The p-adic group ring of $\mathrm{SL}_2(p^f)$

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Abstract

In this article we show that the $\mathbb{Z}_p[\zeta_{p^f-1}]$-order $\mathbb{Z}_p[\zeta_{p^f-1}]\,\mathrm{SL}_2(p^f)$ can be recognized among those orders whose reduction modulo p is isomorphic to $\mathbb{F}_{p^f}\,\mathrm{SL}_2(p^f)$ using only ring-theoretic properties. In other words we show that $\mathbb{F}_{p^f}\,\mathrm{SL}_2(p^f)$ lifts uniquely to a $\mathbb{Z}_p[\zeta_{p^f-1}]$-order, provided certain reasonable conditions are imposed on the lift. This proves a conjecture made by Nebe in [Neb00a] concerning the basic order of $\mathbb{Z}_2[\zeta_{2^f-1}]\,\mathrm{SL}_2(2^f)$.

Keywords: Orders, Integral Representations, Derived Equivalences

1. Introduction

Let p be a prime and let $(K, \mathcal{O}, k)$ be a p-modular system. This article is concerned with the group ring $\mathcal{O}\,\mathrm{SL}_2(p^f)$, where $f \in \mathbb{N}$. Hence we are dealing with the discrete valuation ring version of what is typically referred to as representation theory in "defining characteristic". Our aim in this paper is to prove a conjecture made by Nebe in [Neb00a] which claims to describe the group ring of $\mathrm{SL}_2(2^f)$ over sufficiently large extensions $\mathcal{O}$ of $\mathbb{Z}_2$. We are also interested in the question of whether the results in [Neb00b], which deal with the case $p \neq 2$, are sufficient to describe the group ring $\mathcal{O}\,\mathrm{SL}_2(p^f)$. Here, "to describe the group ring" means to describe its basic order. Our proof of Nebe's conjecture is indirect, and consists essentially of showing that a "unique lifting theorem" (see Corollary 7.15) holds for the group ring of $\mathrm{SL}_2(p^f)$. Basically this unique lifting theorem asserts that any $\mathcal{O}$-order reducing to $k\,\mathrm{SL}_2(p^f)$ which has semisimple K-span and is self-dual has to be isomorphic to $\mathcal{O}\,\mathrm{SL}_2(p^f)$. Note however that some details have been omitted in this short explanation. Namely, there are some technical conditions on the bilinear form with respect to which the $\mathcal{O}$-order is self-dual, and we also need to assume $k \supseteq \mathbb{F}_{p^f}$. Nebe's conjecture is an immediate consequence of this theorem, but the theorem may well be considered an interesting result in its own right.

This work is a continuation of the author's work in [Eis12], where a "unique lifting theorem" similar to the one mentioned above is proved for 2-blocks with dihedral defect group. Our approach is, as in [Eis12], based on the idea that, provided it is properly formulated, such a theorem holds for a k-algebra if and only if it holds for all k-algebras derived equivalent to the original one. By the abelian defect group conjecture, which is known to be true in the special case encountered in the present paper, the blocks of $k\,\mathrm{SL}_2(p^f)$ are derived equivalent to their Brauer correspondents. Technically, we must assume k to be algebraically closed for this, but we manage to work around that in this article. And, as it turns out, proving a "unique lifting theorem" for these Brauer correspondents is fairly easy due to their simple structure. In particular we prove Nebe's conjecture without ever having to put up with the complicated combinatorics that arises in the representation theory of $\mathrm{SL}_2(p^f)$.

The article is structured as follows: In section 2 we introduce our notation and remind the reader of some basic definitions and facts on orders over discrete valuation rings. Section 3 gives a short summary of the results of [Kos94], [Neb00a] and [Neb00b]. In particular that section addresses the question of how our results extend the results of Nebe in [Neb00a] and [Neb00b], and actually lead to a complete description of

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the group ring $\mathcal{O}\,\mathrm{SL}_2(p^f)$. In section 4 we explain how $\mathcal{O}$-orders reducing to a k-algebra A correspond to $\mathcal{O}$-orders reducing to a k-algebra B which is derived equivalent to A. This correspondence was introduced in [Eis12], and we use it as a technical tool to deduce results about $\mathcal{O}$-orders reducing to the blocks of $k\,\mathrm{SL}_2(p^f)$ from analogous results about the Brauer correspondents of these blocks. Section 5 deals with the Brauer correspondents of the blocks of $k\,\mathrm{SL}_2(p^f)$. The main result of that section is Theorem 5.16, which is a unique lifting theorem for the aforementioned Brauer correspondents. Section 6 applies the correspondence of lifts introduced in section 4 to the derived equivalence between the blocks of $k\,\mathrm{SL}_2(p^f)$ and their Brauer correspondents. This yields Corollary 6.4, which implies that a unique lifting theorem holds for $k\,\mathrm{SL}_2(p^f)$, where k is assumed to be algebraically closed. Section 7 deals with the case of non-algebraically closed fields k. Corollary 7.15 states a unique lifting theorem for the blocks of $k\,\mathrm{SL}_2(p^f)$ where $k \supseteq \mathbb{F}_{p^f}$. As an additional result we also obtain Corollary 7.17, which shows that there is a derived equivalence between the blocks of $\mathcal{O}\,\mathrm{SL}_2(p^f)$ and their Brauer correspondents for $\mathcal{O} = \mathbb{Z}_p[\zeta_{p^f-1}]$.

2. Notation and technical prerequisites

Throughout this article, p will denote a prime and $(K, \mathcal{O}, k)$ will denote a p-modular system such that K is a complete and unramified extension of $\mathbb{Q}_p$. We let $\bar{K}$ and $\bar{k}$ denote the respective algebraic closures. By $\nu_p: K \longrightarrow \mathbb{Z}$ we denote the p-valuation on K.

Notation 2.1. *We are going to use the following notations (all of which are more or less standard):*

- $\mathbf{mod}_A$ *and* $\mathbf{proj}_A$*: the categories of finitely-generated modules respectively finitely-generated projective modules over the ring A.*
- $\mathcal{D}^b(A)$*,* $\mathcal{D}^-(A)$*: the bounded respectively right bounded derived category of A-modules.*
- $\mathcal{K}^b(\mathbf{proj}_A)$*: the homotopy category of bounded complexes with finitely generated projective terms.*
- $-\otimes_A^{\mathbb{L}} =$*: the left derived tensor product.*
- $\mathrm{Out}_k(A)$*: the outer automorphism group of the k-algebra A. To keep notation simple we will not differentiate between elements of* $\mathrm{Out}_k(A)$ *and representatives for those elements in* $\mathrm{Aut}_k(A)$*.*
- $\mathrm{Out}_k^0(A)$ *(assuming k is algebraically closed): the identity component of the algebraic group* $\mathrm{Out}_k(A)$*.*
- $\mathrm{Aut}_k^s(A)$ *and* $\mathrm{Out}_k^s(A)$*: These denote the subgroups of* $\mathrm{Aut}_k(A)$ *respectively* $\mathrm{Out}_k(A)$ *which stabilize all isomorphism classes of simple A-modules (with the action of* $\mathrm{Aut}_k(A)$ *and* $\mathrm{Out}_k(A)$ *on isomorphism classes of modules being given by twisting).*
- *If A, B and C are rings, and $\alpha: A \to C$ as well as $\beta: B \to C$ are ring homomorphisms, then we denote by ${}_\alpha C_\beta$ the A-B-bimodule which coincides with C as a set, where $a \in A$ and $b \in B$ act on $c \in C$ by the formula $a \cdot c \cdot b := \alpha(a) \cdot c \cdot \beta(b)$.*

Definition 2.2 (Orders and lifts of elements)**.**

1. *An $\mathcal{O}$-algebra Λ is called an* order *if it is free and finitely-generated as an $\mathcal{O}$-module.*
2. *If an $\mathcal{O}$-order Λ is contained in a finite-dimensional K-algebra A, then we call Λ a* full order *in A if it contains a K-basis of A.*
3. *Assume we are given an $\mathcal{O}$-order Λ and a k-algebra $\bar{\Lambda}$ which is isomorphic to $k \otimes \Lambda$ by means of a given isomorphism $\varphi: k \otimes \Lambda \xrightarrow{\sim} \bar{\Lambda}$. Then we say that an element $x \in \bar{\Lambda}$* lifts to *an element $y \in \Lambda$ if $\varphi(1_k \otimes y) = x$.*

One important property of group rings over integral domains which we are going to exploit in this article is that they are self-dual with respect to a bilinear form of the kind defined in the following definition.

Definition 2.3 (Trace bilinear form)**.** *Let*

$$A = \bigoplus_{i=1}^{l} D_i^{n_i \times n_i} \tag{1}$$

be a finite-dimensional semisimple K-algebra given in its Wedderburn decomposition (i. e. the D_i are division algebras over K and the n_i are certain natural numbers). Given an element $u = (u_1, \ldots, u_l) \in Z(A) = Z(D_1) \oplus \ldots \oplus Z(D_l)$ we define a map

$$T_u : \; A \longrightarrow K : \; a = (a_1, \ldots, a_l) \mapsto \sum_{i=1}^{l} \mathrm{tr}_{Z(D_i)/K} \operatorname{tr.red.}_{D_i^{n_i \times n_i}/Z(D_i)}(u_i \cdot a_i) \tag{2}$$

and (by abuse of notation) a bilinear form of the same name: $T_u : \; A \times A \longrightarrow K : \; (a, b) \mapsto T_u(a \cdot b)$. Here "$\mathrm{tr}_{\mathrm{Z(D_i)/K}}$" denotes the trace map in the sense of Galois theory, and "$\operatorname{tr.red.}_{D_i^{n_i \times n_i}/Z(D_i)}$" denotes the reduced trace as defined for central simple algebras. A definition of the reduced trace can be found in [Rei75, Chapter 9a]. The maps "$\mathrm{tr}_{Z(D_i)/K} \operatorname{tr.red.}_{D_i^{n_i \times n_i}/Z(D_i)}$" appearing in (2) are also called "reduced traces relative to K". A definition of these can be found in [Rei75, Definition 9.13].

For a full $\mathcal{O}$-lattice $L \subset A$ we define its dual *as follows*

$$L^{\sharp, u} := \{a \in A \mid T_u(a, L) \subseteq \mathcal{O}\} \tag{3}$$

We call L self-dual (with respect to T_u) if $L^{\sharp, u} = L$ (the "u" may be omitted when its choice is clear from context).

Remark 2.4.

1. *The definition of T_u as given above is compatible with extensions of scalars in the following sense: If K' is a field extension of K, $\mathcal{O}'$ is the integral closure of $\mathcal{O}$ in K' and Λ is a full $\mathcal{O}$-order in the semisimple K-algebra A, then Λ is self-dual in A with respect to T_u if and only if $\mathcal{O}' \otimes \Lambda$ is self-dual in $K' \otimes A$ with respect to $1 \otimes u$. Therefore we will often think of u as an element of $Z(\bar{K} \otimes A)$.*
2. *An order $\Lambda \subset A$ is self-dual with respect to some form T_u if and only if Λ is a symmetric $\mathcal{O}$-order. But of course, the element $u \in Z(A)$ such that $\Lambda = \Lambda^{\sharp, u}$ contains more information than merely that the order in question is symmetric.*
3. *Group rings $\mathcal{O}G$ (for finite groups G) are self-dual orders. Let $\chi_1, \ldots, \chi_l$ denote the (absolutely) irreducible $\bar{K}$-valued characters of G. Hence*

$$\bar{K}G \cong \bigoplus_{i=1}^{l} \bar{K}^{\chi_i(1) \times \chi_i(1)} \tag{4}$$

is the Wedderburn decomposition of $\bar{K}G$. Then $\mathcal{O}G = \mathcal{O}G^{\sharp, u}$, where

$$u = \left(\frac{\chi_1(1)}{|G|}, \ldots, \frac{\chi_l(1)}{|G|} \right) \in Z(KG) \subset Z(\bar{K}G) \cong \bigoplus_{i=1}^{l} \bar{K} \tag{5}$$

We will be using the following definition of decomposition numbers:

Definition 2.5. *Let Λ be an $\mathcal{O}$-order with semisimple K-span. The decomposition matrix of Λ is a matrix whose rows are labeled by the isomorphism classes of simple $K \otimes \Lambda$-modules and whose columns are labeled by the isomorphism classes of simple Λ-modules. If S is a simple Λ-module, P is the projective indecomposable Λ-module with top S and V is a simple $K \otimes \Lambda$-module, then we define the entry $D_{V,S}$ to be the multiplicity of V as a direct summand of $K \otimes P$.*

3. Koshita's and Nebe's descriptions of the group ring

In this section we are going to have a quick look at the descriptions of the basic algebra of the group algebra of $\mathrm{SL}_2(p^f)$ as given by Koshita and later, in the p-adic case, by Nebe. Our main focus lies on the case $p = 2$. Here our aim is to explain how to write down explicitly the description of the basic order of $\mathcal{O}\,\mathrm{SL}_2(2^f)$ conjectured in [Neb00a] (assuming as known the combinatorial description of the decomposition matrix of this order given in [Bur76]), and to exhibit exactly which parts of it were actually of conjectural nature. This is technically not a prerequisite to understanding the rest of this paper, since we will be dealing exclusively with the Brauer correspondents of the blocks of $k\,\mathrm{SL}_2(p^f)$.

In [Kos94] respectively [Kos98], Koshita gave a description of the basic algebra of $\bar{k}\,\mathrm{SL}_2(p^f)$ as quiver algebra modulo relations, using the description of the projective indecomposable $\mathrm{SL}_2(p^f)$-modules given in [Alp79] as his starting point. Koshita's presentation is given in Theorem 3.2 below.

Notation 3.1. *Let N be a set and let $X, Y \subseteq N$ be subsets. Then denote by $X+Y$ the symmetric difference between X and Y, that is, $X + Y = X \cup Y - X \cap Y$.*

Theorem 3.2 (Koshita)**.** *Let Q be the quiver defined as follows:*

1. *the vertices of Q are labeled by the subsets of $N := \mathbb{Z}/f\mathbb{Z}$.*
2. *for any $I \subseteq N$ and any $i \in N$ such that $i-1 \notin I$ there is an arrow $\alpha_{i,I} : I + \{i\} \longrightarrow I$.*

Then the basic algebra of $\bar{k}\,\mathrm{SL}_2(2^f)$ is isomorphic to the quotient of $\bar{k}Q$ by the ideal generated by the following families of elements:

1. $\alpha_{i,I} \cdot \alpha_{j,I+\{i\}} - \alpha_{j,I} \cdot \alpha_{i,I+\{j\}}$ *where $i-1$ and $j-1$ are not in I and $j \notin \{i-1, i, i+1\}$*
2. $\alpha_{i,I} \cdot \alpha_{i,I+\{i\}}$ *where i and $i-1$ are not in I.*
3. $\alpha_{i+1,I} \cdot \alpha_{i,I+\{i+1\}} \cdot \alpha_{i,I+\{i\}+\{i+1\}} - \alpha_{i,I} \cdot \alpha_{i,I+\{i\}} \cdot \alpha_{i+1,I}$ *where $i-1$ and i are not in I.*
4. $\alpha_{i,I+\{i+1\}} \cdot \alpha_{i+1,I+\{i,i+1\}} \cdot \alpha_{i,I+\{i\}}$ *where $i \in I$ but $i-1 \notin I$.*

Definition 3.3. *We denote the $\bar{k}$-algebra constructed in Theorem 3.2 by $\bar{\Lambda}$. Moreover we let $\{\bar{e}_I\}_{I\subseteq N}$ be a system of pair-wise orthogonal primitive idempotents (where the indices correspond to the respective vertices in Q that the idempotents are associated with). For $I, J \subseteq N$ we define $\bar{\Lambda}_{IJ} := \bar{e}_I \bar{\Lambda} \bar{e}_J$.*

Remark 3.4. *While our notation for the arrow $\alpha_{i,I}$ specifies the vertex from which it originates, this information is usually redundant when specifying a path, since the origin of an arrow must coincide with the target of the arrow preceding it in the path. Therefore we make the following notational convention:*

$$\alpha_i := \sum_{I\subseteq N-\{i-1\}} \alpha_{i,I} \tag{6}$$

In [Neb00a], Nebe describes an $\mathcal{O}$-order which reduces to a k-algebra with quiver and relations as in the foregoing theorem. The constructed order is self-dual, and its K-span is semisimple. We will now outline this description. We assume for the remainder of this section that $\mathcal{O}$ is an (unramified) extension of $\mathbb{Z}_2[\zeta_{2^f-1}]$, in order to ensure that both k and K are splitting fields for the group $\mathrm{SL}_2(2^f)$.

Let $\mathcal{R}$ be the set of subsets of $N = \mathbb{Z}/f\mathbb{Z}$. As seen in Theorem 3.2 the elements of $\mathcal{R}$ are in bijection with the (isomorphism classes of) simple $\bar{k}\,\mathrm{SL}_2(2^f)$-modules. Let $\mathcal{C}$ be an index set for the irreducible ordinary representations of $\mathrm{SL}_2(2^f)$. We make the following two definitions:

1. Given $R \in \mathcal{R}$, denote by $\mathcal{C}_R$ the subset of $\mathcal{C}$ corresponding to the irreducible ordinary representations which have non-zero decomposition number with the simple module associated with R.
2. Given $C \in \mathcal{C}$, denote by $\mathcal{R}_C$ the subset of $\mathcal{R}$ corresponding to the simple modules having non-zero decomposition number with the irreducible ordinary representation associated with C.

Then the basic order of $\mathcal{O}\,\mathrm{SL}_2(2^f)$ – which we henceforth will refer to as Λ – is a full $\mathcal{O}$-order in the split semisimple K-algebra

$$A := \bigoplus_{C\in\mathcal{C}} K^{\mathcal{R}_C \times \mathcal{R}_C} \tag{7}$$

We may assume that we have a complete set $\{e_R\}_{R\in\mathcal{R}}$ of pair-wise orthogonal primitive idempotents in $\Lambda \subset A$ such that each e_R is diagonal in each of the matrix rings $K^{\mathcal{R}_C\times\mathcal{R}_C}$. The fact that all decomposition numbers of $\mathrm{SL}_2(2^f)$ are either zero or one implies that e_R is simply a diagonal matrix unit in the direct summands of A labeled by the elements of $\mathcal{C}_R$. Consequently, $\Lambda_{RR} := e_R\Lambda e_R$ is a commutative $\mathcal{O}$-order, whose K-span may be identified with the commutative split semisimple K-algebra $K^{\mathcal{C}_R}$ (addition and multiplication in this algebra work component-wise). Similarly we may think of the set $\Lambda_{LR} := e_L\Lambda e_R$ for $R, L \in \mathcal{R}$ as sitting inside $K^{\mathcal{C}_R\cap\mathcal{C}_L}$. The set Λ_{LR} may be regarded as a Λ_{LL}-Λ_{RR}-bimdoule. In short, in [Neb00a] Nebe succeeds in describing the $\mathcal{O}$-orders Λ_{RR} and the sets Λ_{LR} as Λ_{LL}-Λ_{RR}-bimodules. However, the bimodule structure of Λ_{LR} is not sufficient to describe Λ, since the multiplication maps $\Lambda_{LR} \times \Lambda_{RS} \longrightarrow \Lambda_{LS}$ cannot be fully recovered from the bimodule structure on the involved sets Λ_{LR}, Λ_{RS} and Λ_{LS}.

The first step in [Neb00a] is to lift a $\bar{k}$-basis of $\bar{\Lambda}_{RR}$ to an $\mathcal{O}$-basis of Λ_{RR} (for each $R \in \mathcal{R}$). The k-basis used for this purpose was given in [Kos94] as follows:

Theorem 3.5 (Koshita). *Let $I \subset N$ and let $i \in N - I$. Let $j = j(i, I)$ be the unique integer $\leq i$ such that $j - 1 \notin I$ but $l \in I$ for all $j \leq l < i$. Define*

$$\omega_{i,I} := \alpha_{j,I} \cdot \alpha_{j+1} \cdots \alpha_{i-1} \cdot \alpha_i \cdot \alpha_i \cdot \alpha_{i-1} \cdots \alpha_{j+1} \cdot \alpha_j \in \bar{\Lambda}_{II} \tag{8}$$

For a subset $T \subset N - I$ define

$$\omega_{I,T} := \prod_{i\in T} \omega_{I,i} \in \bar{\Lambda}_{II} \tag{9}$$

This product is well-defined independent of the order of the factors since $\bar{\Lambda}_{II}$ is commutative. The elements $\omega_{I,T}$ form a $\bar{k}$-basis of $\bar{\Lambda}_{II}$.

Let $\widehat{\alpha}_{i,I} \in \Lambda_{I,I+\{i\}}$ be lifts of the elements $\alpha_{i,I}$. One key observation in [Neb00a] is that since each Λ_{IJ} sits inside $K^{\mathcal{C}_I\cap\mathcal{C}_J}$ (which we may in turn view as a subset of $K^{\mathcal{C}}$ by simply extending vectors by zero) we can reorder elements in a product arbitrarily and always obtain the same result (this is only partially reflected in the commutativity relations in Koshita's presentation of $\bar{\Lambda}$, since we may also reorder the elements in a product in such a way that the start and endpoint of the corresponding path changes). The reason is of course that the ring $K^{\mathcal{C}}$ (with component-wise multiplication) is commutative, and we may consider all products as being taken within this ring (we will do this frequently below). So for instance $\widehat{\alpha}_{i,I} \cdot \widehat{\alpha}_{i,I+\{i\}}$ is equal to $\widehat{\alpha}_{i,I+\{i\}} \cdot \widehat{\alpha}_{i,I}$ inside $K^{\mathcal{C}}$. Now [Neb00a, Lemma 3.10] states that $\frac{1}{2} \cdot \widehat{\alpha}_{i,I+\{i\}} \cdot \widehat{\alpha}_{i,I}$ lies in $\Lambda_{I+\{i\},I+\{i\}}$ (since $\alpha_{i,I+\{i\}} \cdot \alpha_{i,I} = 0$ in $\bar{\Lambda}$), and is in fact a unit in this ring. Let $u_{i,I} \in \Lambda_{I+\{i\},I+\{i\}}$ denote its inverse. Then $u_{i,I} \cdot \widehat{\alpha}_{i,I+\{i\}} \cdot \widehat{\alpha}_{i,I} = 2 \cdot \varepsilon_{I+\{i\}}$, where $\varepsilon_{I+\{i\}}$ denotes the element in $K^{\mathcal{C}}$ which has entry equal to one in the components indexed by elements of $\mathcal{C}_{I+\{i\}}$, and entries equal to zero elsewhere. Since we may reorder elements in the product we obtain that $\widehat{\alpha}_{i,I} \cdot u_{i,I} \cdot \widehat{\alpha}_{i,I+\{i\}} = 2 \cdot \varepsilon_{I+\{i\}}$ (note that this is now an element of $\Lambda_{I,I}$). The same principle is applied to the elements $\omega_{I,i}$ defined above. First observe that

$$\widehat{\alpha}_{j,I} \cdot \widehat{\alpha}_{j+1} \cdots \widehat{\alpha}_{i-1} \cdot \widehat{\alpha}_i \cdot \widehat{\alpha}_i \cdot \widehat{\alpha}_{i-1} \cdots \widehat{\alpha}_{j+1} \cdot \widehat{\alpha}_j = (\widehat{\alpha}_{j,I}\widehat{\alpha}_{j,I+\{j\}}) \cdots (\widehat{\alpha}_{i,I+\{j,\dots,i-1\}}\widehat{\alpha}_{i,I+\{j,\dots,i\}}) \tag{10}$$

where the product on the right hand side is formed within $K^{\mathcal{C}}$. As we saw above, for each $j \leq l \leq i$ there is a unit u_l in $\Lambda_{I+\{j,\dots,l\},I+\{j,\dots,l\}}$ such that

$$\widehat{\alpha}_{l,I+\{j,\dots,l-1\}} \cdot u_l \cdot \widehat{\alpha}_{l,I+\{j,\dots,l\}} = 2 \cdot \varepsilon_{I+\{j,\dots,l\}} \tag{11}$$

We have hence found an explicit description of some element in $\Lambda_{I,I}$ which is analogous to the element $\omega_{i,I} \in \bar{\Lambda}_{I,I}$ (however, it does not necessarily reduce to this element upon reduction modulo two):

$$\beta_{i,I} := \widehat{\alpha}_{j,I} \cdot u_j \cdot \widehat{\alpha}_{j+1} \cdot u_{j+1} \cdots \widehat{\alpha}_{i-1} \cdot u_{i-1} \cdot \widehat{\alpha}_i \cdot u_i \cdot \widehat{\alpha}_{i-1} \cdot \widehat{\alpha}_{i-2} \cdots \widehat{\alpha}_{j+1} \cdot \widehat{\alpha}_j \tag{12}$$

By reordering the factors and using the definition of the u_l one easily sees that

$$\beta_{i,I} = 2^{i-j+1} \cdot \varepsilon_I \cdot \varepsilon_{I+\{j,\dots,i\}} \tag{13}$$

Theorem 3.6 ([Neb00a, Theorem 3.12]). *For any subset $I \subseteq N$ and any subset $T \subseteq N - I$ define*

$$\beta_{T,I} := \prod_{i \in T} \beta_{I,i} \tag{14}$$

where the empty product is defined to be ε_I. Then the set $\{\beta_{T,I} \mid I \subseteq N, T \subseteq N - I\}$ forms an $\mathcal{O}$-basis of the $\mathcal{O}$-order $\Lambda_{I,I}$.

Thanks to formula (13) this description of $\Lambda_{I,I}$ is perfectly explicit. Now let $I, J \subseteq N$ be two distinct subsets. Then we get the following information on the $\Lambda_{I,J}$:

Theorem 3.7 ([Neb00a, Theorem 3.12]). *If $\bar{\Lambda}_{I,J} \neq 0$ then*

$$\Lambda_{I,J} \cong \varepsilon_I \cdot \Lambda_{I \cap J, I \cap J} \cdot \varepsilon_J \tag{15}$$

as a $\Lambda_{I,I}$-$\Lambda_{J,J}$-bimdoule.

For a full description of the order Λ, we need more than a bimodule-isomorphism in (15). In fact, (15) fixes $\Lambda_{I,J}$ exactly up to a $K \otimes \Lambda_{I,I}$-$K \otimes \Lambda_{J,J}$-bimodule-automorphism of $K \otimes \Lambda_{I,J} \cong K^{\mathcal{C}_I \cap \mathcal{C}_J}$. These bimodule automorphisms of $K^{\mathcal{C}_I \cap \mathcal{C}_J}$ may be identified with elements of $(K - \{0\})^{\mathcal{C}_I \cap \mathcal{C}_J}$ acting on $K^{\mathcal{C}_I \cap \mathcal{C}_J}$ by component-wise multiplication. Thus, $\Lambda_{I,J} \cong \mu_{I,J} \cdot \varepsilon_I \cdot \Lambda_{I \cap J, I \cap J} \cdot \varepsilon_J$ with $\mu_{I,J} \in (K - \{0\})^{\mathcal{C}_I \cap \mathcal{C}_J}$. In [Neb00a] the following information on $\mu_{I,J}$ is obtained (one should keep in mind though that the $\mu_{I,J}$ are not uniquely determined; the main source of the ambiguity is that the order Λ is only well-defined up to conjugation)

Theorem 3.8. *We may choose $\mu_{I,J}$ such that*

$$\mu_{I,J} = u_{I,J} \cdot 2^{|I-J|} \cdot \varepsilon_I \cdot \varepsilon_J \tag{16}$$

where $u_{I,J} \in (\mathcal{O}^\times)^{\mathcal{C}_I \cap \mathcal{C}_J}$.

Nebe conjectured the following:

Conjecture 3.9 ([Neb00a, Conjecture following Theorem 3.15]). *We may choose all of the $u_{I,J}$ in Theorem 3.8 to have all entries equal to one.*

This would describe the order Λ up to isomorphism. By construction, the order obtained by setting all entries of all $u_{I,J}$ equal to one has semisimple K-span and the same decomposition matrix as the basic order of $\mathcal{O}\,\mathrm{SL}_2(2^f)$. [Neb00a] also notes that it reduces to a k-algebra which, upon tensoring with $\bar{k}$, becomes isomorphic to the basic algebra of $\bar{k}\,\mathrm{SL}_2(2^f)$ as described by Koshita. As we show in Proposition 3.10 below it is also self-dual with respect to the appropriate trace bilinear form. In the present article we confirm Conjecture 3.9. We also deal with the case of an odd prime p, although the article [Neb00b], which deals with $\mathrm{SL}_2(p^f)$ for odd p, does not explicitly state a similarly precise conjecture. We will need the following proposition to explain how exactly our results can be combined the ones obtained in [Neb00a] and [Neb00b].

Proposition 3.10. *Let A be a semisimple K-algebra and let $e_1, \ldots, e_n \in A$ be a system orthogonal idempotents in A. We do not require the e_i to be primitive. Let Λ_1 and Λ_2 be two full $\mathcal{O}$-orders in A which both contain all of the idempotents $e_1, \ldots, e_n$. Assume moreover that*

1. $e_i \Lambda_1 e_i = e_i \Lambda_2 e_i$ *for all* $1 \leq i \leq n$
2. $e_i \Lambda_1 e_j \cong e_i \Lambda_2 e_j$ *as* $e_i\Lambda_1 e_i$-$e_j \Lambda_1 e_j$*-bimodules for all* $1 \leq i, j \leq n$ *with* $i \neq j$
3. $\varepsilon \Lambda_1 = \varepsilon \Lambda_2$ *for all central primitive idempotents* $\varepsilon \in A$

Then, given any element $u \in Z(A)$, Λ_1 is self-dual with respect to T_u if and only if Λ_2 is self-dual with respect to T_u.

Proof. Assume Λ_1 is self-dual with respect to T_u. Note that due to the cyclic property of the trace we get $T_u(e_i a e_j) = T_u(a e_i e_j) = 0$ for any $a \in A$ and $i \neq j$. So for any $a \in A$ we have $T_u(a) = T_u\left(\sum_{i,j=1}^n e_i a e_j\right) = \sum_{i,j=1}^n T_u(e_i a e_j) = \sum_{i=1}^n T_u(e_i a e_i)$. Since by assumption $e_i \Lambda_1 e_i = e_i \Lambda_2 e_i$ for all $1 \leq i \leq n$ it follows that $T_u(\Lambda_2) \subset \mathcal{O}$, since the same holds true for $T_u(\Lambda_1)$ by virtue of Λ_1 being self-dual. It follows that $\Lambda_2 \subseteq \Lambda_2^{\sharp,u}$, with equality if and only if the determinant of the Gram matrix of T_u with respect to a basis of Λ_2 is a unit in $\mathcal{O}$. Under a base change the determinant of the Gram matrix gets multiplied with the square of the determinant of the base change matrix. So if we could find a linear transformation of A that maps Λ_1 to Λ_2 and whose determinant is a unit in $\mathcal{O}$, then that would show that Λ_2 is indeed self-dual. Hence we proceed by choosing a linear transformation $\alpha: A \longrightarrow A$ which sends a basis of Λ_1 to a basis of Λ_2. By our first two assumptions we may choose α in such a way that it induces the identity on $e_i A e_i$ for each $1 \leq i \leq n$ and an $e_i \Lambda e_i$-$e_j \Lambda_1 e_j$-bimodule homomorphism on $e_i A e_j$ for all $1 \leq i, j \leq n$ with $i \neq j$. Such an α will satisfy $\alpha(\varepsilon \cdot a) = \varepsilon \cdot \alpha(a)$ for all $a \in A$ and each central idempotent $\varepsilon \in A$. In particular, using the third assumption, we get that

$$\alpha\left(\bigoplus_{\varepsilon \in A \text{ c.p.i.}} \varepsilon \Lambda_1\right) \subseteq \bigoplus_{\varepsilon \in A \text{ c.p.i.}} \varepsilon \Lambda_1 \quad \text{where c.p.i. stands for "`central primitive idempotent"} \tag{17}$$

This means that α preserves a lattice, and therefore has integral determinant. We can repeat the same argument to show that α^{-1} has integral determinant. It follows that the determinant of α is a unit in $\mathcal{O}$, which, as seen above, implies that Λ_2 is self-dual. ☐

Remark 3.11. *Both [Neb00a] and [Neb00b] give partial descriptions of the basic order Λ of $\mathcal{O}\,\mathrm{SL}_2(p^f)$ as a full order in some semisimple K-algebra A. Let $e_1, \ldots, e_n$ denote a full system of orthogonal primitive idempotents in Λ. Nebe gives a description of the $e_i \Lambda e_i$ for all i, the $e_i \Lambda e_j$ for all $i \neq j$, and the projections of Λ to the Wedderburn components of A. The preceding proposition tells us that any full $\mathcal{O}$-order in A with the same data is self-dual with respect to the same symmetrizing element as Λ. This will be enough to apply Corollary 7.15 to such an order, which will imply that there is a unique one which reduces to the basic algebra of $k\,\mathrm{SL}_2(p^f)$.*

4. Transfer of unique lifting via derived equivalences

In this section we cite the necessary theorems from [Eis12]. They establish the main technical tool used in this paper: a bijection between the sets of lifts (in the sense of the definition below) of two derived equivalent k-algebras. This bijection will allow us to shift the problem of proving that a given k-algebra lifts uniquely to an $\mathcal{O}$-order to an analogous problem over a simpler algebra which is derived equivalent to the original one.

Definition 4.1. *For a finite-dimensional k-algebra $\overline{\Lambda}$ define its* set of lifts *as follows:*

$$\widehat{\mathfrak{L}}(\overline{\Lambda}) := \left\{(\Lambda, \varphi) \mid \Lambda \text{ is an } \mathcal{O}\text{-order and } \varphi: k \otimes \Lambda \xrightarrow{\sim} \overline{\Lambda} \text{ is an isomorphism}\right\} \Big/ \sim \tag{18}$$

where we say $(\Lambda, \varphi) \sim (\Lambda', \varphi')$ if and only if

1. *There is an isomorphism $\alpha: \Lambda \xrightarrow{\sim} \Lambda'$*
2. *There is a $\beta \in \mathrm{Aut}_k(\overline{\Lambda})$ such that the functor $- \otimes_{\overline{\Lambda}} {}_\beta\overline{\Lambda}_{\mathrm{id}}$ fixes all isomorphism classes of tilting complexes in $\mathcal{K}^b(\mathbf{proj}_{\overline{\Lambda}})$*

such that the following diagram commutes:

$$\begin{array}{ccc} k \otimes \Lambda & \xrightarrow{\varphi} & \bar{\Lambda} \\ \downarrow{\scriptstyle \mathrm{id}_k \otimes \alpha} & & \downarrow{\scriptstyle \beta} \\ k \otimes \Lambda' & \xrightarrow{\varphi'} & \bar{\Lambda} \end{array} \tag{19}$$

Moreover we define

$$\mathfrak{L}(\bar{\Lambda}) := \{ \text{ Isomorphism classes of } \mathcal{O}\text{-orders } \Lambda \text{ with } k \otimes \Lambda \cong \bar{\Lambda} \ \} \tag{20}$$

and the projection map

$$\Pi: \ \widehat{\mathfrak{L}}(\bar{\Lambda}) \longrightarrow \mathfrak{L}(\bar{\Lambda}) \tag{21}$$

Finally, we define the set of lifts with semisimple K-span

$$\widehat{\mathfrak{L}}_s(\overline{\Lambda}) := \{(\Lambda, \varphi) \in \widehat{\mathfrak{L}}(\bar{\Lambda}) \mid K \otimes \Lambda \text{ is semisimple } \} \tag{22}$$

and similarly

$$\mathfrak{L}_s(\bar{\Lambda}) := \{\Lambda \in \mathfrak{L}(\bar{\Lambda}) \mid K \otimes \Lambda \text{ is semisimple } \} \tag{23}$$

Theorem 4.2 ([Eis12, Theorem 5.2])**.** *Let $\overline{\Lambda}$ and $\overline{\Gamma}$ be finite-dimensional k-algebras that are derived equivalent. Let the derived equivalence be afforded by the two-sided tilting complex X. Then there is a bijective map*

$$\Phi_X: \ \widehat{\mathfrak{L}}(\bar{\Lambda}) \longrightarrow \widehat{\mathfrak{L}}(\bar{\Gamma}) \tag{24}$$

such that all of the following properties hold:

(i) If $(\Lambda, \varphi) \in \widehat{\mathfrak{L}}(\bar{\Lambda})$ and $(\Gamma, \psi) = \Phi_X(\Lambda, \varphi)$, then there is a derived equivalence between Λ and Γ.

(ii) Φ_X induces a bijection

$$\widehat{\mathfrak{L}}_s(\overline{\Lambda}) \longleftrightarrow \widehat{\mathfrak{L}}_s(\overline{\Gamma}) \tag{25}$$

(iii) Set $\Phi := \Pi \circ \Phi_X$. If $(\Lambda, \varphi), (\Lambda', \varphi') \in \widehat{\mathfrak{L}}(\overline{\Lambda})$ are two lifts with $Z(K \otimes \Lambda) \cong Z(K \otimes \Lambda')$, then

$$Z(K \otimes \Phi(\Lambda, \varphi)) \cong Z(K \otimes \Phi(\Lambda', \varphi')) \tag{26}$$

and every choice of an isomorphism $\gamma : Z(K \otimes \Lambda) \to Z(K \otimes \Lambda')$ gives rise to an isomorphism $\Phi(\gamma) : Z(K \otimes \Phi(\Lambda, \varphi)) \to Z(K \otimes \Phi(\Lambda', \varphi'))$ depending only on γ and X.

(iv) If $(\Lambda, \varphi), (\Lambda', \varphi') \in \widehat{\mathfrak{L}}(\overline{\Lambda})$ are two lifts and $\gamma : Z(\Lambda) \xrightarrow{\sim} Z(\Lambda')$ is an isomorphism of $\mathcal{O}$-algebras, then the isomorphism $\Phi(\mathrm{id}_K \otimes \gamma)$ which exists according to (iii) restricts to an isomorphism of $\mathcal{O}$-algebras $\Phi(\gamma) : Z(\Phi(\Lambda, \varphi)) \to Z(\Phi(\Lambda', \varphi'))$.

(v) If $(\Lambda, \varphi), (\Lambda', \varphi') \in \widehat{\mathfrak{L}}_s(\overline{\Lambda})$ are two lifts, and $\gamma : Z(K \otimes \Lambda) \xrightarrow{\sim} Z(K \otimes \Lambda')$ is an isomorphism such that $D^{\Lambda} = D^{\Lambda'}$ up to permutation of columns (where rows are identified via γ), then $D^{\Phi(\Lambda,\varphi)} = D^{\Phi(\Lambda',\varphi')}$ up to permutation of columns (where rows are identified via $\Phi(\gamma)$). Here "D" always stands for the decomposition matrix.

(vi) If $(\Lambda, \varphi), (\Lambda', \varphi') \in \widehat{\mathfrak{L}}_s(\overline{\Lambda})$ are two lifts with $D^{\Lambda} = D^{\Lambda'}$ up to permutation of rows and columns then $D^{\Phi(\Lambda,\varphi)} = D^{\Phi(\Lambda',\varphi')}$ up to permutation of rows and columns.

Theorem 4.3 (see [Eis12, Theorem 4.7])**.** *Let Λ and Γ be two derived-equivalent $\mathcal{O}$-orders with semisimple K-span. Then we may identify $Z(K \otimes \Lambda)$ and $Z(K \otimes \Gamma)$. The order Λ is self-dual with respect to T_u (with $u \in Z(K \otimes \Lambda)$) if and only if Γ is self-dual with respect to $T_{\tilde{u}}$, where $\tilde{u} \in Z(K \otimes \Gamma)$ is obtained from u by flipping the signs in some Wedderburn components.*

In the setting of Theorem 4.2 the following holds: Let $(\Lambda, \varphi) \in \widehat{\mathfrak{L}}(\bar{\Lambda})$ and $(\Gamma, \psi) := \Phi_X(\Lambda, \varphi)$. By the first point of the preceding theorem there is an isomorphism $\gamma : Z(K \otimes \Lambda) \to Z(K \otimes \Gamma)$. Then Λ is self-dual with respect to $u \in Z(K \otimes \Lambda)$ if and only if Γ is self-dual with respect to $\tilde{u} \in Z(K \otimes \Gamma)$, where $\tilde{u}$ is obtained from $\gamma(u)$ by flipping signs in certain Wedderburn components.

A remark may be in order about the fact that the above theorem states in two places that some signs may need flipping, but fails to specify which signs exactly. [Eis12, Theorem 4.7] does in fact specify which signs need flipping, depending on the chosen derived equivalence between Λ and Γ. However, this will not matter in the present paper, and was therefore omitted.

We are actually interested in isomorphism classes of orders which reduce to a given k-algebra $\bar{\Lambda}$, i.e. the set $\mathfrak{L}(\bar{\Lambda})$. However, Theorem 4.2 only relates the sets $\widehat{\mathfrak{L}}(\bar{\Lambda})$ among derived equivalent algebras. Proposition 4.7 below relates $\mathfrak{L}(\bar{\Lambda})$ and $\widehat{\mathfrak{L}}(\bar{\Lambda})$ with each other in a special case (which will be sufficient for us). It generalizes [Eis12, Proposition 3.12] to the case where k is no longer required to be algebraically closed.

Proposition 4.4 (see [Eis12, Corollary 2.14])**.** *Assume k is algebraically closed and let A be a finite-dimensional k-algebra. Let $T \in \mathcal{K}^b(\mathbf{proj}_A)$ be a one-sided tilting complex. Then*

$$T \otimes_{A} {}_{\mathrm{id}}A_\gamma \cong T \quad \text{for all } \gamma \in \mathrm{Out}_k^0(A) \tag{27}$$

Proposition 4.5. *Let A be a finite-dimensional k-algebra and let S and T be two tilting complexes over A. Then $S \cong T$ (in $\mathcal{D}^b(A)$) if and only if $\bar{k} \otimes S \cong \bar{k} \otimes T$ in $\mathcal{D}^b(\bar{k} \otimes A)$.*

Proof. This is a special case of [Zim12, Theorem 4]. □

Note that for any k-algebra $\bar{\Lambda}$ there is a left action of $\mathrm{Out}_k(\bar{\Lambda})$ on $\widehat{\mathfrak{L}}(\bar{\Lambda})$. If $(\Lambda, \varphi) \in \widehat{\mathfrak{L}}(\bar{\Lambda})$ and $\alpha \in \mathrm{Out}_k(\bar{\Lambda})$ we simply set $\alpha \cdot (\Lambda, \varphi) := (\Lambda, \alpha \circ \varphi)$. It is proved in [Eis12, Proposition 3.7] that this is indeed well-defined (i. e. independent of the choice of a representative for α).

Corollary 4.6. *Let $\bar{\Lambda}$ be an finite-dimensional k-algebra, and let $G \leq \mathrm{Out}_k(\bar{\Lambda})$ be a subgroup such that the $\bar{k}$-linear extensions of the elements of G all lie in $\mathrm{Out}_{\bar{k}}^0(\bar{k} \otimes \bar{\Lambda})$. Then G acts trivially on $\widehat{\mathfrak{L}}(\bar{\Lambda})$.*

Proof. Since G acts trivially on isomorphism classes of tilting complexes in $\mathcal{K}^b(\mathbf{proj}_{\bar{k}\otimes\bar{\Lambda}})$ by Proposition 4.4, it follows using Proposition 4.5 that G acts trivially on isomorphism classes of tilting complexes in $\mathcal{K}^b(\mathbf{proj}_{\bar{\Lambda}})$. But by definition of the equivalence relation "$\sim$" this means that G acts trivially on $\widehat{\mathfrak{L}}(\bar{\Lambda})$. □

Proposition 4.7 (cf. [Eis12, Proposition 3.12])**.** *Let $\Lambda \in \mathfrak{L}(\overline{\Lambda})$, and let $\gamma : k \otimes \Lambda \xrightarrow{\sim} \overline{\Lambda}$. be an isomorphism. Now assume*

$$\overline{\mathrm{Aut}_{\mathcal{O}}(\Lambda)} \cdot G = \mathrm{Out}_k(\overline{\Lambda}) \tag{28}$$

where $\overline{\mathrm{Aut}_{\mathcal{O}}(\Lambda)}$ is the image of $\mathrm{Aut}_{\mathcal{O}}(\Lambda)$ in $\mathrm{Out}_k(\overline{\Lambda})$ (here we identify $k \otimes \Lambda$ with $\overline{\Lambda}$ via γ) and $G \leq \mathrm{Out}_k(\overline{\Lambda})$ is a subgroup such that the $\bar{k}$-linear extensions of all elements of G lie in $\mathrm{Out}_{\bar{k}}^0(\bar{k} \otimes_k \overline{\Lambda})$. Then the fiber $\Pi^{-1}(\{\Lambda\})$ has cardinality one.

Proof. Let $(\Lambda, \varphi) \in \widehat{\mathfrak{L}}(\overline{\Lambda})$ for some $\varphi : k \otimes \Lambda \xrightarrow{\sim} \bar{\Lambda}$ (i. e. (Λ, φ) is an arbitrary element in $\Pi^{-1}(\{\Lambda\})$). We intend to show $(\Lambda, \varphi) \sim (\Lambda, \gamma)$, since this will imply that $\Pi^{-1}(\{\Lambda\})$ contains indeed only a single element. Now if (28) holds, we can write $\gamma \circ \varphi^{-1} = \gamma \circ (\mathrm{id}_k \otimes \hat{\alpha}) \circ \gamma^{-1} \circ \beta$ for some $\hat{\alpha} \in \mathrm{Aut}_{\mathcal{O}}(\Lambda)$ and $\beta \in G$. Hence $\gamma \circ (\mathrm{id}_k \otimes \hat{\alpha}^{-1}) = \beta \circ \varphi$. Corollary 4.6 (together with the definition of "$\sim$") implies $(\Lambda, \gamma) \sim (\Lambda, \beta^{-1} \circ \gamma \circ (\mathrm{id}_k \otimes \hat{\alpha}^{-1})) = (\Lambda, \varphi)$. □

5. The algebra $k\Delta_2(p^f)$ and unique lifting

We define $\Delta_2(p^f)$ to be the following group:

$$\Delta_2(p^f) := \left\{ \left[\begin{array}{cc} a & b \\ 0 & a^{-1} \end{array} \right] \;\middle|\; a, b \in \mathbb{F}_{p^f},\ a \neq 0 \right\} \cong C_p^f \rtimes C_{p^f-1} \tag{29}$$

Note that $\Delta_2(p^f)$ is the normalizer of a p-Sylow subgroup of $\mathrm{SL}_2(p^f)$, namely of the group of unipotent upper triangular 2×2-matrices. Also note that k splits $\mathrm{SL}_2(p^f)$ and $\Delta_2(p^f)$ if and only if $k \supseteq \mathbb{F}_{p^f}$.

In this section we will write $k\Delta_2(p^f)$ explicitly as a quotient of a quiver algebra, where k is assumed to split $\Delta_2(p^f)$. We then use this presentation to show that $k\Delta_2(p^f)$ lifts uniquely to an $\mathcal{O}$-order satisfying certain properties. At least the first part of this, that is, finding a presentation as a quotient of a quiver algebra, is relatively straightforward. The reason for looking at the group algebra of $\Delta_2(p^f)$ is that its blocks are the Brauer correspondents of the blocks of maximal defect of the group algebra of $\mathrm{SL}_2(p^f)$. Other than those blocks of maximal defect, the group algebra of $\mathrm{SL}_2(p^f)$ only has a block of defect zero. All questions we are concerned with can be answered trivially for a block of defect zero, since such a block is just a matrix ring over a field or a skew-field. Hence the block of defect zero of $k\,\mathrm{SL}_2(p^f)$ will not be of interest to us.

In what follows we will use the notation "$\mathrm{Jac}(A)$" for the Jacobson radical of an algebra A.

Definition 5.1. *Assume that A is an abelian p'-group such that kA is split. Denote by $\hat{A}$ the character group of A, that is,* $\mathrm{Hom}(A, k^\times)$ *(abstractly we will have $A \cong \hat{A}$). Assume moreover that A is acting on a p-group P by automorphisms. Let*

$$\mathrm{Jac}(kP)/\,\mathrm{Jac}^2(kP) \cong \bigoplus_{i=1}^{l} S_i \tag{30}$$

be a decomposition of $\mathrm{Jac}(kP)/\,\mathrm{Jac}^2(kP)$ *as a direct sum of simple kA-modules $S_1, \ldots, S_l$. We define the set $X(P, A)$ to be the disjoint union*

$$\biguplus_{i=1}^{l} \{\chi_{S_i}\} \tag{31}$$

where $\chi_{S_i} \in \hat{A}$ denotes the character of A associated to S_i.

Lemma 5.2. *Let $P = C_p^f$ be the elementary abelian p-group of rank f and let A be a group acting on P by automorphisms. View P as an $\mathbb{F}_p$-vector space by identifying C_p^f with $(\mathbb{F}_p^f, +)$. Under this identification, P becomes an $\mathbb{F}_pA$ module. Then*

$$\mathrm{Jac}(kP)/\,\mathrm{Jac}^2(kP) \cong_{kA} k \otimes_{\mathbb{F}_p} P \tag{32}$$

Proof. First note that after identifying P with $\mathbb{F}_p^f$, the fact that A acts on P by automorphisms translates into A acting linearly on $\mathbb{F}_p^f$, as each automorphism of $(\mathbb{F}_p^f, +)$ is automatically $\mathbb{F}_p$-linear. This turns P into an $\mathbb{F}_pA$-module (in fact, the isomorphism type of this module is independent of the choice of the identification of P with $\mathbb{F}_p^f$). Let $x_1, \ldots, x_f$ be a minimal generating system for $P = C_p^f$. Then $1 \otimes x_1, \ldots, 1 \otimes x_f$ is a k-basis for $k \otimes_{\mathbb{F}_p} P$. Now define a k-linear map

$$\Phi : \; k \otimes_{\mathbb{F}_p} P \to \mathrm{Jac}(kP)/\,\mathrm{Jac}^2(kP) : \; 1 \otimes x_i \mapsto x_i - 1 \tag{33}$$

Since the $x_i - 1$ lie in $\mathrm{Jac}(kP)$ and they are a minimal generating set for kP as a k-algebra, they form a $k = kP/\,\mathrm{Jac}(kP)$ basis of $\mathrm{Jac}(kP)/\,\mathrm{Jac}^2(kP)$. Hence Φ is an isomorphism of vector spaces. We only need to check that Φ is A-equivariant (or, more generally, $\mathrm{Aut}(P)$-equivariant). This amounts to showing that for all $n_1, \ldots, n_f \in \mathbb{Z}_{\geq 0}$ the following holds:

$$x_1^{n_1} \cdots x_f^{n_f} - 1 \equiv \sum_{i=1}^{f} n_i \cdot (x_i - 1) \mod \mathrm{Jac}^2(kP) \tag{34}$$

Let $x, y \in P$. Then clearly $(x-1)(y-1) \in \mathrm{Jac}^2(P)$, and hence $xy - x - y + 1 \equiv 0 \mod \mathrm{Jac}^2(kP)$. This can be rewritten as $xy - 1 \equiv (x-1) + (y-1) \mod \mathrm{Jac}^2(kP)$. Iterated application of this equality clearly implies (34). ☐

Proposition 5.3. *Let $G = P \rtimes A$ with $P \cong C_p^f$ and A an abelian p'-group acting on P. If k splits G then*

$$kG \cong kQ/I \tag{35}$$

where Q is the quiver which has vertices e_χ in bijection with the elements $\chi \in \hat{A}$, and an arrow $e_\chi \xrightarrow{s_{\chi,\psi}} e_{\chi\cdot\psi}$ for each $\chi \in \hat{A}$ and $\psi \in X(P, A)$. I is the ideal generated by the relations

$$s_{\chi,\psi} \cdot s_{\chi\cdot\psi,\varphi} = s_{\chi,\varphi} \cdot s_{\chi\cdot\varphi,\psi} \quad \textit{for all } \chi \in \hat{A} \textit{ and } \psi, \varphi \in X(P, A) \tag{36}$$

and

$$\prod_{i=0}^{p-1} s_{\chi\cdot\psi^i,\psi} = 0 \quad \textit{for all } \chi \in \hat{A} \textit{ and } \psi \in X(P, A) \tag{37}$$

Proof. We first look at kP. We have $kC_p \cong k[T]/\langle T^p\rangle$, and

$$kP \cong \bigotimes^{f} kC_p \cong k[T_1,\dots,T_f]/(T_1^p,\dots,T_f^p) \tag{38}$$

Given any minimal generating set $t_1,\dots,t_f$ of kP contained in $\mathrm{Jac}(kP)$, the epimorphism $k[T_1,\dots,T_f] \twoheadrightarrow kP$ sending T_i to t_i has the same kernel $(T_1^p,\dots,T_f^p)$. This is simply because any automorphism of $k[T_1,\dots,T_f]$ mapping the ideal $(T_1,\dots,T_f)$ into itself will map the ideal $(T_1^p,\dots,T_f^p)$ into itself as well.

Now consider the action of A on $\mathrm{Jac}(kP)$ by conjugation. Since kA is abelian and split semisimple, there is a basis $t_1,\dots,t_{p^f-1}$ of $\mathrm{Jac}(kP)$ such that for each i the set $\{u^{-1}t_iu \mid u \in A\}$ generates a 1-dimensional vector space. We may choose a minimal generating set for kP from said t_i's, say (after reindexing) $t_1,\dots,t_f$. As the images of $t_1,\dots,t_f$ in $\mathrm{Jac}(kP)/\mathrm{Jac}^2(kP)$ form a basis, there is a bijective map

$$S : X(P,A) \longrightarrow \{t_1,\dots,t_f\} \tag{39}$$

such that $u^{-1}\cdot S(\psi)\cdot u = \psi(u)\cdot S(\psi)$ for all $u \in A$. In what follows we will write s_ψ for the image of $\psi \in X(P,A)$ under the map S. Define furthermore for each $\chi \in \hat{A}$ the corresponding primitive idempotent $e_\chi \in kA$ via the standard formula

$$e_\chi = \frac{1}{|A|}\sum_{a\in A}\chi(a)\cdot a^{-1} \tag{40}$$

This is a full set of orthogonal primitive idempotents in kG. Furthermore

$$e_\chi \cdot s_\psi = \frac{1}{|A|}\sum_{a\in A}\chi(a)\cdot a^{-1}s_\psi\cdot a\cdot a^{-1} = s_\psi\cdot\frac{1}{|A|}\sum_{a\in A}\chi(a)\psi(a)\cdot a^{-1} = s_\psi\cdot e_{\chi\cdot\psi} \tag{41}$$

Hence define

$$s_{\chi,\psi} := e_\chi\cdot s_\psi \quad \text{for all } \chi\in\hat{A}, \psi\in X(P,A) \tag{42}$$

The fact that the s_ψ commute implies the relation (36), and the fact that $s_\psi^p = 0$ implies relation (37). What we have to verify though is that the s_ψ and e_χ generate kG as a k-algebra, and that there are no further relations (i. e. $\dim_k kG = \dim_k kQ/I$).

The s_ψ generate kP as a k-algebra and the e_χ generate kA even as a k-vector space. Hence together they generate $kP\cdot kA = kG$ as a k-algebra. Now on to the dimension of kQ/I. We can use relation (36) to rewrite a path involving the arrows $s_{\chi_1,\psi_1},\dots,s_{\chi_l,\psi_l}$ (in that order) as a path $s_{\tilde{\chi}_1,\tilde{\psi}_1}\cdots s_{\tilde{\chi}_l,\tilde{\psi}_l}$ for any chosen reordering $(\tilde{\psi}_1,\dots,\tilde{\psi}_l)$ of $(\psi_1,\dots,\psi_l)$. Notice that necessarily $\chi_1 = \tilde{\chi}_1$, and all other $\tilde{\chi}_i$ are determined by $\tilde{\chi}_1$ and the $\tilde{\psi}_i$. Also we may assume, due to relation (37), that no p of the ψ_i are equal. So ultimately, there are at most $|\hat{A}|\cdot p^{|X(P,A)|}$ linearly independent paths ($|\hat{A}|$ choices for the starting point χ_1, p choices for the number of occurrences of each element of $X(P,A)$ in the sequence $(\psi_1,\dots,\psi_l)$). Hence

$$\dim kQ/I \le |\hat{A}|\cdot p^{|X(P,A)|} = |A|\cdot p^f = \dim_k kG \tag{43}$$

and thus the epimorphism $kQ/I \twoheadrightarrow kG$ is in fact an isomorphism. □

Remark 5.4. *It seems practical to keep on using the notation*

$$s_\psi = \sum_{\chi\in\hat{A}} s_{\chi,\psi} \tag{44}$$

With this notation we may just write

$$kG \cong kQ/\left\langle s_\psi s_\varphi - s_\varphi s_\psi,\ s_\psi^p \mid \psi,\varphi\in X(P,A)\right\rangle \tag{45}$$

Proposition 5.5. *Let $G = \Delta_2(p^f)$, $P = \mathbb{G}_a(\mathbb{F}_{p^f}) \cong C_p^f$ and $A = \mathbb{G}_m(\mathbb{F}_{p^f}) \cong C_{p^f-1}$ (we view P as the subgroup of G consisting of diagonal matrices and A as the subgroup of G consisting of unipotent matrices). Assume $\mathbb{F}_{p^f} \subseteq k$ and identify $\hat{A} = \mathbb{Z}/(p^f-1)\mathbb{Z}$ (where we identify i with the character that sends $a \in A$ to $a^i \in k^\times$) and write the group operation in $\hat{A}$ additively. Then*

$$X(P, A) = \{2 \cdot p^q \mid q = 0, \dots, f-1\} \tag{46}$$

In particular, the Ext*-quiver Q of $k\Delta_2(p^f)$ has $p^f - 1$ vertices e_i labeled by elements $i \in \mathbb{Z}/(p^f-1)\mathbb{Z}$. There are precisely f arrows $s_{i,2\cdot p^q}$ (for $q \in \{0, \dots, f-1\}$) emanating from each vertex e_i.*

Proof. $G = P \rtimes A$ is a semidirect product. The action of A on P is given by

$$P \times A \to P : \ (b, a) \mapsto b \cdot a^2 \quad \text{where we identified } A = \mathbb{F}_{p^f}^\times,\ P = \mathbb{F}_{p^f} \tag{47}$$

When thinking of A as diagonal matrices of the form $\mathrm{diag}(a^{-1}, a)$, and of P as upper triangular unipotent matrices with top right entry u, then this action corresponds to the conjugation action of A on P. Let us denote the $\mathbb{F}_pA$ module $\mathbb{F}_{p^f}$ with the action of A specified above by M. According to Lemma 5.2 we have to determine the simple constituents of $k \otimes_{\mathbb{F}_p} M$ as a kA-module. Note that there is a (one-dimensional) $\mathbb{F}_{p^f}A$-module $\tilde{M}$ with $\tilde{M}|_{\mathbb{F}_pA} \cong M$. So clearly

$$k \otimes_{\mathbb{F}_p} M \cong \bigoplus_{\gamma \in \mathrm{Gal}(\mathbb{F}_{p^f}/\mathbb{F}_p)} k \otimes_{\mathbb{F}_{p^f}} \tilde{M}^\gamma \tag{48}$$

Now $\mathrm{Gal}(\mathbb{F}_{p^f}/\mathbb{F}_p) \cong C_f$ is generated by the Frobenius automorphism. So the simple constituents of $k \otimes_{\mathbb{F}_p} M$ are just copies of k on which $a \in A$ acts as $a^{2\cdot p^q}$ for $q \in \{0, \dots, f-1\}$. This shows that $X(P, A)$ is as claimed. The shape of the Ext-quiver is now immediate from Lemma 5.2. □

Notation 5.6. *We define symbols*

$$[\mathbf{q}] := 2 \cdot p^q \tag{49}$$

to refer to the elements of $X(P, A)$ in the situation of the above proposition.

Lemma 5.7. *Assume k splits $\Delta_2(p^f)$. Then $k\Delta_2(p^f)$ consists of a single block if $p = 2$, and two isomorphic blocks otherwise. In the case $p = 2$, the Cartan matrix is given by $I + J$, where I is the identity matrix, and J is the matrix that has all entries equal to one. In the case p odd, the Cartan matrix of either one of the two blocks is $I + 2 \cdot J$.*

Proof. The (i, j)-entry of the Cartan matrix is, by definition, the k-dimension of $e_i \cdot kQ/I \cdot e_j$. Let $E = \langle e_1, \dots, e_{p^f-1}\rangle_k$ be the subspace of kQ/I spanned by the idempotents. Clearly, $kQ/I = E \oplus \mathrm{Rad}(kQ/I)$ as a vector space. So $\dim_k e_i \cdot kQ/I \cdot e_j = \delta_{ij} + \dim_k e_i \,\mathrm{Rad}(kQ/I)e_j$, where δ_{ij} denotes the Kronecker symbol. Now, using the quiver relations from Proposition 5.3, we can deduce that $\dim_k e_i \,\mathrm{Rad}(kQ/I)e_j$ is equal to the number of vectors $(0, \dots, 0) \neq (n_0, \dots, n_{f-1}) \in \{0, \dots, p-1\}^f$ such that

$$2 \cdot \sum_{q=0}^{f-1} n_q \cdot p^q \equiv i - j \mod (p^f - 1) \tag{50}$$

If p is odd and $i - j$ is odd as well, then (since $p^f - 1$ will be even) the congruence cannot be satisfied by any sequence of n_q's. So the corresponding entries in the Cartan matrix are zero. Now assume that p is odd and $i - j$ is even. Then the above congruence is equivalent to

$$\sum_{q=0}^{f-1} n_q \cdot p^q \equiv \frac{i-j}{2} \mod \left(\frac{p^f-1}{2}\right) \tag{51}$$

By uniqueness of the p-adic expansion of an integer, the analogous equation modulo $p^f - 1$ has a unique solution in which not all of the n_q's are zero. Hence the equation above has precisely two solutions.

Now if $p = 2$, the factor "2" in (50) is a unit in the ring $\mathbb{Z}/(2^f - 1)\mathbb{Z}$, and hence the number of solutions of (50) is equal to the number of solutions of

$$\sum_{q=0}^{f-1} n_q \cdot 2^q \equiv \frac{i-j}{2} \mod (2^f - 1) \tag{52}$$

This equation has a unique non-zero solution thanks to the uniqueness of the 2-adic expansion of an integer. □

Remark 5.8. *By counting conjugacy classes in the group $\Delta_2(2^f)$, one easily obtains that*

$$\dim_K Z(K\Delta_2(2^f)) = 2^f \tag{53}$$

In the same way one obtains for p odd that

$$\dim_K Z(K\Delta_2(p^f)) = p^f + 3 \tag{54}$$

Since $k\Delta_2(p^f)$ is the direct sum of two isomorphic blocks, the dimension of the center of either one of these blocks is $(p^f + 3)/2$.

For reasons that will become apparent in the section on descent to smaller fields, we would like to investigate a slightly larger class of algebras than the blocks of $k\Delta_2(p^f)$, namely those (split) k-algebras which become isomorphic to $k\Delta_2(p^f)$ upon extension of the ground field.

Definition 5.9. *We call a split k-algebra $\overline{\Lambda}$ with $\bar{k} \otimes \overline{\Lambda} \cong B_0(\bar{k}\Delta_2(p^f))$ a* split k-form *of the principal block $B_0(\bar{k}\Delta_2(p^f))$ of $\bar{k}\Delta_2(p^f)$.*

Remark 5.10. *If $\overline{\Lambda}$ is a split k-form of $B_0(\bar{k}\Delta_2(p^f))$, then $\overline{\Lambda}$ has the same* Ext*-quiver and the same Cartan matrix as $B_0(\bar{k}\Delta_2(p^f))$. Moreover, the k-dimension of the center of $\overline{\Lambda}$ is equal to the $\bar{k}$-dimension of the center of $B_0(\bar{k}\Delta_2(p^f))$.*

Remark 5.11. *The quiver relations given in (36) and (37) are defined over $\mathbb{F}_p$. In particular, even if k is no splitting field for $\Delta_2(p^f)$, the blocks of kQ/I are split k-forms of $B_0(\bar{k}\Delta_2(p^f))$.*

Proposition 5.12 (Shape of split k-forms)**.** *Let $\overline{\Lambda}$ be a split k-form of $B_0(\bar{k}\Delta_2(p^f))$. By Q we now denote the* Ext*-quiver of $B_0(\bar{k}\Delta_2(p^f))$ (as opposed to the entire group ring $\bar{k}\Delta_2(p^f)$, which it was before). Denote (as before) the vertices of Q by e_{2i} and the arrows by $s_{2i,q}$. Then $\overline{\Lambda}$ is isomorphic to kQ/I' for some ideal I' which contains all the relations*

$$\prod_{j=0}^{p-1} s_{2i+j\cdot[\mathbf{q}],q} \quad \text{for all } i \in \mathbb{Z} \text{ and } q \in \{0, \ldots, f-1\} \tag{55}$$

and the relations

$$s_{2i,q} \cdot s_{2i+[\mathbf{q}],q'} - \alpha_{2i,q,q'} \cdot s_{2i,q'} \cdot s_{2i+[\mathbf{q}'],q} \tag{56}$$

with i ranging over $\mathbb{Z}$, q and q' ranging over $\{0, \ldots, f-1\}$ and the $\alpha_{2i,q,q'}$ being of the form

$$c_{2i,q,q'} \cdot e_{2i} + r_{2i,q,q'} \tag{57}$$

for some $c_{2i,q,q'} \in k^\times$ and some k-linear combination $r_{2i,q,q'}$ of closed paths of positive length starting and ending in e_{2i} (hence, by construction, the $\alpha_{2i,q,q'}$ will lie in $(e_{2i} \cdot kQ/I' \cdot e_{2i})^\times$).

The relations given in (55) and (56) together with all paths of length $|\Delta_2(p^f)|$ generate I'.

Proof. We can assume that $\overline{\Lambda} \cong kQ/I'$ for some ideal I' contained in the ideal of kQ generated by the paths of length at least two. We need to show that I' is of the desired form. Choose an embedding $\varphi: \ kQ/I' \hookrightarrow \bar{k}Q/I$ that maps each idempotent e_{2i} to itself such that the $\bar{k}$-span of the image of φ is all of $\bar{k}Q/I$. Then for each i and q the image $\varphi(s_{2i,q})$ has to be equal to $x_{2i,q} \cdot s_{2i,q}$ for some $x_{2i,q} \in (e_{2i} \cdot \bar{k}Q/I \cdot e_{2i})^\times$. Indeed, the relations in I can be used to show that $e_{2i} \cdot \bar{k}Q/I \cdot e_{2i+[\mathbf{q}]} = e_{2i} \cdot \bar{k}Q/I \cdot e_{2i} \cdot s_{2i,q}$. If $x_{2i,q}$ were no unit in $e_{2i} \cdot \bar{k}Q/I \cdot e_{2i}$, then $\varphi(s_{2i,q})$ would be contained in $\mathrm{Jac}^2(\bar{k}Q/I)$ and therefore the $\varphi(s_{2i,q})$ together with the e_{2i} could not generate $\bar{k}Q/I$ as a $\bar{k}$-algebra. Since the relations in I imply that $e_{2i} \cdot \bar{k}Q/I \cdot e_{2i} \cdot s_{2i,q} = s_{2i,q} \cdot e_{2i+[\mathbf{q}]} \cdot \bar{k}Q/I \cdot e_{2i+[\mathbf{q}]}$, the relations in (55) follow immediately from the corresponding relation in I by application of φ.

Analogous to the above discussion, we can also deduce that for all $i \in \mathbb{Z}$ and $q, q' \in \{0, \ldots, f-1\}$

$$\varphi(s_{2i,q}) \cdot \varphi(s_{2i+[\mathbf{q}],q'}) = \beta_{2i,q,q'} \cdot \varphi(s_{2i,q'}) \cdot \varphi(s_{2i+[\mathbf{q'}],q}) \tag{58}$$

for some $\beta_{2i,q,q'} \in (e_{2i} \cdot \bar{k}Q/I \cdot e_{2i})^\times$. Now take $\alpha'_{2i,q,q'} := (\mathrm{id}_{\bar{k}} \otimes_k \varphi)^{-1}(\beta_{2i,q,q'}) \in \bar{k} \otimes_k kQ/I'$. Choose a k-vector space complement V of k in $\bar{k}$ and choose $\alpha_{2i,q,q'} \in e_{2i} \cdot kQ/I' \cdot e_{2i}$ such that $\alpha'_{2i,q,q'} = \alpha_{2i,q,q'} +$ (Sum of paths with coefficients in V) . Now clearly the following holds:

$$s_{2i,q} \cdot s_{2i+[\mathbf{q}],q'} = \alpha_{2i,q,q'} \cdot s_{2i,q'} \cdot s_{2i+[\mathbf{q'}],q} + \text{ (Sum of paths with coefficients in } V) \tag{59}$$

in $\bar{k} \otimes_k kQ/I'$. Since a sum of paths with coefficients in V must be k-linearly independent from kQ/I', the relation (56) must hold with this choice of $\alpha_{2i,q,q'}$. To see that the coefficient of e_{2i} in $\alpha_{2i,q,q'}$ is non-zero we could simply map the relation back into $\bar{k}Q/I$ using φ and subtract it from relation (58). This implies $(\beta_{2i,q,q'} - \varphi(\alpha_{2i,q,q'})) \cdot s_{2i,q'} \cdot s_{2i+[\mathbf{q'}],q} = 0$, and hence $\beta_{2i,q,q'} - \varphi(\alpha_{2i,q,q'})$ is no unit in $e_{2i} \cdot \bar{k}Q/I \cdot e_{2i}$, which forces $\varphi(\alpha_{2i,q,q'})$ to be a unit.

The claim that the given relations together with all paths of some sufficiently large length generate I' can be verified by showing that they can be used to rewrite any path as a linear combination of paths of the form

$$s_{2i,q_1} \cdot s_{2i+[\mathbf{q_1}],q_2} \cdots s_{2i+[\mathbf{q_1}]+\ldots+[\mathbf{q_{l-1}}],q_l} \tag{60}$$

such that $q_1 \leq q_2 \leq \ldots \leq q_l$ and no p of the q_j's are equal. This last statement follows from relation (55). The latter requirement can be met using relation (55). If the q_j's are not ordered as claimed, relation (56) can be used to permute them. This will however produce some summands of strictly greater length. So one can apply a rewriting strategy where one starts with the paths of smallest length which are not already in the desired standard form, rewrites those (possibly altering or adding some summands of strictly greater length) and then repeats the process until the shortest paths not in standard form are bigger than the cut-off length and therefore equal to zero. □

Lemma 5.13. *Let $\overline{\Lambda}$ be a split k-form of $B_0(\bar{k}\Delta_2(p^f))$*

1. *Assume $p = 2$. Then any lift $\Lambda \in \mathfrak{L}_s(\overline{\Lambda})$ with $\dim_K Z(K \otimes \Lambda) = \dim_k Z(\overline{\Lambda})$ has the following decomposition matrix over a splitting field*

$$\begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \\ 1 & 1 & \cdots & 1 \end{bmatrix} \tag{61}$$

   up to permutation of rows.
2. *Assume $p \neq 2$. If $\Lambda \in \mathfrak{L}_s(\overline{\Lambda})$ with $\dim_K Z(K \otimes \Lambda) = \dim_k Z(\overline{\Lambda})$, then the decomposition matrix of Λ over a splitting field looks as follows:*

$$\begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \\ 1 & 1 & \cdots & 1 \\ 1 & 1 & \cdots & 1 \end{bmatrix} \tag{62}$$

up to permutation of rows.

3. *Fix a* $\Lambda \in \mathfrak{L}_s(\overline{\Lambda})$ *subject to the same condition on the center as in the first respectively the second point. Assume that there is some totally ramified extension of* K *that splits* Λ*.*
   (a) *If* $p = 2$*, then* K *already splits* Λ*.*
   (b) *If* p *is odd then all one-dimensional representations of* $\bar{K} \otimes \Lambda$ *are already defined over* $K \otimes \Lambda$*. If* K *does not split* $K \otimes \Lambda$*, then* $K \otimes \Lambda$ *has a unique representation of dimension greater than one, and its endomorphism ring is a totally ramified extension of* K *of degree two. In particular, in that case, the decomposition matrix of* Λ *is as in (62) with the last row removed.*

Proof. Concerning the first two parts: Let D be the decomposition matrix of Λ (over a splitting system). First note that all entries of D must be ≤ 1, as $D^\top \cdot D$ is equal to the Cartan matrix C of $\bar{k}\Delta_2(p^f)$, which has "2"'s (respectively "3"'s) on the diagonal. It is straightforward to prove that the only solutions (with non-negative integer entries ≤ 1) to the equation $D^\top \cdot D = C$ are, up to permutation of rows and columns, the ones given in statement of this lemma.

Now we have a look at the assertions in the non-splitting case. First assume that there is a simple $K \otimes \Lambda$-module V such that $\mathrm{End}_{K\otimes\Lambda}(V)$ is non-commutative. Let P be a projective indecomposable Λ-lattice (note that $k \otimes \Lambda \cong \overline{\Lambda}$ is split, so indecomposable projectives are absolutely indecomposable) such that V occurs as a composition factor of $K \otimes P$. Since the endomorphism ring of V is non-commutative, $\bar{K} \otimes V$ is not multiplicity-free, but it is still a composition factor of $\bar{K} \otimes P$. Hence there is some simple $\bar{K} \otimes \Lambda$-module which occurs in $\bar{K} \otimes P$ with multiplicity greater than one. This is the same as saying that (over a splitting system) there is a decomposition number greater than one, which, as we have seen above, is impossible. Now let V be any simple $K \otimes \Lambda$-module. As we have seen $E := \mathrm{End}_{K\otimes\Lambda}(V)$ is commutative, and therefore it is necessarily contained in any splitting field for $K \otimes \Lambda$. Since by assumption there is a splitting field that is totally ramified over K, the field E must be totally ramified over K as well. Now we look at how the decomposition matrix over K relates to the decomposition matrix over a splitting field. $\mathrm{End}_{\bar{K}\otimes\Lambda}(\bar{K} \otimes V) \cong \bar{K} \otimes_K E \cong \bigoplus^{\dim_K E} \bar{K}$. This implies that $\bar{K} \otimes V$ decomposes into $e := \dim_K E$ non-isomorphic absolutely irreducible modules $V_1, \ldots, V_e$. Whenever P is a projective indecomposable Λ-module, the multiplicity of any V_i in $\bar{K} \otimes P$ is the same as the multiplicity of V in $K \otimes P$. Hence, the decomposition matrix of Λ over a splitting field arises from the decomposition matrix over K by repeating certain rows. Namely, if the endomorphism ring of the simple module associated with a row in the decomposition matrix of Λ has dimension e, then that row is repeated e times in the decomposition matrix over a splitting field. If $p = 2$, then the decomposition matrix over a splitting field contains no repeated rows, and therefore all simple $K \otimes \Lambda$-modules must be split. If $p \neq 2$, then the last two rows of the decomposition matrix over a splitting field are identical, and therefore it is possible that the decomposition matrix of Λ over K contains this row only once. If this is the case, then the endomorphism ring of the simple $K \otimes \Lambda$-module associated with that row must have dimension $e = 2$. The other possibility is that the decomposition matrix of Λ over K is identical to the decomposition matrix over a splitting field. In that case, all simple $K \otimes \Lambda$-modules must be split, because otherwise the rows associated with non-split simple $K \otimes \Lambda$-modules would occur multiple times in the decomposition matrix over a splitting field. □

Notation 5.14. *Let* Λ *be an* $\mathcal{O}$*-order with semisimple* K*-span and let* $\varepsilon_1, \ldots, \varepsilon_n \in Z(K \otimes \Lambda)$ *be the central primitive idempotents. So, in particular, we have fixed a bijection* $\{1, \ldots, n\} \leftrightarrow \{$ *central primitive idempotents* $\}$*.*

1. *Given an element* $u \in Z(K \otimes \Lambda)$ *we set*

$$u_i := \varepsilon_i \cdot u \quad \textit{for all } i \in \{1, \ldots, n\} \tag{63}$$

2. *When dealing with orders* Λ *which have a decomposition matrix like the one in (61) or (62), we make the following convention concerning the ordering of the central primitive idempotents: We choose indices so that the idempotents associated with rows in the decomposition matrix with more than one non-zero entry come last.*

Remark 5.15. *If $\Lambda = \mathcal{O}G$ for some finite group G (or a block thereof), then the symmetrizing element u may be chosen so that*

$$u_i = \frac{\chi_i(1)}{m_i \cdot |G|} \in \mathbb{Q}^\times \tag{64}$$

where χ_i is the i-th irreducible K-character of G (or in the block under consideration), and m_i is the number of absolutely irreducible characters it splits up into when passing from K to its algebraic closure $\bar{K}$ (see Remark 2.4). In particular two of the u_i are equal if (and only if) the corresponding absolutely irreducible characters have equal degree. The equality of two rows in the decomposition matrix is a sufficient criterion for the corresponding characters to have equal degree, and therefore for the corresponding u_i to be equal. Note that we potentially have two equal rows in the decomposition matrix of the principal block of $\mathcal{O}\,\mathrm{SL}_2(p^f)$ if p is odd (to be precise, this happens if f is even).

Theorem 5.16 (Unique lifting)**.** *Let A be a finite-dimensional semisimple K-algebra with $\dim_K Z(A) = \dim_{\bar{k}} Z(B_0(\bar{k}\Delta_2(p^f)))$. Assume A is split by some totally ramified extension of K.*

(a) Assume we are given an element $u \in Z(A)^\times$ which has p-valuation $-f$ in every Wedderburn component of $Z(\bar{K} \otimes A)$. Then any two full $\mathcal{O}$-orders Λ_u and Λ_u' in A satisfying the following two conditions are conjugate:
  - *(1) Λ_u and Λ_u' are self-dual with respect to T_u.*
  - *(2) $k \otimes \Lambda_u$ and $k \otimes \Lambda_u'$ are split k-forms of $B_0(\bar{k}\Delta_2(p^f))$*

(b) Assume u and u' are two symmetrizing elements subject to the same conditions as in (a), such that Λ_u and $\Lambda_{u'}$ both exist. Then:
  - *(1) If $p = 2$: Λ_u and $\Lambda_{u'}$ are conjugate.*
  - *(2) If $p \neq 2$ and K splits A: Let $\kappa = \frac{p^f-1}{2}$. If $\frac{u_{\kappa+1}}{u_{\kappa+2}} = \frac{u'_{\kappa+1}}{u'_{\kappa+2}}$, then Λ_u and $\Lambda_{u'}$ are conjugate.*
  - *(3) If $p \neq 2$ and K does not split A: If $u_{\kappa+1} \cdot \mathcal{O}^\times = u'_{\kappa+1} \cdot \mathcal{O}^\times$, then Λ_u and $\Lambda_{u'}$ are conjugate.*

  Here κ denotes the number of isomorphism classes of simple modules in $B_0(\bar{k}\Delta_2(p^f))$.

Proof. We assume that we are given an order $\Lambda = \Lambda_u$ satisfying the conditions given in (a). In order to prove the theorem we will try to conjugate Λ into a kind of "standard form" depending on u. This will prove the claim made in point (a). By looking at how this "standard form" depends on u we will also be able to prove (b). We let I' be an ideal in kQ as described in Proposition 5.12 such that $k \otimes_{\mathcal{O}} \Lambda \cong kQ/I'$ (we will assume that we have fixed an isomorphism and identify the two). Also, as before, we denote the idempotents in kQ by e_{2i} and the arrows by $s_{2i,q}$. We wish to treat the case where K splits A and the case where K does not split A as well as the cases p even and p odd (essentially) uniformly. So assume that

$$A = \left(\bigoplus_{i=1}^{\kappa} K\right) \oplus \tilde{K}^{\kappa\times\kappa} \quad \text{with } \kappa = \begin{cases} \dfrac{p^f-1}{2} & \text{if } p \neq 2 \\ 2^f - 1 & \text{if } p = 2 \end{cases} \tag{65}$$

where $\tilde{K}$ is isomorphic to K if $p = 2$, to $K \oplus K$ if $p \neq 2$ and A is K-split, or to a fully ramified extension of K of degree two if $p \neq 2$ and A is not K-split. By $\tilde{\varepsilon}$ we denote the idempotent in $Z(A) = K \oplus \ldots \oplus K \oplus \tilde{K}$ which has entry "1" in the summand $\tilde{K}$ and entry "0" in all other summands. For each i let $\hat{e}_{2i} \in \Lambda$ be a lift of $e_{2i} \in kQ/I'$, and assume without loss that $\tilde{\varepsilon}\hat{e}_{2i}$ is the i-th diagonal idempotent in $\tilde{K}^{\kappa\times\kappa}$ (this may certainly be achieved by conjugating Λ by an element of $A^\times$). Assume furthermore that $(1-\tilde{\varepsilon}) \cdot \hat{e}_{2i}$ has non-zero entry in the i-th direct summand of the decomposition (65). Hence we have fixed the elements $\hat{e}_{2i}$ as elements of the algebra A as described in (65). Now, using the fact that Λ is supposed to be symmetric with respect to T_u, it follows that

1. If p is odd and K splits A:

$$\hat{e}_i\Lambda\hat{e}_i = \left\langle [1,1,1], [0, p^{\frac{f}{2}}, -c \cdot p^{\frac{f}{2}}], [0,0,p^f] \right\rangle_{\mathcal{O}} \subset \mathcal{O} \oplus \mathcal{O} \oplus \mathcal{O} \quad \text{where } c = \frac{u_{\kappa+1}}{u_{\kappa+2}} \tag{66}$$

This follows simply from the fact that a self-dual order (with respect to T_u) in $\mathcal{O} \oplus \mathcal{O} \oplus \mathcal{O}$ must have elementary divisors $1, p^{\frac{f}{2}}, p^f$ (as an $\mathcal{O}$ lattice in $\mathcal{O} \oplus \mathcal{O} \oplus \mathcal{O}$) and all traces with respect to T_u must be integral. Note that this also implies that f must be even (in this situation, i. e. when K splits A and p is odd).

2. If p is odd and K does not split A:

$$\hat{e}_i \Lambda \hat{e}_i = \left\langle [1,1], [0, c \cdot \pi^f], [0, c^2 \cdot \pi^{2f}] \right\rangle_{\mathcal{O}} \quad \text{for some } c \in \mathcal{O}[\pi]^\times \tag{67}$$

where π is some uniformizer for the integral closure of $\mathcal{O}$ in $\tilde{K}$, which is a fully ramified extension of K of degree two in this case. Up to this point, we have used two facts: First, that the elementary divisors of $\mathcal{O}[\pi] \otimes \hat{e}_i \Lambda \hat{e}_i$ as a lattice in $\mathcal{O}[\pi] \oplus \mathcal{O}[\pi] \oplus \mathcal{O}[\pi]$ must be $1, \pi^f, \pi^{2f}$, and second, that $\hat{e}_i \Lambda \hat{e}_i$ is generated by a single element as an $\mathcal{O}$-order since $e_i \cdot kQ/I \cdot e_i \cong k[T]/(T^3)$ is generated by a single element as a k-algebra. In this case we need to put in some additional information to show that $\hat{e}_i \Lambda \hat{e}_i$ is uniquely determined, since different choices of c may give rise to different orders. Note that $T_u(\{0\} \oplus p^f \mathcal{O}[\pi]) \subseteq \mathcal{O}$, and hence necessarily $\{0\} \oplus p^f \mathcal{O}[\pi] \subset (\hat{e}_i \Lambda \hat{e}_i)^\sharp = \hat{e}_i \Lambda \hat{e}_i$. Moreover an element $[0, \tilde{c} \cdot \pi^f]$ lies in $\hat{e}_i \Lambda \hat{e}_i$ if and only if $T_u([0, \tilde{c} \cdot \pi^f]) \in \mathcal{O}$. To see this, let $x = [r_1, r_1 + r_2 \cdot c \cdot \pi^f + r_3 \cdot c^2 \cdot \pi^{2f}]$ with $r_1, r_2, r_3 \in \mathcal{O}$ be an arbitrary element of $\hat{e}_i \Lambda \hat{e}_i$. Then

$$T_u(x \cdot [0, \tilde{c} \cdot \pi^f]) = r_1 \cdot T_u([0, \tilde{c} \cdot \pi^f]) + r_2 \cdot T_u([0, c \cdot \tilde{c} \cdot \pi^{2f}]) + r_3 \cdot T_u([0, \tilde{c} \cdot c^2 \cdot \pi^{3f}]) \tag{68}$$

Since $\tilde{K}$ is fully ramified of degree two, we have $\pi^{2f}\mathcal{O}[\pi] = p^f \mathcal{O}[\pi]$, and therefore the last two summands are traces of elements in $\hat{e}_i \Lambda \hat{e}_i$, which must be integral. So (68) is integral for all values of $r_1, r_2, r_3 \in \mathcal{O}$ if and only if $T_u([0, \tilde{c} \cdot \pi^f])$ is integral. This characterizes $\hat{e}_i \Lambda \hat{e}_i$ as

$$\hat{e}_i \Lambda \hat{e}_i = \mathcal{O}\left[[0, \tilde{c} \cdot \pi^f] \;\middle|\; T_u([0, \tilde{c} \cdot \pi^f]) \in \mathcal{O} \right] \tag{69}$$

which is obviously uniquely determined by u and the extension $\tilde{K}/K$.

3. If $p = 2$ then

$$\hat{e}_i \Lambda \hat{e}_i = \left\langle [1,1], [0, 2^f] \right\rangle_{\mathcal{O}} \tag{70}$$

by the same argument as in the first point.

In the above considerations we have used that each u_i has p-valuation $-f$. In the case $p = 2$ we have not used any further information on u. In the case $p \neq 2$ we have used the value of the quotient $u_{\kappa+1}/u_{\kappa+2}$ if K splits A and the class $u_{\kappa+1} \cdot \mathcal{O}^\times$ if it does not (since the characterization in (69) depends only on $u_{\kappa+1} \cdot \mathcal{O}^\times$; note that $u_{\kappa+1}$ is an element of $\tilde{K}$ in this case while in the split case $u_{\kappa+1}$ and $u_{\kappa+2}$ are both elements of K). Since we will not make any further use of the symmetrizing element u below, this will prove part (b) of the theorem once part (a) has been proved.

Note that in either case the $\hat{e}_i \Lambda \hat{e}_i$ are equal when we identify the unique maximal orders containing them. In particular, the subset of $\mathrm{End}_K(\tilde{K})$ consisting of the endomorphisms induced by elements of $\hat{e}_i \Lambda \hat{e}_i$ acting on $\hat{e}_i \Lambda \hat{e}_j \subset \tilde{K}$ by multiplication from the left is the same as the subset of $\mathrm{End}_K(\tilde{K})$ consisting of the endomorphisms induced by elements of $\hat{e}_j \Lambda \hat{e}_j$ acting by multiplication from the right. Hence the submodule structure of $\hat{e}_i \Lambda \hat{e}_j$ is independent of whether it is regarded as a left $\hat{e}_i \Lambda \hat{e}_i$-module or a right $\hat{e}_j \Lambda \hat{e}_j$-module. Now we consider the $e_i \Lambda e_{i+[\mathbf{q}]}$ for arbitrary i and q. We know from the Cartan matrix that the dimension of $e_i \cdot kQ/I' \cdot e_{i+[\mathbf{q}]}$ is equal to one if $p = 2$ and equal to two if $p \neq 2$. We want to show that it is generated by a single element as an $e_i \cdot kQ/I' \cdot e_i$-module. In the case $p = 2$ this is trivial, since it is one-dimensional as a k-vector space. If $p \neq 2$, then consider the elements $s_{i,q}$ and $e_i \cdot \left(\prod_{r=0}^{f-1} s_r^{(p-1)/2} \right) \cdot s_q$, where we use notational convention made in Remark 5.4. Those two elements lie in a two-dimensional k-vector space, and they are not scalar multiples of each other since the relations in I' are homogeneous. Also neither of them is zero since the relations (55) and (56) only allow for a product of arrows to be zero if it contains at least p arrows of the type $s_{j,r}$ for fixed r. Hence they form a k-basis of $e_i \cdot kQ/I' \cdot e_{i+[\mathbf{q}]}$. Since the second of the two elements is obtained from the first one by multiplying with an element of $e_i \cdot kQ/I' \cdot e_i$, it follows that $e_i \cdot kQ/I' \cdot e_{i+[\mathbf{q}]}$ is generated by the first element as an $e_i \cdot kQ/I' \cdot e_i$-module. Independent of whether p is even or odd it follows that $e_i \cdot kQ/I' \cdot e_{i+[\mathbf{q}]}$ is isomorphic to $e_i \cdot kQ/I' \cdot e_i/J$ for some ideal J in $e_i \cdot kQ/I' \cdot e_i$. But since $e_i \cdot kQ/I' \cdot e_i$ is a split commutative local symmetric k-algebra, its socle is its unique one-dimensional submodule. This implies that $e_i \cdot kQ/I' \cdot e_{i+[\mathbf{q}]} \cong e_i \cdot kQ/I' \cdot e_i/\operatorname{Soc}(e_i \cdot kQ/I' \cdot e_i)$. There is an epimorphism of k-algebras from $e_i \cdot kQ/I' \cdot e_i \cong k \otimes \hat{e}_i\Lambda\hat{e}_i$ to $k \otimes \tilde{\varepsilon}\hat{e}_i\Lambda\hat{e}_i$, and therefore $k \otimes \tilde{\varepsilon}\hat{e}_i\Lambda\hat{e}_i \cong e_i \cdot kQ/I' \cdot e_i/\operatorname{Soc}(e_i \cdot kQ/I' \cdot e_i)$ as k-algebras. It follows that $k \otimes \hat{e}_i\Lambda\hat{e}_{i+[\mathbf{q}]}$ is free as a left $k \otimes \tilde{\varepsilon}\hat{e}_i\Lambda\hat{e}_i$-module, and therefore $\hat{e}_i\Lambda\hat{e}_{i+[\mathbf{q}]}$ is free as a left $\tilde{\varepsilon}\hat{e}_i\Lambda\hat{e}_i$-module. This implies (when $\hat{e}_iA\hat{e}_{i+[\mathbf{q}]}$ is identified with $\tilde{K}$ in the natural way)

$$\hat{e}_i\Lambda\hat{e}_j = x_{ij} \cdot \tilde{\varepsilon}\hat{e}_i\Lambda\hat{e}_i \quad \text{for some } x_{ij} \in \tilde{K}^\times \text{ where } j = i + [\mathbf{q}] \text{ for some } q \tag{71}$$

In addition, we may and will assume that the x_{ij} are integral over $\mathcal{O}$. For each i and q we have

$$\prod_{l=0}^{p-1} e_{i+l\cdot[\mathbf{q}]} \cdot kQ/I' \cdot e_{i+(l+1)\cdot[\mathbf{q}]} = 0 \tag{72}$$

and hence

$$\prod_{l=0}^{p-1} \hat{e}_{i+l\cdot[\mathbf{q}]} \cdot \Lambda \cdot \hat{e}_{i+(l+1)\cdot[\mathbf{q}]} \subseteq p \cdot \hat{e}_i \cdot \Lambda \cdot \hat{e}_{i+[\mathbf{q+1}]} \tag{73}$$

Everything from here down to (91) below is about showing that the inclusion in (73) is in fact an equality. The significance of this is that it can then be used as a formula to compute the $\hat{e}_i \cdot \Lambda \cdot \hat{e}_{i+[\mathbf{q+1}]}$ from the $\hat{e}_i \cdot \Lambda \cdot \hat{e}_{i+[\mathbf{q}]}$, showing that Λ is determined by the $\hat{e}_i \cdot \Lambda \cdot \hat{e}_{i+[\mathbf{0}]}$.

We define a "normalized index" for full $\mathcal{O}$-lattices $L_1 \supseteq L_2$ in $\tilde{K}$ as follows:

$$\operatorname{idx}(L_1, L_2) := \frac{\operatorname{length}_{\mathcal{O}} L_1/L_2}{\operatorname{length}_{\mathcal{O}} L_1/pL_1} \tag{74}$$

Note that the denominator is equal to the $\mathcal{O}$-rank of L_1, which is in turn equal to the K-dimension of $\tilde{K}$. Hence the denominator is independent of the choice of L_1. For arbitrary lattices $L_1, L_2 \subset \tilde{K}$ (neither of which necessarily contains the other) we define $\operatorname{idx}(L_1, L_2) := \operatorname{idx}(L_1 + L_2, L_2) - \operatorname{idx}(L_1 + L_2, L_1)$. Now, if L is any full lattice in $\tilde{K}$, and $x_1, x_2 \in \tilde{K}^\times$, then

$$\operatorname{idx}(L, x_1 \cdot x_2 \cdot L) = \operatorname{idx}(L, x_1 \cdot L) + \operatorname{idx}(L, x_2 \cdot L) \tag{75}$$

because $\operatorname{idx}(L, x_i \cdot L)$ equals a constant multiple of the p-valuation of the determinant of "multiplication with x_i" regarded as a K-vector space automorphism of $\tilde{K}$. Now define

$$m_{i,q} := \operatorname{idx}\left(\tilde{\varepsilon}\hat{e}_i\Lambda\hat{e}_i, \hat{e}_i\Lambda\hat{e}_{i+[\mathbf{q}]}\right) \tag{76}$$

where we view $\hat{e}_i\Lambda\hat{e}_{i+[\mathbf{q}]}$ as a subset of $\tilde{K}$ as in (71). Define furthermore

$$a_{i,q} := \operatorname{idx}\left(\hat{e}_i \cdot \Lambda \cdot \hat{e}_{i+[\mathbf{q+1}]}, \prod_{l=0}^{p-1} \hat{e}_{i+l\cdot[\mathbf{q}]} \cdot \Lambda \cdot \hat{e}_{i+(l+1)\cdot[\mathbf{q}]}\right) = \left(\sum_{l=0}^{p-1} m_{i+l\cdot[\mathbf{q}],q}\right) - m_{i,q+1} \tag{77}$$

Clearly $a_{i,q} \geq 1$ for all i and q. We have for any $q \neq r$

$$e_i \cdot kQ/I' \cdot e_{i+[\mathbf{q}]} \cdot kQ/I' \cdot e_{i+[\mathbf{q}]+[\mathbf{r}]} = e_i \cdot kQ/I' \cdot e_{i+[\mathbf{q}]+[\mathbf{r}]} \tag{78}$$

and hence in particular

$$\hat{e}_i\Lambda\hat{e}_{i+[\mathbf{q}]}\Lambda\hat{e}_{i+[\mathbf{q}]+[\mathbf{q+1}]} = \hat{e}_i\Lambda\hat{e}_{i+[\mathbf{q}]+[\mathbf{q+1}]} = \hat{e}_i\Lambda\hat{e}_{i+[\mathbf{q+1}]}\Lambda\hat{e}_{i+[\mathbf{q}]+[\mathbf{q+1}]} \tag{79}$$

which implies for all i and q that

$$m_{i,q} + m_{i+[\mathbf{q}],q+1} = m_{i,q+1} + m_{i+[\mathbf{q+1}],q} \tag{80}$$

Now

$$a_{i,q} - a_{i+[\mathbf{q}],q} = \left(\sum_{l=0}^{p-1} m_{i+l\cdot[\mathbf{q}],q}\right) - \left(\sum_{l=1}^{p} m_{i+l\cdot[\mathbf{q}],q}\right) - m_{i,q+1} + m_{i+[\mathbf{q}],q+1}$$
$$= m_{i,q} - m_{i+[\mathbf{q+1}],q} - m_{i,q+1} + m_{i+[\mathbf{q}],q+1} \overset{(80)}{=} 0 \tag{81}$$

Since p is relatively prime to κ, this implies that $a_{i,q} = a_q$ for some a_q independent of i. Now we sum up (77) over all κ values of i, and get

$$\sum_{i=1}^{\kappa} m_{2i,q+1} = p \cdot \sum_{i=1}^{\kappa} m_{2i,q} - \kappa \cdot a_q \tag{82}$$

Plugging this formula into itself f times yields (for all values of q)

$$\sum_{i=1}^{\kappa} m_{2i,q} = p^f \cdot \sum_{i=1}^{\kappa} m_{2i,q} - \kappa \sum_{i=1}^{f} p^{f-i} \cdot a_{q+i-1} \tag{83}$$

which implies

$$\sum_{i=1}^{\kappa} m_{2i,q} = \frac{\kappa}{p^f - 1} \cdot \sum_{i=1}^{f} p^{f-i} \cdot a_{q+i-1} \geq \frac{\kappa}{p-1} \tag{84}$$

with equality if and only if all a_q are equal to 1. We will now show that

$$\mathrm{Jac}(e_i \cdot kQ/I' \cdot e_i) = \prod_{q=0}^{f-1} \prod_{j=1}^{\frac{p-1}{2}} e_{i+\frac{1}{2}\cdot([\mathbf{q}]-[\mathbf{0}])+(j-1)\cdot[\mathbf{q}]} \cdot kQ/I' \cdot e_{i+\frac{1}{2}\cdot([\mathbf{q}]-[\mathbf{0}])+j\cdot[\mathbf{q}]} \quad (p \neq 2)$$
$$\mathrm{Jac}(e_i \cdot kQ/I' \cdot e_i) = \prod_{q=0}^{f-1} e_{i+[\mathbf{q}]-[\mathbf{0}]} \cdot kQ/I' \cdot e_{i+[\mathbf{q+1}]-[\mathbf{0}]} \quad (p = 2) \tag{85}$$

In both cases it is clear that the right hand sides are ideals in $e_i \cdot kQ/I' \cdot e_i$. They are in fact proper ideals in $e_i \cdot kQ/I' \cdot e_i$, since they only contain paths of strictly positive length. The ring $e_i \cdot kQ/I' \cdot e_i$ is an algebra of dimension two if $p = 2$ and dimension three if $p \neq 2$. In the case $p = 2$, $\mathrm{Jac}(e_i \cdot kQ/I' \cdot e_i)$ is the only non-zero proper ideal in $e_i \cdot kQ/I' \cdot e_i$, because $e_i \cdot kQ/I' \cdot e_i$ is local. Hence it suffices to show that the given ideal contains a non-zero element. We may choose $e_i \cdot \prod_{q=0}^{f-1} s_q$ for this purpose. This element is non-zero since no arrow of type s_q for fixed q occurs more than once. In the case $p \neq 2$, the ideal $\mathrm{Jac}(e_i \cdot kQ/I' \cdot e_i)$ is the unique ideal of dimension two, and since $e_i \cdot kQ/I' \cdot e_i$ is symmetric, $\mathrm{Soc}(e_i \cdot kQ/I' \cdot e_i)$ is the unique ideal of dimension 1. So in order to show that the module defined in (85) is in fact as claimed, we just need to specify an element with non-zero square, since every element in the socle squares to zero. The element $e_i \cdot \prod_{q=0}^{f-1} s_q^{(p-1)/2}$ has this property, since its square involves no p arrows of type s_q for any q and is therefore non-zero. Note that in the upper equation we used that $\frac{1}{2}([\mathbf{q}] - [\mathbf{0}]) = \sum_{r=0}^{q-1} \frac{p-1}{2}[\mathbf{r}]$ to simplify the indices.

Now $\tilde{\varepsilon}\hat{e}_i\Lambda\hat{e}_i \cap \hat{e}_i\Lambda\hat{e}_i$ is a pure sublattice of $\hat{e}_i\Lambda\hat{e}_i$. The k-dimension of its image in $e_i \cdot kQ/I' \cdot e_i$ must therefore be equal to its $\mathcal{O}$-rank (which is one if $p = 2$ and two otherwise), which implies that said image is equal to $\mathrm{Jac}(e_i \cdot kQ/I' \cdot e_i)$. Another consequence of $\tilde{\varepsilon}\hat{e}_i\Lambda\hat{e}_i \cap \hat{e}_i\Lambda\hat{e}_i$ being a pure sublattice of $\hat{e}_i\Lambda\hat{e}_i$ is that any proper sublattice of it maps to a proper subspace of $\mathrm{Jac}(e_i \cdot kQ/I' \cdot e_i)$. Hence (85) implies the following:

$$\tilde{\varepsilon}\hat{e}_i\Lambda\hat{e}_i \cap \hat{e}_i\Lambda\hat{e}_i = \prod_{q=0}^{f-1} \prod_{j=1}^{\frac{p-1}{2}} \hat{e}_{i+\frac{1}{2}\cdot([\mathbf{q}]-[\mathbf{0}])+(j-1)\cdot[\mathbf{q}]} \Lambda \hat{e}_{i+\frac{1}{2}\cdot([\mathbf{q}]-[\mathbf{0}])+j\cdot[\mathbf{q}]} \quad (p \neq 2)$$
$$\tilde{\varepsilon}\hat{e}_i\Lambda\hat{e}_i \cap \hat{e}_i\Lambda\hat{e}_i = \prod_{q=0}^{f-1} \hat{e}_{i+[\mathbf{q}]-[\mathbf{0}]} \Lambda \hat{e}_{i+[\mathbf{q+1}]-[\mathbf{0}]} \quad (p = 2) \tag{86}$$

This, in turn, implies that the following holds for any index i:

$$\begin{aligned}\frac{f}{2} &= \operatorname{idx}(\tilde{\varepsilon}\hat{e}_i\Lambda\hat{e}_i, \tilde{\varepsilon}\hat{e}_i\Lambda\hat{e}_i \cap \hat{e}_i\Lambda\hat{e}_i) &= \sum_{q=0}^{f-1}\sum_{j=1}^{\frac{p-1}{2}} m_{i+\frac{1}{2}\cdot([\mathbf{q}]-[\mathbf{0}])+(j-1)\cdot[\mathbf{q}],q} \quad (p\neq 2)\\ f &= \operatorname{idx}(\tilde{\varepsilon}\hat{e}_i\Lambda\hat{e}_i, \tilde{\varepsilon}\hat{e}_i\Lambda\hat{e}_i \cap \hat{e}_i\Lambda\hat{e}_i) &= \sum_{q=0}^{f-1} m_{i+[\mathbf{q}]-[\mathbf{0}],q} \quad (p=2)\end{aligned} \tag{87}$$

Summing this up over all κ different values of i yields (regardless of whether p is even or odd)

$$\kappa\cdot\frac{f}{2} = \sum_{q=0}^{f-1}\frac{p-1}{2}\sum_{i=1}^{\kappa} m_{2i,q} \tag{88}$$

Now we plug in (84) to get

$$\kappa\cdot\frac{f}{2} = \frac{p-1}{2}\cdot\frac{\kappa}{p^f-1}\cdot\sum_{q=0}^{f-1}\sum_{i=1}^{f} p^{f-i}\cdot a_{q+i-1} = \frac{p-1}{2}\cdot\frac{\kappa}{p^f-1}\cdot\frac{p^f-1}{p-1}\cdot\sum_{q=0}^{f-1} a_q \tag{89}$$

We conclude

$$\sum_{q=0}^{f-1} a_q = f \tag{90}$$

which implies that all a_q are equal to one. This implies that the $\hat{e}_{2i}\Lambda\hat{e}_{2i+[\mathbf{0}]}$ determine Λ in the sense that the formula

$$\hat{e}_{2i}\Lambda\hat{e}_{2i+[\mathbf{q+1}]} = \frac{1}{p}\cdot\hat{e}_{2i}\Lambda\hat{e}_{2i+[\mathbf{q}]}\cdots\hat{e}_{2i+(p-1)\cdot[\mathbf{q}]}\Lambda\hat{e}_{2i+p\cdot[\mathbf{q}]} \tag{91}$$

shows how to calculate $\hat{e}_{2i}\Lambda\hat{e}_{2i+[\mathbf{q+1}]}$ from the knowledge of the $\hat{e}_{2j}\Lambda\hat{e}_{2j+[\mathbf{q}]}$ (for all j).

Now we may replace Λ by $y^{-1}\cdot\Lambda\cdot y$, where

$$y := \left[1,\ldots,1,\operatorname{diag}\left(\prod_{j=0}^{i-1} x_{2j,2j+[\mathbf{0}]}\;\middle|\; i=1,\ldots,\kappa\right)\right]\in A^\times \tag{92}$$

(the x_{ij} were defined in (71)) and so we may assume without loss that all $x_{2i,2i+[\mathbf{0}]}$ are equal to 1, except possibly $x_{2\kappa-[\mathbf{0}],2\kappa}$. In other words, we have fixed all but one of the $\hat{e}_{2i}\Lambda\hat{e}_{2i+[\mathbf{0}]}$. But we have

$$\hat{e}_{2\kappa-[\mathbf{0}]}\Lambda\hat{e}_{2\kappa} = \left\{v\in\hat{e}_{2\kappa-[\mathbf{0}]}A\hat{e}_{2\kappa}\mid \hat{e}_{2\kappa}\Lambda\hat{e}_{2\kappa-[\mathbf{0}]}\cdot v\subseteq\hat{e}_{2\kappa}\Lambda\hat{e}_{2\kappa}\right\} \tag{93}$$

which is a consequence of the fact that $\hat{e}_{2\kappa-[\mathbf{0}]}\Lambda\hat{e}_{2\kappa}$ is the dual of $\hat{e}_{2\kappa}\Lambda\hat{e}_{2\kappa-[\mathbf{0}]}$ with respect to the bilinear pairing induced by T_u (this is a general fact on self-dual orders independent of the concrete symmetrizing form T_u; in fact u does not even show up in (93)). Now in the above formula, $\hat{e}_{2\kappa}\Lambda\hat{e}_{2\kappa}$ is explicitly known, and $\hat{e}_{2\kappa}\Lambda\hat{e}_{2\kappa-[\mathbf{0}]}$ can be calculated by repeated application of (91) from the $\hat{e}_{2i}\Lambda e_{2i+[\mathbf{0}]}$ with $0\leq i<\kappa-1$ (which were fixed above by means of conjugation). This can be seen by realizing that $e_{2\kappa}\cdot kQ/I'\cdot e_{2\kappa-[\mathbf{0}]}$ can be written as a product of various $e_{2i}\cdot kQ/I'\cdot e_{2i+[\mathbf{q}]}$ with $0\leq 2i<2i+[\mathbf{q}]\leq 2(\kappa-1)$ and hence $\hat{e}_{2\kappa}\Lambda\hat{e}_{2\kappa-[\mathbf{0}]}$ can be written as a product of various $\hat{e}_{2i}\Lambda\hat{e}_{2i+[\mathbf{q}]}$ with the same restriction in i and q. But the restriction on i and q ensures that these $\hat{e}_{2i}\Lambda\hat{e}_{2i+[\mathbf{q}]}$ can be computed by means of (91) using only those $\hat{e}_{2i}\Lambda e_{2i+[\mathbf{0}]}$ with $0\leq i<\kappa-1$. Hence, Λ is determined in the sense that we have conjugated Λ to some fixed order determined by the data given in the statement of the theorem. This concludes the proof. □

Remark 5.17. *Situation as in the last theorem. Assume furthermore that the (unique) lift $\Lambda=\Lambda_u$ exists. Then the above proof also implies the following: If $\alpha\in\operatorname{Aut}_k(k\otimes\Lambda)$ is an automorphism of $k\otimes\Lambda$ permuting the set of idempotents $\{e_i\}_i$, then there exists an element $\hat{\alpha}\in\operatorname{Aut}_{\mathcal{O}}(\Lambda)$ inducing the corresponding permutation on the set of idempotents $\{\hat{e}_i\}_i$. This follows simply from the fact that we fixed the idempotents at the beginning of the proof of the Theorem and then only used conjugation by elements of $A^\times$ that commuted with all $\hat{e}_i$ to conjugate Λ to any potential other lift of $k\otimes\Lambda$ (also containing the same fixed set of idempotents $\{\hat{e}_i\}_i$).*

6. Transfer to $\mathcal{O}\mathrm{SL}_2(p^f)$

Now we will generalize the result of Theorem 5.16 to all algebras derived equivalent to a split k-form of $B_0(\bar{k}\Delta_2(p^f))$. This will in particular include the two non-semisimple blocks of $k\,\mathrm{SL}_2(p^f)$.

Lemma 6.1. *Let k be algebraically closed and let B be the principal block of $k\Delta_2(p^f)$. There is an epimorphism of algebraic groups*

$$\prod_{i=1}^{f} Z(B)^\times \twoheadrightarrow \mathrm{Out}_k^s(B) \tag{94}$$

In particular, $\mathrm{Out}_k^s(B)$ is connected as an algebraic group, and hence equal to $\mathrm{Out}_k^0(B)$.

Proof. We retain the notations of the previous section, and in particular we identify B with a block of kQ/I (with Q and I as defined in Proposition 5.3). First define a homomorphisms of algebraic groups

$$\psi:\ \prod_{i=1}^{f} Z(B)^\times \to \mathrm{Aut}_k^s(B) \tag{95}$$

which sends $(z_1,\ldots,z_f)$ to the automorphism given by $s_{i,q}\mapsto z_q\cdot s_{i,q}$ and fixes each e_i. It is clear that these are automorphisms by checking that the images satisfy the relations given in Proposition 5.3. We claim that the composition of ψ with the natural epimorphism $\mathrm{Aut}_k^s(B)\twoheadrightarrow \mathrm{Out}_k^s(B)$ is surjective. Note that $Z(B)^\times = Z(B) - \mathrm{Jac}(Z(B))$, and therefore $Z(B)^\times$ is Zariski-dense in $Z(B)$. Since $Z(B)$ is a k-vector space, it is connected as an algebraic variety and therefore so is the dense subset $Z(B)^\times$. So the claimed surjectivity would indeed imply the connectedness of $\mathrm{Out}_k^s(B)$.

We first prove the following claim, which will be used below: If $n\in\mathbb{N}$ is relatively prime to p, then the equation $T^n - z$ for $z\in Z(B)^\times$ has a solution in $Z(B)^\times$. This follows from the fact that a full set of n orthogonal primitive idempotents can be lifted from $k[T]/(T^n-\overline{z})$ to $Z(B)[T]/(T^n-z)$ (where $\overline{z}$ is the image of z in $Z(B)/\mathrm{Jac}(Z(B))=k$). This yields a decomposition of algebras $Z(B)[T]/(T^n-z)\cong A_1\oplus\ldots\oplus A_n$. Since the A_i are, in particular, $Z(B)$-modules, and $Z(B)[T]/(T^n-z)$ is free of rank n as a $Z(B)$-module, we must have that each A_i is a $Z(B)$-algebra that is free of rank one as a $Z(B)$-module. Hence each A_i is canonically isomorphic to $Z(B)$ as a k-algebra, and the image of T in any of the $A_i\cong Z(B)$ is a solution of $T^n-z=0$.

Now we come to the actual proof of surjectivity of the composition of ψ with the natural epimorphism $\mathrm{Aut}_k^s(B)\twoheadrightarrow \mathrm{Out}_k^s(B)$. Assume that $\alpha\in\mathrm{Aut}_k(B)$ is an automorphism such that $P\otimes{}_{\mathrm{id}}A_\alpha\cong P$ for all projective indecomposables P. All full sets of orthogonal primitive idempotents in B are conjugate (see, for instance, [CR81, Introduction §6, Exercise 14]), and hence we may compose α with an inner automorphism of B such that the resulting automorphism fixes all idempotents. So we may and will assume that α fixes e_i for each i. Since the canonical map $Z(B)\to e_iBe_i$ is surjective, and $s_{i,q}$ is a generator for the e_iBe_i module $e_iBe_{i+[\mathbf{q}]}$, we will have $\alpha(s_{i,q})=z_{i,q}\cdot s_{i,q}$ for certain elements $z_{i,q}\in Z(B)^\times$. Moreover, the $z_{i,q}$ determine α. Now consider conjugation with elements v of the form $v=\sum_i c_ie_i$ for certain $c_i\in Z(B)^\times$:

$$v^{-1}\cdot\alpha(s_{i,q})\cdot v = \underbrace{\frac{c_{i+[\mathbf{q}]}}{c_i}\cdot z_{i,q}}_{=:\tilde{z}_{i,q}}\cdot s_{i,q} \tag{96}$$

With $\tilde{z}_{i,q}$ defined as in the above equation we have

$$\prod_i \tilde{z}_{i,0} = \prod_i z_{i,0} =: \lambda \tag{97}$$

Furthermore we can choose the c_i in the definition of v to assign prescribed values to all but one of the $\tilde{z}_{i,0}$. Choose the c_i so that all but possibly one of the $\tilde{z}_{i,0}$ become equal to a κ-th root of λ where κ is the number of simple modules in the block, which is relatively prime to p. Then by the invariance of the product given in (97), all $\tilde{z}_{i,0}$ will be equal. Without loss of generality we replace α by the composition of α with conjugation by this v, that is, we assume that all $z_{i,0}$ are equal. We claim that this α, which differs from the α we started with only by an inner automorphism, lies in the image of ψ with ψ as defined in (95). To show this first notice that for $q \neq r$ the product $s_{i,q} \cdot s_{i+[\mathbf{q}],r}$ is a generator for the e_iBe_i-module $e_iBe_{i+[\mathbf{q}]+[\mathbf{r}]}$, which is isomorphic to the e_iBe_i-module $e_iBe_{i+[\mathbf{q}]}$. Hence for any $c, \tilde{c} \in Z(B)^\times$ we have $c \cdot s_{i,q} = \tilde{c} \cdot s_{i,q}$ if and only if $c \cdot s_{i,q}s_{i+[\mathbf{q}],r} = \tilde{c} \cdot s_{i,q}s_{i+[\mathbf{q}],r}$. Furthermore, in order for α to be an automorphism, the following relation must hold:

$$\begin{aligned} z_{i,q} \cdot z_{i+[\mathbf{q}],q+1} \cdot s_{i,q}s_{i+[\mathbf{q}],q+1} &= z_{i,q+1} \cdot z_{i+[\mathbf{q+1}],q} \cdot s_{i,q+1}s_{i+[\mathbf{q+1}],q} \\ &= z_{i,q+1} \cdot z_{i+[\mathbf{q+1}],q} \cdot s_{i,q}s_{i+[\mathbf{q}],q+1} \end{aligned} \tag{98}$$

So if we assume as an induction hypothesis that all $z_{i,q}$ are equal for some fixed value of q, then this implies that $z_{i+[\mathbf{q}],q+1} \cdot s_{i,q} = z_{i,q+1} \cdot s_{i,q}$, and hence we may set $z_{i+[\mathbf{q}],q+1} = z_{i,q+1}$. Consequentially, all $z_{i,q+1}$ are equal. Therefore α agrees with an element in $\mathrm{Im}(\psi)$ on the generators $s_{i,q}$. But this implies $\alpha \in \mathrm{Im}(\psi)$. $\square$

Remark 6.2. *By determining the kernel of the epimorphism in (94) one can deduce that*

$$\mathrm{Out}_k^s(B) \cong \prod^f k[T]/(T^2)^\times \cong (\mathbb{G}_m^f \times \mathbb{G}_a^f)(k) \quad \text{if } p \neq 2 \tag{99}$$

and

$$\mathrm{Out}_k^s(B) \cong \mathbb{G}_m^f(k) \quad \text{if } p = 2 \tag{100}$$

Lemma 6.3. *Let $\overline{\Lambda}$ be a split k-form of the principal block $\bar{k}\Delta_2(p^f)$, and assume there is a lift Λ of $\overline{\Lambda}$ subject to conditions as in Theorem 5.16. Then if $\alpha \in \mathrm{Aut}_k(\overline{\Lambda})$, then there exists a $\beta \in \mathrm{Aut}_{\mathcal{O}}(\Lambda)$ such that $\alpha \circ \overline{\beta} \in \mathrm{Aut}_k^s(\overline{\Lambda})$, where $\overline{\beta}$ denotes the image of β in $\mathrm{Aut}_k(\overline{\Lambda})$.*

Proof. Since any two full sets of orthogonal primitive idempotents in $\bar{\Lambda}$ are conjugate we can find an inner automorphism γ such that $\gamma \circ \alpha$ induces a permutation on some full set of orthogonal primitive idempotents in $\bar{\Lambda}$. Now Remark 5.17 implies the existence of a $\beta \in \mathrm{Aut}_{\mathcal{O}}(\Lambda)$ such that $\gamma \circ \alpha \circ \bar{\beta}$ fixes a full set of orthogonal primitive idempotents. This implies in particular that $\gamma \circ \alpha \circ \bar{\beta}$ lies in $\mathrm{Aut}_k^s(\bar{\Lambda})$. Since γ is an inner automorphism, it fixes all simple modules. Therefore we also have $\alpha \circ \bar{\beta} \in \mathrm{Aut}_k^s(\bar{\Lambda})$. $\square$

Corollary 6.4. *Let $\overline{\Gamma}$ be a k-algebra that is derived equivalent to a split k-form $\overline{\Lambda}$ of $B_0(\bar{k}\Delta_2(p^f))$. Moreover let B be a finite-dimensional semisimple K-algebra with $\dim_K Z(B) = \dim_{\bar{k}} Z(B_0(\bar{k}\Delta_2(p^f)))$ and assume B is split by some totally ramified extension of K. Given an element $u \in Z(B)^\times$ which has p-valuation $-f$ in every Wedderburn component of $Z(\overline{K} \otimes B)$, there is at most one conjugacy class of full $\mathcal{O}$-orders $\Gamma_u \subset B$ satisfying the following conditions:*

1. *Γ_u is self-dual with respect to T_u.*
2. *$k \otimes \Gamma_u$ is isomorphic to $\overline{\Gamma}$.*

Proof. Recall the result of Proposition 4.7, which stated that if Λ is a lift of $\overline{\Lambda}$ for which every outer automorphism of $\overline{\Lambda}$ may be written as a composition of (the reduction of) an automorphism of Λ and an element the $\bar{k}$-linear extension of which lies in $\mathrm{Out}_{\bar{k}}^0(\bar{k} \otimes_k \overline{\Lambda})$, then Λ corresponds to a single equivalence class of lifts in $\widehat{\mathfrak{L}}(\overline{\Lambda})$. This proposition is applicable to $\overline{\Lambda}$ and the unique lift Λ of $\overline{\Lambda}$ subject to conditions as in Theorem 5.16, since we have verified in Lemma 6.1 and Lemma 6.3 above that the conditions of the proposition are met. Theorem 4.2 shows that the equivalence classes in $\widehat{\mathfrak{L}}(\overline{\Lambda})$ subject to the conditions of Theorem 5.16 (with a modified u, depending on the choice of the derived equivalence; see Theorem 4.3) are in bijection with the equivalence classes in $\widehat{\mathfrak{L}}(\overline{\Gamma})$ subject to the conditions given in the statement of this corollary. Therefore there is at most one equivalence class of lifts of $\overline{\Gamma}$ satisfying our assumptions. In particular there is at most one isomorphism class of orders satisfying the assumptions. $\square$

Remark 6.5. *Broué's abelian defect conjecture states the following: Let k be an algebraically closed field, G a group, B a block of kG, P a defect group of B, and b the Brauer correspondent of B in $kN_G(P)$. Then b and B are derived equivalent.*

Broué's conjecture has been proven (in defining characteristic) for the principal block of $\mathrm{SL}_2(q)$ in [Oku00]. It has also been shown to hold for the unique non-principal block of maximal defect of $\mathrm{SL}_2(q)$, which exists if q is odd, in [Yos09].

Corollary 6.6. *Assume that $p = 2$ and that k is algebraically closed. Then Conjecture 3.9 holds in that case, that is, the generators for a basic order of $\mathcal{O}\,\mathrm{SL}_2(2^f)$ as conjectured in [Neb00a] define an $\mathcal{O}$-order which is Morita equivalent to $\mathcal{O}\,\mathrm{SL}_2(2^f)$. This is because Corollary 6.4 holds for the blocks of $k\,\mathrm{SL}_2(2^f)$ (due to the abelian defect conjecture), guaranteeing unique lifting.*

7. Rationality of tilting complexes

Our goal in this section is to perform a "Galois descent for derived equivalences" to the degree up to which this is possible. This will allow us to state a unique lifting theorem for the group ring $\mathbb{F}_{p^f}\,\mathrm{SL}_2(p^f)$, thus ridding us of the necessity to assume an algebraically closed coefficient field.

Concerning notation: In this section we often use field extensions $\tilde{K}$ and K' of K. We will always assume that $\tilde{K}$ and K' are (possibly infinite) algebraic extensions of K of finite ramification. We denote by $\tilde{O}$ respectively $\mathcal{O}'$ the corresponding discrete valuation rings and by $\tilde{k}$ respectively k' their respective residue fields.

Definition 7.1. *An $\mathcal{O}$-order Λ is* split *if the k-algebra $k \otimes \Lambda$ is split and the K-algebra $K \otimes \Lambda$ is split.*

Lemma 7.2. *Let k be finite. Let Λ be an $\mathcal{O}$-order such that $K \otimes \Lambda$ is split semisimple. Assume that there is a field extension $\tilde{K}/K$ of finite degree such that $\tilde{\mathcal{O}} \otimes \Lambda$ is split and its decomposition matrix has full row rank (that is, its rank is equal to its number of columns). Then Λ is already split.*

Proof. Assume S is a simple Λ-module that is not absolutely irreducible. Since there are no non-commutative finite-dimensional division algebras over k, $\mathrm{End}_\Lambda(S)$ is commutative and hence $\mathrm{End}_{\tilde{k}\otimes\Lambda}(\tilde{k}\otimes S) \cong \tilde{k}\otimes\mathrm{End}_\Lambda(S)$ is a direct sum of copies of $\tilde{k}$. Therefore $\tilde{k} \otimes S$ is a direct sum of non-isomorphic simple $\tilde{\mathcal{O}} \otimes \Lambda$-modules $\tilde{S}_1, \ldots, \tilde{S}_l$ (for some $l > 1$). Each simple $\tilde{K} \otimes \Lambda$-module is of the form $\tilde{K} \otimes V$ for some simple $K \otimes \Lambda$-module V. Let L be a Λ-lattice in V. Then $\tilde{\mathcal{O}} \otimes L$ is a $\tilde{\mathcal{O}} \otimes \Lambda$-lattice in $\tilde{K} \otimes V$, and the multiplicities of $\tilde{S}_1, \ldots \tilde{S}_l$ in $\tilde{k} \otimes L$ are all equal to the multiplicity of S in $k \otimes L$. Therefore, the columns in the decomposition matrix of $\tilde{\mathcal{O}} \otimes \Lambda$ associated to the simple modules $\tilde{S}_1, \ldots, \tilde{S}_l$ are all equal, in contradiction to the assumption that the decomposition matrix of $\tilde{\mathcal{O}} \otimes \Lambda$ has full row rank. Therefore all simple Λ-modules are absolutely simple, that is, Λ is split. □

Lemma 7.3. *Assume that $\tilde{K}$ is totally ramified over K. If Λ is an $\mathcal{O}$-order such that $\tilde{k} \otimes \Lambda$ is split, then $k \otimes \Lambda$ is split.*

In particular, under the assumption that k is finite, $\tilde{K} \otimes \Lambda$ is split semisimple and the decomposition matrix of Λ over a splitting system has full row rank, $k \otimes \Lambda$ will be split.

Proof. Since $\tilde{K}$ is assumed to be totally ramified over K, we have $\tilde{k} = k$. Therefore the assertion is trivial. □

Remark 7.4. *We should note that*

1. *If the Cartan matrix of an algebra is non-degenerate (which is a known fact in the case of group rings), then the decomposition matrix has full row rank.*
2. *The absolute value of the determinant of the Cartan matrix is preserved under derived equivalences (even under stable equivalences of Morita type). In particular, non-degeneracy of the Cartan matrix is preserved under derived equivalences.*

Definition 7.5. *Let A be a ring. We say a tilting complex $T \in \mathcal{C}^b(\mathbf{proj}_A)$ is* determined by its terms, *if any tilting complex $T' \in \mathcal{C}^b(\mathbf{proj}_A)$ with $T^i \cong T'^i$ for all $i \in \mathbb{Z}$ is isomorphic to T in $\mathcal{K}^b(\mathbf{proj}_A)$.*

Remark 7.6. *By [JSZ05, Corollary 8] two-term tilting complexes defined over algebras over a field are determined by their terms. By unique lifting of tilting complexes (see [Ric91b]), the same is true for two-term tilting complexes defined over orders over complete discrete valuation rings.*

Definition 7.7. *Let $\tilde{\Lambda}$ be an $\tilde{\mathcal{O}}$-order. We call an $\mathcal{O}$-order $\Lambda \subseteq \tilde{\Lambda}$ an* $\mathcal{O}$-form of $\tilde{\Lambda}$ *if* $\operatorname{rank}_{\mathcal{O}} \Lambda = \operatorname{rank}_{\tilde{\mathcal{O}}} \tilde{\Lambda}$ *and $\tilde{\mathcal{O}} \cdot \Lambda = \tilde{\Lambda}$. We define a k-form of a finite-dimensional $\tilde{k}$-algebra analogously.*

Lemma 7.8. *Let Λ be an $\mathcal{O}$-order and let $\tilde{K}$ be an unramified finite extension of K. Furthermore, let $\tilde{C} \in \mathcal{C}^b(\mathbf{mod}_{\tilde{\mathcal{O}}\otimes\Lambda})$ be a complex of $\tilde{\mathcal{O}} \otimes \Lambda$-modules and let C be the restriction of $\tilde{C}$ to Λ. Then, in the category $\mathcal{C}^b(\mathbf{mod}_{\tilde{\mathcal{O}}\otimes\Lambda})$,*

$$\tilde{\mathcal{O}} \otimes C \cong \bigoplus_{i=1}^{[\tilde{K}:K]} \tilde{C}^{\alpha_i} \tag{101}$$

for certain $\alpha_i \in \operatorname{Aut}_{\mathcal{O}}(\tilde{\mathcal{O}})$. Here, for an $\alpha \in \operatorname{Aut}_{\mathcal{O}}(\tilde{\mathcal{O}})$, $\tilde{C}^\alpha$ denotes the complex of $\tilde{\mathcal{O}} \otimes \Lambda$-module the terms of which are (as sets) equal to the terms of $\tilde{C}$, with differential equal to that of $\tilde{C}$, but with the following twisted action of $\tilde{\mathcal{O}} \otimes \Lambda$ on the terms:

$$\tilde{C}^i \times \tilde{\mathcal{O}} \otimes \Lambda \longrightarrow \tilde{C}^i : (m, a \otimes b) \mapsto m \cdot \alpha(a) \otimes b \tag{102}$$

We claim furthermore that at least one of the α_i may be chosen to be the identity automorphism of $\tilde{\mathcal{O}}$.

Proof. First note that $\tilde{\mathcal{O}} \otimes_{\mathcal{O}} \tilde{\mathcal{O}} \cong \bigoplus^{[\tilde{K}:K]} \tilde{\mathcal{O}}$, since $\tilde{K}$ is unramified over K. For $i \in \{1, \ldots, [\tilde{K} : K]\}$ denote by ε_i the epimorphism from $\tilde{\mathcal{O}} \otimes_{\mathcal{O}} \tilde{\mathcal{O}}$ to $\tilde{\mathcal{O}}$ given by projection to the i-th component of $\bigoplus^{[\tilde{K}:K]} \tilde{\mathcal{O}}$ (of course, the ordering of the ε_i is not canonical). By abuse of notation, we also denote by ε_i the unique primitive idempotent in $\tilde{\mathcal{O}} \otimes_{\mathcal{O}} \tilde{\mathcal{O}}$ that gets mapped to 1 under the projection ε_i. Now we consider the complex of $\tilde{\mathcal{O}} \otimes_{\mathcal{O}} \tilde{\mathcal{O}} \otimes_{\mathcal{O}} \Lambda$-modules $\tilde{\mathcal{O}} \otimes_{\mathcal{O}} \tilde{C}$. We can decompose this complex as follows:

$$\tilde{\mathcal{O}} \otimes_{\mathcal{O}} \tilde{C} = \bigoplus_{i=1}^{[\tilde{K}:K]} \tilde{\mathcal{O}} \otimes_{\mathcal{O}} \tilde{C} \cdot (\varepsilon_i \otimes 1_\Lambda) \tag{103}$$

Now consider the embedding

$$\eta : \tilde{\mathcal{O}} \hookrightarrow \tilde{\mathcal{O}} \otimes_{\mathcal{O}} \tilde{\mathcal{O}} : a \mapsto a \otimes 1 \tag{104}$$

If we turn $\tilde{\mathcal{O}} \otimes_{\mathcal{O}} \tilde{C}$ into a complex of $\tilde{\mathcal{O}} \otimes \Lambda$-modules via the embedding $\eta \otimes \mathrm{id}_\Lambda$ we get, by definition, $\tilde{\mathcal{O}} \otimes_{\mathcal{O}} C$. If we turn $\tilde{\mathcal{O}} \otimes_{\mathcal{O}} \tilde{C} \cdot (\varepsilon_i \otimes 1_\Lambda)$ into a complex of $\tilde{\mathcal{O}} \otimes \Lambda$-modules via the embedding $\eta \otimes \mathrm{id}_\Lambda$ we get $\tilde{C}^{\varepsilon_i \circ \eta}$. So our first claim follows (with $\alpha_i := \varepsilon_i \circ \eta$). As for the claim that one of the α_i may be chosen equal to the identity, just note that there is an epimorphism $\tilde{\mathcal{O}} \otimes_{\mathcal{O}} \tilde{\mathcal{O}} \twoheadrightarrow \tilde{\mathcal{O}} : a \otimes b \mapsto a \cdot b$. Since the ε_i are in fact all epimorphisms from $\tilde{\mathcal{O}} \otimes_{\mathcal{O}} \tilde{\mathcal{O}}$ to $\tilde{\mathcal{O}}$, this epimorphism needs to be equal to some ε_i. But then $\alpha_i = \mathrm{id}$. ☐

Proposition 7.9 (Reduction to finite field extensions)**.** *Let Λ and Γ be two $\mathcal{O}$-orders such that $\tilde{\mathcal{O}} \otimes \Lambda$ and $\tilde{\mathcal{O}} \otimes \Gamma$ are derived equivalent, and let $\tilde{T}$ be a tilting complex over $\tilde{\mathcal{O}} \otimes \Lambda$ with endomorphism ring $\tilde{\mathcal{O}} \otimes \Gamma$. Then there exists a finite extension K' of K which is contained in $\tilde{K}$ such that $\mathcal{O}' \otimes \Lambda$ is derived equivalent to an $\mathcal{O}'$-form Γ' of $\tilde{\mathcal{O}} \otimes \Gamma$, and there is a tilting complex T' over $\mathcal{O}' \otimes \Lambda$ with endomorphism ring Γ' such that $\tilde{\mathcal{O}} \otimes_{\mathcal{O}'} T' \cong \tilde{T}$ in $\mathcal{K}^b(\mathbf{proj}_{\tilde{\mathcal{O}}\otimes\Lambda})$.*

Proof. There is some invertible complex $\tilde{X} \in \mathcal{D}^b((\tilde{\mathcal{O}} \otimes \Lambda)^{\mathrm{op}} \otimes_{\tilde{\mathcal{O}}} (\tilde{\mathcal{O}} \otimes \Gamma))$ with inverse $\tilde{Y} \in \mathcal{D}^b((\tilde{\mathcal{O}} \otimes \Gamma)^{\mathrm{op}} \otimes_{\tilde{\mathcal{O}}} (\tilde{\mathcal{O}} \otimes \Lambda))$ such that the restriction of $\tilde{Y}$ to $\tilde{\mathcal{O}} \otimes \Lambda$ is isomorphic to $\tilde{T}$ in $\mathcal{D}^b(\tilde{\mathcal{O}} \otimes \Lambda)$. We can find a finite extension K' of K (contained in $\tilde{K}$) such that there are bounded complexes X' and Y' such that $\tilde{\mathcal{O}} \otimes_{\mathcal{O}'} X' \cong \tilde{X}$ and $\tilde{\mathcal{O}} \otimes_{\mathcal{O}'} Y' \cong \tilde{Y}$. This is simply because $\tilde{X}$ and $\tilde{Y}$ can be represented by bounded complexes of finitely generated modules, and so K' needs only be big enough for all terms of these complexes to be defined over $\mathcal{O}'$ and for the differentials (which are made up of finitely many homomorphisms) to be defined. Looking at the construction of the derived tensor product, it is clear that

$$\tilde{\mathcal{O}} \otimes^{\mathbb{L}}_{\mathcal{O}'} (X' \otimes^{\mathbb{L}}_{\mathcal{O}'\otimes\Gamma} Y') \cong \tilde{X} \otimes^{\mathbb{L}}_{\tilde{\mathcal{O}}\otimes\Gamma} \tilde{Y} \text{ and } \tilde{\mathcal{O}} \otimes^{\mathbb{L}}_{\mathcal{O}'} (Y' \otimes^{\mathbb{L}}_{\mathcal{O}'\otimes\Lambda} X') \cong \tilde{Y} \otimes^{\mathbb{L}}_{\tilde{\mathcal{O}}\otimes\Lambda} \tilde{X} \tag{105}$$

But the right hand terms in (105) have homology concentrated in degree zero. This means that $X' \otimes^{\mathbb{L}}_{\mathcal{O}' \otimes \Gamma} Y'$ and $Y' \otimes^{\mathbb{L}}_{\mathcal{O}' \otimes \Lambda} X'$ are isomorphic to stalk complexes in $\mathcal{D}^-((\mathcal{O}' \otimes \Lambda)^{\mathrm{op}} \otimes_{\mathcal{O}'} (\mathcal{O}' \otimes \Lambda))$ respectively $\mathcal{D}^-((\mathcal{O}' \otimes \Gamma)^{\mathrm{op}} \otimes_{\mathcal{O}'} (\mathcal{O}' \otimes \Gamma))$. Since tensoring with $\tilde{\mathcal{O}}$ renders them isomorphic to $0 \to \tilde{\mathcal{O}} \otimes \Lambda \to 0$ respectively $0 \to \tilde{\mathcal{O}} \otimes \Gamma \to 0$ it follows from the Noether-Deuring theorem for modules that they are isomorphic to $0 \to \mathcal{O}' \otimes \Lambda \to 0$ respectively $0 \to \mathcal{O}' \otimes \Gamma \to 0$. Therefore X' and Y' are invertible, and thus the restriction of Y' to $\mathcal{O}' \otimes \Lambda$ is a tilting complex T' with $\tilde{\mathcal{O}} \otimes_{\mathcal{O}'} T' \cong \tilde{T}$.

By [Ric91a, Theorem 2.1] it follows that the endomorphism ring of T' in $\mathcal{D}^b(\mathcal{O}' \otimes \Lambda)$ is an $\mathcal{O}'$-form of $\tilde{\mathcal{O}} \otimes \Lambda$. □

Remark 7.10. *We should mention the following trivial supplement to the above proposition: If $\tilde{\mathcal{O}}$ splits Λ and/or Γ, we may choose an $\mathcal{O}'$ which splits Λ and/or Γ. Similarly, if $\tilde{k}$ splits $k \otimes \Lambda$ and/or $k \otimes \Gamma$, we may choose an $\mathcal{O}'$ such that $k' = \mathcal{O}'/\operatorname{Jac}(\mathcal{O}')$ splits $k \otimes \Lambda$ and/or $k \otimes \Gamma$.*

Lemma 7.11. *Let Λ be an $\mathcal{O}$-order and let $T \in \mathcal{C}^b(\mathbf{mod}_\Lambda)$ be a complex with differential $d : T \longrightarrow T[-1]$. If $\tilde{\mathcal{O}} \otimes T$ is a tilting complex for $\tilde{\mathcal{O}} \otimes \Lambda$ (in particular $\tilde{\mathcal{O}} \otimes T \in \mathcal{C}^b(\mathbf{proj}_{\tilde{\mathcal{O}} \otimes \Lambda})$), then T is a tilting complex for Λ.*

Proof. First note that by Proposition 7.9 we may assume that $\tilde{K}/K$ is a field extension of finite degree. If M is a (finitely-generated) Λ-module such that $\tilde{\mathcal{O}} \otimes M$ is a projective $\tilde{\mathcal{O}} \otimes \Lambda$-module, M must itself be projective. This follows easily from the fact that $\tilde{\mathcal{O}} \otimes M$ is projective if and only if it is a direct summand of some free module, and so the restriction of $\tilde{\mathcal{O}} \otimes M$, which is just a direct sum of copies of M, is a summand of a restriction of a free module, which is again a free module. This shows that $\tilde{\mathcal{O}} \otimes T \in \mathcal{C}^b(\mathbf{proj}_{\tilde{\mathcal{O}} \otimes \Lambda})$ implies $T \in \mathcal{C}^b(\mathbf{proj}_\Lambda)$.

By [Zim12, Lemma 4] we have $\tilde{\mathcal{O}} \otimes \operatorname{Hom}_{\mathcal{D}^b(\Lambda)}(T, T[i]) \cong \operatorname{Hom}_{\mathcal{D}^b(\tilde{\mathcal{O}} \otimes \Lambda)}(\tilde{\mathcal{O}} \otimes T, \tilde{\mathcal{O}} \otimes T[i])$ for each $i \in \mathbb{Z}$. Since we assume that $\tilde{\mathcal{O}} \otimes T$ is a tilting complex, we get $\operatorname{Hom}_{\mathcal{D}^b(\tilde{\mathcal{O}} \otimes \Lambda)}(\tilde{\mathcal{O}} \otimes T, \tilde{\mathcal{O}} \otimes T[i]) = \{0\}$ for all $i \neq 0$. Therefore, $\tilde{\mathcal{O}} \otimes \operatorname{Hom}_{\mathcal{D}^b(\Lambda)}(T, T[i]) = \{0\}$ for all $i \neq 0$, which in turn implies $\operatorname{Hom}_{\mathcal{D}^b(\Lambda)}(T, T[i]) = \{0\}$ for all $i \neq 0$.

Now we show that T generates $\mathcal{K}^b(\mathbf{proj}_\Lambda)$. To see this we look at the functor

$$\operatorname{Res} : \mathcal{K}^-(\mathbf{proj}_{\tilde{\mathcal{O}} \otimes \Lambda}) \longrightarrow \mathcal{K}^-(\mathbf{proj}_\Lambda) \tag{106}$$

which, by definition, simply restricts the terms of the complexes from $\tilde{\mathcal{O}} \otimes \Lambda$-modules to Λ-modules. Since this is an exact functor, and $\operatorname{Res}(\tilde{\mathcal{O}} \otimes T)$ is just a direct sum of copies of T, $\operatorname{add}(T) \supseteq \operatorname{Res}(\operatorname{add}(\tilde{\mathcal{O}} \otimes T))$. But $0 \to \tilde{\mathcal{O}} \otimes \Lambda \to 0$ lies in $\operatorname{add}(\tilde{\mathcal{O}} \otimes T)$, and therefore $0 \to \Lambda \to 0$ lies in $\operatorname{add}(T)$ (since $\operatorname{Res}(0 \to \tilde{\mathcal{O}} \otimes \Lambda \to 0)$ is isomorphic to a direct sum of copies of $0 \to \Lambda \to 0$). □

Theorem 7.12. *Assume k is finite and $\tilde{K}$ is unramified over K. Let $\tilde{\Lambda}$ be an $\tilde{\mathcal{O}}$-order such that $\tilde{k} \otimes \tilde{\Lambda}$ is split, $\tilde{K} \otimes \tilde{\Lambda}$ is semisimple and the decomposition matrix of $\tilde{\Lambda}$ over a splitting system has full row rank. Let $\tilde{T} \in \mathcal{C}^b(\mathbf{proj}_{\tilde{\Lambda}})$ be a tilting complex that is determined by its terms. Set*

$$\tilde{\Gamma} := \operatorname{End}_{\mathcal{D}^b(\tilde{\Lambda})}(\tilde{T}) \tag{107}$$

If Λ is an $\mathcal{O}$-form of $\tilde{\Lambda}$ such that $k \otimes \Lambda$ is split and there is a totally ramified extension of K that splits $K \otimes \Lambda$, then there is an $\mathcal{O}$-form Γ of $\tilde{\Gamma}$ with the same properties that is derived equivalent to Λ.

Proof. Let T be the restriction of $\tilde{T}$ to $\mathcal{C}^b(\mathbf{proj}_\Lambda)$. By Lemma 7.8 the complex $\tilde{\mathcal{O}} \otimes T$ is isomorphic to a direct sum of complexes of the form $\tilde{T}^\alpha$ for certain $\alpha \in \operatorname{Aut}_{\mathcal{O}}(\tilde{\mathcal{O}})$. Now note that since $k \otimes \Lambda$ is split, the projective indecomposable $\tilde{\Lambda}$-modules $\tilde{P}$ are of the form $\tilde{\mathcal{O}} \otimes P$ for projective indecomposable Λ-modules P. Therefore they are isomorphic to their Galois twists. In particular, the terms of $\tilde{T}^\alpha$ and $\tilde{T}$ are isomorphic for all $\alpha \in \operatorname{Aut}_{\mathcal{O}}(\tilde{\mathcal{O}})$. Since $\tilde{T}$ is by assumption determined by its terms, we must have $\tilde{T}^\alpha \cong \tilde{T}$ for all $\alpha \in \operatorname{Aut}_{\mathcal{O}}(\tilde{\mathcal{O}})$. This shows that $\tilde{\mathcal{O}} \otimes T$ is a tilting complex, and therefore so is T (by Lemma 7.11). It is clear by [Ric91a, Theorem 2.1] that the endomorphism ring of T is an $\mathcal{O}$-form of the endomorphism ring of $\tilde{\mathcal{O}} \otimes T$, and of course it is derived equivalent to Λ. We have

$$\tilde{\mathcal{O}} \otimes \mathrm{End}_{\mathcal{D}^b(\Lambda)}(T) \cong \mathrm{End}_{\mathcal{D}^b(\tilde{\Lambda})}(\tilde{\mathcal{O}} \otimes T) \cong \tilde{\Gamma}^{[\tilde{K}:K]\times[\tilde{K}:K]} \tag{108}$$

The first isomorphism is a special case of [Zim12, Lemma 4]. The second isomorphism follows from the fact that $\tilde{\mathcal{O}} \otimes T \cong \tilde{T}^{[\tilde{K}:K]}$, which we just proved. Equation (108) shows that $\mathrm{End}_{\mathcal{D}^b(\Lambda)}(T)$ is on $\mathcal{O}$-form of $\tilde{\Gamma}^{[\tilde{K}:K]\times[\tilde{K}:K]}$. This will yield on $\mathcal{O}$-form of $\tilde{\Gamma}$ with the desired properties (simply by applying a Morita equivalence) once we see that $k \otimes \mathrm{End}_{\mathcal{D}^b(\Lambda)}(T)$ is split. Let K' be a totally ramified extension of K such that $K' \otimes \Lambda$ is split. Since $K' \otimes \mathrm{End}_{\mathcal{D}^b(\Lambda)}(T) \cong \mathrm{End}_{\mathcal{D}^b(K'\otimes\Lambda)}(K' \otimes T)$ is Morita equivalent to $K' \otimes \Lambda$, it follows by Lemma 7.3 that $k \otimes \mathrm{End}_{\mathcal{D}^b(\Lambda)}(T)$ is split. □

Corollary 7.13. *The assertion of the preceding Theorem remains true if $\tilde{\Lambda}$ and $\tilde{\Gamma}$ are linked by a series of derived equivalences which all are afforded by tilting complexes that are determined by their terms.*

Proof. This follows by iterated application of the preceding theorem. □

Corollary 7.14. *Let $\mathcal{O}$ be the p-adic completion of the maximal unramified extension of $\mathbb{Q}_p$. The blocks of defect C_p^f of the group ring $\mathbb{Z}_p[\zeta_{p^f-1}]\,\mathrm{SL}_2(p^f)$ are derived equivalent to a $\mathbb{Z}_p[\zeta_{p^f-1}]$-form (split over $\mathbb{F}_{p^f}$) of their respective Brauer correspondent in $\mathcal{O}\Delta_2(p^f)$ with $\mathbb{Q}_p[\zeta_{p^f-1}]$-span isomorphic to the $\mathbb{Q}_p[\zeta_{p^f-1}]$-span of the corresponding block of $\mathbb{Z}_p[\zeta_{p^f-1}]\Delta_2(p^f)$.*

Proof. The respective blocks of $k\,\mathrm{SL}_2(p^f)$ and $k\Delta_2(p^f)$ are linked by a series of two-term complexes (see [Oku00] respectively [Yos09]). Hence the first claim follows from Theorem 7.12 and Corollary 7.13. The assertion concerning the $\mathbb{Q}_p[\zeta_{p^f-1}]$-spans follows from the fact that the $\mathbb{Q}_p[\zeta_{p^f-1}]$-spans of the blocks of $\mathbb{Z}_p[\zeta_{p^f-1}]\,\mathrm{SL}_2(p^f)$ and $\mathbb{Z}_p[\zeta_{p^f-1}]\Delta_2(p^f)$ which are Brauer correspondents are Morita equivalent. □

Corollary 7.15. *Assume $k \supseteq \mathbb{F}_{p^f}$ and B is a block of $k\,\mathrm{SL}_2(p^f)$ of maximal defect. Let A be a finite-dimensional semisimple K-algebra with $\dim_K Z(A) = \dim_k Z(B)$. Assume A is split by some totally ramified extension of K. Given an element $u \in Z(A)^\times$ which has p-valuation $-f$ in every Wedderburn component of $Z(\bar{K} \otimes A)$, there is, up to conjugacy, at most one full $\mathcal{O}$-order $\Lambda_u \subset A$ satisfying the following conditions:*

1. *Λ_u is self-dual with respect to T_u.*
2. *$k \otimes \Lambda_u$ is isomorphic to B.*

Proof. By Corollary 7.14 the block B is derived equivalent to a split k-form $\overline{\Gamma}$ of $B_0(\bar{k}\Delta_2(p^f))$. Thus the assertion follows directly from Corollary 6.4. □

Corollary 7.16. *Conjecture 3.9 holds true in the case $\mathcal{O} = \mathbb{Z}_2[\zeta_{2^f-1}]$, that is, the generators for a basic order of $\mathbb{Z}_2[\zeta_{2^f-1}]\,\mathrm{SL}_2(2^f)$ as conjectured in [Neb00a] define a $\mathbb{Z}_2[\zeta_{2^f-1}]$-order which is Morita-equivalent to $\mathbb{Z}_2[\zeta_{2^f-1}]\,\mathrm{SL}_2(p^f)$.*

As a corollary we can also prove that a discrete valuation ring version of the abelian defect conjecture holds for $\mathbb{Z}_p[\zeta_{p^f-1}]\,\mathrm{SL}_2(p^f)$.

Corollary 7.17. *The non-semisimple blocks of $\mathbb{Z}_p[\zeta_{p^f-1}]\,\mathrm{SL}_2(p^f)$ are derived equivalent to their Brauer-correspondents in $\mathbb{Z}_p[\zeta_{p^f-1}]\Delta_2(p^f)$.*

Proof. As we have already seen, any non-semisimple block Γ of $\mathbb{Z}_p[\zeta_{p^f-1}]\,\mathrm{SL}_2(p^f)$ is derived equivalent to the unique lift $\Lambda_u \subset \mathbb{Q}_p[\zeta_{p^f-1}] \otimes B_0(\mathbb{Z}_p[\zeta_{p^f-1}]\Delta_2(p^f)) =: A$ of a split $\mathbb{F}_{p^f}$-form of $B_0(\bar{\mathbb{F}}_p\Delta_2(p^f))$ with respect to some $u \in Z(A)$ satisfying the conditions of Theorem 5.16 (this is just putting Corollary 7.14 and Theorem 5.16 together). For the rest of this proof we will use the same notational conventions as in Theorem 5.16, including Notation 5.14. Theorem 5.16 (b) tells us that if $p = 2$, then $\Lambda_u \cong B_0(\mathbb{Z}_2[\zeta_{2^f-1}]\Delta_2(2^f))$ which implies the assertion of this corollary. If $p \neq 2$ and $\mathbb{Q}_p[\zeta_{p^f-1}]$ does not split $\mathrm{SL}_2(p^f)$, then Theorem 5.16 (b) tells us that Λ_u depends only on $u_{\kappa+1} \cdot \mathcal{O}^\times$, which we may assume to be equal to $p^{-f} \cdot \mathcal{O}^\times$ by virtue of $u_{\kappa+1}$ being rational. So again, $\Lambda_u \cong B_0(\mathbb{Z}_p[\zeta_{p^f-1}]\Delta_2(p^f))$ follows and we are done. Now if p is odd and $\mathbb{Q}_p[\zeta_{p^f-1}]$ does split $\mathrm{SL}_2(p^f)$, then Λ_u depends only on the quotient $u_{\kappa+1}/u_{\kappa+2}$. Assume for the rest of the proof that we are in this case. We also fix some tilting complex T in $\mathcal{K}^b(\mathbf{proj}_{\Lambda_u})$ with endomorphism ring Γ. Furthermore let $V_{\kappa+1}$ and $V_{\kappa+2}$ be the $(\kappa+1)$-st and $(\kappa+2)$-nd simple $\bar{\mathbb{Q}}_p \otimes A$-module. Note that the symmetrizing element u for Λ_u arises from the symmetrizing element u' we use for Γ by flipping signs in certain Wedderburn components. As mentioned in Remark 5.15, u' may be chosen so that $u'_{\kappa+1} = u'_{\kappa+2}$, since the corresponding rows in the decomposition matrix are equal. We do not need any particular knowledge of the decomposition matrix of $\mathrm{SL}_2(p^f)$ to establish this. Indeed, we can simply use the fact that the $(\kappa+1)$-st and $(\kappa+2)$-nd row of the decomposition matrix of $\Delta_2(p^f)$ over a splitting system are equal, which implies that the corresponding rows in the decomposition matrix of a derived equivalent order will also be equal. The sign of $u'_{\kappa+1}$ respectively $u'_{\kappa+2}$ is flipped upon passage to Λ_u depending on the sign of $[V_{\kappa+1}]$ respectively $[V_{\kappa+2}]$ as a coefficient of

$$\sum_i (-1)^i \cdot [\bar{\mathbb{Q}}_p \otimes_{\mathbb{Z}_p[\zeta_{p^f-1}]} T^i] \in K_0(\mathbf{mod}_{\bar{\mathbb{Q}}_p\Delta_2(p^f)}) \tag{109}$$

These signs are equal, since all of the T^i are projective modules and therefore $V_{\kappa+1}$ and $V_{\kappa+2}$ occur in their $\bar{\mathbb{Q}}_p$-span with the same multiplicities. This follows from the fact that the corresponding rows in the decomposition matrix are equal. We conclude that $u_{\kappa+1} = u_{\kappa+2}$, and therefore $\Lambda_u \cong B_0(\mathbb{Z}_p[\zeta_{p^f-1}]\Delta_2(p^f))$, which is what we wanted to prove. □

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