An Investigation of the PAYG (Pay-As-You-Go) Financing Method using a Contingency Fund and Optimal Control Techniques

Abstract

In many countries, ageing populations are expected to lead to substantial rises in the cost of public pension systems financed by the pay-as-you-go (PAYG) method. These systems will need to be adapted to cope with these changes. In this paper, we consider one approach to reform, described in the literature as 'parametric' (see, for example, Disney (2000)). We develop a model for adapting the PAYG method using a contingency fund and optimal control techniques. The solution of the original model is investigated within two different frameworks: a deterministic-continuous one and a stochastic-discrete one. Finally, we discuss a worked example applied to Greece, leading to a potentially acceptable proposal of a smooth path for contribution rates and the age of eligibility for the normal retirement pension.

Keywords

PAYG method, Ageing population, Optimal Control

1. Introduction

1.1 The Ageing Population

Many developed countries are experiencing, or are expected to experience over the next 40 to 60 years, the demographic phenomenon called the "ageing population". This describes the upward trend over time in the aged dependency ratio - the number of retired persons relative to the number of working persons in a population. As is well known, there are two demographic causes of this trend – the decline in birth rates and the increase in life expectancies (Banks and Emmerson, 2000). As an illustration we quote an extract from the relevant table of Brown (1992), which presents an international comparison of aged population ratios combined with the corresponding data for the population of Greece, (Table 1.1). Here, the aged population ratio (65+) is defined as the ratio of the number of persons aged above 65 to the total population of persons.

Table 1.1

Aged Population Ratios for a range of countries (%)

Country	1985			2005			2025		
%	65+	75+	85+	65+	75+	85+	65+	75+	85+
UK	15.1	6.3	3.1	15.3	6.9	3.8	18.7	8.1	4.0
USA	12.0	4.9	2.6	13.1	6.7	4.1	19.5	8.5	4.8
China	5.1	1.4	0.5	7.4	2.4	1.0	12.8	4.1	1.8
India	4.3	1.1	0.4	6.1	1.8	0.7	9.7	3.1	1.3
Japan	10.0	3.7	1.7	16.5	6.4	3.0	20.3	8.0	4.9
Greece	13.4	5.3	1.2	18.4	7.8	1.6	22.2	11.5	3.2

Sources: 1) U.S. Department of Commerce (1987), 46-62;

2) National Statistical Service of Greece (1998), Tables 1995;

3) Eurostat (1999), Demographic Statistics, Data 1960-1999

1.2 Basic Principles of the PAYG Financing Method

The PAYG financing method requires no accumulation of funds, although normally there is a small fund for liquidity purposes. The insured group of lives is split in two subgroups, the active lives (or workers or contributors) and the retired lives (or pensioners). A specific time period is chosen, usually one calendar year, and a balance between contribution income and benefit outgo is required to hold, i.e.

$$cWL = bP \tag{1.1}$$

where c is the required contribution rate, W is the average wage(salary), L is the number of contributors, P the number of eligible pensioners and b the level of average pension.

Each active life pays contributions to meet the benefits of the current retired lives while relying on the goodwill of future generations of active lives to contribute to the cost of his/her retirement benefits. Inter-generational solidarity and equity are two necessary requirements for operating a successful PAYG system.

Solidarity is defined as the willingness of different groups of people (in the context of public pension systems, the concept refers both to young and old generations) to participate in a common pool, sharing actual experience, including any losses emerging (see Wilkie (1997) for further discussion).

Inter-generational equity is defined as the situation where, under a certain type of measurement (normally a financial one) all generations are equal to each other. A widely used metric is the implied rate of return, defined by Lapkoff (1991) to be "the interest rate that equalizes the stream of contributions to the stream of benefits and would have been the interest rate applicable had the contributions actually been invested". This rate of return can be calculated either for individuals or cohorts of lives. Samuelson (1958) demonstrates that, under certain conditions, the implied rate of return equals the population growth plus the growth rate of real wages.

1.3 PAYG Public Pension Systems: Rising Costs

If we apply the international demographic trend described in subsection (1.1), it is clear that the public pension systems of all countries operating under the PAYG model will be faced with rising costs over the next years. Equation (1.1) demonstrates that, to maintain balance, c will have to increase if the decline in fertility rates leads to a decrease in L at the same time as a decrease in mortality rates leads to an increase in P (assuming also constant values for the parameters Wand b).

As noted by Disney (2000), a range of proposals has been put forward and these can be categorized broadly under four headings.

a) "parametric" reforms, where the nature and structure of the PAYG system are maintained but changes are made to the key variables in equation (1.1), for example, increasing the retirement age, raising contribution rates, limiting the generosity of retirement benefits;

b) "actuarially fair" reforms where a direct link between contribution and benefit at the individual level is introduced (as in a defined contribution pension scheme);

c) privatization;

d) a multi-pillar approach, involving a combination of partial privatization and a reduced (residual) public system (i.e. a combination of a) and c)). The World Bank (1994) has advocated this specific approach.

Banks and Emmerson (2000) provide a fuller discussion of these options in the context of the UK. Our proposal represents a reform of type a), which involves controlling two of the important variables of the system. i.e. the contribution rate and the age of eligibility for normal retirement. Of course, the control and consequently the modification of these variables is not an easy exercise as any solution or action should be acceptable by society in general and by the lives who are going to meet the costs.

2. Motivation for a New Approach to the PAYG Method

Normally, in the PAYG method there is no reserve fund. In our model, we abandon this basic assumption and introduce a contingency fund, which operates as a buffer absorbing fluctuations in mortality or fertility patterns. The existence of this fund improves significantly the performance of the system with respect to the concept of inter-generational equity by smoothing the rates of return produced for each cohort of lives. This non-zero reserve fund has two basic characteristics.

(a) It fluctuates deliberately (in the short run) in order to absorb fluctuations in mortality, fertility rates or other random events (e.g. stochastic interest rates).

(b) It returns to zero when the fluctuations disappear leaving the system at a new equilibrium point regarding the two major control variables mentioned above.

In order to achieve (a) and (b), we build a control model in which the reserve fund has the ability to distinguish the varying or constant nature of a certain demographic pattern. The age of normal retirement and the contribution rate are to be controlled through a smooth path over time. The smoothness of the path is determined by a functional, which weights changes in the two variables. The weights are key parameters, which reflect the expectations of all participants in the pension system as well as the underlying demographic trends. Below, we briefly comment on each of the two control variables under consideration and also put forward some practical aspects with respect to the management of such a contingency fund.

2.1 The Age of Normal Retirement

The age of normal retirement determines the actual number of active and retired lives and consequently the potential wages and pensions. This variable could be linked with the post retirement life expectancy. We could argue that it would be fair that each cohort of lives should live in retirement an equal percentage of its total lifetime. That means cohorts with high life expectancy should retire at a higher age and cohorts with low life expectancy should retire earlier. This would also comply with the view of Dilnot et al (1994) who point that "If the population is ageing because of increased longevity then individuals will need a longer period in the labor force to obtain a given level of average consumption over their lifetime. This might lead to individuals prolonging their working lives by postponing retirement...".

As noted by Banks and Emmerson (2000), care is needed with the parametric reform approach in terms of defining precisely the meaning of each parameter or component of the model. Thus, the age of retirement could be “the age at which one is first entitled to receive benefits at a reduced rate, or the age at which one is entitled to receive full benefits, or the age at which one actually retires”. Since our overriding purpose is to discuss and present a methodology, we overlook such detailed points, while acknowledging that they would need to be allowed for in any real, practical implementation.

2.2 The Contribution Rate

Similarly, the contribution rate could be linked with the fertility rates and the respective population growth rate. The higher is the population growth rate, then the smaller is the contribution rate which would be applied to the PAYG model. A rule (similar to that proposed for the age of normal retirement and life expectancy) may be put forward in order to link and control the contribution rate with respect to changes in the population growth rate. Analogously, the contribution rate may be also linked with the growth rate of real wages.

2.3 The Contingency Fund

The contingency fund may fluctuate deliberately and can take positive or negative values. In the first case (positive) there is a surplus which can be invested (normally conservatively, e.g. buying Government Bonds) while in the second case (negative) there is a deficit, which may be covered by borrowing (e.g. the supervisory authority of the Social Security System issues Bonds with a level of Governmental Guarantee).

3. <u>Formulation of the Model using Optimal Control Techniques</u>

In this section, we proceed with the presentation of our proposed model, translating the general discussion and motivation of the last section into equations. The model is initially established as a deterministic one in a continuous framework and then reformulated as a stochastic one in a discrete background.

3.1 Notation of the Model

F(t) : reserve (accumulated) fund at time t ("contingency fund")

c(t) : contribution rate at time t

r(t) : normal retirement age at time t

pl(t,y) : population density aged y ~~(exact)~~ at time t

s(t,y) : total wage (salary) rate ~~received by~~ for a person aged y (exact) at time t

b(t,y) : total pension benefit rate ~~paid to~~ for a life aged y (exact) at time t

a : age of entry to the labor force

ω : limiting age of the life table ($l_\omega=0$)

W(t,y): Total wage rate (where the contribution rate c(t) is applicable to) at time t if the relevant retirement age has been fixed at age y

$$W(t,y)=\int_a^y pl(t,z)\cdot s(t,z)dz.$$

B(t,y) : Total rate of benefits paid at time t if the relevant retirement age has been fixed at age y

$$B(t,y)=\int_y^\omega pl(t,z)\cdot b(t,z)dz.$$

c_0 : A standard value for the contribution rate. We may consider it as an initial value or average value near to which we aim to place the path of the future consecutive values of c(t) using the smoothing process.

r_0 : A standard value for the retirement age, with similar meaning to c_0.

δ : The force of investment rate of return applicable to the reserve fund (assumed to be constant).

θ : The weight applicable to any change of the contribution rate from the standard value c_0 and $(1-\theta)$ is the weight applicable to any change of the retirement age from the standard age r_0.

p : The growth rate of real wages, corresponding to improvements in productivity.

We assume that functions B(t,y) and W(t,y) for benefits and wages are known for each time t, $t \in [0, \infty]$ and for any age, y. These forms may be derived from demographic projections.

3.2 Formulation of the Respective Equations

The first equation describes the development of F(t) i.e.

$$F'(t) = \delta F(t) + c(t) W(t, r(t)) - B(t, r(t)). \tag{3.1}$$

This is a differential equation for F, which determines a dynamic system where F(t) is the state variable and c(t), r(t) are the control (input) variables.

The supervisory authority of a public pension system has to determine a "smooth" path for the control variables guiding the system over time and targeting a zero (or almost zero) fund value. The zero value has to be achieved at the end of a specific time period say, T (i.e., the ending time point is denoted by T). The existence of a "smooth" path requires that the relevant choices for c(t) and r(t) will comply with people's expectations and the other technical criteria described in section (2).

Then, the optimal path may be derived by minimization of the following expression

$$\min_{c(t),r(t)} \int_0^T \left\{ \theta \cdot \left[100 \cdot \left(c(t) - c_0 \right) \right]^2 + (1-\theta) \cdot \left(r(t) - r_0 \right)^2 \right\} dt \qquad (3.2)$$

The coefficient of 100 has been applied to the deviation of the contribution rate in an attempt to deal with the metric problems which exist as c(t) is a percentage less than unity and r(t) is a number greater than unity (near 65). The weights θ and $1-\theta$ measure the impact which occurs when the control variables c(t) and r(t) respectively are changed. The parameter θ would be obtained after research and negotiations with all the parties involved in the public pension system (i.e. government, employers, employees). Hence, expression (3.2) minimizes the effects induced in the public pension system by changes in the contribution rate and normal retirement age.

The functional in expression (3.2) is a special case of the more general proposal of Haberman and Sung (1994), which in this case would be defined as follows:

$$\min_{c(t),r(t)} \int_0^T \left\{ \theta \cdot \left[100 \cdot \left(c(t) - \tau c_t \right) \right]^2 + (1-\theta) \cdot \left(r(t) - \tau r_t \right)^2 \right\} dt \qquad (3.3)$$

where τc_t and τr_t are the desired contribution rate and normal retirement age at time t. In our context, $\tau c_t = c_0$ and $\tau r_t = r_0$ for any value of t.

The functional expression (3.2) does not contain the fund values as we assume that F(t) will be varied deliberately in order to absorb fluctuations in experience. Of course, we require a small value for the fund level at the end of the respective period of examination at time t=T

i.e. $$F(T) \in (-\gamma, +\gamma),\ \gamma > 0 \qquad (3.4)$$

or more strictly $$F(T) = 0\ . \qquad (3.5)$$

Combining equations (3.1) (3.2) and (3.5) we obtain the typical form of an optimal deterministic control problem as below.

$$\left.\begin{array}{c}\min_{c(t),r(t)}\int_0^T \left\{\theta\left[100\cdot\left(c(t)-c_0\right)\right]^2+(1-\theta)\cdot\left(r(t)-r_0\right)^2\right\}dt \\ F'(t)=\delta F(t)+c(t)W(t,r(t))-B(t,r(t)) \\ F(0)=F_0 \quad \text{and} \quad F(T)=0.\end{array}\right\} \qquad (3.6)$$

3.3 The equilibrium point of the model under a stable population pattern

Before going further with the manipulation of system (3.6), we investigate the equilibrium point of our model under a stable demographic pattern. We assume that

$$s(t,y)=s(0,y)e^{pt}=se^{pt} \text{ assuming } s(0,y)=s, \forall y\in[a,r_0) \qquad (3.7)$$

$$b(t,y)=b(0,y)e^{pt}e^{-p(y-r_0)}=be^{pt}e^{pr_0}e^{-py} \text{ assuming } b(0,y)=b, \forall y\in[r_0,\omega] \qquad (3.8)$$

i.e. we assume that wages increase over time exponentially at a rate p. Pension benefits at the age of eligibility for normal retirement r_0 are linked to the final wage/salary through the replacement ratio b/s, and subsequently the pension benefit remains constant over time.

The stable population (in the sense of Keyfitz (1985)) may be considered as a generalization of the stationary population. The number of lives is not constant as time passes but increases or decreases at a constant rate, say g (actually the number of lives entering the population at the starting age a increases or decreases at a constant rate, say g). A negative g (for a finite period) would be consistent with "demographic ageing" (while g=0 would imply stationary population). So, following Keyfitz (1985), we obtain the following formula for pl(t,y):

$$pl(t,a)=pl(0,a)e^{gt}=l_a e^{gt} \qquad (3.9)$$

assuming, without loss of generality, that $pl(0,a)=l_a$ where l_a represents the underlying life table of the stable population. Then,

$$pl(t,y)=pl(t-(y-a),a)\frac{l_y}{l_a}=l_a e^{g(t-(y-a))}\frac{l_y}{l_a}=e^{g(t+a)}e^{-gy}l_y. \qquad (3.10)$$

Then, we choose the “ideal” pair for (c_0,r_0) respectively using the formula below, based on (1.1):

$$c_0 \cdot \int_a^{r_0} pl(t,z)s(t,z)dz = \int_{r_0}^{\omega} pl(t,z)\cdot b(t,z)dz \Rightarrow c_0 \cdot s \cdot \int_a^{r_0} l_z \cdot e^{-gz}dz = be^{pr_0}\int_{r_0}^{\omega} l_z \cdot e^{-gz} \cdot e^{-pz}dz \tag{3.11}$$

(see Keyfitz (1985) for further details) and if we consider g and g+p as forces of interest we may rewrite equation (3.11) in terms of continuous certain annuity values i.e.

$$c_0 s\, \bar{a}^{(g)}_{a:\overline{r_0-a}|} = b\frac{l_{r_0}}{l_a}e^{-g(r_0-a)}\, \bar{a}^{(g+p)}_{r_0:\overline{\omega-r_0}|}. \tag{3.12}$$

It is clear from the expression (3.12) above that an increase in the productivity rate, p, leads to a reduction in c_0 given a constant value for r_0, or similarly a reduction in r_0 for a constant value of c_0.

Assuming that $F_0 = 0$, then $c(t) = c_0$ and $r(t) = r_0$ for all $t \geq 0$ is the optimal path that satisfies the system (3.6). This is because the expression (3.2) is minimized with value equal to zero. Then equation (3.1) with the condition (3.5) is satisfied by the function $F(t) = 0$ for all $t \geq 0$.

3.4 Non-Stable Population

In a non-stable population we may distinguish two cases.

(i) The demographic pattern may be approximated by a combination of stationary or stable populations. For example, we may have a population with a decreasing growth rate say $g(t)$ where $g(0) = g_0$ and $\lim_{t\to\infty} g(t) = g_\infty$. Consequently, the population will be asymptotically stable with a growth rate of g_∞. In such cases we may use similar reasoning as before and imagine the optimal path lying in the area described by the lines of the initial and ultimate growth rates.

(ii) The demographic pattern cannot be described by a combination of stable and stationary patterns. Then, in such cases, we have to solve the system (3.6) and design the respective optimal path according to the analysis described in section 3 for our model.

4. The solution of the model within a deterministic - continuous framework

The continuous version of the model is described exactly by the system of equations (3.6). The solution is complex as it requires a functional optimization. Rather than tackle the general case, we investigate its solution under a deterministic framework while using (for practical reasons) linear approximations for the Wages and Benefits functions W(t,y) and B(t,y). This linearization technique is widely used to solve control problems (see – Kamien & Schwartz (1981) and Benjamin (1989)).

We use the result of Kamien & Schwartz (1981) which states that at least one solution exists if the integrand of the first expression of system (3.6) and the right hand side of the respective differential equation of the system are convex in (F,c,r). The solution of the problem may be determined using the Hamiltonian of the system which is defined as:

$$H(t) = \theta\left[100\left(c(t) - c_0\right)\right]^2 + (1-\theta)\cdot\left[r(t) - r_0\right]^2 + p(t)\delta F(t) + p(t)c(t)W(t, r(t)) - p(t)B(t, r(t)) \quad (4.1)$$

where p(t) is the relevant costate vector of the system and is given by the following equation

$$p' = -\frac{\partial H}{\partial F}, \qquad \text{so we obtain} \qquad p'(t) = -\delta p(t) \Leftrightarrow p(t) = p_0 e^{-\delta t}. \quad (4.2)$$

The optimal c(t) and r(t) controls can be found as the solution of the following system

$$\frac{\partial H}{\partial c} = 0 \quad \text{and} \quad \frac{\partial H}{\partial r} = 0. \quad (4.3)$$

Differentiating the Hamiltonian according to (4.3) and equating to zero we obtain

$$2\theta 100^2\left[c(t)-c_0\right]+p(t)W(t,r(t))=0 \qquad (4.4)$$

$$2(1-\theta)\left[r(t)-r_0\right]+p(t)c(t)\cdot\frac{\partial}{\partial r(t)}\cdot W(t,r(t))-p(t)\frac{\partial}{\partial r(t)}\cdot B(t,r(t))=0 \qquad (4.5)$$

We use the following linear approximations for W(t,y) and B(t,y)

$$W(t,r(t))=\lambda_1 t+\lambda_2 r(t)+\lambda_3 \qquad (4.6)$$

$$B(t,r(t))=k_1 t+k_2 r(t)+k_3 \qquad (4.7)$$

where $k_1,k_2,k_3,\lambda_1,\lambda_2,\lambda_3$ are constant coefficients obtained by a standard linearization procedure

$\lambda_1=\frac{\partial}{\partial t}W(t,r(t))\Big|_{t=r_0}$, $\lambda_2=\frac{\partial}{\partial r(t)}W(t,r(t))\Big|_{r(t)=r_0}$ $\lambda_3=W(t_0,r(t_0))\Big|$, and $k_1, k_2\, k_3$ are similarly defined.

Substituting equations (4.6) and (4.7) into (4.3) and (4.4) we obtain

$$2\theta 100^2\cdot\left[c(t)-c_0\right]+p(t)\cdot\left[\lambda_1 t+\lambda_2 r(t)+\lambda_3\right]=0 \qquad (4.8)$$

$$2(1-\theta)\cdot\left[r(t)-r_0\right]+\lambda_2 p(t)\cdot c(t)-k_2 p(t)=0\ . \qquad (4.9)$$

We rearrange the terms of the equations above in order to obtain the system in the standard format and solve it by using the relevant determinant

$$2\theta 100^2 c(t)+\lambda_2 p(t)r(t)=2\theta 100^2 c_0-\lambda_1 t p(t)-\lambda_3 p(t) \qquad (4.10)$$

$$\lambda_2 p(t)c(t)+2(1-\theta)r(t)=2(1-\theta)\, r_0+k_2 p(t). \qquad (4.11)$$

The solution of the system is given by equations (4.12) and (4.13) below. These are obtained as follows. We define:

$$c(t) = \frac{D_{C(t)}}{D} \quad \text{and} \quad r(t) = \frac{D_{r(t)}}{D}, \qquad D = \det\begin{bmatrix} 2\theta 100^2 & \lambda_2 p(t) \\ \lambda_2 p(t) & 2(1-\theta) \end{bmatrix} = 4\theta(1-\theta)100^2 - [\lambda_2 p(t)]^2 \neq 0$$

so that $D_{C(t)}$ and $D_{r(t)}$ are the determinants which are produced by substituting the column of the relevant index in the determinant D. Allowing for equation (4.2) for the costate variable p(t), we finally obtain

$$c(t) = \frac{[2(1-\theta)] \cdot \left[2\theta 100^2 c_0 - \lambda_1 t p_0 e^{-\delta t} - \lambda_3 p_0 e^{-\delta t}\right] - \left[\lambda_2 p_0 e^{-\delta t}\right]\left[2(1-\theta) r_0 + k_2 p_0 e^{-\delta t}\right]}{4\theta(1-\theta)100^2 - \left[\lambda_2 p_0 e^{-\delta t}\right]^2} \quad (4.12)$$

$$r(t) = \frac{\left[2\theta 100^2\right] \cdot \left[2(1-\theta)\, r_0 + k_2 p_0 e^{-\delta t}\right] - \left[\lambda_2 p_0 e^{-\delta t}\right]\left[2\theta 100^2 c_0 - \lambda_1 t p_0 e^{-\delta t} - \lambda_3 p_0 e^{-\delta t}\right]}{4\theta(1-\theta) \cdot 100^2 - \left[\lambda_2 p_0 e^{-\delta t}\right]^2} \quad (4.13)$$

Equations (4.12) and (4.13) should be combined with the second equation of system (3.6) and using the third condition of (3.6) we obtain the value of p_0.

The sufficiency conditions for the existence of the minimum requires the following matrix A to be positive definite

$$A = \begin{bmatrix} \frac{\partial^2 H}{\partial c^2} & \frac{\partial^2 H}{\partial r \partial c} \\ \frac{\partial^2 H}{\partial c \partial r} & \frac{\partial^2 H}{\partial r^2} \end{bmatrix} = \begin{bmatrix} 2\theta 100^2 & \lambda_2 p(t) \\ \lambda_2 p(t) & 2(1-\theta) \end{bmatrix}.$$

This is equivalent to requiring the minors to be greater than zero, which leads to the two inequalities below

$$2\theta \cdot 100^2 > 0 \quad \text{and} \quad \det(A_1) = D = 4 \cdot 100^2 \theta(1-\theta) - [\lambda_2 p(t)]^2 > 0.$$

The last inequality holds if and only if $p(t) \in \left(-\frac{200}{\lambda_2}\sqrt{\theta(1-\theta)}, +\frac{200}{\lambda_2}\sqrt{\theta(1-\theta)} \right).$

Closing this section, we comment on the solution of the system (see equations (4.12), (4.13)) and its behaviour as each parameter changes. Firstly, we observe that

$$\lim_{t\to\infty} c(t) = c_0 \quad \text{and} \quad \lim_{t\to\infty} r(t) = r_0 . \tag{4.14}$$

Hence, the ultimate values for c(t) and r(t) converge to the equilibrium point (c_0, r_0). As regards the behavior of the solution with respect to the different parameters involved we may draw some conclusions observing the position of each parameter (whether it is in the numerator or denominator in equations (4.12), (4.13)), its sign (positive or negative) and its relevant coefficient. Thus, we note the following:

(1) The higher is δ, the force of interest, the faster the solution converges to its ultimate state of (c_0, r_0).

(2) c_0 appears in the numerator of c(t) with positive sign, so as c_0 increases so does the solution for c(t). c_0 also appears in the numerator of r(t) but with a negative sign and consequently the magnitude of r(t) decreases as c_0 increases.

(3) A similar situation (as with comment (2)) applies to the parameter r_0. As r_0 increases, then r(t) increases while c(t) decreases.

(4) Parameters λ_1, λ_3 appear in the numerator of c(t) with a negative sign, so higher values will result in smaller values for c(t). The opposite result holds for r(t) as λ_1, λ_3 appear in the numerator but with positive sign. These results are as expected since λ_1, λ_3 are parameters in the wages function (4.6). So the bigger are λ_3 (constant term in the function) or λ_1 (the slope of the function with respect to time) the lower is the contribution required.

(5) Parameter k_2 (the slope of benefits with respect to age of normal retirement in (4.7)) appears in the numerator of c(t) with a negative sign and in the numerator of r(t) with a positive sign (and so similar results as before may be drawn).

(6) Parameter λ_2 (the slope of wages with respect to age of normal retirement in (4.6)) appears both in the numerator and denominator of c(t) and r(t) with a negative sign, so its effect is the same for both control variables while the direction of the effect depends on the magnitude of the parameters in the expressions.

(7) Finally parameter θ appears both in the numerator and denominator of c(t) and r(t). From its pattern we may conclude that c(t) increases and r(t) decreases as θ increases, a result that would be expected by general reasoning as θ is the weight given to c(t) in (3.6).

5. <u>The solution of the model within a stochastic - discrete framework</u>

The general model in the continuous form (apart from being very difficult to solve, especially within a stochastic background) is not practical because it describes the contribution rate c(t) and the age of normal retirement r(t) as continuous functions. Obviously, it is impossible in practice to change these variables "continuously". In this section, we reformulate the problem within a stochastic – discrete framework in order to produce an improved approximation to the real world. This new version of the model needs certain adaptations in notation and equations. Firstly, we define the equivalent annual accumulation factor J_n for the n^{th} year assuming a non-constant rate of interest i_n

$$J_n = 1 + i_n \, , \tag{5.1}$$

Additionally and in order to improve the realistic approach of our model we assume that i_n is a stochastic variable (consequently. J_n is a stochastic variable).

Then, the other symbols are defined in a similar manner to section 3, keeping in mind the discrete format of the process (with the variable t replaced by n) i.e.

F_n: reserve fund at time n (at the end of the n-th year).

c_n: contribution rate during the n-th year, (assumed to be constant for the whole year)

r_n: age of eligibility for normal retirement during the n-th year (assumed to be constant for the whole year).

W(n,y): Total wages during the n-th year if the retirement age has been fixed at age y

B(n,y): Total benefits paid during the n-th year ~~at time n~~ if the retirement age has been fixed at age y

The differential equation (3.1) then becomes the difference equation

$$F_{n+1} = J_n F_n + c_n W(n, r_n) - B(n, r_n). \tag{5.2}$$

The minimization criterion of (3.2) becomes (substituting the integral with the summation operator, changing the notation of the ending point from T to m -i.e. the minimization criterion should be fulfilled within a time period of m years- and adding the expectation operator E)

$$\min_{c_n, r_n} \mathrm{E}\left\{ \sum_{n=1}^{m} \left\{ \theta\left[100\left(c_n - c_0\right)\right]^2 + (1-\theta)\left[r_n - r_0\right]^2 \right\} \right\}. \tag{5.3}$$

Again, we assume a linear format for the development of the Wages and Benefits as described by equation (4.6) and (4.7). Now, in order to improve the realistic approach of our model we assume that the constant terms $\lambda_{3,n,}$ and $k_{3,n}$ are stochastic variables. Under this modification, we incorporate a stochastic element into our linear projections for Wage W(n,y) and Benefit B(n,y) functions. We restrict the stochastic nature only for $\lambda_{3,n,}$ and $k_{3,n}$ although in general $\lambda_{1,}$ λ_2 , k_1 and k_2 could also be stochastic variables or of a time–varying format. This restriction secures a viable solution for our model and focus on the effect of the constant terms of the linear functions. So, we obtain

$$F_{n+1} = J_n F_n + c_n(\lambda_1 n + \lambda_2 r_n + \lambda_{3,n}) - (k_1 n + k_2 r_n + k_{3,n}) . \quad (5.4)$$

~~At this point, we must stress that J_n, $\lambda_{3,n}$, and $k_{3,n}$ are assumed to be stochastic variables.~~

The last difference equation (5.4) is still difficult to solve, as it remains a non-linear one. We proceed with a standard linearization procedure considering the Δ operator, the equilibrium point $(n_0, F_{n_0}, J_{n_0}, c_{n_0}, r_{n_0}, \lambda_3, k_3)$ and small changes for the parameters i.e.

$$F_n = F_{n_0} + \Delta F_n \qquad c_n = c_{n_0} + \Delta c_n \qquad r_n = r_{n_0} + \Delta r_n \quad (5.5)$$

$$J_n = J_{n_0} + \Delta J_n \qquad \lambda_{3,n} = \lambda_3 + \Delta\lambda_n \qquad k_{3,n} = k_3 + \Delta k_n \quad (5.6)$$

As we have stated before, J_n, $\lambda_{3,n}$ and $k_{3,n}$ are stochastic variables so, the terms ΔJ_n, $\Delta\lambda_n$ and Δk_n ~~$\Delta\lambda_{3,n}$ and $\Delta k_{3,n}$~~ are also stochastic variables.

$$\Delta F_{n+1} = J_{n_0}\Delta F_n + F_{n_0}\Delta J_n + \lambda_1 c_{n_0} + \lambda_1 n_0 \Delta c_n + \lambda_2 r_{n_0}\Delta c_n + \lambda_2 c_{n_0}\Delta r_n + \lambda_3 \Delta c_n + c_{n_0}\Delta\lambda_n - k_1 - k_2\Delta r_n - \Delta k_n \Rightarrow$$

$$\Delta F_{n+1} = J_{n_0}\Delta F_n + \left[\lambda_1 n_0 + \lambda_2 r_{n_0} + \lambda_3\right]\Delta c_n + \left[\lambda_2 c_{n_0} - k_2\right]\Delta r_n + \left[\lambda_1 c_{n_0} - k_1\right] + F_{n_0}\Delta J_n + c_{n_0}\Delta\lambda_n - \Delta k_n$$

$$\Delta F_{n+1} = J_{n_0}\Delta F_n + \left[\lambda_1 n_0 + \lambda_2 r_{n_0} + \lambda_3 \quad \lambda_2 c_{n_0} - k_2\right]\begin{bmatrix}\Delta c_n \\ \Delta r_n\end{bmatrix} + \left[\lambda_1 c_{n_0} - k_1\right] + \left[F_{n_0}\Delta J_n + c_{n_0}\Delta\lambda_n - \Delta k_n\right] \quad (5.7)$$

The last equation may be rewritten as a matrix difference equation

$$x_{n+1} = Ax_n + Bu_n + \phi + \xi_n \quad (5.8)$$

where, $x_n = \Delta F_n$, $A = J_{n_0}$, $B = [B_1 B_2] = \left[\lambda_1 n_0 + \lambda_2 r_{n_0} + \lambda_3 \quad \lambda_2 c_{n_0} - k_2\right]$

$$u_n = \begin{bmatrix} \Delta c_n \\ \Delta r_n \end{bmatrix}, \qquad \phi = \lambda_1 c_{n_0} - k_1 \quad , \quad \xi_n = F_{n_0} \Delta J_n + c_{n_0} \Delta \lambda_n - \Delta k_n$$

We note that ξ_n is a stochastic variable because it is written as a linear combination of the stochastic terms ΔJ_n, $\Delta \lambda_n$ and Δk_n ~~$\Delta \lambda_{3,n}$ and $\Delta k_{3,n}$~~.

Substituting the last two expressions of (5.5) into the objective function (5.3) we obtain

$$\min_{\Delta c_n, \Delta r_n} E \sum_{n=1}^{m} \left\{ 100^2 \theta \Delta c_n^2 + (1-\theta) \Delta r_n^2 \right\} = \min_{\Delta c_n, \Delta r_n} E \sum_{n=1}^{m} [\Delta c_n \Delta r_n] \begin{bmatrix} 100^2 \theta & 0 \\ 0 & 1-\theta \end{bmatrix} \begin{bmatrix} \Delta c_n \\ \Delta r_n \end{bmatrix} \tag{5.9}$$

and assuming $Q = \begin{bmatrix} 100^2 \theta & 0 \\ 0 & 1-\theta \end{bmatrix}$ we derive $\min_{\Delta c_n, \Delta r_n} E \sum_{n=1}^{m} [\Delta c_n \Delta r_n] Q \begin{bmatrix} \Delta c_n \\ \Delta r_n \end{bmatrix}$ (5.10)

The optimization (minimization) problem described by (5.10) under the restriction imposed by the difference equation (5.8) is fully investigated in Appendix II and the respective solution (in the steady state) is the following

$$u_n = M x_{n-1} + \zeta \tag{5.11}$$

where $M = \begin{bmatrix} M_1 \\ M_2 \end{bmatrix}$, $g = \begin{bmatrix} \zeta_1 \\ \zeta_2 \end{bmatrix}$ and H is a real number, satisfying the system below

$$M = -\left(B^T H B + Q\right)^{-1} \left(B^T H A\right) \tag{5.12}$$

$$H = (A + BM)^T H (A + BM) + M^T Q M \tag{5.13}$$

$$\zeta = -\left(B^T H B + Q\right)^{-1} \left(B^T H \phi\right) \tag{5.14}$$

A certain body of text (and the respective computations) has been moved from this point down to the end of Appendix II. We

have also modified (slightly) the remaining text (below) of the current section 5.

So, the optimal choice (as derived in Appendix II, see formulae (II.15), (II.16)) for the control vector $u_n = \begin{bmatrix} \Delta c_n \\ \Delta r_n \end{bmatrix}$ i.e. for the contribution rate c_n and the age of eligibility for normal retirement r_n is obtained via a feedback mechanism and described by the following formulae

$$\Delta c_n = -\frac{J_{n_0}^{\,2} - 1}{J_{n_0}(\lambda_1 n_0 + \lambda_2 r_{n_0} + \lambda_3)\left[1 + \dfrac{100^2\theta}{(1-\theta)} \dfrac{(\lambda_2 c_{n_0} - k_2)^2}{(\lambda_1 n_0 + \lambda_2 r_{n_0} + \lambda_3)^2}\right]} \left(\Delta F_{n-1} + \frac{\lambda_1 c_{n_0} - k_1}{J_{n_0}}\right) \qquad (5.15)$$

$$\Delta r_n = -\frac{100^2\theta}{(1-\theta)} \frac{(\lambda_2 c_{n_0} - k_2)(J_{n_0}^{\,2} - 1)}{J_{n_0}(\lambda_1 n_0 + \lambda_2 r_{n_0} + \lambda_3)^2\left[1 + \dfrac{100^2\theta}{(1-\theta)} \dfrac{(\lambda_2 c_{n_0} - k_2)^2}{(\lambda_1 n_0 + \lambda_2 r_{n_0} + \lambda_3)^2}\right]} \left(\Delta F_{n-1} + \frac{\lambda_1 c_{n_0} - k_1}{J_{n_0}}\right) \quad (5.16)$$

6. Application of the stochastic-discrete model to the population of Greece

As a case study, we apply the stochastic-discrete model and the respective results to the public pension system of Greece, using a recent population projection, (1995 – 2020)

	Total Population in thousands					
	Years --->					
Age Bands	**1995**	**2000**	**2005**	**2010**	**2015**	**2020**
20-24	790,6	779,3	707,5	588,7	540,3	568,3
25-29	798,1	800,9	783,5	711,9	593,7	545,4
30-34	751,4	811,0	806,3	788,8	717,7	599,8
35-39	728,1	759,9	813,6	808,9	791,8	721,0
40-44	689,7	733,4	760,6	814,1	809,8	792,8
45-49	655,3	687,5	729,4	756,4	810,1	805,8
50-54	592,1	652,4	682,4	723,7	751,4	804,3
55-59	640,6	584,9	642,3	671,8	713,8	740,9
60-64	642,0	621,4	568,2	623,7	654,2	695,1
65-69	567,1	601,7	587,5	537,3	592,7	621,8
70-74	400,2	504,3	544,3	531,3	490,9	541,4
75-79	274,7	327,5	423,4	456,4	453,4	419,2
80-84	210,8	191,2	241,1	311,0	345,4	342,7
85-89	99,9	111,8	112,5	142,1	193,2	214,0
90+	43,2	47,1	62,2	66,7	89,0	120,5

Table 6.1 , National Statistical Service of Greece (1998)

The steps and the respective assumptions of the application are described below.

1) The starting date of our simulation is the year 2000 while the closing date is the year 2020.

2) The weight θ has been fixed over the whole period equal to 0.5. In other words people are indifferent between a change of 1% in the contribution rate and a change of one year of age for the age of eligibility for normal retirement.

3) The entry age has been fixed equal to twenty years old (a=20) and the labor force participation rate equal to 100%.

4) The accumulation factor is a stochastic variable (the same as the interest rates). We assume that the fund is conservatively invested (e.g. in Government Bonds) and

$$J_n = 1 + i_n \approx Uniform\ \ Distr.\ \ over\ \ the\ \ \text{int}erval\ \ (1.035, 1.045) \quad \text{i.e.}$$

$$J_{n_0} = 1 + i_{n_0} = 1.04\ \ and\ \ \Delta J_n \approx Uniform\ \ Distr.\ \ over\ \ the\ \ \text{int}erval\ \ (-0.005, +0.005).$$

5) Using population data (as shown in Table 6.1) and the following two simplifying assumptions:

a/ Each life (active) above 20 years old and up to the age of normal retirement receives an annual salary of one money unit. Actually, we assume a labor force participation rate equal to 100%; and

b/ Each life (pensioner) above the age of normal retirement receives an annual pension of half (0.5) money unit. (This assumption is in line with the typical situation in Greece);

We apply linear regression techniques and obtain estimates of the λ_1, λ_2, λ_3, k_1, k_2, k_3 parameter values.

For the regression, we use 30 points for each of W(n,y) and B(n,y) taken from the projected values of these functions (according to the data of table (6.1)) for values of n=1995,2000,2005,2010,2015,2020 and y=55,60,65,65,70. Regarding the goodness of fit of the regressions, we conclude that this is adequate for our purposes since the coefficient of determination R^2 is very close to 1 in both cases (for the Wage and Benefit functions, we have values respectively of 0,9795 and 0,9943). Also the F-statistic values and the corresponding tests provide satisfactory results revealing that the high value of the coefficient of determination R^2 has not occurred by chance.

Consequently, we derive approximately linear functions for the total Wages and Benefits, with parameter estimates:

λ_1= -2.08, λ_2 = 119.84, λ_3 = 2728.90, k_1= 13.37, k_2 = -59.92, k_3 = -21922,95

In order to incorporate a stochastic element into the functions for W(n,y) and B(n,y) we assume that

$$\Delta \lambda_n \approx Uniform \;\; Distr. \;\; over \;\; the \;\; interval \;\; (-200, +200)$$

$$\Delta k_n \approx Uniform \;\; Distr. \;\; over \;\; the \;\; interval \;\; (-100, +100)$$

6) The initial contribution rate is chosen to be 14.35287%, the initial corresponding age of eligibility for normal retirement is chosen to be 65 (exactly) and the initial fund value is chosen to be zero. This pair, (14.35287%, 65) almost equalizes the total contributions and the total benefits so that the fund at the end of the first year is almost zero, and is chosen as the first equilibrium point.

7) After the establishment of the first equilibrium point, we proceed with the design of the path for c_{n_0} and r_{n_0}, for the years 2001 till 2020. Firstly, we observe the overall trend for the W(n,y) and B(n,y) functions and anticipate the respective trend of the path (for c_{n_0} and r_{n_0}), which should be an increasing one. Then, we use a trial and error

procedure increasing marginally the consecutive values of c_{n_0} and r_{n_0} , (initially by 0,00097 and 0,097 for the years 2001 till 2004 then 0,00096 and 0,096 for the years 2005 till 2014 and finally 0,00095 and 0,095 for the years 2015-2020 – see the second and third column of table (6.2)) targeting to a zero annual cash flow,a zero accumulated reserve at the end of 2020 and assuming a constant accumulation factor of 1.04 with no stochastic elements for the W(n,y) and B(n,y) functions. This certain pattern for the annual increase of the c_{n_0} and r_{n_0} coincides with assumption 2) for the θ factor, i.e. that people are indifferent between a change in the c_{n_0} from 14.35287% to 14.44987% and a change in the r_{n_0} from 65.000 to 65.097. We then calculate the actual contribution and age of eligibility from equations (5.5), (5.15) and (5.16) and then execute 500 simulations and the results for the Expectations and Standard Deviations for c_n, r_n and F_n which are presented in Table 6.2.

Table 6.2

Simulation Results for the Greek population (2000-2020)

n	c_{n0}	r_{n0}	$E(c_n)$	$E(r_n)$	$E(F_n)$	$St.Dev(c_n)$	$St.Dev(r_n)$	$St.Dev(F_n)$
2000	14,35287%	65,000	14,35287%	65,000	0,51	0,00000%	0,000	59,37
2001	14,44987%	65,097	14,45618%	65,105	-0,88	0,02968%	0,036	85,93
2002	14,54687%	65,194	14,55388%	65,202	-0,92	0,04295%	0,052	101,65
2003	14,64387%	65,291	14,65090%	65,300	-2,33	0,05081%	0,062	113,47
2004	14,74087%	65,388	14,74860%	65,397	-0,10	0,05672%	0,069	128,41
2005	14,83687%	65,484	14,84348%	65,492	0,18	0,06419%	0,078	135,61
2006	14,93287%	65,580	14,93935%	65,588	-0,62	0,06778%	0,082	141,21
2007	15,02887%	65,676	15,03575%	65,684	-4,77	0,07058%	0,086	151,31
2008	15,12487%	65,772	15,13382%	65,783	-9,76	0,07563%	0,092	155,29
2009	15,22087%	65,868	15,23231%	65,882	-8,29	0,07762%	0,094	158,70
2010	15,31687%	65,964	15,32758%	65,977	-2,19	0,07933%	0,096	161,95
2011	15,41287%	66,060	15,42053%	66,069	2,09	0,08095%	0,098	171,75
2012	15,50887%	66,156	15,51439%	66,163	3,81	0,08585%	0,104	175,28
2013	15,60487%	66,252	15,60953%	66,258	4,82	0,08761%	0,106	177,39
2014	15,70087%	66,348	15,70502%	66,353	3,78	0,08867%	0,107	179,00
2015	15,79587%	66,443	15,80055%	66,449	1,29	0,08948%	0,108	176,35
2016	15,89087%	66,538	15,89679%	66,545	-0,67	0,08815%	0,107	179,73
2017	15,98587%	66,633	15,99277%	66,641	4,91	0,08984%	0,109	182,43
2018	16,08087%	66,728	16,08498%	66,733	0,64	0,09119%	0,111	187,61
2019	16,17587%	66,823	16,18212%	66,831	-0,33	0,09378%	0,114	191,99
2020	16,27087%	66,918	16,27760%	66,926	-1,25	0,09597%	0,116	191,98

8) The procedure described in step 7) is then repeated step-wise as we move forward year by year.

In the simulations, the optimal path for c_n and r_n is smooth, while the resulting path for the reserve (contingency) fund exhibits oscillations, which reflects the fund's absorbing of the stochastic fluctuations in mortality patterns, salary inflation and investment performance. This is evident from the development of the expectations and the magnitude of the standard deviations and is explained briefly in the following paragraph.

We observe that the paths of expectations for c_n and r_n are smooth (and very near to the designed paths of the equilibrium points) while the $St.Dev(c_n)$ and $St.Dev(r_n)$ are small for each of the years (2000-2020). The path of $E(F_n)$ is also smooth and remains near the zero region (because of the special design of the equilibrium points) but the $St.Dev(F_n)$ is quite large, reflecting a highly oscillatory pattern for F_n.

7. Conclusions

In this paper, we attempt an alternative approach to the standard version of the PAYG model. After a brief discussion for the international demographic trend of "ageing populations" and the impact on public pension systems financed by the PAYG method, we concentrate on three principal variables -the reserve fund, the contribution rate and the age of eligibility for normal retirement.

We observe that "inter-generational equity" concept may be well served with the existence of a contingency fund. Such a fund can absorb the fluctuations in the mortality pattern, the fertility rates or the investment performance and consequently can smooth the contribution rates and the rates of return for each cohort of lives.

Under the lines described above, we construct a general continuous model in order to design an optimal control path for the contribution rate and age of eligibility for normal retirement assuming a fund, which operates as a buffer. We have overcome the difficulty of the complicated version of the problem by considering linear functions for wages/salaries and benefits obtaining an analytical solution for the two control variables within a deterministic framework.

Then the initial model is reformulated by adopting a more realistic stochastic–discrete approach. The full solution of this version is obtained by means of linear stochastic control theory. The optimal path for the contribution rate and the age of eligibility for normal retirement is calculated through a series of equilibrium points and a feedback mechanism.

Finally, the second discrete version of the model is applied to the projected population of Greece for the years 2000 – 2020. The simulation results are fully compatible with the theoretical analysis. An optimal (in terms of the smoothness defined by the respective functional) path for contribution rate and the age of eligibility for normal retirement is designed starting from (14.35287%, 65) and ending with (16.27760%, 66,926) in 2020.

Appendix I (Deterministic Optimal Control)

Here, we discuss the functional minimization procedure within a deterministic framework (further analysis is provided in Athans & Falb (1966) p. 237-363).

Let us consider the dynamic differential equation $\dot{x}(t) = f(x(t), u(t))$ and search for the optimal u*(t) which minimizes the functional L , i.e. $\min_u L = \min_u \int_{t_0}^{t_1} L(x(t), u(t))dt$ given that $x(t_0) = x_0$ and $x(t_1) = x_1$.

We introduce the Hamiltonian H of the system

$$H(t) = L(x(t), u(t)) + p(t) \cdot f(x(t), u(t))$$

Then $H(x*(t), p*(t)u, t)$ has an absolute minimum as a function of u when

$u = u*(t)$ $t \in [t_0, t_1]$ where $u^*(t)$ is obtained from the following equation

$$\frac{\partial H}{\partial u} = 0 \quad \text{or} \quad \frac{\partial H(t)}{\partial u(t)} = 0 \quad \forall t \in [t_0, t_1] \quad \text{while} \quad p*(t) = -\frac{\partial H(t)}{\partial x(t)}, \quad x'(t) = \frac{\partial H(t)}{\partial p}$$

Appendix II (Stochastic Linear Optimal Control)

We have modified Appendix II incorporating all computations with respect to the minimization problem which appears at the end of section 5.

Here, we discuss the functional minimization procedure within a stochastic framework (further analysis is provided in Kushner (1969) p. 228-236).

Let us consider the linear dynamic difference equation

$$x_{n+1} = Ax_n + Bu_n + \phi + \xi_n \qquad \text{(II.1)}$$

where x_n is the state variable, u_n is a control variable. A and B are given constant matrices, ϕ is constant vector and ξ_n denotes a random vector with zero mean and finite second moment that is independent of x_{n-1}.

and search for the optimal control u (i.e determine u_1, u_2, …u_m) which minimizes the following expression :

$$E\left[\sum_{n=1}^{m} u_n^T \cdot Q \cdot u_n\right] \qquad \text{(II.2)}$$

If the system is stable (i.e. all the characteristic roots of (A+BM) are smaller than 1 in absolute value), then the solution will converge and will be given by

$$u_n = Mx_{n-1} + \zeta \qquad \text{(II.3)}$$

where $M = \begin{bmatrix} M_1 \\ M_2 \end{bmatrix}$, $g = \begin{bmatrix} \zeta_1 \\ \zeta_2 \end{bmatrix}$ and H is a real number, satisfying the system below

$$M = -\left(B^T HB + Q\right)^{-1}\left(B^T HA\right) \qquad \text{(II.4)}$$

$$H = (A + BM)^T H (A + BM) + M^T QM \qquad \text{(II.5)}$$

$$\zeta = -\left(B^T HB + Q\right)^{-1}\left(B^T H\phi\right) \qquad \text{(II.6)}$$

We proceed with the solution of the system of matrix equations (II.4) and (II.5)

$$\left.\begin{aligned} M = \begin{bmatrix} M_1 \\ M_2 \end{bmatrix} = -\left(\begin{bmatrix} B_1 \\ B_2 \end{bmatrix} H \left[B_1 B_2\right] + \begin{bmatrix} 100^2\theta & 0 \\ 0 & 1-\theta \end{bmatrix}\right)^{-1}\left(\begin{bmatrix} B_1 \\ B_2 \end{bmatrix} HA\right) \quad & (II.7) \\ H = \left(A + \left[B_1 B_2\right]\begin{bmatrix} M_1 \\ M_2 \end{bmatrix}\right)^T H \left(A + \left[B_1 B_2\right]\begin{bmatrix} M_1 \\ M_2 \end{bmatrix}\right) + \left[M_1 M_2\right]\begin{bmatrix} 100^2\theta & 0 \\ 0 & 1-\theta \end{bmatrix}\begin{bmatrix} M_1 \\ M_2 \end{bmatrix} \quad & (II.8) \end{aligned}\right\}$$

Firstly, we solve equation (II.8) i.e.

$$H = H\left[A + B_1 M_1 + B_2 M_2\right]^2 + \left[100^2\theta M_1^2 + (1-\theta)M_2^2\right] \Rightarrow H = \frac{100^2\theta M_1^2 + (1-\theta)M_2^2}{1-(A+B_1M_1+B_2M_2)^2} \quad (II.9)$$ T

Then from equation (II.7) we derive the following analysis

$$\begin{bmatrix} M_1 \\ M_2 \end{bmatrix} = -\left(H\begin{bmatrix} B_1^2 & B_1B_2 \\ B_1B_2 & B_2^2 \end{bmatrix} + \begin{bmatrix} 100^2\theta & 0 \\ 0 & 1-\theta \end{bmatrix}\right)^{-1}\left(\begin{bmatrix} B_1 \\ B_2 \end{bmatrix} HA\right) \Rightarrow$$

$$\begin{bmatrix} M_1 \\ M_2 \end{bmatrix} = -\begin{bmatrix} HB_1^2 + 100^2\theta & HB_1B_2 \\ HB_1B_2 & H\mathrm{B}_2^2 + 1-\theta \end{bmatrix}^{-1}\begin{bmatrix} B_1 \\ B_2 \end{bmatrix} HA \Rightarrow$$

$$\begin{bmatrix} M_1 \\ M_2 \end{bmatrix} = -\frac{1}{100^2\theta \mathrm{HB}_2^2 + (1-\theta)\mathrm{HB}_1^2 + 100^2\theta(1-\theta)}\begin{bmatrix} HB_2^2 + 1-\theta & -HB_1B_2 \\ -HB_1B_2 & HB_1^2 + 100^2\theta \end{bmatrix}\begin{bmatrix} B_1 \\ B_2 \end{bmatrix} HA \Rightarrow$$

$$\begin{bmatrix} M_1 \\ M_2 \end{bmatrix} = -\frac{HA}{100^2\theta \mathrm{HB}_2^2 + (1-\theta)\mathrm{HB}_1^2 + 100^2\theta(1-\theta)}\begin{bmatrix} (1-\theta)B_1 \\ 100^2\theta B_2 \end{bmatrix} \Rightarrow$$

$$\left.\begin{aligned} M_1 &= -\frac{(1-\theta)A}{100^2\theta B_2^2 + (1-\theta)B_1^2 + \frac{100^2\theta(1-\theta)}{H}} B_1 \\ M_2 &= -\frac{100^2\theta A}{100^2\theta B_2^2 + (1-\theta)B_1^2 + \frac{100^2\theta(1-\theta)}{H}} B_2 \end{aligned}\right\} \Rightarrow$$

$$\left.\begin{array}{l} M_2=\dfrac{100^2\theta B_2}{(1-\theta)B_1}M_1 \\ M_1=-\dfrac{(1-\theta)A}{(1-\theta)B_1^2+100^2\theta B_2^2+100^2\theta(1-\theta)\dfrac{1-(A+B_1M_1+B_2M_2)^2}{100^2\theta M_1^2+(1-\theta)M_2^2}}B_1 \end{array}\right\}\begin{array}{l}(II.10)\\ \\ (II.11)\end{array}$$

$$(II.11)\Rightarrow M_1=-\frac{(1-\theta)AB_1\left(100^2\theta M_1^2+(1-\theta)\dfrac{100^4\theta^2B_2^2}{(1-\theta)^2B_1^2}M_1^2\right)}{\left(100^2\theta M_1^2+(1-\theta)M_2^2\right)\left((1-\theta)B_1^2+100^2\theta B_2^2\right)+100^2\theta(1-\theta)\left[1-\left(A+B_1M_1+B_2M_2\right)^2\right]}$$

$$-M_1=\frac{AB_1\left((100^2\theta(1-\theta)+100^4\theta^2\ \dfrac{B_2^2}{B_1^2}\right)M_1^2}{\left(100^2\theta+\dfrac{100^4\theta^2}{1-\theta}\ \ \dfrac{B_2^2}{B_1^2}\right)\left[(1-\theta)\,\mathrm{B}_1^2+100^2\theta\,\mathrm{B}_2^2\right]M_1^2+100^2\theta(1-\theta)\left(1-\left(A+B_1M_1+\dfrac{100^2\theta B_2^2}{(1-\theta)B_1}M_1\right)^2\right)}$$

$$-1=\frac{AB_1\left(100^2\theta(1-\theta)+100^4\theta^2+\dfrac{B_2^2}{B_1^2}\right)M_1}{\left(100^2\theta(1-\theta)B_1^2+2\cdot100^4\theta^2B_2^2+\dfrac{100^6\theta^3B_2^4}{(1-\theta)B_1^2}\right)M_1^2+100^2\theta(1-\theta)\left[1-\left[A+\left(B_1+\dfrac{100^2\theta B_2}{(1-\theta)B_1}\right)M_1\right]^2\right]}\Rightarrow$$

$$-AB_1\left(100^2\theta(1-\theta)+100^4\theta^2\ \frac{B_2^2}{B_1^2}\right)M_1=$$

$$=\left(100^2\theta(1-\theta)\,B_1^2+2\cdot100^4\theta^2\mathrm{B}_2^2+\frac{100^6\theta^3B_2^4}{(1-\theta)B_1^2}\right)M_1^2+100^2\theta(1-\theta)\left[1-\left[A+\left(B_1+\frac{100^2\theta B_2^2}{(1-\theta)B_1}\right)M_1\right]^2\right]\Rightarrow$$

$$-AB_1\left(1+100^2\theta\frac{1}{1-\theta}\frac{B_2^2}{B_1^2}\right)M_1 =$$

$$=\left(B_1^2+2\cdot 100^2\frac{\theta}{1-\theta}\mathrm{B}_2^2+\frac{100^4\theta^2B_2^4}{(1-\theta)^2B_1^2}\right)M_1^2+1-\left(A^2+2A\left(B_1+\frac{100^2\theta B_2^2}{(1-\theta)B_1}\right)M_1+\left(B_1+\frac{100^2\theta B_2^2}{(1-\theta)B_1}\right)^2M_1^2\right)\Rightarrow$$

$$-\mathrm{A}B_1\left[1+\frac{100^2\theta}{(1-\theta)}\frac{\mathrm{B}_2^2}{\mathrm{B}_1^2}\right]\mathrm{M}_1+1-\mathrm{A}^2=0 \qquad \text{(II.12)}$$

Combining equations (II.10) and (II.12) we obtain

$$\mathrm{M}_1=-\frac{\mathrm{A}^2-1}{\mathrm{A}B_1\left[1+\frac{100^2\theta}{(1-\theta)}\frac{\mathrm{B}_2^2}{\mathrm{B}_1^2}\right]} \qquad \mathrm{M}_2=-\frac{100^2\theta\,\mathrm{B}_2}{(1-\theta)\mathrm{B}_1}\cdot\frac{\mathrm{A}^2-1}{\mathrm{A}B_1\left[1+\frac{100^2\theta}{(1-\theta)}\frac{\mathrm{B}_2^2}{\mathrm{B}_1^2}\right]} \qquad \text{(II.13)}$$

Combining equations (II.4) and (II.6) we derive that

$$g=\frac{\phi}{\mathrm{A}}\mathrm{M} \qquad \text{and consequently,} \qquad g_1=\frac{\phi}{\mathrm{A}}\mathrm{M}_1 \qquad g_2=\frac{\phi}{\mathrm{A}}\mathrm{M}_2 \qquad \text{(II.14)}$$

So, the optimal choice for the control vector $\mathrm{u_n}=\begin{bmatrix}\Delta \mathrm{c_n}\\ \Delta \mathrm{r_n}\end{bmatrix}$ i.e. for the contribution rate c_n and the age of eligibility for normal retirement r_n is obtained via a feedback mechanism

$$\Delta c_n=-\frac{\mathrm{A}^2-1}{\mathrm{A}B_1\left[1+\frac{100^2\theta}{(1-\theta)}\frac{\mathrm{B}_2^2}{\mathrm{B}_1^2}\right]}(\Delta F_{n-1}+\frac{\phi}{A})$$

or equivalently substituting A,B_1,B_2 and ϕ.

$$\Delta c_n=-\frac{J_{n_0}^2-1}{J_{n_0}(\lambda_1 n_0+\lambda_2 r_{n_0}+\lambda_3)\left[1+\frac{100^2\theta}{(1-\theta)}\frac{(\lambda_2 c_{n_0}-k_2)^2}{(\lambda_1 n_0+\lambda_2 r_{n_0}+\lambda_3)^2}\right]}(\Delta F_{n-1}+\frac{\lambda_1 c_{n_0}-k_1}{J_{n_0}}) \qquad \text{(II.15)}$$

$$\Delta r_n = -\frac{100^2\theta}{(1-\theta)}\cdot\frac{B_2(\mathrm{A}^2-1)}{\mathrm{A}B_1^2\left[1+\frac{100^2\theta}{(1-\theta)}\frac{\mathrm{B}_2^2}{\mathrm{B}_1^2}\right]}(\Delta F_{n-1}+\frac{\phi}{\mathrm{A}})$$ again substituting A,B_1,B_2 and ϕ.

$$\Delta r_n = -\frac{100^2\theta}{(1-\theta)}\frac{(\lambda_2 c_{n_0}-k_2)(J_{n_0}^2-1)}{J_{n_0}(\lambda_1 n_0+\lambda_2 r_{n_0}+\lambda_3)^2\left[1+\frac{100^2\theta}{(1-\theta)}\frac{(\lambda_2 c_{n_0}-k_2)^2}{(\lambda_1 n_0+\lambda_2 r_{n_0}+\lambda_3)^2}\right]}(\Delta F_{n-1}+\frac{\lambda_1 c_{n_0}-k_1}{J_{n_0}}) \quad \text{(II.16)}$$

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