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**Citation:** Fring, A., Dey, S. and Gouba, L. (2015). Milne quantization for non-Hermitian systems. *Journal of Physics A: Mathematical and Theoretical*, 48, p. 40. doi: 10.1088/1751-8113/48/40/40FT01

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# Milne quantization for non-Hermitian systems

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ABSTRACT: We generalize the Milne quantization condition to non-Hermitian systems. In the general case the underlying nonlinear Ermakov-Milne-Pinney equation needs to be replaced by a nonlinear integral differential equation. However, when the system is  $\mathcal{PT}$ -symmetric or/and quasi/pseudo-Hermitian the equations simplify and one may employ the original energy integral to determine its quantization. We illustrate the working of the general framework with the Swanson model and two explicit examples for pairs of supersymmetric Hamiltonians. In one case both partner Hamiltonians are Hermitian and in the other a Hermitian Hamiltonian is paired by a Darboux transformation to a non-Hermitian one.

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## 1. Introduction

As one of the first phase amplitude methods Milne provided in 1930 [1] a relation between the time-independent Schrödinger equation and a non-linear integrable equation referred to these days as the Ermakov-Milne-Pinney (EMP) equation [2, 1, 3] or variants thereof. Solving either of the two equations for any generic energy will provide a solution for the other. In addition, the interrelation involves an auxiliary equation whose solutions lead to the exact energy quantization in a very general fashion. It should be emphasized that the Milne quantization is exact and the more popular WKB-approximation is obtained as a limiting case when the second order derivative term in the EMP-equation is neglected. While the latter method has been generalized [4] to non-Hermitian  $\mathcal{PT}$ -symmetric systems, this task has not been carried out for the more general Milne quantization procedure. The main purpose of this manuscript is to perform the first step in this direction and to demonstrate that a successful application of the Milne quantization procedure is indeed possible.

We analyze two types of non-Hermitian systems, in one case we exploit the fact that the model is quasi/pseudo-Hermitian, the Swanson model, and in the other that it is  $\mathcal{PT}$ -symmetric, a supersymmetric pair in which one of the partner Hamiltonians is non-Hermitian.

Our manuscript is organized as follows: In section 2 we recall the key features of the Milne quantization procedure and generalize it to a general non-Hermitian setting. In section 3 we discuss the Swanson model and in section 4 we provide two explicit examples for pairs of supersymmetric Hamiltonians, where in one case both partner Hamiltonians are Hermitian and in the other only one of them. Our conclusions and outlook are stated in section 5.

## 2. The Milne quantization for Hermitian and non-Hermitian systems

We commence by briefly recalling the key idea of the solution procedure and quantization method proposed originally by Milne in 1930 [1]. Its starting point is the time-independent Schrödinger equation in the form

$$\psi''(x) + k^2(x)\psi(x) = 0, \quad (2.1)$$

where the continuous energy parameter  $E$  and the potential  $V(x)$  are combined into the local wavevector  $k^2(x) = \hbar^2/2m[E - V(x)]$ . Assuming the solution to equation (2.1) to be of the general form

$$\psi(x) = N\rho(x) \sin[\phi(x) + \alpha], \quad (2.2)$$

with normalization constant  $N$ , constant phase  $\alpha$ , amplitude  $\rho(x)$  and variable phase  $\phi(x)$  a direct substitution leads to the constraining equations

$$\rho''(x) + k^2(x)\rho(x) = \frac{\lambda^2}{\rho^3(x)}, \quad \text{and} \quad \rho^2(x)\phi'(x) = \lambda, \quad (2.3)$$

with  $\lambda$  being some arbitrary constant. The first equation in (2.3) is known as the Ermakov-Milne-Pinney (EMP) equation [2, 1, 3]. From (2.2) it is clear that its solution together with a solution for the auxiliary equation for the phase function will lead to an exact solution for the time-independent Schrödinger equation (2.1) for generic values of  $E$ . Notice that when we neglect  $\rho''(x)$ , the two equations in (2.3) combine into  $\phi'(x) = k(x)$  which corresponds to the WKB approximation. In what follows we will employ Pinney's [3] general solution for the EMP-equation<sup>1</sup>

$$\rho(x) = \sqrt{\psi_1^2(x) + \frac{\lambda^2}{W^2}\psi_2^2(x)}, \quad (2.4)$$

with  $\rho(x_0) = \rho_0 \neq 0$ ,  $\rho'(x_0) = \rho'_0$ ,  $-\infty < x_0 < \infty$ . Here  $\psi_1, \psi_2$  are the two fundamental solutions of the Schrödinger equation (2.1) and  $W := W(\psi_1, \psi_2) = \psi_1\psi_2' - \psi_1'\psi_2$  denotes the corresponding Wronskian. Integrating the second equation in (2.3) directly and taking

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<sup>1</sup>There exist other types of solutions, such as for instance the one reported in [5] involving two free constants, which we may, however, suitable chose.

the initial conditions to be  $\psi_1(x_0) = 1$ ,  $\psi_2(x_0) = 0$ ,  $\psi_1'(x_0) = 1$ ,  $\psi_2'(x_0) = \lambda$  implies  $W = \lambda$  and leads to the general solution of equation (2.1) expressed in terms of the solutions to the EMP-equation

$$\psi(x) = N\rho(x) \sin \left[ W \int_{x_0}^x \rho^{-2}(s) ds + \alpha \right]. \quad (2.5)$$

Next we implement the boundary conditions. Demanding the wavefunction  $\psi(x)$  to vanish at the boundaries then implies the quantization condition

$$I(E) = \frac{W(E)}{\pi} \int_{-\infty}^{\infty} \rho^{-2}(s, E) ds = n \in \mathbb{N}, \quad (2.6)$$

when  $\rho(x)$  is non-vanishing, meaning that any solution  $E_n$  to  $I(E_n) = n$  constitutes a bound state energy. Note that the value of  $I(E)$  is not sensitive to the normalization factors in the fundamental solution.

When the potential and possibly also the energy eigenvalues are complex the general treatment is more involved. In that case we can make the Ansatz

$$\psi(x) = N\rho(x)e^{i\phi(x)}, \quad (2.7)$$

with  $\rho(x), \phi(x) \in \mathbb{R}$  and separate the wavevector into its real and imaginary part  $k^2 = \kappa + i\tau$ . The substitution of (2.7) into the time-independent Schrödinger equation then yields the two constraining equations when reading off the real and imaginary parts

$$\rho''(x) + \kappa(x)\rho(x) = \rho(x)\phi'(x), \quad \text{and} \quad \phi''(x)\rho(x) + 2\phi'(x)\rho'(x) + \tau(x)\rho(x) = 0. \quad (2.8)$$

Combining these two equations generalizes the EMP-equation (2.3) to

$$\rho''(x) + \kappa(x)\rho(x) = \frac{1}{\rho^3(x)} \left( \lambda - \int^x \tau(s)\rho^2(s) ds \right)^2 \quad (2.9)$$

with

$$\phi(x) = \lambda \int^x \rho^{-2}(s) ds - \int^x \rho^{-2}(t) \left( \int^t \tau(s)\rho^2(s) ds \right) dt. \quad (2.10)$$

Evidently when  $\tau = 0$  we recover (2.3). These equations are difficult to solve, even in an approximate fashion. However, we may assume that the quantization condition (2.6) still holds when  $W(E) \in \mathbb{R}$  and  $\text{Im}[\rho^2(s, E)]$  is an odd function in  $s$ . We will demonstrate below that these properties can be attributed to the  $\mathcal{PT}$ -symmetry of the models.

### 3. A quasi-Hermitian model, the Swanson Hamiltonian

Quasi/pseudo-Hermitian Hamiltonian systems constitute a large subclass of non-Hermitian systems [6, 7, 8]. They are characterized by the fact that their non-Hermitian Hamiltonian  $H$  can be mapped to an isospectral Hermitian counterpart  $h$  by means of a similarity transformation  $h = \eta H \eta^{-1}$ . The map  $\eta$  is sometimes referred to as the Dyson map [9] and satisfies certain properties. A prime example for which this map and all other relevant quantities are known in its explicit analytic form is the Swanson model [10]

$$H_S = \omega \left( a^\dagger a + 1/2 \right) + \alpha a^2 + \beta \left( a^\dagger \right)^2, \quad \omega, \alpha, \beta \in \mathbb{R}, \quad (3.1)$$

with  $a = \sqrt{\omega/2}x + i/\sqrt{2\omega}p$ ,  $a^\dagger = \sqrt{\omega/2}x - i/\sqrt{2\omega}p$ . Evidently  $H_S$  is only Hermitian when  $\alpha = \beta$ , but its isospectral Hermitian counterpart is known to be [11]

$$h_S = \frac{\mu_+}{2}p^2 + \frac{\mu_-}{2}x^2, \quad (3.2)$$

with

$$\mu_{\pm} = \frac{-\lambda(\alpha + \beta) + \omega \mp (\alpha + \beta - \lambda\omega)\sqrt{1 - \frac{(1-\lambda^2)(\alpha-\beta)^2}{(\alpha+\beta-\lambda\omega)^2}}}{(1 \pm \lambda)\omega^{\pm 1}}, \quad \lambda \in [-1, 1]. \quad (3.3)$$

The eigenvalue spectrum for both Hamiltonians is

$$E_n = \left(n + \frac{1}{2}\right) \sqrt{\omega^2 - 4\alpha\beta}, \quad n \in \mathbb{N}, \quad (3.4)$$

and thus real for  $\omega^2 \geq 4\alpha\beta$ . The corresponding time-independent Schrödinger equations are exactly solvable for both Hamiltonians. The two fundamental solutions for the one corresponding to  $h_S$  can be expressed in terms of parabolic cylinder functions, but in the current context it is more convenient to employ the solutions in terms of the closely related Whittaker functions

$$\psi_1(x) = \frac{1}{\sqrt{x}} M_{\frac{E}{2\sqrt{\mu_- \mu_+}}, -\frac{1}{4}} \left( \sqrt{\frac{\mu_-}{\mu_+}} x^2 \right) \Theta(x) + \frac{i}{\sqrt{x}} M_{\frac{E}{2\sqrt{\mu_- \mu_+}}, -\frac{1}{4}} \left( \sqrt{\frac{\mu_-}{\mu_+}} x^2 \right) \Theta(-x), \quad (3.5)$$

$$\psi_2(x) = \frac{1}{\sqrt{x}} W_{\frac{E}{2\sqrt{\mu_- \mu_+}}, -\frac{1}{4}} \left( \sqrt{\frac{\mu_-}{\mu_+}} x^2 \right) \Theta(x) + \frac{i}{\sqrt{x}} W_{\frac{E}{2\sqrt{\mu_- \mu_+}}, -\frac{1}{4}} \left( \sqrt{\frac{\mu_-}{\mu_+}} x^2 \right) \Theta(-x). \quad (3.6)$$

We neglect here normalization factors for the above mentioned reason. Unlike the solutions in terms of parabolic cylinder functions this choice guarantees that  $\psi_{1,2}(x) \in \mathbb{R}$  or  $\psi_{1,2}(x) \in i\mathbb{R}$ , such that  $\rho(x), W(E) \in \mathbb{R}$ . Using these expressions we compute the energy integral  $I(E)$  in (2.6) and depict our results in figure 1.

The energy eigenvalues are located precisely at the expected values at points of inflection of the function  $I(E)$ .

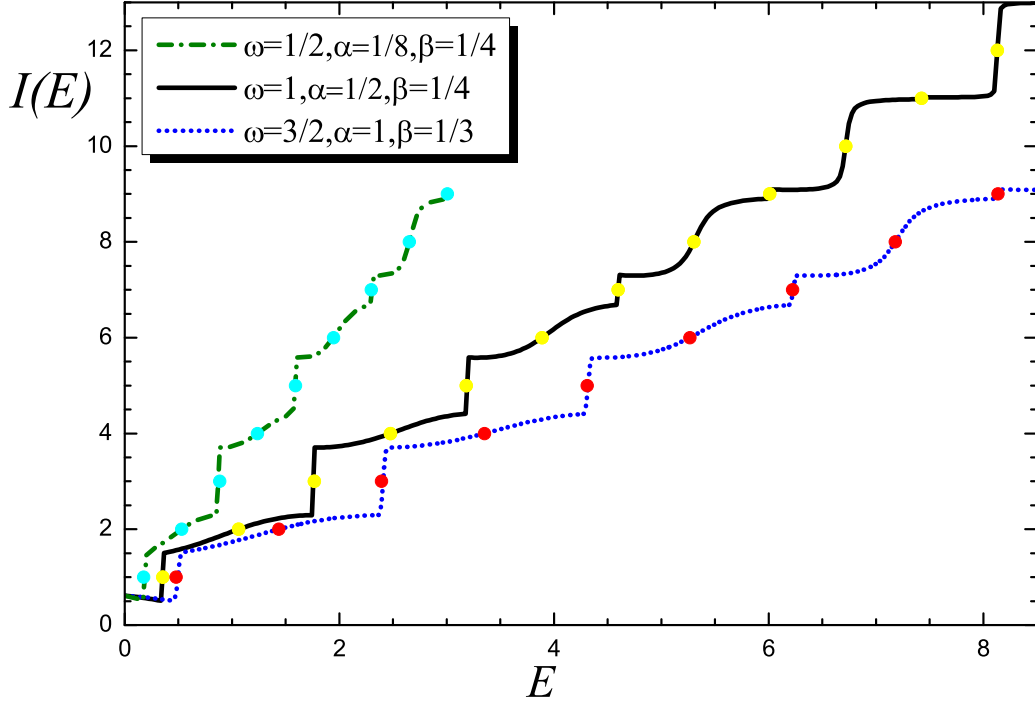
#### 4. Non-Hermitian models with supersymmetric Hermitian counterparts

Now we study a model in which we exploit the  $\mathcal{PT}$ -symmetry of the system. We consider a pair of supersymmetric quantum mechanical [12, 13, 14, 15] models described by the two Hamiltonians

$$H_{\pm} = L_{\pm}L_{\mp} = -\frac{d^2}{dx^2} + U^2(x) \pm U'(x) = -\frac{d^2}{dx^2} + V_{\pm}(x), \quad (4.1)$$

involving the so-called superpotential  $U(x)$ . It is easily verified that the solutions to the time-independent Schrödinger equations  $H_{\pm}\psi_{\pm} = E\psi_{\pm}$  are related to each other by means of the two intertwining operators  $L_{\pm}$

$$L_{\pm} := \pm \frac{d}{dx} + U(x), \quad \psi_{\pm} = \frac{1}{\sqrt{E}} L_{\pm} \psi_{\mp}. \quad (4.2)$$



**Figure 1:** Energy integrals  $I(E)$  for the Swanson model, with  $I((2n-1)/4\sqrt{2}) = I(E_{n-1}) = n \in \mathbb{N}$  for  $\omega = 1/2, \alpha = 1/8, \beta = 1/4$ ,  $I((2n-1)/2\sqrt{2}) = I(E_{n-1}) = n \in \mathbb{N}$  for  $\omega = 1, \alpha = 1/2, \beta = 1/4$  and  $I((2n-1)11/4\sqrt{2}) = I(E_{n-1}) = n \in \mathbb{N}$  for  $\omega = 3/2, \alpha = 1, \beta = 1/3$

Denoting now the two fundamental solutions to the Schrödinger equation by  $\psi$  and  $\chi$ , Ioffe and Korsch [16] found that the corresponding Wronskians and solutions to the EMP-equations

$$W_{\pm} := W(\psi_{\pm}, \chi_{\pm}), \quad \rho_{\pm} = \sqrt{\psi_{\pm}^2 + \chi_{\pm}^2} \quad (4.3)$$

are related to each other as

$$W_{+} = W_{-}, \quad \text{and} \quad E\rho_{\pm}^2 = (L_{\pm}\rho_{\mp})^2 + \frac{W_{\mp}}{\rho_{\mp}^2}. \quad (4.4)$$

The first identity follows from a direct substitution of the wavefunction in (4.2) into the defining relation for the Wronskian, the use of the Schrödinger equation and recalling that  $dW/dx = 0$ . The derivation of the second identity follows from a direct evaluation. We also add here for later use an intermediate relation from that computation

$$E\rho_{+}^2 = U^2\rho_{-}^2 + U(\rho_{-}^2)' + (\psi'_{-})^2 + (\chi'_{-})^2. \quad (4.5)$$

Let us now select our superpotentials to be of a very specific type, such that one of the partner Hamiltonians is Hermitian whereas the other one is not. Such a setting allows us to test our assertions from section 2. Bagchi and Roychoudhury [17] provided a necessary condition for such type of pairs and noted that one may even construct solvable models in

this case. Separating the real and imaginary parts in the superpotentials in the form

$$U(x) = a(x) + ib(x), \quad \text{with } a(x), b(x) \in \mathbb{R}, \quad a(x) = \frac{1}{2} \frac{d}{dx} \ln b(x), \quad (4.6)$$

they observed that one obtains a real and a complex partner potential

$$V_-(x) = \frac{3b'^2}{4b(x)^2} - \frac{b''(x)}{2b(x)} - b(x)^2 \in \mathbb{R}, \quad (4.7)$$

$$V_+(x) = \frac{b''(x)}{2b(x)} - \frac{b'^2}{4b(x)^2} - b(x)^2 + 2ib'(x) \notin \mathbb{R}. \quad (4.8)$$

In the following it will be important to utilize the effect of the parity operator  $\mathcal{P}$  and time-reversal operator  $\mathcal{T}$  on the various quantities involved. Our main requirement is that  $V_+$  becomes  $\mathcal{PT}$ -symmetric, which is achieved as follows

$$\mathcal{PT} : a(x) \rightarrow -a(x), b(x) \rightarrow b(x); \quad \mathcal{PT} : U(x) \rightarrow -U(x), V_{\pm}(x) \rightarrow V_{\pm}(x). \quad (4.9)$$

In order to obtain real eigenvalues  $E \in \mathbb{R}$ , usually referred to as the spontaneously unbroken  $\mathcal{PT}$ -symmetric regime, we also require the wavefunctions to be symmetric with regard to the anti-linear  $\mathcal{PT}$ -operator [18, 19]

$$\mathcal{PT} : \psi_{\pm}(x), \rightarrow \psi_{\pm}(x), \chi_{\pm}(x) \rightarrow \chi_{\pm}(x), W_{\pm}(x) \rightarrow -W_{\pm}(x), \rho_{\pm}(x) \rightarrow \rho_{\pm}(x). \quad (4.10)$$

When assuming that  $\psi_-, \chi_- \in \mathbb{R}$ , it follows from (4.5) and the subsequent use of the second relation in (4.4) that

$$\text{Im}(E\rho_+^2) = \text{Im}\left[(L_+\rho_-)^2\right] = \frac{d}{dx}(b\rho_-^2). \quad (4.11)$$

This implies that for real energies there will not be any contribution to the integral in (2.6) from the imaginary part of the integrand  $1/\rho_+^2$  as it will be an odd function. The assumption  $\psi_-, \chi_- \in \mathbb{R}$  also guarantees that  $W_- \in \mathbb{R}$  and therefore by the first relation in (4.4)  $W_+ \in \mathbb{R}$ , which are the requirements mentioned at the end of section 2.

#### 4.1 A Hermitian/Hermitian supersymmetric pair

As an illustration for the working of the conventional Milne quantization for supersymmetric pairs we first consider a well studied exactly solvable in the mathematical physics literature, [20, 21, 22], the Pöschl-Teller model [23]. Taking the superpotential to be of the form

$$U(x) = \lambda \tan x - \kappa \cot x, \quad \kappa, \lambda \in \mathbb{R}, 0 \leq x \leq \pi/2. \quad (4.12)$$

equation (4.1) yields the pair of potentials

$$V_{\pm}(x) = \lambda(\lambda \pm 1) \sec^2 x + \kappa(\kappa \pm 1) \csc^2 x - (\lambda + \kappa)^2, \quad (4.13)$$

with  $V_-(x)$  being the standard Pöschl-Teller potential. The fundamental solutions are well known. We have

$$\psi_1^-(x) = \sin^{\kappa} x \cos^{\lambda} x {}_2F_1 \left[ \frac{\kappa + \lambda - \tilde{E}}{2}, \frac{\kappa + \lambda + \tilde{E}}{2}; \kappa + \frac{1}{2}; \sin^2 x \right], \quad (4.14)$$

$$\psi_2^-(x) = \sin^{1-\kappa} x \cos^{\lambda} x {}_2F_1 \left[ \frac{1 - \kappa + \lambda - \tilde{E}}{2}, \frac{1 - \kappa + \lambda + \tilde{E}}{2}; \frac{3}{2} - \kappa; \sin^2 x \right], \quad (4.15)$$

and

$$\psi_1^+(x) = \sin^{\kappa+1} x \cos^{\lambda+1} x {}_2F_1 \left[ \frac{2 + \kappa + \lambda - \tilde{E}}{2}, \frac{2 + \kappa + \lambda + \tilde{E}}{2}; \kappa + \frac{3}{2}; \sin^2 x \right], \quad (4.16)$$

$$\psi_2^+(x) = \sin^{-\kappa} x \cos^{\lambda+1} x {}_2F_1 \left[ \frac{1 - \kappa + \lambda - \tilde{E}}{2}, \frac{1 - \kappa + \lambda + \tilde{E}}{2}; \frac{1}{2} - \kappa; \sin^2 x \right], \quad (4.17)$$

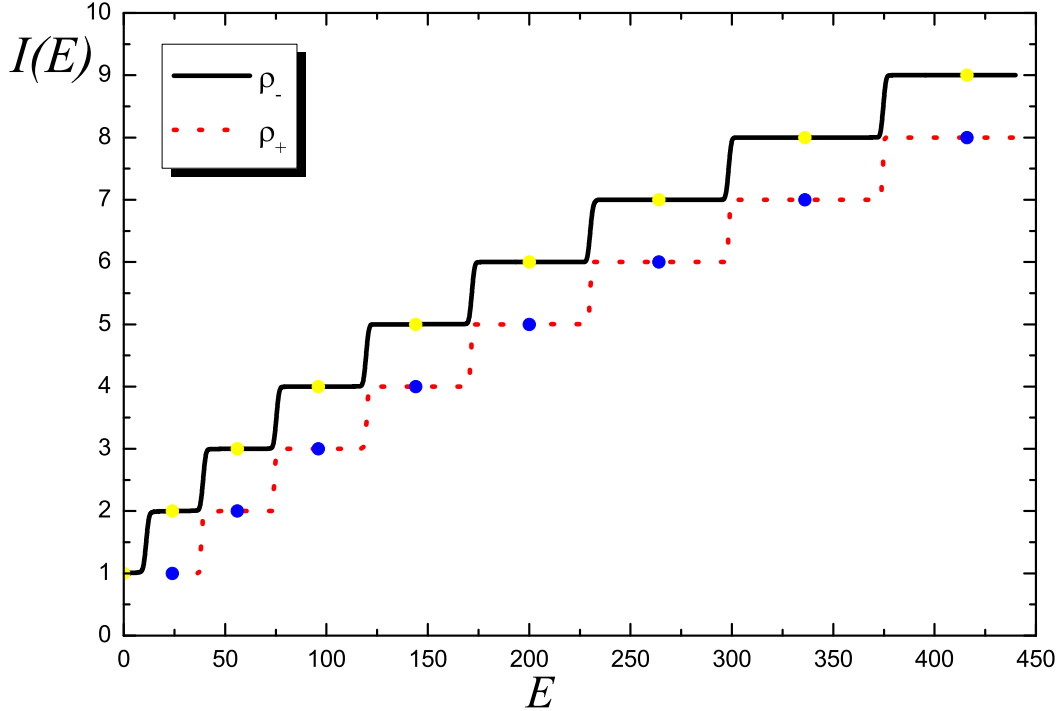
where  ${}_2F_1$  denoted hypergeometric function and we abbreviated  $\tilde{E} := \sqrt{(\kappa + \lambda)^2 + E}$ . Solutions to the EMP-equation are simply obtained from (2.4)

$$\rho_{\pm}(x) = \sqrt{[\psi_1^{\pm}(x)]^2 + [\psi_2^{\pm}(x)]^2}, \quad (4.18)$$

which allows us to compute the energy integrals (2.6) to

$$I_{\pm}(E) = \frac{W_{\pm}(E)}{\pi} \int_0^{\pi/2} \rho_{\pm}^{-2}(s, E) ds. \quad (4.19)$$

Our numerical computations of (4.19) are depicted in figure 2.



**Figure 2:** Energy integrals  $I_{\pm}(E)$  for a supersymmetric pair of Pöschl-Teller potentials for coupling constants  $\kappa = 2$ ,  $\lambda = 3$ , with  $I_-(0) = I_+(24) = 1$ ,  $I_-(24) = I_+(56) = 2$ ,  $I_-(56) = I_+(96) = 3$ ,  $I_-(96) = I_+(144) = 4$ ,  $I_-(144) = I_+(200) = 5$ ,  $I_-(200) = I_+(264) = 6$ ,  $I_-(264) = I_+(336) = 7$ ,  $I_-(336) = I_+(416) = 8$  and  $I_-(416) = 9$ .

For the selected values of the coupling constant  $k = 2$ ,  $\lambda = 3$  the solutions to  $I_{\pm}(E_n^{\pm}) = n + 1$  yield  $E_0^- = 0$ ,  $E_n^+ = E_{n+1}^- = 4(n + 1)(n + 6)$  for  $n = 0, 1, 2, \dots$ . This is of course



the well known quantization condition obtained from demanding that  $\lim_{x \rightarrow 0} \psi_1^\pm(x) = \lim_{x \rightarrow \pi/2} \psi_1^\pm(x) = 0$ , achieved by setting the first entry of the hypergeometric function  ${}_2F_1$  to  $-n$  with  $n = 0, 1, 2, \dots$

#### 4.2 A Hermitian/Non-Hermitian supersymmetric pair

Next we consider a superpotential giving rise to a Hermitian potential paired with a non-Hermitian potential as proposed in [17]. We take the superpotential  $U(x)$  to be of the form

$$U(x) = -\frac{1}{2} \tanh x + \frac{i}{2} (1 - 2\lambda) \operatorname{sech} x, \quad \lambda \in \mathbb{R}, \quad (4.20)$$

such that the real and imaginary parts are related as in (4.6). As expected, when evaluating (4.1) one of the partner potentials turn out to be real

$$V_-(x) = \frac{1}{4} + (\lambda - \lambda^2) \operatorname{sech}^2 x, \quad (4.21)$$

whereas the other one becomes complex

$$V_+(x) = \frac{1}{4} - (1 - \lambda + \lambda^2) \operatorname{sech}^2 x + i(2\lambda - 1) \operatorname{sech} x \tanh x, \quad (4.22)$$

albeit  $\mathcal{PT}$ -symmetric. The fundamental solutions are in this case

$$\psi_1^-(x) = \sinh x \cosh^\lambda x {}_2F_1 \left[ \mu_-, \mu_+; \frac{3}{2}; -\sinh^2 x \right], \quad (4.23)$$

$$\psi_2^-(x) = \cosh^\lambda x {}_2F_1 \left[ \mu_- - \frac{1}{2}, \mu_+ - \frac{1}{2}; \frac{1}{2}; -\sinh^2 x \right], \quad (4.24)$$

and according to (4.2) we obtain the solutions for the partner Hamiltonian as

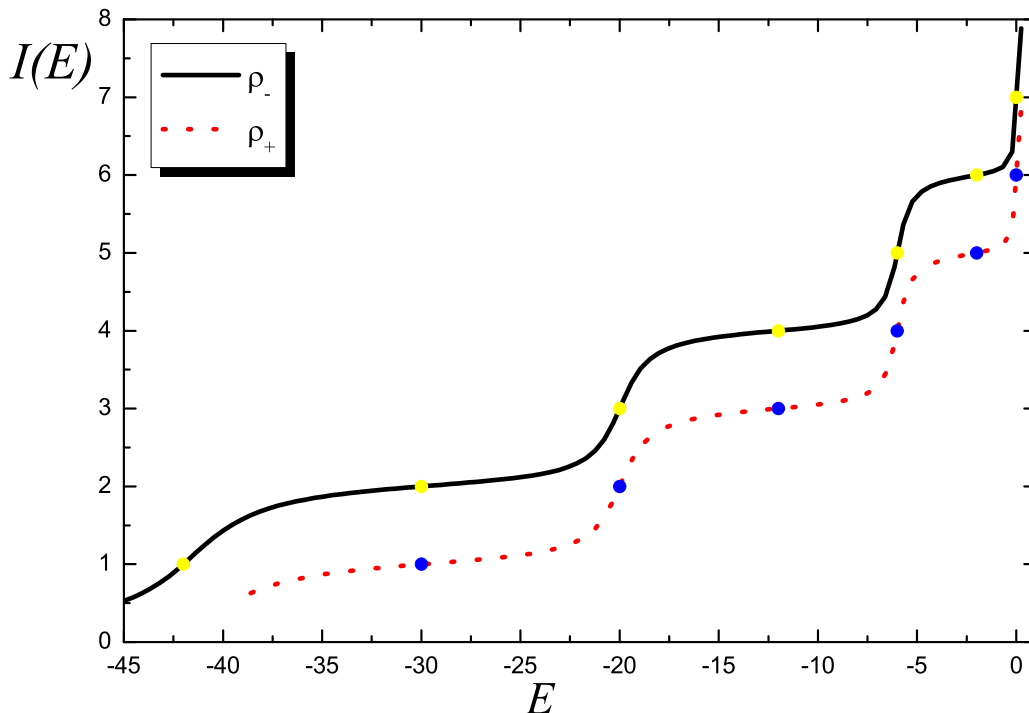
$$\begin{aligned} \psi_1^+(x) = & \frac{\cosh^{\lambda-1}(x)}{12\sqrt{E}} \left[ 6 [2 \cosh^2 x + (2\lambda - 1) \sinh x (\sinh x - i)] {}_2F_1 \left[ \mu_-, \mu_+; \frac{3}{2}; -\sinh^2 x \right] \right. \\ & \left. - \frac{1}{4} \sinh^2(2x) [4E + 4\lambda(\lambda + 2) + 3] {}_2F_1 \left[ \mu_- + 1, \mu_+ + 1; \frac{5}{2}; -\sinh^2 x \right] \right], \quad (4.25) \end{aligned}$$

$$\begin{aligned} \psi_2^+(x) = & \frac{\cosh^{\lambda-1}(x)}{4\sqrt{E}} \left[ 2(2\lambda - 1)(\sinh x - i) {}_2F_1 \left[ \mu_- - \frac{1}{2}, \mu_+ - \frac{1}{2}; \frac{1}{2}; -\sinh^2 x \right] \right. \\ & \left. + (1 - 4E - 4\lambda^2) \sinh x \cosh^2 x {}_2F_1 \left[ \mu_- + \frac{1}{2}, \mu_+ + \frac{1}{2}; \frac{3}{2}; -\sinh^2 x \right] \right], \quad (4.26) \end{aligned}$$

where  $\mu_\pm := (2 + 2\lambda \pm \sqrt{1 - 4E})/4$ .

We have now all the ingredients to evaluate the energy integrals in (4.19). Our results are depicted in figure 3.

For the selected values of the coupling constant  $\lambda$  the solutions to  $I_\pm(E_n^\pm) = n + 1$  yield  $E_0^- = -42$ ,  $E_n^+ = E_{n+1}^- = -(n - 6)(n - 7)$  for  $n = 0, 1, 2, \dots$ . This is again the quantization condition obtained from demanding that  $\lim_{x \rightarrow \pm\infty} \psi_1^\pm(x) = 0$ , achieved by setting the first entry of the hypergeometric function  ${}_2F_1$  to  $-n$  with  $n = 0, 1, 2, \dots$ . The remarkable feature is here that we can still use the standard formula for the Milne quantization even though one of the Hamiltonians is non-Hermitian. Notice that this feature can be attributed entirely to the  $\mathcal{PT}$ -symmetry of the system, which is responsible for the vanishing of the imaginary part in the energy integral.



**Figure 3:** Energy integrals  $I_{\pm}(E)$  for a supersymmetric pair potentials  $V_{\pm}$  in (4.21), (4.22) for the coupling constant  $\lambda = 15/2$ , with  $I_-(-42) = I_+(-30) = 1$ ,  $I_-(-30) = I_+(-20) = 2$ ,  $I_-(-20) = I_+(-12) = 3$ ,  $I_-(-12) = I_+(-6) = 4$ ,  $I_-(-6) = I_+(-2) = 5$ ,  $I_-(-2) = I_+(0) = 6$ , and  $I_-(0) = 7$ .

## 5. Conclusion

We demonstrated that the Milne quantization procedure can be successfully adopted to non-Hermitian systems that are either quasi/pseudo-Hermitian or  $\mathcal{PT}$ -symmetric. For each scenario we provided an explicit example. We proposed some generalized formulae for the generic non-Hermitian case, which are left as a challenge to be solved for some concrete example.

Building on the success, it is to be expected that this method can be applied also to systems for which the quantization is still incompletely understood [24], such as the complex Mathieu system currently of great interest as it corresponds to the eigenvalue equation of the collision operator in a two-dimensional Lorentz gas.

**Acknowledgements:** SD is supported by the Postdoctoral Fellowship jointly funded by the Laboratory of Mathematical Physics of the Centre de Recherches Mathématiques (CRM) and by Prof. Syed Twareque Ali, Prof. Marco Bertola and Prof. Véronique Hussin. LG is supported by the Abdus Salam International Centre for Theoretical Physics (ICTP).

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