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Verifying Inevitability of Phase-locking in a Charge Pump Phase Lock Loop using Sum of Squares Programming

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ABSTRACT
Phase-locking in a charge pump (CP) phase lock loop (PLL) is said to be inevitable if all possible states of the CP PLL eventually converge to the equilibrium where the input and output phases are in lock. We verify this property for a CP PLL using a mixed deductive and bounded verification methodology. This involves a positivity check of polynomial inequalities (which is an NP-Hard problem) so we use the sound but incomplete Sum of Squares (SOS) relaxation algorithm to provide a numerical solution.

1. INTRODUCTION
Formal methods are in their infancy in Analog and Mixed Signal (AMS) circuits verification. Start up problems have been very common in PLL circuits, i.e., for certain initial states of voltages, the circuits do not converge to the desired behaviour. In addition, while in phase-locking state and disturbed by an external input, it is important to know whether the PLL circuit retains its locking state. Verifying both these properties are closely related to the verification of inevitability property.

Hybrid systems are well known modelling paradigm for a CP PLL [10], [2], [6]. Techniques for the verification of hybrid systems can be classified as, reach-set methods, abstraction based methods, and certificate based methods. Reachability has been used to prove the above stated property. Hundreds of discrete transitions are required by the hybrid model of the CP PLL before it reaches the locking state. This results in large number of continuous set computations followed by the guard conditions describing the switching laws, making the reachability computation prohibitively expensive. In this paper, we use a mixture of certificate based deductive and bounded (reach-set) verification methodology. We verify the inevitability of the phase-locking in a CP PLL by adopting a two-pronged verification approach. Due to complexity of the property, we essentially divide the inevitability property in to the conjunction of two sub-properties. These Permission to make digital or hard copies of all or part of this work for personal or classroom use is granted without fee provided that copies are not made or distributed for profit or commercial advantage and that copies bear this notice and the full citation on the first page. Copyrights for components of this work owned by others than ACM must be honored. Abstracting with credit is permitted. To copy otherwise, or republish, to post on servers or to redistribute to lists, requires prior specific permission and/or a fee. Request permissions from permissions@acm.org

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two properties determine the truth value of the inevitability property in two disjoint subsets of the state space. The first property specifies, that in a compact set, all system trajectories eventually converge to the equilibrium locking state. The second property specifies, such that the set where the first property holds, is reachable from the second subset of the state space. The first property is verified by computing an attractive region (AR) utilizing the deductive Lyapunov stability theory for hybrid systems [5]. We construct multiple Lyapunov certificates for different modes of the CP PLL hybrid system. The maximized level curves of these Lyapunov certificates characterize the level sets whose union is the AR. We verify the second property utilizing bounded advection of level sets [15], and deductive Escape certificate method, showing the reachability of this AR from the states outside it. The deductive and bounded verification approaches involve checking positivity of polynomial inequalities, which is an NP-hard problem. We use the sound but incomplete SOS relaxation for the positivity verification of polynomial inequalities.

1.1 Related Work
A survey of the formal verification of AMS circuits can be found in [17]. In [9], the author verified the ‘time to locking’ property for a digitally extensive PLL. [10] verified ‘global convergence’ property for an all digital PLL. They divided the state space into linear and non-linear regions, and applied linear Lyapunov stability theory (Using Quadratic Lyapunov Certificate) for linear and reachability analysis for non-linear regions respectively. Time-outs of the reachability tool has been reported by the author due to the large number of discrete transitions needed by the PLL hybrid automata. To avoid discrete jumps, [2] presented a continuization technique and verified the ‘time to locking’ property of a CP PLL.

Hybrid models have different flavours that can be found in [8], [11]. Here we consider the framework outlined in [4]. In the last decade, SOS programming has been the major tool used in the algorithmic construction of Lyapunov certificates for continuous, as well as hybrid systems [9], [11]. Deductive verification of continuous and hybrid systems have been demonstrated in [4], [13]. Recently an advection algorithm for polynomial level sets based on SOS programming has been presented in [15]. This advection algorithm has been used for reachable sets estimation for continuous dynamical systems. We extend this approach to bounded verification of the hybrid systems.
This paper is organized as follows: In Sec. II, we introduce the preliminaries of this paper. Sec. III illustrates verification of the inevitability of phase-locking in CP PLL. Experimental results are shown in Sec. IV. Sec. V concludes the paper.

2. PRELIMINARIES

2.1 Hybrid Systems Model

We use the hybrid system formalism described in [4]. We consider a hybrid system described by the tuple \((\mathcal{C}, F, D, G)\). Here, \(\mathcal{C} = \bigcup_{i \in \mathbb{I}} C_i \subseteq \mathbb{R}^n\), and \(D = \bigcup_{i \in \mathbb{I}} D_i \subseteq \mathbb{R}^m\) are the flow set and jump set for \(i \in \mathbb{N}\), respectively. \(I_c\) and \(I_D\) are finite disjoint index sets and it is possible that \(C_i \cap D_i \neq \emptyset\). The flow and jump maps are, respectively, \(F = \bigcup_{i \in \mathbb{I}} F_i\), and \(G = \bigcup_{i \in \mathbb{I}} G_i\), where each \(F_i : \mathbb{R}^n \to \mathbb{R}^{m+n}\), and \(G_i : \mathbb{R}^m \to \mathbb{R}^n\). These two mappings characterize the continuous and discrete evolution of the system, whereas \(C_i\) and \(D_i\) describe subsets of \(\mathbb{R}^n\) where such evolution may occur. We represent a hybrid system \(H\) as

\[
H = \begin{cases}
\dot{x} = F_i(x, u) \in F & x \in C, u \in U \\
x^+ = G_i(x) \in G & x \in D
\end{cases}
\]  

(1)

Here \(u \in U \subseteq \mathbb{R}^m\) is a vector of uncertain parameters. The state of the hybrid system consists of alternate flows in jumps through \(\mathcal{C}\) and \(\mathcal{D}\) according to \(F_i\) and \(G_i\), respectively. This hybrid phenomena can be described by a notion of time called hybrid time.

Definition 1 (Hybrid Time Domain).

A set \(\mathcal{T} \subset \mathbb{R}_{\geq 0} \times \mathbb{N}\) is a hybrid time domain if

\[
\mathcal{T} = \bigcup_{j=0}^{j-1} \{[t_j, t_{j+1}], j\}
\]

where \(0 = t_0 \leq t_1 \leq t_2 \leq \ldots\), with the last interval possibly of the form \([t_j, t_{j+1}] \times \{j\}, [t_j, t_{j+1}] \times \{j\}\), or \([t_j, \infty) \times \{j\}\).

Definition 2 (Hybrid Arc).

A mapping \(x : \mathcal{T} \to \mathbb{R}^n\) is a hybrid arc if \(x : \mathcal{T}\) is a hybrid time domain and for each \(j \in \mathbb{N}\), the function \(t \mapsto x(t, j)\) is locally absolutely continuous on the interval \(I_j = \{t : (t, j) \in \mathcal{T}\}\).

Assumption 1.

The flow maps \(F_i(x, u)\) and the jump maps \(G_i(x)\) are polynomials.

Definition 3 (Equilibrium point).

A point \((x, j) \in \mathcal{C} \cup \mathcal{D}\) is called an equilibrium, if \(\exists t, \exists j, F_j(x(t, j), u) = 0\).

Definition 4 (Inevitability of Equilibrium).

The equilibrium point \(x_e\) is said to be inevitable, if \(\forall x(0, j) \in \mathcal{C} \cup \mathcal{D}\) and bounded time, \(t(x(t, j)) \to x_e\).

2.2 CP PLL Model

A CP PLL circuit consists of a reference signal, a phase frequency detector (PFD), a charge pump (CP), a loop filter (LF), and a voltage controlled oscillator (VCO). In this paper we consider a single path higher order (Third and fourth) CP PLL shown in Fig. 1. We use a behavioural model of the PLL, where we consider a linear model for VCO, a linear model for the third order LF, and a non-linear model for the PFD. We denote by \(\phi_{\text{ref}}\), and \(\phi_{\text{VCO}}\), the phases of the reference and VCO output feedback signals respectively. We model the CP PLL as a hybrid system such that the non-linearities of the PFD is modelled as a piecewise continuous signal. Ignoring the cycle slip phenomena, the PFD output in the form of the charge pump current \(I_p\), is given by the following piecewise linear inclusion:

\[
I_p = \begin{cases}
\{0\} & \text{UP}=1, \text{Down}=0, 0 \leq \phi_{\text{VCO}} < 2\pi \leq \phi_{\text{ref}} \\
\{0\} & \text{UP}=0, \text{Down}=1, 0 \leq \phi_{\text{ref}} < 2\pi \leq \phi_{\text{VCO}} \\
\{0\} & \text{UP}=0, \text{Down}=0, 0 \leq \phi_{\text{VCO}}, \phi_{\text{ref}} < 2\pi
\end{cases}
\]

(2)

We denote the three modes as model 1 (UP=0, Down=0), model2 (UP=1, Down=0) and model3 (UP=0, Down=1). The transition from one mode to another is based on the reference and feedback signals hitting the 2\pi threshold. Due to the cyclic behaviour of the PLL and to keep the analysis modulo 2\pi, we need to ensure the phases remain in the range 0 \leq \phi_{\text{VCO}}, \phi_{\text{CO}} < 2\pi (Similarly \(\phi_{\text{ref}}\)) after resetting the PFD. This is achieved by resetting the two phases such that \(\phi_{\text{ref}} = 0\), \(\phi_{\text{VCO}} = \phi_{\text{VCO}} - 2\pi\), and \(\phi_{\text{ref}} = \phi_{\text{ref}} - 2\pi\), \(\phi_{\text{VCO}} = 0\), while taking transitions from model1 to model2 and model2 to model3, respectively. Identity resets are used for transitions from model2 to model1 and model3 to model1.

Our model consists of the state variables, \(\phi_{\text{VCO}}, \phi_{\text{ref}},\) voltage \(v_1\) across the capacitor \(C_1\), and the voltage \(v_2\) across the capacitor \(C_2\) (Fourth order has an additional voltage variable across the third capacitor). Let \(f_{\text{VCO}}\), and \(f_{\text{ref}}\), represent the frequencies of the VCO output and the reference signal respectively. If \(K_p\) is the gain of the LF, then \(f_{\text{VCO}} = K_p v_2/2\pi + f_0\), where \(f_0\) is the free running frequency of the VCO. Therefore, \(\phi_{\text{VCO}} = 2\pi f_{\text{VCO}} / N, \phi_{\text{ref}} = 2\pi f_{\text{ref}}\). By Kirchhoff’s current law and using the three modes of the PFD, we get the following hybrid system of the third order CP PLL

\[
\dot{x} = A x + B I_p + c x \in \mathcal{C},
\]

(3)

where \(A = \begin{pmatrix}
-1/R C_1 & 1/R C_1 & 0 & 0 \\
0 & -1/R C_2 & 0 & 0 \\
0 & 0 & K_p / N & 0 \\
-2\pi f_{\text{ref}} & 2\pi f_{\text{ref}} & 0 & 0
\end{pmatrix}\), \(B = \begin{pmatrix}
0 \\
0 \\
0 \\
0
\end{pmatrix}\), \(c = \begin{pmatrix}
0 \\
0 \\
0 \\
0
\end{pmatrix}\), \(I_p\) as given by Eq. 2

Figure 1: Charge Pump (CP) Phase Lock Loop (PLL)
From the three modes and their invariants, we can easily find the sets $C, D,$ and the jump maps $G_i(x)$. Here, we use the difference $\phi_{ref} - \phi_{VCO}$ as a state variable instead of $\phi_{ref}$. We use the following theorem of AS and version of the stability theorems for hybrid systems based on $\phi_{ref}$ and $\phi_{VCO}$.

**Remark 1.**
This change of state variables transforms all jump maps $G_i$ into identity maps $(G_i(x) = x)$, as the same constant $2\pi$ is subtracted from $\phi_{VCO}$ and $\phi_{ref}$, leaving their difference $\phi_{ref} - \phi_{VCO}$ before and after the jumps unchanged.

### 2.3 Attractive Invariants in Hybrid Systems using Lyapunov Certificates

Contrary to safety properties where existence of an invariant set is sufficient for proving/disproving the property, for inevitability property, we use the concept of "attractive invariants". Attractive invariants are compact semi-algebraic sets where if the CP PLL hybrid system starts, will always remain there and will eventually converge to the equilibrium state. Stability and attractivity concepts for an equilibrium point, $x_e = 0$, is called asymptotically stable (AS) if it is both stable and attractive. There are several versions of the stability theorems for hybrid systems based on the global Lyapunov function, and multiple Lyapunov certificates [12]. We use the following theorem of AS and define the attractive invariant.

**Theorem 1.**

Let $I_0 \subseteq I_e$ be the set of indices that contain the equilibrium. For a hybrid system $H$ having an equilibrium point $x_e = 0$, if there exist Lyapunov certificates $V_i$ such that,

1. $V_i(x) > 0$, $\forall i \in I_C$, $\forall x \in C \setminus x_e$,
2. $V_i(0) = 0$, $\forall i \in I_0$,
3. $\frac{\partial V_i}{\partial x}(x)F_i(x,u) < 0$, $\forall i \in I_C$, $\forall x \in C \setminus x_e$, $F_i \in F$,
4. $V_j(G_i(x)) - V_j(x) < 0$, $\forall j \in I_C$, $j \neq j'$, $\forall i \in I_0$, $\forall x \in D \setminus x_e$, $G_i \notin G$,

then $x_e$ is asymptotically stable. Furthermore, the set $\mathcal{A} = \bigcup_i (V_i \leq c_{max}) \subseteq C \cup D$ is an "attractive invariant" set.

**Proof.** Similarly to [12].

We also use another important characteristic of the trajectories in a semi algebraic set and term it as the Escape property. This property ensures that trajectories in a compact set can not converge to an invariant set (Equilibrium, Limit Cycle) and will eventually leave that set.

**Proposition 1.**

For a compact set $X \subset C$, if there is a differentiable Escape certificate, $E : \mathbb{R}^n \to \mathbb{R}$, and $\epsilon > 0$, such that

$$\frac{\partial E}{\partial x}(x)F_i(x,u) \leq -\epsilon, \forall x \in X, u \in U$$

then $\forall x(t,i) \in X, x(t+T,i) \notin X, T > t$.

Proof. Assume that there exists $x_0 \in X$ such that $x(t,i)$ starting at $x_0$ remain in $X$ as $t \to \infty$. From equation (4)

$$E(x) = \int_0^t \frac{\partial E}{\partial x}(x)F_i(x,u) \leq -\epsilon.$$

This contradicts the assumption as $E(x)$ should be bounded if $x(t,j)$ has to be in the bounded set $X$. Therefore, $x(t,j)$ has to eventually escape $X$ in finite time.

### 2.4 SOS Programming

Our hybrid deductive and bounded verification approach involves checking the positivity of polynomials in semi-algebraic sets. To solve this NP-Hard problem, a sound relaxation method based on SOS programming has been discussed in [10], [12]. A sufficient condition for a multivariate polynomial $p(x)$ to be non-negative everywhere, is that it can be decomposed as a sum of squares of polynomials. A polynomial $p(x)$ is a sum of squares, if there exist polynomials $p_1(x), \ldots, p_m(x)$ such that $p(x) = \sum_i p_i(x)^2$. We denote the set of polynomials in $n$ variables with real coefficients by $P_n$. A subset of this set is the set of SOS polynomials in $n$ variables denoted by $P_n$. For a polynomial $q : \mathbb{R}^n \to \mathbb{R}$, differentiable scalar function, we define the $0$-sub-level-set of $q$ as $Z(q) = \{x \in \mathbb{R}^n | q(x) \leq 0\}$. We present an important lemma to be used for polynomial level sets operations such as intersection, union, and set inclusion [13].

**Lemma 1.**

For polynomials $p_1, p_2 \in P_n$, if there exist SOS polynomials $s_0, s_1 \in P_n$ such that

$$s_0 - s_1 p_1 + p_2 = 0 \quad \forall x \in \mathbb{R}^n$$

Then $Z(p_1) \subseteq Z(p_2)$

**Proof.** See for example [13] and the references there in.

### 2.5 Bounded Advection of Level Sets

Let we define a flow map $\psi : \mathbb{R}^n \times \mathbb{R}_{\geq 0} \to \mathbb{R}^n$ for the hybrid system $H$. A time $t$ advection operator $A_t$ is a map $A_t : Q(\mathbb{R}^n, \mathbb{R}) \to Q(\mathbb{R}^n, \mathbb{R})$, such that $U = A_t V$ for $U, V \in Q(\mathbb{R}^n, \mathbb{R})$, and $U(x) = V(\psi_{-t}(x))$ for all $x \in C \cup D$. This advection operator has an important property of linearity. For polynomial functions $U, U_2 \in Q(\mathbb{R}^n, \mathbb{R})$, if $U_2 = A_t U_1$, then $Z(U_2) = A_t(Z(U_1))$. For a detailed discussion on the advection operator see [13].

### 3. VERIFICATION METHODOLOGY

To verify inevitability of the CP PLL equilibrium, we introduce two compact sets $S_1$ and $S_2$, such that $S_1 \cap S_2 = \emptyset$, and $S_1 \cup S_2 = C \cup D$. We define two properties whose verification implies verification of the inevitability of the equilibrium.

**Property 1.**

$\forall x(0,j) \in S_1, x(t,j) \rightarrow x_1$, for $t \to \infty$.

**Property 2.**

$\forall x(0,j) \in S_2 = (C \cup D) \setminus S_1, x(t,j) \in S_1$ for $t \to b \in \mathbb{R}_{>0}$.

If we denote the inevitability property by $\varphi$, Property 1 by $\varphi_1$ and Property 2 by $\varphi_2$, then $\varphi = \varphi_1 \land \varphi_2$. A hybrid arc $x$ satisfies $\varphi$ iff it satisfies $\varphi_1$ in $S_1$ and $\varphi_2$ in $S_2$ i.e., $x \in C \cup D, x = \varphi \iff (x = \varphi_1 \forall x \in S_1) \land (x = \varphi_2 \forall x \in S_2)$. We verify property $\varphi_1$ by the deductive Lyapunov theory, and use a combination of bounded advection and Escape certificate for the verification of property $\varphi_2$. 

Theorem 2.
If there are feasible Lyapunov certificates (fulfilling Th. 1), \( \{V_1, V_2, V_3\} \), then, \( x \equiv \varphi_1, \forall x(0,j) \in S1 \) \( \{V_1 \leq \gamma_1 \text{max}\} \cup \{V_2 \leq \gamma_2 \text{max}\} \cup \{V_3 \leq \gamma_3 \text{max}\} \), and \( A2 = S1 \) is the attractive invariant set.

Proof. Follows directly from Th. 1 since the level sets defined by the level curves of the Lyapunov certificates represent attractive invariant sets with the negative Lie-derivative along the system trajectories. Therefore, eventually all system trajectories starting in these level sets converge to the equilibrium point.

Following Th. 2 we encode the verification of \( \varphi_1 \) as two SOS programs. The truth value of \( \varphi_1 \) depends on the existence of the attractive invariant set \( S1 \). The set \( S1 \) is computed from the maximized level sets defined by the three candidate Lyapunov certificates \( V_1, V_2, V_3 \). We compute these certificates using a sound mathematical technique called the S-procedure [3]. The first SOS program is given below:

(a) \( V_1(x) - \sum_{k=1}^{n_{c_1}} s_1^{(i_k)}(x)g_k(x) \in S_n, \forall x \neq 0, i \in \{1,2,3\}, \forall k \in \{1,...,n_{c_1}\}, s_1^{(i_k)} \in S_n \),

(b) \[-\sum_{j=1}^{m_{c_1}} s_2^{(j)}(x)h_{io}(x) - \sum_{j=1}^{m_{c_1}} s_2^{(j)}(x)h_{io}(x) \in S_n, \forall j \in \{1,2,3,4\}, \forall k \in \{1,...,n_{c_2}\}, s_2^{(j)} \in S_n \],

(c) \[ V_3(x) - V_j((G_i(x)) - s_4^{(i)}(x)h_{io}(x) - \sum_{k=1}^{m_{c_3}} s_5^{(i_k)}(x)h_{ik}(x) \in S_n, \forall j \neq j', i \in \{1,2,3,4\}, \forall k \in \{1,...,n_{c_4}\}, s_4^{(i)} \geq 0, s_5^{(i_k)} \in S_n \].

Here \( V_1(x), V_j(x), s_1^{(i_k)}, s_2^{(j)}, s_3^{(j)}, s_4^{(i)}, s_5^{(i_k)} \) are polynomials of degree \( d \).

SOS constraints (a) and (b) enforce positive definiteness on the Lyapunov certificates, and negative semi-definiteness on their Lie-derivatives respectively. Furthermore, these constraints have to be satisfied in their respective domains \( C_i, s \), where, \( C_i = \{ x \in R^n | g_k \geq 0, \forall k \in \{1,...,n_{c_i}\}, i \in \{1,2,3\} \} \). Constraint (b) also ensures parameters \( u \) to belong to the set, \( \{a(u) \geq 0, \forall j \in \{1,...,m\} \} \). Constraint (c) ensures that Lyapunov certificates \( V_j(x) \) decrease along the discrete jumps in the sets, \( D_i = \{ x \in R^n | h_{ik} \geq 0, \forall k \in \{1,...,n_{D_i}\}, i \in \{1,2,3,4\} \} \), through the mappings \( G_i(x)s \). SOS polynomials \( s_1^{(i_k)}, s_2^{(j)}, s_3^{(i)}, s_4^{(i)}, s_5^{(i_k)} \) are used to map the domain constraints through the S-procedure. A feasible solution of the above SOS program results in Lyapunov certificates \( V_i \). If this SOS program is infeasible, then either the program is repeated for an increased degree \( d \) of the polynomials, or we conclude that the truth value of the property \( \varphi_1 \) can not be established.

The second SOS program for maximizing the level curves for every \( V_i \leq \gamma_i \in R_{\geq 0}, i \),

maximize \( \gamma_i \)
subject to \( s_5 + \sum_{k=1}^{n_{c_1}} s_6^{(i_k)}(-g_k) - (V_i - \gamma_i) + \epsilon = 0, \)
\((s_5, s_6^{(i_k)}) \in S_n, i \in \{1,2,3\}, k \in \{1,...,n_{c_1}\}\).

This algorithm maximizes the level curves of the Lyapunov certificates \( V_i \) such that \( Z(V_i - \gamma_i) \subseteq Z(-g_k) \), for \( k \in \{1,...,n_{c_i}\} \) (Lemma 1). The set \( S1 = \bigcup_{i=1}^{3} V_i \leq \gamma_i \text{max} \). The non-emptiness of the set \( S1 \) shows that \( x \equiv \varphi_1, \forall x \in S1 \).

To verify property \( \varphi_2 \), we extend the advection of level sets [13] to hybrid systems, and show that for all \( x(0,j) \in S2, x(t,j) \to S1 \) for some bounded \( t \). This is given in Alg. 1 where Line-5 is the set advection implemented as a SOS program. Let the sets, \( S2 = Z(P_{\text{initial}}), \) and \( S2_{\text{next}} = Z(P_{\text{next}}), \) where, \( (P_{\text{initial}}, P_{\text{next}}): R^n \rightarrow R \) are polynomial functions. The SOS algorithm for bounded advection of level sets is given below,

minimize \( \eta \)
s.t. \( P_{\text{next}}(0) < 0, \)
\( \nabla P_{\text{next}}(v_1, v_2, \varphi_{ref} - \varphi_{CC}) \in T > 0, \)
\( s_{1i} - s_{2i} \mu_{\text{Initial}} + A_{-h} P_{\text{next}} + \eta + \sum_{k=1}^{m_{c_1}} s_{3i} g_k = 0, \)
\( s_{4i} + s_{5i} (P_{\text{Initial}} - \mu) - A_{-h} P_{\text{next}} + \eta + \sum_{k=1}^{m_{c_1}} s_{6i} g_k = 0, \)
\( s_{7i} - s_{8i} (P_{\text{Initial}} - \mu) + \sum_{k=1}^{m_{c_1}} s_{9i} g_k + \nabla^2 P_{\text{next}} h^2 \leq 0, \)
\( s_{10i} - s_{11i} (P_{\text{Initial}} - \mu) + \sum_{k=1}^{m_{c_1}} s_{12i} g_k \nabla^2 P_{\text{next}} h^2 \leq 0, \)
\( i \in \{1,2,3,4\}, k \in \{1,...,n_{c_1}\}, \)
\( s_{1i}, s_{2i}, s_{3i}, s_{4i}, s_{5i}, s_{6i}, s_{7i}, s_{8i}, s_{9i}, s_{10i}, s_{11i}, s_{12i} \in S_n. \)

Here \( P_{\text{next}} \) is of degree \( d_{i}, \mu > 0, \eta > 0, h > 0, \)
\( u \in \{L,U\}, \) and \( s_{1i}, s_{2i}, s_{3i}, s_{4i}, s_{5i}, s_{6i}, s_{7i}, s_{8i}, s_{9i}, s_{10i}, s_{11i}, s_{12i} \), are polynomials of degree \( d \).

The first two constraints of this SOS program ensure the advection level sets are closed and connected (See [15] and the references there in). The next two constraints search for a polynomial \( P_{\text{next}} \), such that when the set \( Z(P_{\text{next}}) \) is backward advected by the first order Taylor advection map \( A_{-h} \), we obtain a set such that,

\( Z(P_{\text{Initial}}) \subseteq Z(A_{-h} P_{\text{next}} + \eta) \subseteq \psi_{\text{lin}}(Z(P_{\text{next}})) \subseteq Z(A_{-h} P_{\text{next}} - \eta) \subseteq Z(P_{\text{Initial}} - \mu). \)

Here \( \mu \) is used as a precision parameter determining, how closely we want the set \( Z(P_{\text{Initial}}) \) to be approximated by the set \( Z(A_{-h} P_{\text{next}} + ||\mu||) \). The last two constraints enforce the truncation error of the first order Taylor approximation such that, \( ||\nabla^2 P_{\text{next}} h^2 \| \leq \eta \), in the set \( Z(P_{\text{Initial}} - \mu) \). Note that we solve the above optimization SOS program by using bisection on \( \eta \). To be conservative, and use an over-approximation to the set \( \psi_{\text{lin}}(Z(P_{\text{Initial}})) \), the set membership in Line-6 of the Alg. 1 is encoded as a SOS program utilizing Lemma 1 for the sets \( Z(P_{\text{next}} - \eta) \) and \( S1 \).
Algorithm 1 Verification of Property \( \varphi_2 \)

**INPUT:** : Hybrid System Model of CP PLL, Sets \( S_1, S_2 \)

**OUTPUT:** : \( \varphi_2 \) Verified in Bounded Time/No-answer

1: \( S_{2next} \leftarrow \emptyset \)
2: \( S_{2adveect} \leftarrow \emptyset \)
3: \( S_{2adveect} \leftarrow S_2 \)
4: for \( j \leftarrow 1 \) to \( j \leftarrow m \) do
5: \( S_{2next} \leftarrow Adveect(S_{2adveect}) \)
6: if \( S_{2next} \not\subseteq S_1 \) then
7: \( S_{2adveect} \leftarrow S_{2next} \)
8: else
9: \( x \models \varphi_2, \forall x \in S_2 \)
10: break
11: end if
12: end for
13: Try a large value of \( m \)
14: if \( S_{2next} \not\subseteq S_1 \) then
15: For \( S_{2next} \setminus (S_{2next} = S_1 \cap S_{2next}) \) find the Escape Certificate \( E \).
16: if \( E \) exists then
17: \( x \models \varphi_2, \forall x \in S_2 \)
18: break
19: else
20: No Answer about \( \varphi_2 \)
21: end if
22: end if

Remark 2.
As for the transformed CP PLL hybrid system (Remark.1) we have identity jump maps, there is therefore no need of constraints on the level sets due to discrete jumps.

After each iteration of the advection of level sets, if the set inclusion \( S_{2next} \subseteq S_1 \) is true, then property \( \varphi_2 \) is verified. Alternatively, the algorithm keeps on advecting the set \( S_{2next} \) for a user defined bounded number of iterations (Line 7-13). If the property \( \varphi_2 \) is still not verified (This can happen when the advection of the level sets is unsymmetrical and a subset of the set \( S_2 \) is not immersed in \( S_1 \)), we compute the Escape certificate \( E \) for the set, \( S_{2next} \setminus S_{2next} \), showing that trajectories in this set will eventually leave and reach \( S_1 \) (Line 14-18, as they can not reach \( S_{2next} \)). This either results in the verification of the property \( \varphi_2 \) (respectively \( \varphi \)) in the set \( S_2 \) or we conclude inconclusiveness about the truth value of \( \varphi_2 \) (respectively \( \varphi \)). Line 15 is implemented by the following SOS program,

\[ -\frac{\partial E_2}{\partial x}(x)F_i(x,u) - s_1(x)g_2(x) + s_2(x)g_2'(x) + \varepsilon \in S_n \]

\[ (s_1, s_2) \in S_n \]

where, \( S_{next} := g_2(x) \geq 0 \), and \( S_{2next} := g_2'(x) \geq 0 \).

4. EXPERIMENTAL EVALUATION

We used YALMIP [7] solvers within MATLAB for the verification of the inevitability property (respectively sub-properties) on a 2.6 GHz Intel Core i5 machine with 4 GB of memory. The CP PLL parameters are listed in Table 1 with all phases normalized by 2\( \pi \). We computed degree-6 multiple Lyapunov certificates for the third order, and degree-4 multiple Lyapunov certificates for the fourth order CP PLL. Their attractive invariant sets as projected on different planes are shown in Fig. 2 and Fig. 3. Results of their bounded

![Figure 2: 3-Order AZ Projected onto (v1, v2), and (v2, \( \Phi_{ref} - \Phi_{VCO} \))](image1)

![Figure 3: 4-Order AZ Projected onto (v1, v2), and (v2, \( \Phi_{ref} - \Phi_{VCO} \))](image2)

Table 1: PLL Parameters used in the Experimentation

<table>
<thead>
<tr>
<th>Parameters</th>
<th>Third Order</th>
<th>Fourth Order</th>
</tr>
</thead>
<tbody>
<tr>
<td>( C_1 )</td>
<td>1.08 2.2( e ) - 12( P )</td>
<td>3.2 3.4( e ) - 12( P )</td>
</tr>
<tr>
<td>( C_2 )</td>
<td>6.1 6.3( e ) - 12( P )</td>
<td>6.2 3.4( e ) - 12( P )</td>
</tr>
<tr>
<td>( C_3 )</td>
<td>1.8 2.2( e ) - 12( P )</td>
<td>[48 52e-3]</td>
</tr>
<tr>
<td>( R )</td>
<td>7.8 8.2( e ) - 30( I )</td>
<td>7.9 9.3( I )</td>
</tr>
<tr>
<td>( RO )</td>
<td>2730MHZ</td>
<td>55MHZ</td>
</tr>
<tr>
<td>( Kp )</td>
<td>495 305e-6A</td>
<td>495 405e-6A</td>
</tr>
</tbody>
</table>

Results show the effectiveness of our approach to the verification of the inevitability property of a complex real circuit. We have proved the inevitability property avoiding hundred of discrete transitions as well as the complex continuation as in [2]. Computation time is comparable to [2], and it is less by an order of at least considering their approach using gridding of the state space for a third order PLL only. Though user input is needed in the formalization of the problem, our Lyapunov and Escape certificate based deductive methods are applicable to infinite domain (oppose to bounded) and avoid approximating (under or over) solutions of the differential equations. Furthermore, our bounded advection of level sets has the advantage of dealing with larger sets in a single iteration as compared to the existing bounded model checking approaches.
<table>
<thead>
<tr>
<th>Verification Step</th>
<th>3-Order Time (Sec)</th>
<th>4-Order Time (Sec)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Attractive Invariant</td>
<td>1.981.7 (Degree 6)</td>
<td>10621 (Degree 3)</td>
</tr>
<tr>
<td>Max-Level Curves</td>
<td>15.5</td>
<td>17</td>
</tr>
<tr>
<td>Advection</td>
<td>100.6857 (14 iterations)</td>
<td>140.678 (7 iterations)</td>
</tr>
<tr>
<td>Checking Set Inclusion</td>
<td>14</td>
<td>n/a</td>
</tr>
<tr>
<td>Escape Certificate</td>
<td>18 (2 Certificates)</td>
<td>n/a</td>
</tr>
</tbody>
</table>

Table 2: Computation Time of the Inevitability Verification

5. CONCLUSION

We have presented a scalable verification methodology benefiting from both deductive and bounded verification approaches. We tailored these approaches to verify the complex inevitability property for a practical AMS CP PLL circuit of higher order. As the problem is known to be NP-hard, we used the sufficient SOS relaxation algorithm for the verification of these properties. Experimental results show the effectiveness of our methodology avoiding expensive discretization and reach set computations.

6. REFERENCES


Figure 4: 3-Order Advection Projected onto (v1, v2), and (v2, (ϕref − ϕVCO))

Figure 5: 4-Order Advection Projected onto (v1, v2), and (v2, (ϕref − ϕVCO))