\beta}(Z) = \inf_{s\in\Re}\left\{s + \frac{1}{1-\beta}E\big[(Z-s)_+\big]\right\},$$

where $(Z-s)_+ = \max(Z-s, 0)$. Using the above definition, the optimization (2.5) becomes:

$$\begin{aligned}
&\min_{s,c_t,\mathbf{x}_t}\ c_t \\
\text{s.t.}\quad & s + \tfrac{1}{1-\beta}E_t\big[(L_{t,t+\tau}-s)_+\big] \le 0, \\
& E_t\big[ROC_{t,t+\tau}\big] \ge ROC^{\beta}, \\
& \mathbf{1}^T\mathbf{x}_t = 1,\ \ \mathbf{x}_t \ge \mathbf{0},\ \ c_t \ge 0.
\end{aligned} \tag{2.6}$$

There are different potential strategies for reformulating the solvency constraint. The traditional method used in the literature is based on approximating the above conditional expectation with a Monte-Carlo type estimator and transforming (2.6) into a Linear Programming (LP) problem. Indeed, under a fully non-parametric prescription, the $CVaR$ constraint can be rewritten as:

$$s + \frac{1}{m(1-\beta)}\sum_{j=1}^{m}\big(Y_{t+\tau}(j) - (p_t + c_t)\mathbf{R}_{t+\tau}^T(j)\mathbf{x}_t - s\big)_+ \le 0,$$

which can be further reformulated as a system of linear inequalities by introducing m additional decision variables (e.g., see Rockafellar and Uryasev, 2000). Despite the attractiveness of having the LP representation, the implementation of (2.6) with standard solvers becomes less efficient when the number of Monte-Carlo paths is large, since the dimension of the problem increases with m. Therefore, alternative convex approximations for the conditional expectation should be investigated to accommodate such scenarios. For example, Alexander *et al.* (2006) use a continuously differentiable piecewise quadratic approximation. As in the RP-constrained optimization case, we propose here a semiparametric approach which reformulates (2.6) as:

$$\begin{aligned} &\min_{s,c_t,\mathbf{x}_t} \; c_t \\ \text{s.t.}\quad & s + \frac{1}{m(1-\beta)} \sum_{j=1}^{m} E_t^{(j)}\left[(Y_{t+\tau} - (p_t + c_t)\mathbf{R}_{t+\tau}^T(j)\mathbf{x}_t - s)_+\right] \leq 0, \\ & E_t\left[ROC_{t,t+\tau}\right] \geq ROC^{\beta}, \\ & \mathbf{1}^T\mathbf{x}_t = 1, \;\; \mathbf{x}_t \geq \mathbf{0}, \;\; c_t \geq 0. \end{aligned} \tag{2.7}$$

A sufficient condition which ensures the convexity of (2.7) is that $E_t^{(j)}\left[\left(Y_{t+\tau} - (p_t + c_t)\mathbf{R}_{t+\tau}^T(j)\mathbf{x}_t - s\right)_+\right]$ is a convex function in s, c_t and $\mathbf{x}_t$. This issue is discussed in Section 3, once the liability r.v. is fully specified.

2.3 Optimization with EPD constraint

The EPD concept was introduced by Butsic (1994) as an alternative method to the ruin probability for measuring insolvency risk, and constitutes a useful tool in establishing the US RBC system. EPD is defined as the expected loss in the event of insolvency, and thus, it is similar to the ES concept (for a detailed discussion, see Cummins and Phillips, 2009). Translating this definition into our setting, we write:

$$EPD(L_{t,t+\tau}) = E_t\left[(Y_{t+\tau} - (p_t + c_t)\mathbf{R}_{t+\tau}^T\mathbf{x}_t)_+\right]. \tag{2.8}$$

The solvency constraint based on this measure can be constructed by imposing a maximum allowance level for EPD. However, since an a priori choice of such threshold is not straightforward and it depends on the insurer expected liability, we introduce a solvency criteria based on a target level for the deficit ratio. Consequently, the EPD constraint is defined as:

$$\frac{EPD\left(L_{t,t+\tau}\right)}{E_t\left[Y_{t+\tau}\right]} \leq f.$$

Here, f is the maximum level for the EPD ratio with $0 \leq f < 1$. Since (2.8) contains a similar expectation term as in the $CVaR$ definition, the discussion on dealing with the $CVaR$-constrained problem applies here as well. For consistency, we only give the semiparametric representation for our EPD-constrained optimization problem:

$$\begin{aligned} &\min_{c_t,\mathbf{x}_t} c_t \\ \text{s.t.}\quad &\frac{1}{m}\sum_{j=1}^{m} E_t^{(j)}\big[(Y_{t+\tau}-(p_t+c_t)\mathbf{R}_{t+\tau}^T(j)\mathbf{x}_t)_+\big] \le fE_t\big[Y_{t+\tau}\big], \\ &E_t\big[ROC_{t,t+\tau}\big] \ge ROC^f, \\ &\mathbf{1}^T\mathbf{x}_t = 1,\;\; \mathbf{x}_t \ge \mathbf{0},\;\; c_t \ge 0. \end{aligned} \tag{2.9}$$

The convexity of the solvency constraint in (2.9) will be discussed in the same manner as in the $CVaR$ case in Section 3.

3 Modelling assets and liabilities

MV-GARCH processes are probably the most popular tools for modelling the variances and covariances of different assets in discrete time. Depending of the conditional covariance matrix structure, a large number of MV-GARCH models have been proposed in the literature. We consider here the class of DCC-GARCH processes of Engle (2002), for modelling the multivariate dynamic of the log-return process. Due to their relative simple estimation procedure, the DCC framework is also convenient for large scale risk management problems.

We assume that the vector of asset log-returns are observed at a higher frequency than solvency is observed. In particular, we sample returns on a daily basis:

$$\log \mathbf{R}_{t+1} = \mathbf{m}_{t+1} + \boldsymbol{\varepsilon}_{t+1}, \qquad \boldsymbol{\varepsilon}_{t+1}|\mathcal{F}_t \sim \mathbf{MVN}(\mathbf{0}, H_{t+1}). \tag{3.1}$$

Here, $\mathbf{m}_{t+1}$ is the n-dimensional $\mathcal{F}_t$-measurable conditional mean log-return vector and $\boldsymbol{\varepsilon}_{t+1} = (\varepsilon_{1,t+1},\ldots,\varepsilon_{n,t+1})^T$ has a conditionally multivariate Gaussian distribution with mean $\mathbf{0}$ and covariance matrix H_{t+1}.

One of the main features of the DCC structure is that it allows for separate dynamics for the individual conditional variances and the time-varying conditional correlation matrix. In the following, we briefly illustrate

Engle's (2002) formulation:

$$H_{t+1} = D_{t+1}^{1/2}\Sigma_{t+1}D_{t+1}^{1/2}, \tag{3.2}$$

$$D_{t+1} = diag(h_{1,t+1},\ldots,h_{n,t+1}), \tag{3.3}$$

$$\Sigma_{t+1} = diag(q_{11,t+1}^{-1/2},\ldots,q_{nn,t+1}^{-1/2})Q_{t+1}diag(q_{11,t+1}^{-1/2},\ldots,q_{nn,t+1}^{-1/2}), \tag{3.4}$$

$$Q_{t+1} = (1-\theta_1-\theta_2)\bar{Q}+\theta_1\mathbf{u}_t\mathbf{u}_t^T+\theta_2 Q_t. \tag{3.5}$$

Here, D_{t+1} is the $n \times n$ diagonal matrix formed with the univariate conditional variances which are assumed to follow a standard GARCH(1,1) process as below:

$$h_{i,t} = \omega_i + \alpha_i\varepsilon_{t-1}^2 + \beta_i h_{i,t-1}, \qquad i = 1,\ldots,n \tag{3.6}$$

The time-varying conditional correlation matrix of $\mathbf{R}_{t+1}$ is denoted by Σ_{t+1} and its elements $\rho_{ij,t+1}$ are of the form $\rho_{ij,t+1} = q_{ij,t+1}q_{ii,t+1}^{-1/2}q_{jj,t+1}^{-1/2}$, for any $1 \le i,j \le n$; $q_{ij,t+1}$ are the elements of Q_{t+1} and are assumed to follow another GARCH(1,1) dynamic given in (3.5). The process $\mathbf{u}_t$ represents the $n \times 1$ vector of devolatilized, but correlated innovations (i.e. $u_{i,t} = h_{i,t}^{-1/2}\varepsilon_{i,t}$) and $\bar{Q}$ is the unconditional covariance matrix of $\mathbf{u}_t$. We assume that all univariate GARCH parameters in (3.6), ω_i, α_i and β_i, and the DCC parameters in (3.5), θ_1, θ_2, satisfy the conditions required for covariance stationarity, and positive definiteness of H_{t+1}, for any t.

In order to investigate the effect of the time-varying conditional correlations between the portfolio's assets, we shall also look at two other models, which can be viewed as particular cases of the DCC-GARCH. The first one is the Conditional Constant Correlation (CCC) model of Bollerslev (1990) that can be obtained by replacing the time-varying correlation matrix by a symmetric positive definite matrix with constant elements (i.e. $\Sigma_t = \Sigma$). The second alternative analyzed assumes the assets are uncorrelated and this is immediately obtained by letting $\Sigma_t = I_n$ in (3.5), where I_n is the $n \times n$ identity matrix. In our numerical applications, we call these models the CCC-GARCH and the UNI-GARCH, respectively.

Historical data for modelling claim amounts is commonly fitted using one-component parametric distributions such as, Pareto, Lognormal, Gamma, Weibull etc., or more recently using composite distributions (see for example, Scollnik and Sun, 2012, and the references therein for Pareto composite models). Since the objective of the paper is not to investigate goodness-of-fit of different alternatives, we restrict our attention only to a one parametric family. In particular, we consider that claims are modelled with a Lognormal distribution. Since our semiparametric method requires the computations of various conditional expectations given historical information on the asset evolutions, we further assume in our numerical examples that $Y_{t+\tau}$ is independent of the enlarged filtration $\mathcal{F}_t \bigcup \sigma(\mathbf{R}_{t+\tau})$, for any time t and a given solvency horizon τ. Although this allows us for a more convenient implementation, the optimization problems can be solved under more general dependence structures between assets and liabilities, as long as the resulting constraints are convex. Therefore, we let:

$$Y_{t+\tau} \sim \mathbf{LGN}(\mu_{t+\tau}, \sigma_{t+\tau}). \tag{3.7}$$

The model parameters are assumed to be time-dependent as they will be re-estimated using a double rolling-window exercise. In the remainder of this section, we discuss the convexity of the solvency constraints under the lognormality assumption from (3.7).

First, we let $\mathbf{z}_t = (p_t + c_t)\mathbf{x}_t$ in all three optimization problems (2.4), (2.7) and (2.9). With this notational change, the new decision variables are c_t and $\mathbf{z}_t$, and the budget constraint becomes $\mathbf{1}^T\mathbf{z}_t = p_t + c_t$ with $\mathbf{z}_t \geq \mathbf{0}$. Under the above assumption, the solvency constraint for the EPD problem can be rewritten as:

$$\frac{1}{m}\sum_{j=1}^{m} E\big[(Y_{t+\tau} - \mathbf{R}^T_{t+\tau}(j)\mathbf{z}_t)_+\big] \leq fE\big[Y_{t+\tau}\big].$$

A sufficient condition for convexity is that $E\big[(Y_{t+\tau} - \mathbf{R}^T_{t+\tau}(j)\mathbf{z}_t)_+\big]$ is convex for any $j = 1, \ldots, m$. We notice that the quantity under the expectation represents the payoff of a European Call option written on $Y_{t+\tau}$. Under the lognormality assumption of $Y_{t+\tau}$, we can write the above expectation as the present value at maturity of a Black-Scholes Call price, $e^{rT}BS(S, K, T, \sigma, r)$, with the following parameter matching:

$$S = 1, \quad K = \mathbf{R}^T_{t+\tau}(j)\mathbf{z}_t, \quad T = 1, \quad \sigma = \sigma_{t+\tau}, \quad r = \mu_{t+\tau} + \frac{\sigma^2_{t+\tau}}{2}.$$

Thus, the solvency constraint can be reformulated as:

$$\sum_{j=1}^{m} \Big[\exp\big(\mu_{t+\tau} + \frac{\sigma_{t+\tau}^2}{2}\big)\Phi\Big(\frac{-\log(\mathbf{R}_{t+\tau}^T(j)\mathbf{z}_t) + \mu_{t+\tau} + \sigma_{t+\tau}^2}{\sigma_{t+\tau}}\Big) - \mathbf{R}_{t+\tau}^T(j)\mathbf{z}_t\Phi\Big(\frac{-\log(\mathbf{R}_{t+\tau}^T(j)\mathbf{z}_t) + \mu_{t+\tau}}{\sigma_{t+\tau}}\Big)\Big] \leq b. \quad (3.8)$$

Here, $b = fm \exp\big(\mu_{t+\tau} + \frac{\sigma_{t+\tau}^2}{2}\big)$ and $\Phi(\cdot)$ is the cumulative distribution function of a standard Gaussian random variable. The convexity of (3.8) follows now from the convexity property of the European Call price with respect to the strike price, which is itself an affine function of $\mathbf{z}_t$.

Using a similar prescription, we can show the convexity of the $CVaR$ optimization problem (2.7) based on the Black-Scholes formula. The only difference consists of having a different strike price $K = \mathbf{R}_{t+\tau}^T(j)\mathbf{z}_t + s$, which is a linear function of the decision variables s and $\mathbf{z}_t$.

We now turn our attention to the RP problem (2.4). The solvency constraint is equivalent to:

$$\frac{1}{m}\sum_{j=1}^{m} \Phi\Big(\frac{-\log(\mathbf{R}_{t+\tau}^T(j)\mathbf{z}_t) + \mu_{t+\tau}}{\sigma_{t+\tau}}\Big) \leq 1 - \alpha. \quad (3.9)$$

Since the standard Gaussian c.d.f. is convex only on its negative domain, a sufficient condition for the convexity of (3.9) is the following:

$$\min_{1 \leq j \leq m} \mathbf{R}_{t+\tau}^T(j)\mathbf{z}_t \geq \exp \mu_{t+\tau}. \quad (3.10)$$

Thus, according to condition (3.10), the convexity of (2.4) is satisfied when the terminal value of the total assets investment in the worst case scenario is greater than the median of the liability distribution. Although this requirement cannot be verified analytically as in the previous two cases, our numerical simulations from Section 4 indicate that (3.10) is never violated.

4 Empirical Analysis

In this section, we investigate the empirical performance of three MV-GARCH models for all optimization problem considered. We provide two main numerical experiments. Firstly, for a specified solvency target, we construct efficient frontiers for (2.4), (2.7) and (2.9) by varying the expected ROC. Secondly, the out-of-sample performance analysis is carried out through a detailed double rolling window estimation exercise.

4.1 Data Description

We consider two 3-asset portfolios. The first portfolio consists of NASDAQ and NYSE Composite indices, and the 3-month US T-Bills, while the second is formed with the S&P 500 Index and two exchange-traded funds (ETF): the iShares Barclays 1-3 Year Treasury Bond (SHY) and the iShares iBoxx $ Investment Grade Corporate Bond (LQD).

The data is recored on a daily basis from January 3, 2005 to July 29, 2011 for a total of $l = 1,656$ observations. Descriptive statistics are provided in Table 4.1.1.

Table 4.1.1: Descriptive statistics for NASDAQ, NYSE, S&P 500 Index, SHY and LQD log-returns from January 3, 2005 - July 29, 2011 for a total of 1,656 observations.

Index	Min	Max	Mean	Std	Skewness	Kurtosis
NASDAQ	-0.0959	0.1116	0.0001	0.0149	-0.1670	10.2725
NYSE	-0.1023	0.1153	0.0001	0.0150	-0.3480	12.7329
S&P 500	-0.0947	0.1096	0.0000	0.0142	-0.2565	13.4443
SHY	-0.0066	0.0071	0.0001	0.0011	-0.1685	7.1727
LQD	-0.0956	0.0932	0.0002	0.0062	-1.1486	82.3485

We notice that there are no significant differences between the three stock index series relative to the first two moments. The NYSE log-returns exhibit a more pronounced negative skewness, while S&P 500 has the highest kurtosis of the three. However, the Corporate Bond ETF displays a very high kurtosis and is more negatively skewed than the stock indexes. Since the T-Bill will not be modelled stochastically, its descriptive statistics are not illustrated in Table 4.1.1.

The data set is divided into two samples: Sample A consists of $l_A = 1,259$ daily observations for a 5-year period from January 3, 2005 through December 31, 2009, and it is used for the in-sample estimation and analysis of the efficient frontiers. Sample B, which covers the period from January 1, 2010 through July 29, 2011 with $l_B = 397$ daily points, is used for testing the out-of-sample performance in the rolling window exercise. The most significant part of the recent financial crisis period is included in Sample A.

For liabilities, we use a data set on property insurance claim amounts provided by a European Union-based insurance company for the same period used in the assets case. However, the main difference is that the sampling frequency is different from the one used for assets. There are 79 observations representing aggregate monthly claim amounts, which are divided into two samples according to a similar prescription (i.e. Sample A' consist of $l_{A'} = 60$

monthly observations and Sample B' has $l_{B'} = 19$ data points which are used for the out-of-sample comparison). The main characteristics of the entire sample are illustrated in Table 4.1.2.

Table 4.1.2: Descriptive statistics for monthly claim amounts from January 3, 2005 - July 29, 2011 for a total of 79 observations (figures are in thousands €).

Min	Max	Mean	StDev	Skewness	Kurtosis
8.2465	2049.2119	603.2802	375.1311	1.2434	5.5068

4.2 Estimation results

We first estimate the parameters for the asset returns. For the first portfolio we estimate a bivariate GARCH structure for NASDAQ and NYSE, while for the second portfolio we estimate a multivariate DCC-GARCH based on S&P 500, SHY and LQD. There are various ways which one can specify the conditional mean vector in the MV-GARCH log-return equation (3.1). For example, Rombouts and Stentoft (2011) use a multivariate risk premium specification for $\mathbf{m}_t$ when pricing options under a DCC-GARCH model, while Hlouskova *et al.* (2009) consider an autoregressive structure for deriving multistep predictions with applications in risk management. Since our objective is to analyze the conditional correlation effect on the optimal capital and its allocation, we perform our estimation ignoring the mean effect.

The estimation procedure follows the two-stage Maximum Likelihood Estimator (MLE) algorithm proposed by Engle and Sheppard (2001). In the first stage, the univariate GARCH parameters are estimated by replacing the conditional correlation matrix of $\mathbf{R}_k$, Σ_k, with the identity matrix in the log-likelihood function below:

$$\log L = -\frac{1}{2}\sum_{k=1}^{l}\left(\log(|H_k|) + \boldsymbol{\varepsilon}_k' H_k^{-1}\boldsymbol{\varepsilon}_k\right),$$

where l represents the number of observations in the dataset. Given the parameters estimated in the first stage, the DCC and CCC parameters are estimated based on the correct log-likelihood specification with Σ_k and Σ, respectively. Thus, at the second stage only θ_1 and θ_2 for DCC, and ρ for CCC are estimated. The results are reported in Table 4.2.1 for Portfolio 1 and in Table 4.2.2 for the second portfolio.

According to all three selection criteria, the DCC specification is preferred to the CCC one. The parameter estimates for the DCC-GARCH are in the same range with the values obtained in other previous studies. Each

Table 4.2.1: Parameter estimates (with corresponding asymptotic variances reported the brackets) using log-returns for NASDAQ and NYSE during January 3, 2005 - July 29, 2011 for a total of 1656 observations. AIC and BIC are the Akaike and Bayesian Information Criteria.

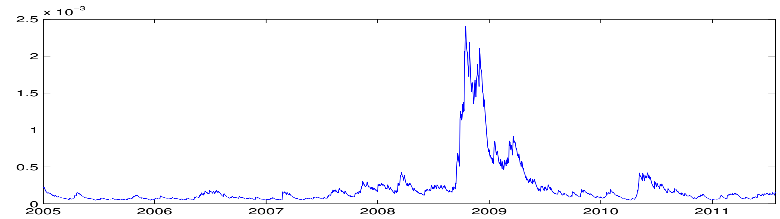
Estimation Stage	Model parameters				Selection criteria		
	Index	ω	α	β			
1	NASDAQ	2.0E-06 (5.06E-13)	0.0736 (1.35E-04)	0.9146 (1.40E-04)			
	NYSE	1.4E-06 (3.99E-13)	0.0856 (1.46E-04)	0.9061 (1.41E-04)			
	Covariance model						
2	DCC	θ_1 0.0432 (5.58E-05)	θ_2 0.9409 (9.28E-05)		log L 11,630	AIC -23,245	BIC -23,201
	CCC	ρ_{12} 0.9061 (1.64E-05)			log L 11,574	AIC -23,134	BIC -23,096

Table 4.2.2: Parameter estimates (with corresponding asymptotic variances reported the brackets) using log-returns for S&P 500, SHY and LQD during January 3, 2005 - July 29, 2011 for a total of 1656 observations. AIC and BIC are the Akaike and Bayesian Information Criteria.

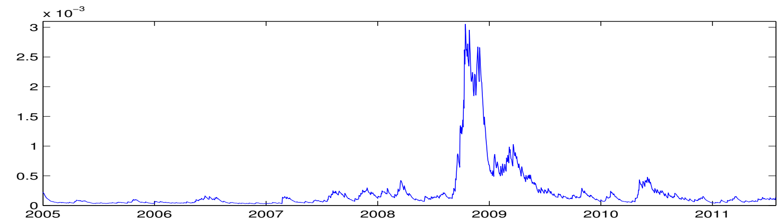
Estimation Stage	Model parameters				Selection criteria		
	Index	ω	α	β			
1	S&P 500	1.4E-06 (4.02E-13)	0.0842 (1.48E-04)	0.9055 (1.55E-04)			
	SHY	1.0E-13 (1.18E-17)	0.0286 (9.23E-07)	0.9713 (2.98E-06)			
	LQD	6.0E-07 (1.16E-13)	0.1942 (1.24E-02)	0.8052 (7.21E-03)			
	Covariance model						
2	DCC	θ_1 0.0325 (1.65E-04)	θ_2 0.9564 (4.64E-04)		log L 21,607	AIC -43,193	BIC -43,133
	CCC	ρ_{12} -0.3192 (7.93E-04)	ρ_{13} -0.0285 (1.04E-03)	ρ_{23} 0.5183 (1.76E-03)	log L 21,458	AIC -43,892	BIC -42,828

univariate series is characterized by a high degree of persistence (e.g., $\alpha+\beta = 0.988$ for NASDAQ, and $\alpha+\beta = 0.9994$ for LQD). However, the volatility clustering effect is less pronounced in the Corporate Bond series than in the others (i.e. smaller value of β). A similar persistence can be observed in the conditional correlation dynamic, since $\theta_1 + \theta_2 = 0.984$ for the assets in the first portfolio, while $\theta_1 + \theta_2 = 0.989$ in the second portfolio. The value of $\rho_{12} = 0.91$ in the CCC case for NASDAQ and NYSE suggests a high degree of positive correlation over the considered period. However, this is no longer the case of the 3-asset portfolio. For example, there is almost no CCC-GARCH implied correlation between the stock index and the Corporate Bond fund ($\rho_{13} = -0.0285$) and the two ETFs are moderately positively correlated ($\rho_{23} = 0.5183$). The implied GARCH conditional variances and DCC-conditional correlation are illustrated in Figures 4.2.1 and Figures 4.2.2.

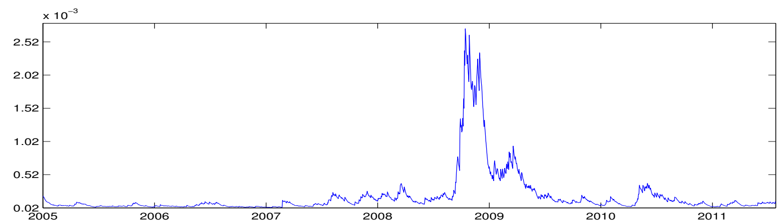
Next, we use the MLE to fit a Lognormal distribution on the monthly claim amounts for the same period. The results are reported in Table 4.2.3. The Kolmogorov-Smirnov test indicates that a Lognormal distribution cannot be rejected at 5% significance level.



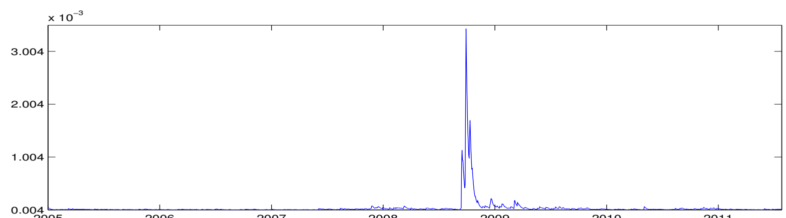
(a) Conditional variance for NASDAQ



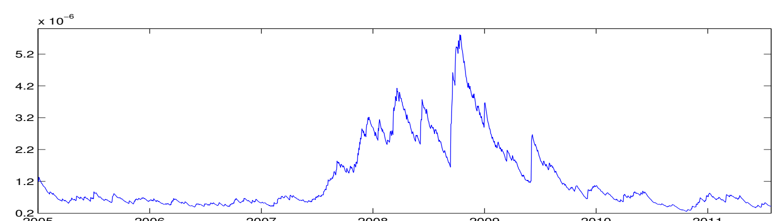
(b) Conditional variance for NYSE



(c) Conditional variance for S&P 500

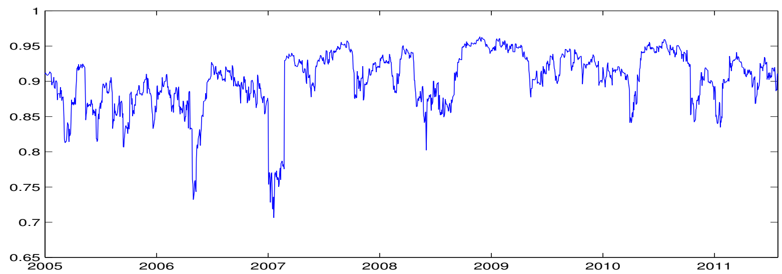


(d) Conditional variance for SHY

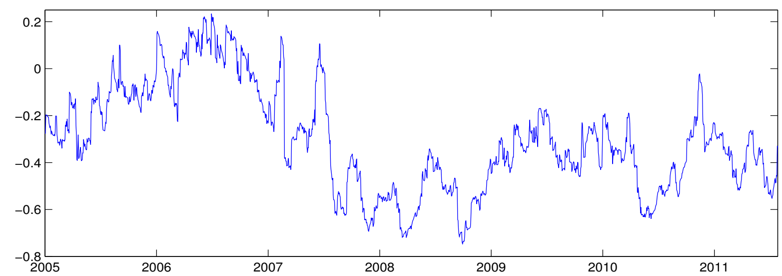


(e) Conditional variance for LQD

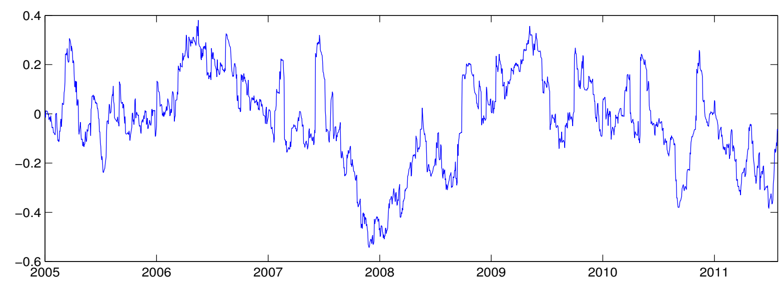
Figure 4.2.1: Conditional variances for the DCC-GARCH models based on the MLE estimates over the period January 3, 2005 - July 29, 2011 for a total of 1656 observations.



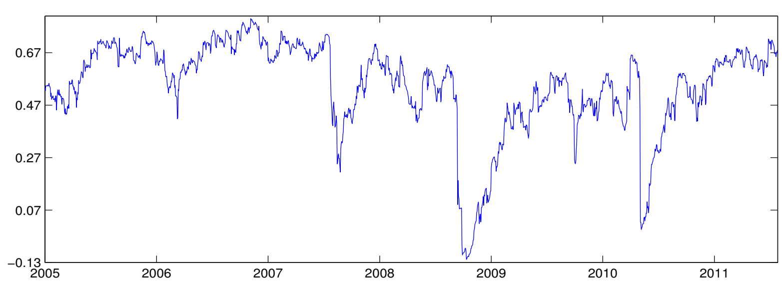
(a) Conditional correlation NASDAQ and NYSE



(b) Conditional correlation S&P 500 and SHY



(c) Conditional correlation S&P 500 and LQD



(d) Conditional correlation SHY and LQD

Figure 4.2.2: Conditional correlations for the DCC-GARCH models based on the MLE estimates over the period January 3, 2005 - July 29, 2011 for a total of 1656 observations.

Table 4.2.3: Parameter estimates (with corresponding asymptotic variances reported the brackets) for Lognormal distribution using monthly claim amounts for property insurance during January 3, 2005 - July 29, 2011 for a total of 79 observations. KStest stands for the Kolmogorov-Smirnov test and its p-value is reported in the brackets.

$\hat{\mu}$	$\hat{\sigma}$	Log L	KStest
6.160460	0.829457	584.00	0.1297
(0.0087)	(0.0044)		(0.1279)

4.3 Implementation of solvency constrained optimization

All three optimization problems (2.4), (2.7) and (2.9), combined with the convex reformulations for the solvency constraints from Section 3, are implemented using Matlab's non-linear optimization routine `fmincon` based on interior-point algorithms. The solvency targets are fixed as follows: $\alpha = 99.5\%$ (the standard value imposed by Solvency II) for the RP-constrained problem, $\beta = 99\%$ (the standard value imposed by SST) for the $CVaR$-constrained problem and $f = 0.25\%$ (arbitrarily chosen) for the EPD-constrained problem.[7] Since losses are sampled on a monthly basis, we let the solvency horizon $\tau = 21$ days. Given all the information up to time t, each optimization is implemented according to the following algorithm:

1. Estimate the asset and liability parameters according to the methodology described in Section 4.2.

2. Compute the insurance premiums using the expected premium principle, so $p_t = (1+\eta)E[Y_{t+\tau}]$, where η is the relative security loading factor fixed at 0.1.

3. Generate $m = 10,000$ Monte-Carlo paths, $\mathbf{R}_{t+\tau}(j),\ j = 1,\ldots,m$, for the asset returns, according to the corresponding covariance structure from equations (3.1)-(3.5); for the T-Bill rate, we use the three month rate corresponding to the period $[t, t+\tau]$.

4. Solve each optimization problem and find the optimal capital required c_t^*, and the optimal portfolio allocation $(x_{i,t}^*,\ i = 1,\ldots,3)$.

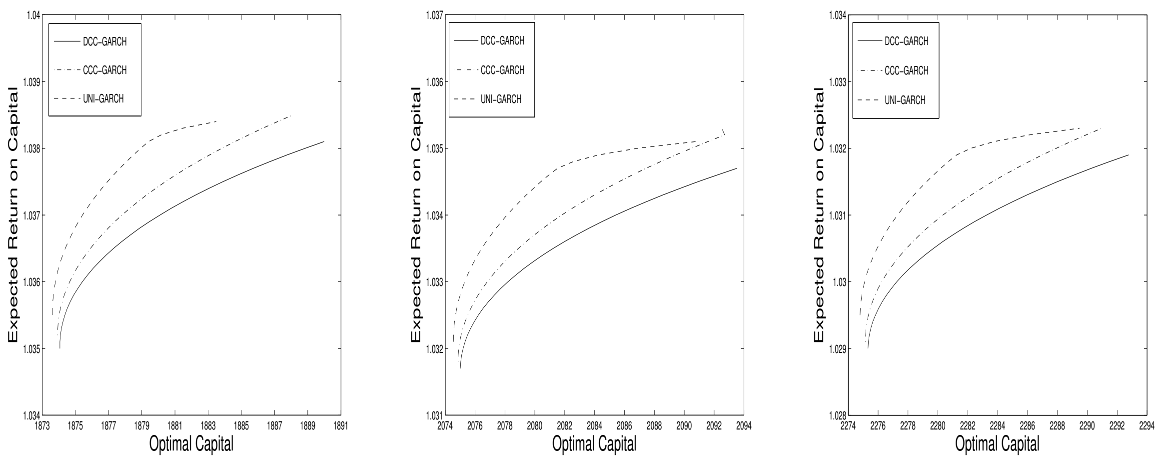
Different choices for the number of Monte-Carlo paths used in the scenario generation step have been discussed in Asimit *et al.* (2012). When liabilities are Pareto distributed, they showed that the semiparametric approximation implemented with Matlab provides very close solutions to the ones obtained via an SOC representation implemented

[7]Unlike in the RP and $CVaR$ cases, there is no generally recommended threshold for the EPD ratio. We only chose $f = 0.25\%$ in order to make the optimal capital requirements comparable to those obtained in the other two optimization problems. This can be achieved by computing EPD ratios based on the optimal solutions for the RP and $CVaR$ constrained problems.

in Mosek when $m = 10,000$. Moreover, in the Gaussian case, this solution converges to the theoretical one.[8]

4.3.1 Efficient Frontier Analysis

Efficient frontiers are constructed only for Portfolio 1, by running the above algorithm for different targets for the expected return on capital. The minimum levels for the expected ROC are obtained by solving the unconstrained version of each optimization (i.e. we discard the performance measure constraint). All parameters are estimated from samples A and A' data.



(a) RP-constrained optimization (b) $CVaR$-constrained optimization (c) EPD-constrained optimization

Figure 4.3.1: Efficient frontiers for Portfolio 1 based on DCC, CCC and UNI-GARCH models under the RP, $CVaR$ and EPF-constrained problems. Solvency constraints are approximated based on 10,000 Monte-Carlo paths and scenarios are generated based on Sample A estimates.

The behaviours of (c_t^*) and $(x_{i,t}^*,\ i = 1, \ldots, 3)$ are analyzed for all three covariance specifications. First, we plot the efficient frontiers in Figure 4.3.1. We notice that all efficient frontiers are smooth for all three optimizations. Moreover, the same pattern can be observed for each of the covariance model considered. On the one hand, the DCC-GARCH, which captures the best the correlation dynamic, is the most conservative model in the sense that it requires the highest minimum optimal capital for the same level of expected ROC. On the other hand, c_t^* has the smallest values for UNI-GARCH, as this model totally ignores the strong positive correlation between the two risky assets. The correlation dynamic seems to have a strong impact on the structure of the optimal portfolio. This is depicted in Figure 4.3.2. The optimal allocation into NASDAQ increases with the expected ROC level for all

[8] In an unreported numerical exercise, we tested the accuracy of the Monte-Carlo approximation when $m = 10,000$ for the $CVaR$ and EPD problems; our results suggested that the standard errors of the optimal solutions are in the same range as those obtained in the RP case.

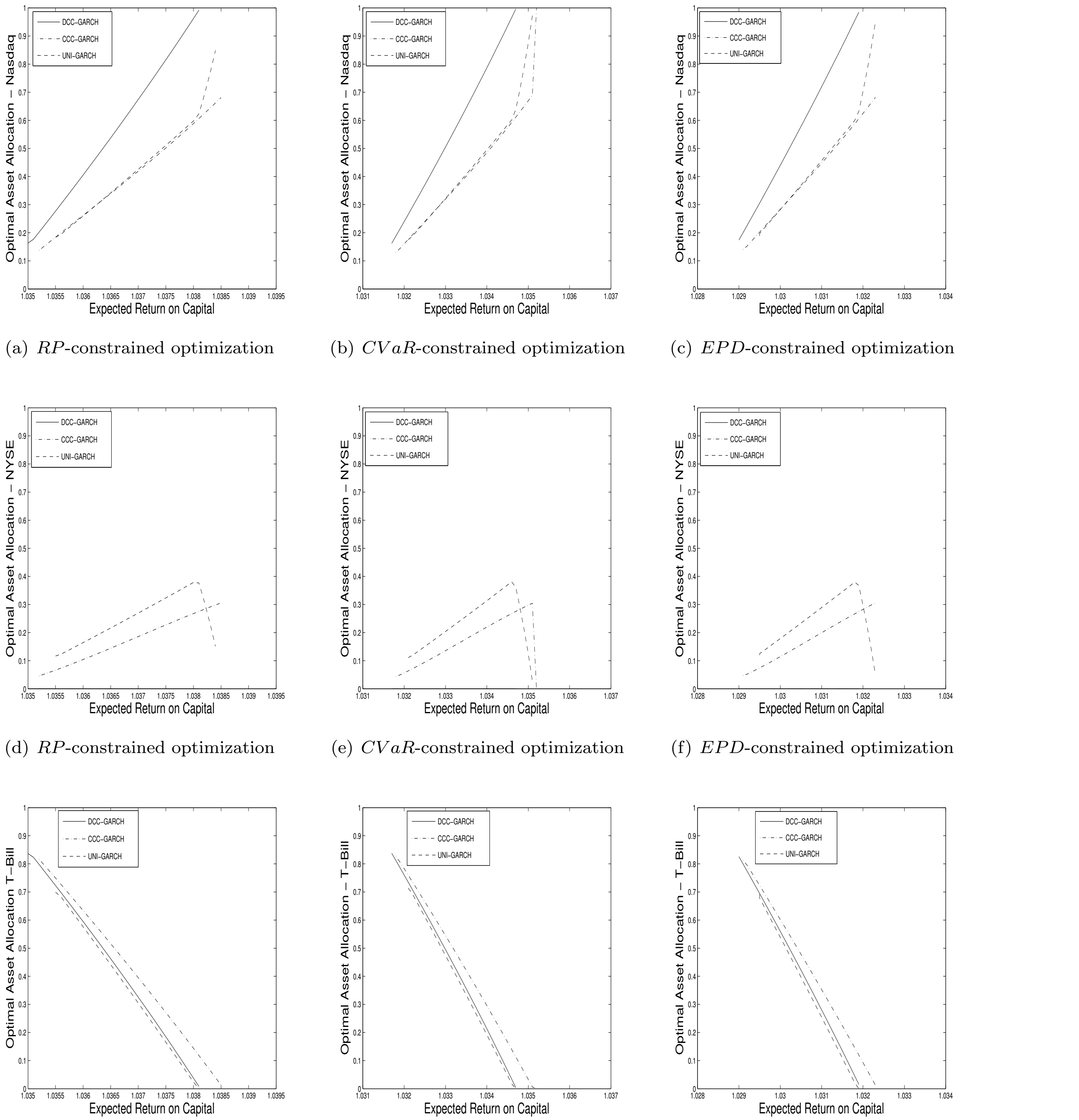
three models and for all of the problems considered, while the optimal allocation in T-Bills decreases in a similar fashion. Indeed, when no expected ROC is imposed, the optimal allocations are around 20% in NASDAQ, and 70% (UNI) and 80% (CCC and DCC) in T-Bills. When the shareholders' expected ROC approaches its maximum feasible value, the optimal portfolios are constructed based almost solely on the NASDAQ index. Interestingly, the optimal allocation in the riskiest asset (NYSE) is almost zero for the DCC-GARCH, while for the other two dynamics it first increases until a maximum is reached, and after that decreases approximately to zero as well.

4.3.2 Out-of-Sample Performance

In this section, we carry out an out-of-sample analysis for the optimal portfolios based on RP, $CVaR$ and EPD constraints. Our approach is similar to the standard rolling window methodology proposed in the portfolio optimization literature (e.g., see Santos *et al.*, 2012). However, since we have two main sources of risk, we further propose and analyze the effect of a double rolling window estimation on our optimal solutions.

We set the length of the rolling window $l_A = 1,259$ for the estimation of asset returns and $l_{A'} = 60$ observations for liability estimation. First, we compute the optimal solutions $(c_t^*, \mathbf{x}_t^*)$ for period $[t, t+\tau]$ using data from Samples A and A'. Next, we construct a new sample for assets by dropping the first $\tau = 21$ observations from Sample A and adding the same number of data points from Sample B. This corresponds to a monthly portfolio rebalancing. Similarly, we construct the new sample for liabilities by discarding the first observation from Sample A' and adding the first observation from Sample B'. With this new data set, we recompute the next period optimal solutions $(c_{t+\tau}^*, \mathbf{x}_{t+\tau}^*)$ based on Step 1 - 4. We repeat this sampling procedure and the corresponding optimization steps until the end of Sample B/Sample B' is reached. In other words, we have computed $(c_{t+(k-1)\tau}^*, \mathbf{x}_{t+(k-1)\tau}^*)$, with $k = 1, \ldots, l_{B'}$, optimal solutions for each solvency constrained problem and MV-GARCH model. In order to avoid potential feasibility issues created by the expected return on capital constraint under liability re-estimation, all optimization problems are implemented without a lower bound for the expected ROC. The results illustrated in Figures 4.3.3-4.3.5.

Figure 4.3.3 plots the evolution of the total optimal assets invested $(p_t + c_t^*)$ for the two portfolios. First, we notice that there are no significant differences among the covariance models for all optimization problems. The portfolio choice does not seem to affect the value of the assets invested. However, there is a large variation in the optimal capital required over the rolling window, and this is mainly caused by the changes in the liability



(a) RP-constrained optimization (b) $CVaR$-constrained optimization (c) EPD-constrained optimization

(d) RP-constrained optimization (e) $CVaR$-constrained optimization (f) EPD-constrained optimization

(g) RP-constrained optimization (h) $CVaR$-constrained optimization (i) EPD-constrained optimization

Figure 4.3.2: Optimal asset allocation for Portfolio 1 for different levels of expected ROC for DCC, CCC and UNI-GARCH models under the RP, $CVaR$ and EPF-constrained problems. Solvency constraints are approximated based on 10,000 Monte-Carlo paths and scenarios are generated based on Sample A estimates.

parameters.[9] The variation in total assets invested is quite large, ranging from 1,887 to 2,728 for RP, 1,986 to 2,949 for $CVaR$ and 1,868 to 3,222 for EPD; thus, the EPD problem has the largest fluctuation, while RP is the smallest.

Figures 4.3.4-4.3.5 suggest that the differences in the optimal portfolio allocations are less pronounced. The portfolio structure also depends on the choice of the MV-GARCH model. For the RP and $CVaR$ Portfolio 1 problems, the variation in optimal allocations for NASDAQ and NYSE are smaller for UNI-GARCH when compared to the DCC and CCC counterparts. We also notice that the largest investment is typically made to the T-Bills, the minimum value of approximately 70% corresponding to the UNI-GARCH for each problem. A similar pattern can be observed in Portfolio 2 where most of the capital is allocated to the lowest risk entity represented by the Treasury Bond ETF.

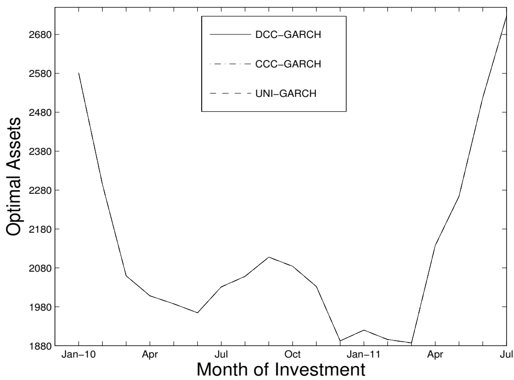
In the remainder of this section, we compute a variety of out-of-sample indicators to provide a comparison between the covariance models relative to the solvency and portfolio performances. In order to measure the solvency requirement performance of the optimal solutions, we consider three metrics: the average assets invested, the average solvency value and the maximum solvency value. All averages are computed over the rolling window period. Depending on the solvency criteria, the average solvency values are computed based on the following expressions:

$$
\begin{aligned}
\widehat{RP} &= \frac{1}{l_{B'}} \sum_{k=1}^{l_{B'}} \Phi\big(d_{t+k\tau}\big), \\
\widehat{CVaR} &= \frac{1}{l_{B'}} \sum_{k=1}^{l_{B'}} \Big(\frac{E[Y_{t+k\tau}]}{1-\beta} \Phi\big(\sigma_{t+k\tau} - \Phi^{-1}(\beta)\big) - \mathbf{R}_{t+k\tau}^T \mathbf{z}_{t+(k-1)\tau}^* \Big), \\
\widehat{EPD} &= \frac{1}{l_{B'}} \sum_{k=1}^{l_{B'}} \Big[E[Y_{t+k\tau}] \Phi\big(d_{t+k\tau} + \sigma_{t+k\tau}^2\big) - \mathbf{R}_{t+k\tau}^T \mathbf{z}_{t+(k-1)\tau}^* \Phi\big(d_{t+k\tau}\big) \Big].
\end{aligned}
$$

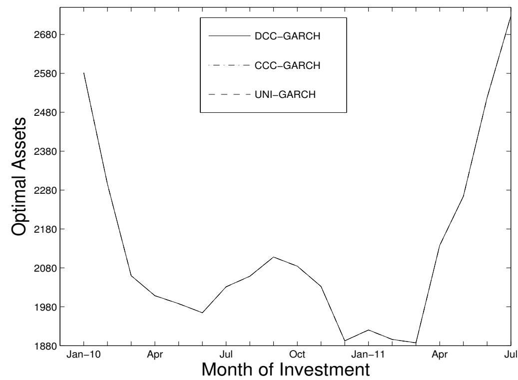
Here,

$$
\begin{aligned}
\mathbf{z}_{t+(k-1)\tau}^* &= (p_{t+(k-1)\tau} + c_{t+(k-1)\tau}^*)\mathbf{x}_{t+(k-1)\tau}^*, \\
d_{t+k\tau} &= \frac{-\log \mathbf{R}_{t+k\tau}^T \mathbf{z}_{t+(k-1)\tau}^* + \mu_{t+k\tau}}{\sigma_{t+k\tau}},
\end{aligned}
$$

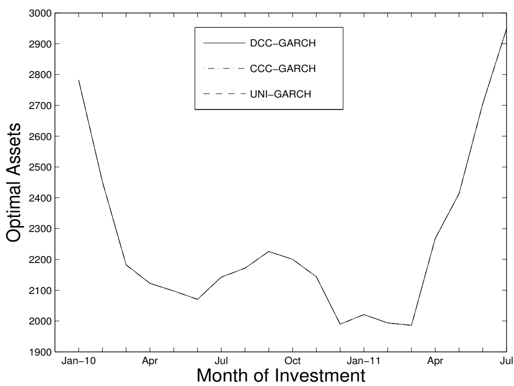
[9] In an unreported simulation exercise, we solved the rolling window optimizations under the assumption of constant liability parameters. Our results showed no significant changes in the optimal required capital for the whole rolling period.



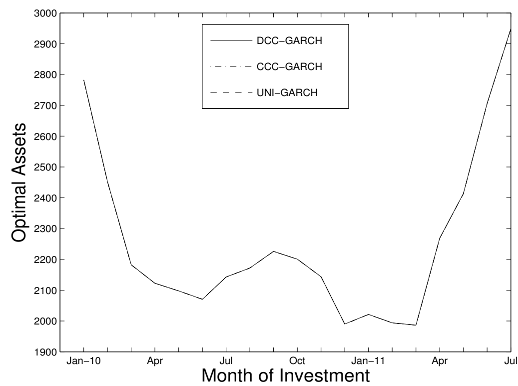
(a) RP-constrained for Portfolio 1



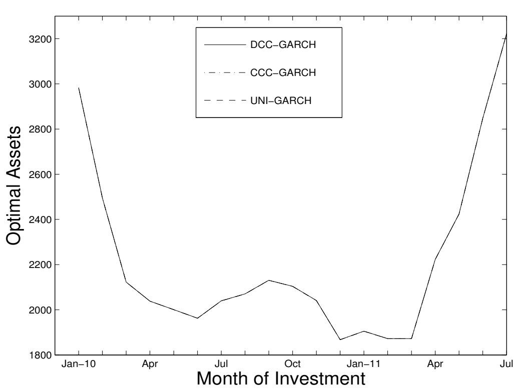
(b) RP-constrained for Portfolio 2



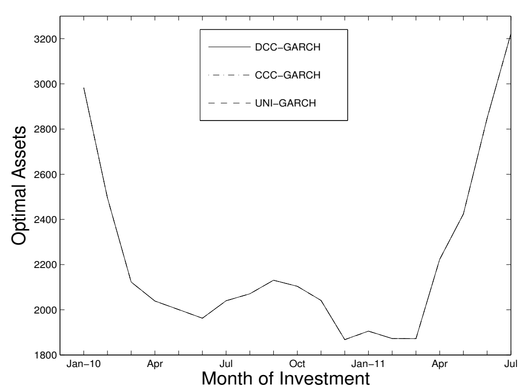
(c) $CVaR$-constrained for Portfolio 1



(d) $CVaR$-constrained for Portfolio 2

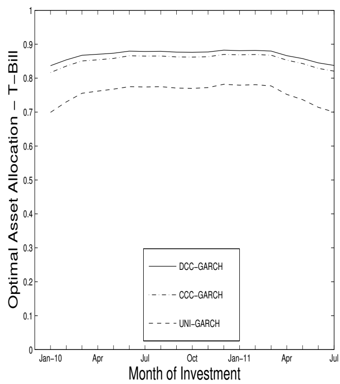


(e) EPD-constrained for Portfolio 1

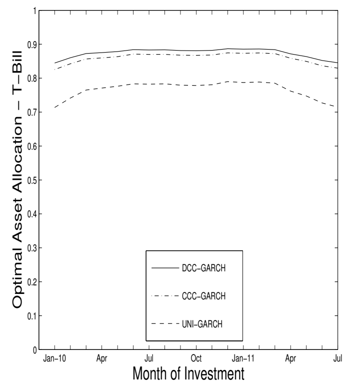


(f) EPD-constrained for Portfolio 2

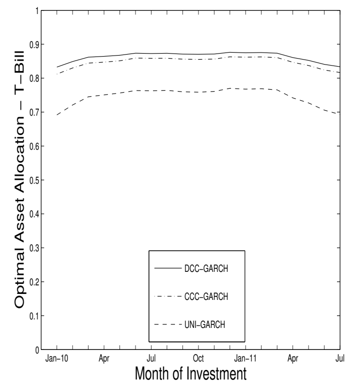
Figure 4.3.3: Optimal total assets invested, $p_t + c_t^*$, for Portfolios 1 and 2.



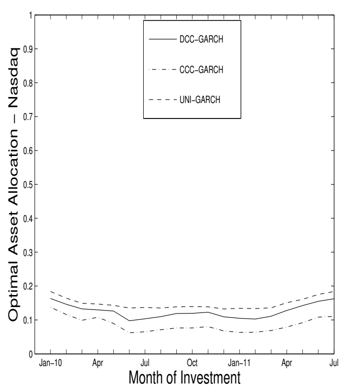
(a) RP-constrained optimization



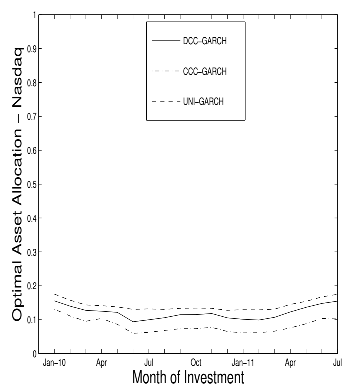
(b) $CVaR$-constrained optimization



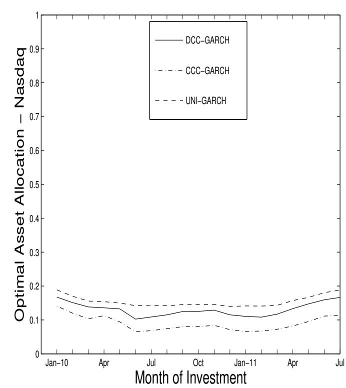
(c) EPD-constrained optimization



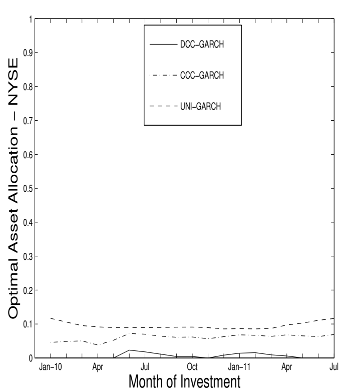
(d) RP-constrained optimization



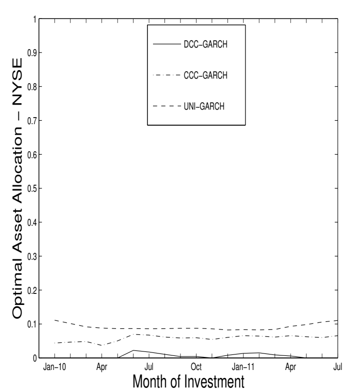
(e) $CVaR$-constrained optimization



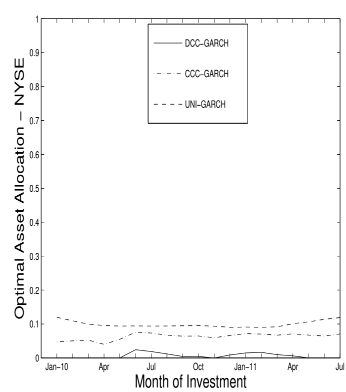
(f) EPD-constrained optimization



(g) RP-constrained optimization

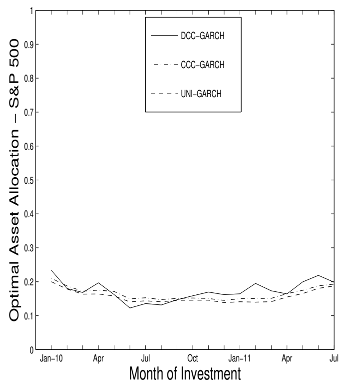


(h) $CVaR$-constrained optimization

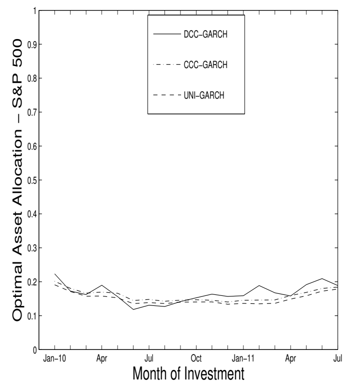


(i) EPD-constrained optimization

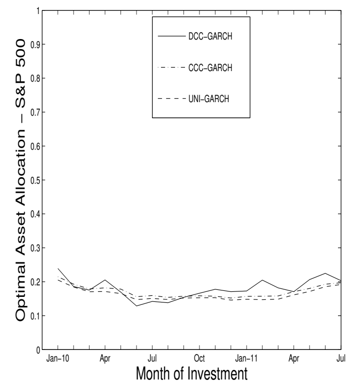
Figure 4.3.4: Optimal asset allocation for Portfolio 1 with double rolling window.



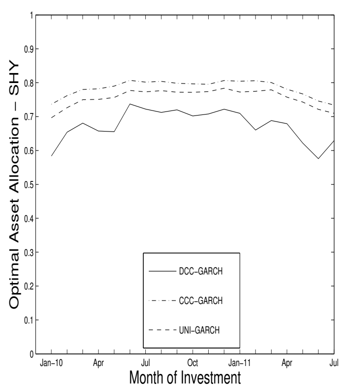
(a) RP-constrained optimization



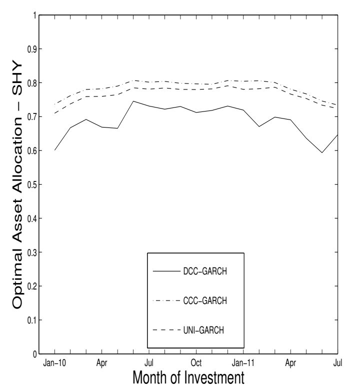
(b) $CVaR$-constrained optimization



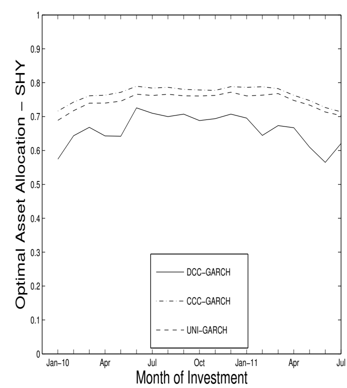
(c) EPD-constrained optimization



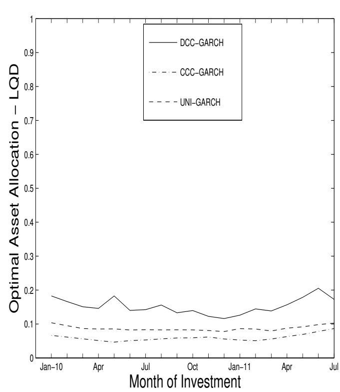
(d) RP-constrained optimization



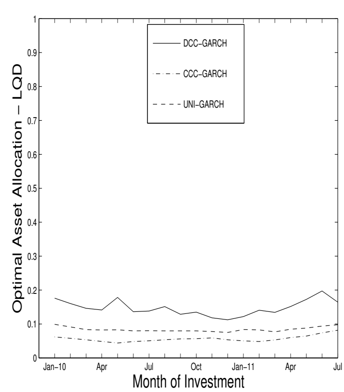
(e) $CVaR$-constrained optimization



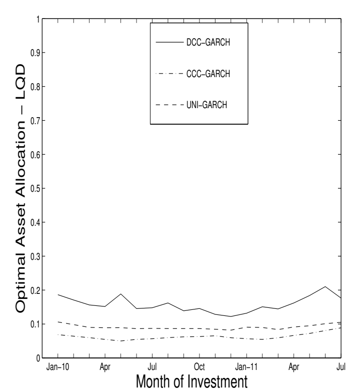
(f) EPD-constrained optimization



(g) RP-constrained optimization



(h) $CVaR$-constrained optimization



(i) EPD-constrained optimization

Figure 4.3.5: Optimal asset allocation for Portfolio 2 with double rolling window.

where $(c^*_{t+(k-1)\tau}, \mathbf{x}^*_{t+(k-1)\tau})$ and $\mathbf{R}_{t+k\tau}$ represents the optimal solution and the gross return vector, respectively, over the period $[t+(k-1)\tau, t+k\tau]$, for any $k = 1, \ldots, l_{B'}$. The average assets invested are calculated by taking averages of all $p_{t+(k-1)\tau} + c^*_{t+(k-1)\tau}$ over the rolling period. The results are reported in the first panel of Tables 4.3.1 and 4.3.2.

Table 4.3.1: Solvency and out-of-sample performance for Portfolio 1.

	Solvency Performance			Portfolio Performance			
	Avg. Assets invested	Avg. Solvency Value	Max. Solvency Value	Avg. AROC (%)	Std. AROC	Sharpe Ratio	Turnover
Problem 1. Ruin Constraint		Avg. Ruin Probability (%)	Max. Ruin Prob (%)				
Covariance Model							
DCC	2129.17	0.497	0.546	5.08	1.46	3.46	0.019
CCC	2129.06	0.497	0.553	5.06	1.56	3.23	0.021
UNI	2128.81	0.496	0.589	5.17	2.25	2.29	0.020
Problem 2. CVaR Constraint		Avg. CVaR	Max. CVaR				
Covariance Model							
DCC	2258.85	-1.773	28.645	4.67	1.38	3.38	0.018
CCC	2258.75	-1.319	32.857	4.65	1.47	3.16	0.020
UNI	2258.48	-2.114	58.813	4.75	2.11	2.24	0.018
Problem 3. EPD Constraint		Avg. EPD Ratio (%)	Max. EPD Ratio (%)				
Covariance Model							
DCC	2222.25	0.248	0.273	4.92	1.67	2.94	0.019
CCC	2222.15	0.249	0.276	4.91	1.77	2.77	0.022
UNI	2221.86	0.248	0.294	5.03	2.46	2.04	0.019

Table 4.3.2: Solvency and out-of-sample performance for Portfolio 2.

	Solvency Performance			Portfolio Performance			
	Avg. Assets invested	Avg. Solvency Value	Max. Solvency Value	Avg. AROC (%)	Std. AROC	Sharpe Ratio	Turnover
Problem 1. Ruin Constraint		Avg. Ruin Probability (%)	Max. Ruin Prob (%)				
Covariance Model							
DCC	2129.43	0.491	0.554	5.39	1.51	3.55	0.068
CCC	2129.18	0.493	0.549	5.29	1.46	3.62	0.023
UNI	2128.14	0.493	0.547	5.31	1.41	3.74	0.022
Problem 2. CVaR Constraint		Avg. CVaR	Max. CVaR				
Covariance Model							
DCC	2259.14	-6.684	33.681	4.97	1.42	3.49	0.065
CCC	2258.88	-5.106	30.659	4.87	1.37	3.55	0.021
UNI	2258.84	-5.418	28.999	4.89	1.33	3.67	0.020
Problem 3. EPD Constraint		Avg. EPD Ratio (%)	Max. EPD Ratio (%)				
Covariance Model							
DCC	2222.53	0.245	0.277	5.24	1.73	3.03	0.068
CCC	2222.26	0.247	0.275	5.13	1.67	3.07	0.021
UNI	2222.21	0.246	0.274	5.16	1.63	3.15	0.021

For all models and for both portfolios the average total investment is almost the same across each covariance model. Table 4.3.1 shows that the mean out-of-sample ruin probability is around 0.497 for all models in Portfolio 1. However, we observe scenarios under which the ruin solvency constraint is violated. Although not reported in the tables, the number of violations is the same across all models and typically corresponds to a negative monthly rate of return for both risky assets. More specifically, the maximum values for the ruin probabilities are attained when the asset monthly rate of returns are -11% for NASDAQ and -12% for NYSE. . The DCC-GARCH is the best model choice in the sense that it gives the lowest maximum ruin probability of 0.546%, as opposed to 0.589% observed in the no-correlation case. A similar pattern can be observed for the maximum levels of $CVaR$ and EPD ratio. The results in Table 4.3.2 indicate that although the DCC-GARCH model has the smallest average solvency values, it produces the highest maximum solvency values observed in the periods of high negative returns. However, the differences between these values are not as high as in the Portfolio 1 case. Potential improvements for reducing the number of constraint violations could be obtained using a more sophisticated conditional mean return (e.g. an autoregressive structure) and estimating the model parameters based on lower frequency data (e.g. weekly or monthly). The latter reduces the number of simulation steps and thus improves the GARCH forecasting performance.

We now analyze the out-of-sample portfolio performance by computing averages, standard deviations and Sharpe ratios based on an adjusted rate of return on capital defined below:

$$AROC_{t,t+\tau} = \frac{(p_t + c_t^*)\mathbf{R}_{t+\tau}^T \mathbf{x}_t^* - E[Y_{t+\tau}]}{c_t} - 1.$$

The following quantities are calculated and reported in the second panel of Tables 4.3.1 and 4.3.2:

$$\begin{aligned}
\hat{\mu}_{AROC} &= \frac{1}{l_{B'}} \sum_{k=1}^{l_{B'}} AROC_{t+(k-1)\tau, t+k\tau}, \\
\hat{\sigma}_{AROC} &= \sqrt{\frac{1}{l_{B'}} \sum_{k=1}^{l_{B'}} (AROC_{t+(k-1)\tau, t+k\tau} - \hat{\mu}_{AROC})^2}, \\
\widehat{SR}_{AROC} &= \frac{\hat{\mu}_{AROC} - E[r_f]}{\hat{\sigma}_{AROC}}, \\
Turnover &= \frac{1}{l_{B'} - 1} \sum_{k=1}^{l_{B'}-1} \sum_{i=1}^{n} |x_{i,t+k\tau}^* - x_{i,t+(k-1)\tau}^*|.
\end{aligned}$$

Here r_f represents the risk-free rate of return given by the 3-month T-Bills. DeMiguel and Nogales (2009) interpret the portfolio turnover as the average percentage of wealth traded in each period. From Table 4.3.1, we observe that the no-correlation GARCH model outperforms the other two covariance specifications in terms of the average $AROC$. The DCC and CCC-GARCH models have slightly lower and approximately equal values for $\hat{\mu}_{AROC}$. The risk-return trade-off is also visible from the fact that the average $AROC$ is a decreasing function of capital invested. The DCC-GARCH provides the highest values of Sharpe Ratio in all of the situations. For example,

$\widehat{SR}_{AROC} = 3.46$ for the RP-constrained optimization, 3.38 for $CVaR$ and 2.94 for EPD. The smallest Sharpe Ratios are recorded for the no-correlation dynamic with values of 2.29, 2.24 and 2.04, respectively. Thus, we can conclude that the incorporation of a dynamic correlation for modelling the two risky assets in Portfolio 1 increases the portfolio performance as measured by its Sharpe Ratio. The turnover ratios have similar values for all models. According to the results in Table 4.3.2, the DCC-GARCH model outperforms the other GARCH counterparts in terms of average $AROC$, but it has the smallest $\widehat{SR}_{AROC}$. However, the differences are much smaller than in the Portfolio 1 case (e.g. the largest difference is for the RP-constrained optimization when the $\widehat{SR}_{AROC} = 3.55$ for the DCC-GARCH as opposed to 3.74 for the UNI-GARCH). Unlike in the previous study, the portfolio turnover is approximately three times higher in the case of the DCC-GARCH. A potential justification for explaining these numerical findings is the presence of relatively small correlations between the Portfolio 2 assets over the time period considered. Therefore, the less complex no-correlation model is preferred in this case.

Finally, we provide a brief discussion regarding the advantages/disadvantages of choosing between the RP and $CVaR$ solvency criteria. On the one hand, we notice that for both portfolios, the $CVaR$-based optimization at 99% requires a higher initial optimal capital than the corresponding VaR at 99.5%. This also results in a higher overall average out-of-sample EPD ($\widehat{EPD} = 2$ for Portfolio 1 and $\widehat{EPD} = 1.98$ for Portfolio 2) for the $CVaR$-constrained problem compared to the RP counterpart ($\widehat{EPD} = 1.45$ for Portfolio 1 and $\widehat{EPD} = 1.44$ for Portfolio 2).[10] On the other hand, shareholders will benefit more from their investment based on the less conservative approach, since the overall average Sharpe Ratio for the RP problems are 3.33 and 3.64, respectively, while the corresponding values for the $CVaR$ optimizations are 2.93 and 3.57, respectively. We do not comment on further comparisons with the EPD-constrained optimization, since the latter is constructed based on an arbitrary upper limit for the EPD ratio.

5 Conclusions

In this paper, we propose three problems to solve jointly for the optimal capital requirement and its optimal portfolio allocation. Each problem is constructed based on two types of constraints. The first set of constraints are dictated by standard solvency insurance requirements such as VaR, $CVaR$ and EPD calculated for a specified

[10] These overall average EPD values are computed across all models for the rolling period using the optimal solutions of RP and $CVaR$ optimization problems and observed returns.

horizon and for a given confidence level. The second constraint represents a performance measure constraint based on a lower bound for the shareholders' expected ROC. We provide a novel semiparametric approach for solving these problems based on a parametric distribution of the liability random variable and the empirical distribution for asset returns. In particular, we assume claim amounts follow a Lognormal distribution and portfolio's asset returns are generated according to a Dynamic Conditional Correlation multivariate GARCH model. We provide sufficient conditions such that each solvency constraint admits a convex representation; these are further implemented using the non-linear optimization Matlab solver based on interior-point algorithms. We examine optimal solutions for 3-asset portfolios (two indices and one risk-free asset) through two numerical experiments.

In the first numerical example, we construct efficient frontiers for the optimal capital based on different levels of expected ROC. The efficient frontiers have the same pattern for all constraints and covariance models considered. The correlation between the two entities plays an important role in the behaviour of the optimal capital required and the portfolio structure. For the same level of expected ROC, the minimum value of c_t^* is obtained for the no-correlation model, while DCC-GARCH is the most conservative model.

The out-of-sample performance of our portfolio is tested in a second detailed numerical example using a double rolling window estimation for both assets and liabilities. On the one hand, we found that the optimal required capital varies substantially across all models and optimization problems. On the other hand, the differences between the optimal portfolio weights are not as pronounced. We computed two types of indicators for assessing the solvency and return on capital performances. Our results suggest that the DCC model outperforms the other candidates (has the smallest value of the maximum RP, $CVaR$ and EPD and provides the highest out-of-sample Sharpe Ratio) when assets are strongly correlated, while the univariate GARCH is slightly preferred for low correlated asset portfolios. Several extensions to our models can be further investigated by including more complex models for assets and liabilities, as well as by extending this work to allow for multiple business lines, friction costs and possibly a multiperiod setting.

Acknowledgments

The authors would like to thank the Committee on Knowledge Extension Research (CKER) and the Financial Reporting Section of the Society of Actuaries for their support.

References

[1] Alexander, S., Coleman, T., and Li, Y. (2006). Minimizing CVaR and VaR for a Portfolio of Derivatives. *Journal of Banking and Finance*, 30, 583-605.

[2] Acerbi, C., and Tasche, D. (2002). On the Coherence of Expected Shortfall. *Journal of Banking and Finance*, 26(7), 1487-1503.

[3] Artzner, P. (1999). Application of Coherent Risk Measures to Capital Requirements in Insurance. *North American Actuarial Journal*. 13, 11-25.

[4] Artzner, P., Delbaen, F., Eber, J. M., and Heath, D. (1999). Coherent measure of risk. *Mathematical Finance*, 9(3), 203-228.

[5] Artzner, P., Delbaen, F. and Koch-Medina, P. (2009). Risk Measures and Efficient Use of Capital. *ASTIN Bulletin*, 39(1), 101-116.

[6] Asimit, V.A., Badescu, A., Siu, T.K., and Zinchenko, Y. (2012). Capital Requirements and Optimal Investment with Solvency Constraints. *IMA Journal of Management Mathematics*, In Press.

[7] Balbas, A. (2008). Capital Requirements: Are They the Best Solutions?. *working paper*, available at `http://ideas.repec.org/p/ner/carlos/infohdl10016-3367.html`.

[8] Barth, M. (2000). A Comparison of Risk-Based Capital Standards under the Expected Policyholder Deficit and the Probability of Ruin Approaches. *Journal of Risk and Insurance*. 67(3), 397-413.

[9] Bauwens, L., Laurent, S., and Rombouts, J.K.V. (2006). Multivariate GARCH models: a survey. *Journal of Applied Econometrics*, 21, 79-109.

[10] Bollerslev, T. (1990). Modeling the Coherence in Short-Run Nominal Exchange Rates: a Multivariate Generalized ARCH Model. *Review of Economics and Statistics*, 72. 498-505.

[11] Boyd, S., and Vandenbergue, L. (2004). Convex Optimization. Cambridge University Press. Cambridge.

[12] Butsic, R.P. (1994). Solvency Measurement for Property-Liability Risk-Based Capital Applications. *Journal of Risk and Insurance*, 61, 656-690.

[13] Cherny, A., and Madan, D. (2009). New Measures for Performance Evaluation. *Review of Financial Studies*, 22(7), 2571-2606.

[14] Cummins, D. and Phillips, R.D. (2009). Capital adequacy and insurance risk-based capital systems, *Journal of Insurance Regulation*, 28(1), 2572.

[15] DeMiguel, V., and Nogales, F.J. (2009). Portfolio Selection with Robust Estimators. *Operations Research*, 57(3), 560-577.

[16] Dhaene J., Vanduffel S., Goovaerts M., Kaas R., Tang Q., and Vyncke D. (2006). Risk measures and comonotonicity: A review. *Stochastic Models*, 22(4), 573-606.

[17] Djehiche, B., and Hörfelt, P. (2005). Standard Approaches to Asset & Liability Risk, *Scandinavian Actuarial Journal*, 5, 377-400.

[18] EIOPA, 2011. "Equivalence Assessment of the Swiss Supervisory System in Relation to Articles 172, 227 and 260 of the Solvency II Directive". *EIOPA-BoS-11-028*, available at `https://eiopa.europa.eu/consultations/consultation-papers/index.html`.

[19] Eling, M., Gazert, N., and Schmeiser, H. (2009). Minimum Standards for Investment Performance: A New Perspective on Non-Life Insurer Solvency, *Insurance; Mathematics and Economics*, 45(1), 113-122.

[20] Engle, R. F. (2002). Dynamic Conditional Correlation - a Simple Class of Multivariate GARCH Models. *Journal of Business and Economic Statistics*, 20, 339-350.

[21] Engle, R. F. and Sheppard, K. (2001). Theoretical and Empirical Properties of Dynamic Conditional Correlation Multivariate GARCH. NBER *working paper No. 8554.*

[22] Engle, R. F. and Sheppard, K. (2008). Evaluating the Specification of Covariance Models for Large Portfolios. *working paper.*

[23] Farkas, W., Koch-Medina, P., and Munari, C-A. (2012). Capital requirements and risk measures with multiple assets, *working paper*, available at `http://ssrn.com/abstract=1989077`.

[24] Föllmer, H., and Schied, A. (2002). Convex Measures of Risk and Trading Constraints. *Finance and Stochastics*, 6, 429-447.

[25] FOPI. (2004). Federal Office of Private Insurance, *Whitepaper on Swiss Solvency Test*.

[26] Gaivoronski, A., and Pflug, G. (2004). Value at Risk in Portfolio Optimization: Properties and Computational Approach. *Journal of Risk*, 7(2), 1-31.

[27] Hlouskova, J., Schmidheiny, K., and Wagner, M. (2009). Multistep Predictions for Multivariate GARCH Models: Closed Form Solution and the Value for Portfolio Management. *Journal of Empirical Finance*, 16, 330-336.

[28] Holzmüller I. (2009). The United States RBC Standards, Solvency II and the Swiss Solvency Test: A Comparative Assessment. *The Geneva Papers*, 34, 56-77.

[29] Hürliman, W. (2003). Conditional Value-at-Risk Bounds for Poisson Risks and a Normal Approximation. *Journal of Applied Mathematics*, 3, 141-153.

[30] Krokhmal. P., Palmquist, J., and S. Uryasev. (2002). Portfolio Optimization with Conditional Value-At-Risk Objective and Constraints. *The Journal of Risk*, 4(2), 11-27.

[31] Krokhmal, P., Zabarankin, M., and Uryasev, S. (2011). Modeling and Optimization of Risk. *Surveys in Operations Research and Management Science*, 16 (2), 49-66.

[32] Larsen, N. Mausser, H. and Uryasev, S. (2002). Algorithms for optimization of value-at-risk. In P. Pardalos and V.K. Tsitsiringos, editors, Financial Engineering, e-Commerce and Supply Chain, 129157. Kluwer Academic Publishers.

[33] Laurent, S., Rombouts, J.V.K., Violante, F. (2012). On the Forecasting Accuracy of Multivariate GARCH Models. *Journal of Applied Econometrics*, 27(6), 934-955.

[34] Luedtke, J., and Ahmed, S. (2008). A sample approximation approach for optimization with probabilistic constraints. *SIAM Journal on Optimization*, 19, 674-699.

[35] National Association of Insurance Commissioners (NAIC). (2009). NAIC Capital Adequacy Task Force - Risk-Based Capital Overview, Kansas City, MO.

[36] Mankai, S., and Bruneau, C. (2012). Optimal Investment and Capital Management decisions for a Non-Life Insurance Company, *Bankers, Markets & Investors*, 119.

[37] Nemirovski, A., and Shapiro, A. (2006). Convex Approximations of Chance Constrained Programs. *SIAM Journal of Optimization*, 17(4), 969-996.

[38] Pflug, G.C. (2000). Some Remarks on the Value-at-Risk and Conditional Value-at-Risk. Uryasev, S.P. ed. *Probabilistic Constrained Optimization: Methodology and Applications*, Kluwer, Norwell. MA, 278-287.

[39] Rockafellar, R.T. and Uryasev, S. (2000). Optimization of Conditional Value-at-Risk. *Journal of Risk*, Number 2, 21-41.

[40] Rockafellar, R.T. and Uryasev, S. (2002). Conditional Value-at-Risk for General Loss Distributions. *Journal of Banking and Finance*, 26(7), 1443-1471.

[41] Rombouts, J. V. K., and Stentoft, L. (2011). Multivariate Option Pricing with Time Varying Volatility and Correlations, *Journal of Banking and Finance*, 35, 2267-2281.

[42] Sandström, A. (2006). *Solvency: Models, Assessment and Regulation*. Chapman & Hall/CRC, Boca Raton.

[43] Santos, A.A.P., Nogales, F.J., Ruiz, E., and Van Dijk, D. (2012). Optimal Portfolios with Minimum Capital Requirements, *Journal of Banking and Finance*, 36, 1928-1942.

[44] Scollnik, D.P.M., and Sun, C. (2012). Modeling with Weibull-Pareto Models. *North American Actuarial Journal*, 16(2), 260-272.

[45] Silvennoinen, A., and Terasvirta, T. (2008). Multivariate GARCH Models. in *Handbook of Financial Time Series*, Springer, Andersen, T., Davis, R., Kreiss, J., and Mikosch, T. (Eds.).

[46] Tian R., Cox, S.H., Lin, Y, and Zuluaga, L.F. (2010). Portfolio Risk Management with CVaR-like Constraints. *North American Actuarial Journal*, 14(1), 86-106.

[47] Wozabal, D., Hochreiter, R., Pflug, G. (2008). A D.C. Formulation of Value-at-Risk Constrained Optimization. Tech. Rep. TR2008-01, Department of Statistics and Decision Support Systems, University of Vienna, Vienna.