Group Rings over the p-Adic Integers

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Chapter 1

Introduction

The representation theory of finite groups is, in modern terminology, the study of the module category of the group ring RG for some finite group G and some commutative ring R. The methods best used for this study do, however, depend strongly on what commutative ring R is being used. If we let $R = K$ be a field of characteristic zero, then we are in the setting of *ordinary representation theory* and the module category of KG will be semisimple. To describe it, it suffices to describe its simple objects and their endomorphism rings. While this may be tricky enough in some concrete cases, there is no theoretical obstruction to giving a complete description. If we let $R = k$ be a field of characteristic $p > 0$, then we are in the setting of *modular representation theory.* If p divides the order of G, describing the module category of kG will be much more difficult, since it will no longer be semisimple. Even worse, in most cases it is considered impossible to classify all indecomposable kG-modules. Since there is no remedy for this, one has to look for alternative ways to understand the module category of kG. One possibility of doing so is to find a different k-algebra A such that the module categories of kG and A are equivalent. An algebra A of minimal dimension with this property is unique up to isomorphism and called a *basic algebra* of kG. And even though by determining a basic algebra we will not necessarily have learned anything about the modules of kG, we will at least be dealing with modules over an algebra which is quite often of much lower dimension. Therefore, whenever we wish to "describe" a group ring in this thesis, this (and nothing more) is what we do: *determine its basic algebra.* Now there is a third choice for the ring of coefficients R, a choice which connects ordinary and modular representation theory: $R = \mathcal{O}$, a complete discrete valuation ring with residue field k of characteristic $p > 0$ and field of fractions K of characteristic zero (our preferred choice would be $\mathcal{O} = \mathbb{Z}_p$, the ring of p-adic integers). At first glance, this choice makes things more difficult. After all, the module category of $\mathcal{O}G$ contains that of kG as a full subcategory, so describing the former seems to require that we describe latter first. However, we have already abandoned the idea of "determining the module category" in favor of determining a basic algebra. And this is where the choice of $R = \mathcal{O}$ proves its merit: The basic algebra Λ of $\mathcal{O}G$ (also, and more commonly, called a "basic order") is a "lift" of the basic algebra A of kG in the sense that it is free as an $\mathcal{O}$-module (such algebras are known as *orders*) and that it satisfies $k \otimes \Lambda \cong A$. And surprisingly enough, we actually have more tools at our disposal to describe Λ than we have to describe A directly. For instance, we can and usually will describe Λ as an order in a semisimple K-algebra, via its *Wedderburn embedding*

$$\Delta: \Lambda \hookrightarrow \bigoplus_i D_i^{n_i \times n_i} \cong K \otimes \Lambda \tag{1.1}$$

where the D_i are certain division algebras over K and the n_i are certain natural numbers. Looking for orders in the right hand algebra has the advantage that *all* orders Γ of maximal $\mathcal{O}$-rank in that algebra satisfy $K \otimes \Gamma \cong K \otimes \Lambda$ *by construction.* On the level of k-algebras there would be no easy way of checking whether some k-algebra "comes from" an order in a given semisimple K-algebra, and so whatever we know about the ordinary representation theory of G could not readily be applied to the determination of a basic algebra.

As the reader may have guessed by now, *the determination of basic orders of group rings of finite groups* is actually the central theme of this thesis. However, we should clarify two points: The first point is that we do not treat all finite groups G, but rather certain *blocks* (that is, direct summands) of certain group rings, or certain classes of blocks that are well understood. Each example requires some ad hoc reasoning, there is no uniform theoretical approach to determine basic orders of $\mathcal{O}G$ for all finite groups G. The second point is that our methods are actually not particularly specific to group rings. In fact, this thesis is essentially an application of the theory of orders to group rings. Our questions all boil down to the following: *What can an order satisfying a bunch of conditions (which are known to hold for some specific block) look like*?

Now let us describe in a little more detail the types of blocks we consider in this thesis and the methods being employed to do so. The blocks we consider are blocks of dihedral defect, blocks of $\mathrm{SL}_2(q)$ in defining characteristic and blocks of symmetric groups of defect two.

The blocks of symmetric groups are "elementary" examples in the sense that their treatment requires no fundamentally new techniques. Using the theory of orders with decomposition numbers zero and one as developed by Plesken in [Ple83], in conjunction with various facts on the representation theory of the symmetric groups, it is possible to give descriptions of the basic orders of such blocks. So even though this part was placed at the very end of this thesis, it might not be the worst part to start for anyone not familiar with the way we look at orders in semisimple algebras.

The main results of this thesis are the descriptions of blocks of dihedral defect and of the integral group ring of $\mathrm{SL}_2(q)$ in defining characteristic. For blocks of dihedral defect defined over an algebraically closed field there is a classification by Erdmann (see [Erd90]), listing possible basic algebras (although this work strictly speaking does not classify blocks of dihedral defect, but rather the larger class of "algebras of dihedral type"). For $k\,\mathrm{SL}_2(q)$ there is a description of the basic algebra by Koshita (see [Kos94] and [Kos98]). Apart from that, blocks of dihedral defect and the group ring of $\mathrm{SL}_2(q)$ do not have a lot in common. What we exploit in this thesis is the fact that in both cases we have *derived equivalences* to different algebras which are easier to handle. The reason why exactly derived equivalences play a role in the modular representation theory of finite groups is still somewhat mysterious, but according to *Broué's abelian defect group conjecture* there should be plenty of them. The derived equivalence we use in the case of $\mathrm{SL}_2(q)$ is in fact due to this conjecture, which is a theorem in that particular case due to Okuyama (see [Oku00]). What we observe in this thesis is that the problem of lifting a k-algebra to an $\mathcal{O}$-order is (essentially) equivalent for any algebra in a derived equivalence class (this is ultimately based on a theorem by Rickard given in [Ric91b]). However, from the order theoretic ("linear algebra") point of view, the lifting problem looks quite different for different algebras in a derived equivalence class. This is due to the fact that the most important measure of difficulty from this point of view is the decomposition matrix of the lift, and many properties of the decomposition matrix (like the size of its entries) are not necessarily preserved under derived equivalences.

Important results obtained from this are proofs of two open conjectures: In the case of dihedral blocks we can show that in the classification given in [Erd90], among the Morita equivalence classes with two simple modules, only those with parameter $c = 0$ actually contain any blocks of group rings (the parameter "c" is an undetermined parameter in the quiver presentations given in the appendix of [Erd90]; it may, a priori, take the values "0" and "1"). In the context of $\mathrm{SL}_2(q)$ we are able to show that the basic order of the integral group ring $\mathbb{Z}_p[\zeta_{p^f-1}]\,\mathrm{SL}_2(p^f)$ is correctly described by Nebe in [Neb00a] respectively [Neb00b].

All of the k-algebras A we consider in Chapter 4 and Chapter 5 exhibit a "unique lifting property": There is a unique $\mathcal{O}$-order Λ which satisfies a certain list of conditions (depending on A) with $k \otimes \Lambda \cong A$. This may be seen as a result in its own right. Also, the existence of a property like this is necessary for order theoretic methods to work (since, as already mentioned, the way we determine basic orders is basically nothing more than asking what an order with a list of prescribed properties may look like).

A last result included in this thesis is a version of *Scopes reduction* (see [Sco91]) which establishes Morita equivalences between certain epimorphic images of certain pairs of blocks of symmetric groups. This works in particular in cases where Scopes' result does not give a Morita equivalence between the blocks, and can for example be used to determine so-called *exponent matrices* of certain Wedderburn components in blocks of symmetric groups (or, rather, reduce their determination to a smaller block). However, the real problems with the determination of basic orders of blocks of symmetric groups lie in different places: namely, in the structure of the decomposition matrix and the endomorphism rings of the projective indecomposables.

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Chapter 2

Foundations

In this chapter we give a short introduction into the theory we are going to use later in this thesis. Since all of this is well documented in literature, we omit proofs, unless they happen to be particularly short and insightful. We presume some knowledge of the representation theory of finite dimensional algebras, for which the first few chapters of [Ben91] may serve as a reference. For some background on local fields and skew-fields, as well as orders in such, the reader may consult [Rei75].

Notation 2.1 (Conventions)**.** *If A is a ring, we denote by $\mathbf{Mod}_A$ the category of right A-modules, and by $\mathbf{Proj}_A$ the category of projective right A-modules. By $\mathbf{mod}_A$ respectively $\mathbf{proj}_A$ we denote the respective subcategories of finitely generated modules. We adopt analogous notations for left A-modules and bimodules (for instance, ${}_A\mathbf{Mod}_B$ denotes the category of A-B-bimodules).*

By a "module" we will always mean a right module (unless we explicitly say otherwise). For composition of homomorphisms we use the convention where $f \circ g$ means "first apply g, then f". In particular, we equip endomorphism rings of modules with this product. As a result of this convention, an A-module M may be construed as a left *$\mathrm{End}_A(M)$-module.*

2.1 Basic Definitions and Facts about Orders

Let $p > 0$ be a fixed prime number. By $\mathbb{Z}_{(p)}$ we denote the localization of $\mathbb{Z}$ at the prime ideal (p). By $\mathbb{Q}_p$ we denote the p-adic completion of the field $\mathbb{Q}$ (called the p-adic numbers), and by $\mathbb{Z}_p \subset \mathbb{Q}_p$ we denote the p-adic completion of $\mathbb{Z}$ (called the p-adic integers). Note that we view $\mathbb{Q}$ as a discretely valued field with valuation function

$$\nu_p : \ \mathbb{Q} \longrightarrow \mathbb{R} \cup \{\infty\} : \ c \mapsto \begin{cases} \infty & \text{if } c = 0 \\ n & \text{if } c = p^n \cdot \frac{a}{b}, \text{ where } a, b \in \mathbb{Z} - p\mathbb{Z} \text{ and } n \in \mathbb{Z} \end{cases} \tag{2.1}$$

This is an additive non-archimedian valuation, and the image of the multiplicative group $\mathbb{Q}^\times$ in the additive group of $\mathbb{R}$ is equal to $\mathbb{Z}$ and thus discrete (where $\mathbb{R}$ is given the usual topology). It has a unique continuous extension to $\mathbb{Q}_p$, and we will view $\mathbb{Q}_p$ as a valued field with that valuation.

Definition 2.2 (p-Modular System)**.** *Let K be a discretely valued field of characteristic zero, with valuation function*

$$\nu : \ K \longrightarrow \mathbb{R} \cup \{\infty\} \tag{2.2}$$

Let $\mathcal{O} := \nu^{-1}(\mathbb{R}_{\geqslant 0} \cup \{\infty\})$ be its valuation ring, and $\mathfrak{p} := \nu^{-1}(\mathbb{R}_{>0} \cup \{\infty\})$ its (unique) maximal ideal. We put k to be the residue field of $\mathcal{O}$, that is, $k := \mathcal{O}/\mathfrak{p}$. If this field k has characteristic p, then we call the triple

$$(K, \mathcal{O}, k) \tag{2.3}$$

a p-modular system.

The ideal $\mathfrak{p}$ is a principal ideal, hence generated by a single element $\pi \in \mathfrak{p}$. We call any such π a uniformizer *for $\mathcal{O}$.*

The triples $(\mathbb{Q}, \mathbb{Z}_{(p)}, \mathbb{F}_p)$ and $(\mathbb{Q}_p, \mathbb{Z}_p, \mathbb{F}_p)$ are the archetypical examples of p-modular systems. Other p-modular systems in this thesis will usually be obtained by taking algebraic extensions of $\mathbb{Q}$ or $\mathbb{Q}_p$. Note that the valuation function ν_p will always extend uniquely to such an extension. We will use the name "ν_p" for the extended valuation function as well.

Definition 2.3 (Algebras, Orders and Lattices)**.** *For a commutative ring R, an* R-algebra *is a (not necessarily commutative) ring A together with a ring homomorphism $\iota : R \longrightarrow Z(A)$ from R into the center of A. An R-algebra A is in particular an R-module (as the image of R lies in the center of A, it does not matter whether we let R act by multiplication from the right or from the left).*

If R is a PID, we call A an R-order *if A is finitely generated and torsion-free (or, equivalently, free) as an R-module. Moreover, if A is an R-order and M is a finitely generated A-module that is torsion-free as an R-module, we call M an A-lattice.*

Our usual setting will be that we have a p-modular system $(K, \mathcal{O}, k)$ and an $\mathcal{O}$-order Λ. As far as notation is concerned, we fix at this point the meaning of the letters K, $\mathcal{O}$ and k; those will always refer to a p-modular system. The following remark shows that an $\mathcal{O}$-order Λ also determines a finite-dimensional k-algebra and a finite-dimensional K-algebra. Note that both the definition of an order and the definition of a lattice already contain the analogue of the "finite-dimensionality"-condition for algebras respectively modules.

Remark 2.4. *If A and B are R-algebras, then $A \otimes_R B$ can be given the structure of an R-algebra. If A is commutative, then $A \otimes_R B$ can also be construed as an A-algebra.*

Note that we usually omit the subscript "R". In particular $k \otimes \Lambda$ is a finite-dimensional k-algebra, and $K \otimes \Lambda$ is a finite-dimensional K-algebra. We sometimes refer to the latter as the "K-span of Λ". This is due to the following universal property of $K \otimes \Lambda$: If A is a K-algebra, and $\varphi : \Lambda \longrightarrow A$ is an $\mathcal{O}$-algebra homomorphism (note that any K-algebra may also be viewed as an $\mathcal{O}$-algebra), then there is a unique K-algebra homomorphism $\tilde{\varphi} : K \otimes \Lambda \longrightarrow A$ such that $\varphi = \tilde{\varphi} \circ \iota$, where $\iota : \Lambda \hookrightarrow K \otimes \Lambda$ is the inclusion map. In particular, if φ is an embedding, then $\tilde{\varphi}$ is a canonical isomorphism between $K \otimes \Lambda$ and the K-span of $\varphi(\Lambda)$ in A.

The orders we are most interested in are group rings of finite groups. If $\Lambda = \mathcal{O}G$ for a finite group G, we will have $k \otimes \Lambda \cong kG$ and $K \otimes \Lambda \cong KG$. Due to the above remarks, there is no harm in identifying $K \otimes \Lambda$ with KG (and neither is there in identifying $k \otimes \Lambda$ with kG).

Theorem 2.5 (Maschke)**.** *KG is a semisimple, and therefore*

$$KG \cong \bigoplus_{i=1}^{n} D_i^{d_i \times d_i} \tag{2.4}$$

for some $n \in \mathbb{N}$, certain finite-dimensional skew-fields D_i over K, and certain $d_i \in \mathbb{N}$

Informally, we call the right hand side of (2.4) the *Wedderburn decomposition* of KG, and the direct summands $D_i^{d_i \times d_i}$ the *Wedderburn components* of KG. The semisimplicity of the K-span of an order has many ramifications, for instance that there is at least one *maximal order* containing it. Also the theory of decomposition numbers at the end of this section and the explicit description of symmetrizing forms given in Section 2.2 depend on the semisimple K-span.

Definition 2.6. *Let R be a PID with field of fractions F. An R-order Λ is called* maximal *if there is no R-order in $F \otimes_R \Lambda$ properly containing Λ.*

Remark 2.7. *(1) Let R be a PID with field of fractions F. If Λ is an R-order with semisimple F-span, then Λ is contained in some maximal order $\Gamma \subset F \otimes_R \Lambda$.*

(2) If $\mathcal{O}$ is a complete, then all maximal $\mathcal{O}$-orders in a finite-dimensional semisimple K-algebra A are conjugate to each other.

Definition 2.8 (Grothendieck Group)**.** *Let $\mathcal{A}$ be an exact category. We associate to each isomorphism class of objects in $\mathcal{A}$ with representative M a symbol $[M]$. Define the* Grothendieck group *of $\mathcal{A}$ to be the free group on the symbols $[M]$ modulo relations $[M] - [N] - [N']$ for each short exact sequence $0 \to N' \to M \to N \to 0$ in $\mathcal{A}$ (we assume that $\mathcal{A}$ is small enough for the isomorphism classes of objects to form a set). We denote the Grothendieck group of $\mathcal{A}$ by $K_0(\mathcal{A})$.*

Remark 2.9. *If R is a field and A is a finite-dimensional R-algebra, then $K_0(\mathbf{mod}_A) \cong \mathbb{Z}^n$, where n is the number of isomorphism classes of simple A-modules. The elements $[S]$, for each isomorphism class of simple modules S, form a basis. The Grothendieck group $K_0(\mathbf{proj}_A)$ is also isomorphic to $\mathbb{Z}^n$, with a basis consisting of the elements $[P]$ for each isomorphism class of projective indecomposable modules P. Note that the natural homomorphism $K_0(\mathbf{proj}_A) \longrightarrow K_0(\mathbf{mod}_A)$ coming from the embedding $\mathbf{proj}_A \longrightarrow \mathbf{mod}_A$ is usually not surjective (to be more precise: it is surjective if and only if A is semisimple).*

Remark 2.10. *Let $\mathcal{A}$ and $\mathcal{B}$ be exact categories. If a functor $\mathcal{F} : \mathcal{A} \longrightarrow \mathcal{B}$ is exact, then it induces a well-defined group homomorphism*

$$\mathcal{F} : K_0(\mathcal{A}) \longrightarrow K_0(\mathcal{B}) : \ [M] \mapsto [\mathcal{F}(M)] \tag{2.5}$$

Remark 2.11. *Let F be a field and A a finite-dimensional F-algebra. The functor*

$$\mathrm{Hom}_A : \ \mathbf{proj}_A \times \mathbf{proj}_A \longrightarrow \mathbf{vect}_F \tag{2.6}$$

is exact in both arguments. By a variation of Remark 2.10, it induces a $\mathbb{Z}$-bilinear map

$$K_0(\mathbf{proj}_A) \times K_0(\mathbf{proj}_A) \longrightarrow K_0(\mathbf{vect}_F) \xrightarrow{\sim} \mathbb{Z} \tag{2.7}$$

where the rightmost arrow is given by taking F-dimensions.

Remark 2.12. *If R is a commutative ring, A is an R-algebra and S is a commutative R-algebra, then*

$$S \otimes_R - : \ \mathbf{mod}_A \longrightarrow \mathbf{mod}_{S \otimes_R A} \tag{2.8}$$

is an additive, right exact functor. It is, however, not exact unless S is flat as an R-module. Nevertheless, its restriction to $\mathbf{proj}_A$:

$$S \otimes_R - : \ \mathbf{proj}_A \longrightarrow \mathbf{proj}_{S\otimes_R A} \tag{2.9}$$

is always exact.

The following theorem, known as the Hensel lemma, is the reason why we prefer p-modular systems where the discrete valuation ring is complete. We state the version for lifting idempotents. There also is a version that ascertains the existence of lifts of factorizations of monic polynomials $f(x) \in \mathcal{O}[x]$. The latter version can easily be recovered from the version for idempotents by realizing that a factorization of $\overline{f}(x) \in k[x]$ corresponds to a set of orthogonal idempotents in the algebra $k[x]/(\overline{f}(x))$.

Theorem 2.13 (Hensel Lemma)**.** *If $\mathcal{O}$ is complete, Λ is an $\mathcal{O}$-order, and $1_{k\otimes\Lambda} = \overline{e}_1 + \ldots + \overline{e}_n$ is a decomposition of the unit element of $k \otimes \Lambda$ into a sum of pairwise orthogonal idempotents $\overline{e}_1, \ldots, \overline{e}_n \in k \otimes \Lambda$, then there exists a set of pairwise orthogonal idempotents $e_1, \ldots, e_n \in \Lambda$ such that $1_\Lambda = e_1 + \ldots + e_n$ and $1_k \otimes e_i = \overline{e}_i$.*

Note that, for instance, $\mathbb{Z}_{(5)}[x]/(x^2+1)$ is indecomposable (that is, it does not decompose into proper blocks; this is simply due to the fact that x^2+1 is irreducible over $\mathbb{Q}$), whereas $\mathbb{F}_5[x]/(x^2+1) \cong \mathbb{F}_5 \oplus \mathbb{F}_5$. So Hensel's lemma fails for non-complete discrete valuation rings. However, if we replace $\mathbb{Z}_{(5)}$ by the 5-adic integers, it shows that $\mathbb{Z}_5[x]/(x^2+1) \cong \mathbb{Z}_5 \oplus \mathbb{Z}_5$.

Theorem 2.14 (Krull-Schmidt)**.** *Let A be an artinian ring or an order over a discrete valuation ring. If M is a finitely generated A-module and*

$$M \cong M_1 \oplus \ldots \oplus M_r \cong N_1 \oplus \ldots \oplus N_s \tag{2.10}$$

are two ways to write M as a direct sum of indecomposable A-modules, then $r = s$ and, after appropriate reordering, $M_i \cong N_i$ for all i.

Finitely generated indecomposable projective modules over a ring A for which the Krull-Schmidt Theorem holds are necessarily of the form eA, where $e \in A$ is a primitive idempotent. So if Λ is an $\mathcal{O}$-order with $\mathcal{O}$ complete, then Hensel's lemma implies that any projective indecomposable $k\otimes\Lambda$-module $\overline{P} \cong \overline{e} \cdot k \otimes \Lambda$ lifts (uniquely up to isomorphism) to a projective indecomposable Λ-module $P \cong e \cdot \Lambda$. It should be noted that the definition of projective modules also implies that P is up to isomorphism the only Λ-lattice which reduces to $\overline{P}$.

Corollary 2.15. *The map*

$$k \otimes - : \ K_0(\mathbf{proj}_\Lambda) \longrightarrow K_0(\mathbf{proj}_{k\otimes\Lambda}) \tag{2.11}$$

is an isomorphism and preserves the bilinear form (2.7).

Another important isomorphism between Grothendieck groups is $K_0(\mathbf{mod}_{k\otimes\Lambda}) \cong K_0(\mathbf{proj}_\Lambda)$ for any $\mathcal{O}$-order Λ with $\mathcal{O}$ complete. This isomorphism follows from a slightly stronger version of Hensel's lemma than the one we stated. Concretely, this isomorphism is given by sending $[P] \in K_0(\mathbf{proj}_\Lambda)$ to $[P/\operatorname{Rad}(P)] \in K_0(\mathbf{mod}_{k\otimes\Lambda})$ or, more abstractly, by the exact functor $- \otimes_\Lambda (\Lambda/\operatorname{Jac}(\Lambda)) : \ \mathbf{proj}_\Lambda \longrightarrow \mathbf{mod}_{k\otimes\Lambda}$ (Note that this functor is only exact because we restricted the domain of definition to projective modules). The following is the reason why "taking the top of projective modules" really induces an isomorphism between $K_0(\mathbf{proj}_\Lambda)$ and $K_0(\mathbf{mod}_{k\otimes\Lambda})$:

Remark 2.16. *Let A be an artinian ring or an $\mathcal{O}$-order with $\mathcal{O}$ complete. Then for each finitely generated A-module M, there exists a unique (up to isomorphism) finitely generated projective A-module P together with an epimorphism $\varphi: P \longrightarrow M$ such that for any other projective A-module P' we have $\mathrm{Hom}_A(P', M) = \varphi \circ \mathrm{Hom}_A(P', P)$. The arrow*

$$P \xrightarrow{\varphi} M \tag{2.12}$$

is unique up to isomorphism in the appropriate category, and called the projective cover *of M (although the term "projective cover" will also be used to refer to the module P itself).*

If M is a simple module, then $P/\mathrm{Rad}(P)$ will be isomorphic to M. Conversely, any projective indecomposable A-module is isomorphic to the projective cover of some simple A-module. So indecomposable projectives are distinguished among all projectives by the fact that they have simple radical quotient (also called "top").

Notation 2.17 (Projective Cover)**.** *In the setting of the above remark, we denote the projective cover of an A-module M by $\mathcal{P}_A(M)$, or $\mathcal{P}(M)$ when the choice of A is clear from the context.*

Remark 2.18 (Projective Modules & Idempotents)**.** *Let A be a ring for which the Krull-Schmidt theorem holds. Then we can decompose the regular right A-module, denoted by A_A, as a direct sum of projective indecomposable modules P_i, i. e.*

$$A_A \cong P_1 \oplus \ldots \oplus P_n \tag{2.13}$$

The projection maps $\pi_1, \ldots, \pi_n$ onto the respective P_i's form a full set of orthogonal primitive idempotents in $\mathrm{End}_A(A_A)$. There is an isomorphism between A and $\mathrm{End}_A(A_A)$ that sends $a \in A$ to the map "left multiplication with a". Let $e_1, \ldots, e_n$ be the preimages of $\pi_1, \ldots, \pi_n$ under said isomorphism. The idempotents $e_1, \ldots, e_n$ form a full set of orthogonal primitive idempotents in A. The following holds:

(1) $P_i = e_i \cdot A$ for $i \in \{1, \ldots, n\}$.

(2) If $e \in A$ is a primitive idempotent, then $e = a^{-1} \cdot e_i \cdot a$ for some i and some unit $a \in A^\times$. With this choice of i we have $e \cdot A \cong e_i \cdot A$.

(3) $e_i \cdot A \cong e_j \cdot A$ if and only if $e_i = a^{-1} \cdot e_j \cdot a$ for some $a \in A^\times$.

Definition 2.19. *Let A be a finite-dimensional algebra over some field R. We call the matrix of the bilinear form (2.7) with respect to the basis of $K_0(\mathbf{proj}_A)$ consisting of the $[\mathcal{P}(S)]$, where S runs over all isomorphism classes of simple A-modules, the* Cartan matrix *of A. Its entries are called the* Cartan numbers *of A.*

Definition 2.20. *Let Λ be an $\mathcal{O}$-order and assume $\mathcal{O}$ is complete. The functor $K \otimes -$ from $\mathbf{proj}_\Lambda$ to $\mathbf{mod}_{K\otimes\Lambda}$ is exact and we call the induced homomorphism of Grothendieck groups*

$$D: \ K_0(\mathbf{proj}_{k\otimes\Lambda}) \cong K_0(\mathbf{proj}_\Lambda) \longrightarrow K_0(\mathbf{mod}_{K\otimes\Lambda}) \tag{2.14}$$

the decomposition map for projective modules.

We call its matrix with respect to the canonical bases of $K_0(\mathbf{proj}_{k\otimes\Lambda})$ *and* $K_0(\mathbf{mod}_{K\otimes\Lambda})$ *the decomposition matrix of* Λ *(the convention is that its columns are indexed by the simple* $k\otimes\Lambda$*-modules and its rows are indexed by the simple* $K\otimes\Lambda$*-modules; so in the context of this definition we should think of the decomposition matrix as acting on column vectors rather than on row vectors). Its entries are called the* decomposition numbers *of* Λ*. We will usually denote the decomposition number associated to the simple* $K\otimes\Lambda$*-module* V *and the simple* $k\otimes\Lambda$*-module* S *by* $D_{V,S}$*.*

Remark 2.21. *If* $K\otimes\Lambda$ *is semisimple, then the decomposition map defined above is in fact an isometric embedding of* $K_0(\mathbf{proj}_{k\otimes\Lambda})$ *into* $K_0(\mathbf{mod}_{K\otimes\Lambda}) = K_0(\mathbf{proj}_{K\otimes\Lambda})$*.*

It is easy to see that for a finite-dimensional semisimple algebra over a field, the Cartan matrix (or, equivalently, the bilinear form (2.7)) is symmetric. Therefore, if $K\otimes\Lambda$ *is semisimple, then the Cartan matrix of* $k\otimes\Lambda$ *will be symmetric as well.*

While the way we introduced Cartan numbers and decomposition numbers is probably the most natural way to do so, there are issues with it when the algebras involved are not split. Namely, if any one of $k\otimes\Lambda$ or $K\otimes\Lambda$ is not split, there will be more than one sensible way to define Cartan numbers and decomposition numbers. This ambiguity will be addressed now.

Proposition 2.22. *Let* F *be a field and let* A *be a finite-dimensional* F*-algebra. Then, for any two simple* A*-modules* S *and* T*:*

$$\dim_F \operatorname{Hom}_A(\mathcal{P}(S),\mathcal{P}(T)) = \dim_F \operatorname{End}_A(S)\cdot\begin{pmatrix} \textit{Multiplicity of } S \\ \textit{as a composition factor of } \mathcal{P}(T)\end{pmatrix} \tag{2.15}$$

In particular, if A *is split, then the Cartan numbers of* A *coincide with the multiplicities of simple modules in projective indecomposable modules.*

Clearly, the above proposition gives us an alternative definition of Cartan numbers, namely, if S and T are two simple A-modules, we could define the Cartan number associated to S and T to be the multiplicity of S in a composition series of $\mathcal{P}(T)$. While this definition is widespread in literature, the downside of it is that the so-defined Cartan matrix is not necessarily symmetric, and does not arise naturally as the matrix of a bilinear form.

Proposition 2.23. *Let* Λ *be an* $\mathcal{O}$ *order with* $\mathcal{O}$ *complete and* $K\otimes\Lambda$ *semisimple. If* S *is a simple* $k\otimes\Lambda$*-module,* V *is a simple* $K\otimes\Lambda$*-module and* L *is any full* Λ*-lattice in* V*, then*

$$D_{V,S} = \frac{\dim_k \operatorname{End}_{k\otimes\Lambda}(S)}{\dim_K \operatorname{End}_{K\otimes\Lambda}(V)}\cdot\begin{pmatrix} \textit{Multiplicity of } S \\ \textit{as a composition factor of } k\otimes L\end{pmatrix} \tag{2.16}$$

Note that another formulation of the definition of $D_{V,S}$ is the following:

$$D_{V,S} = \begin{pmatrix} \text{Multiplicity of } V \\ \text{as a composition factor of } K\otimes\mathcal{P}_\Lambda(S)\end{pmatrix} \tag{2.17}$$

The fact that the right hand side of (2.16) equals the right hand side of (2.17) is commonly referred to as *Brauer reciprocity.* The proof of Proposition 2.22 is an easy consequence of Remark 2.10 applied to $\operatorname{Hom}_A(\mathcal{P}(S),-) : \mathbf{mod}_A \longrightarrow \mathbf{vect}_k$. The proof of Proposition 2.23

follows from the following chain of equalities:

$$\begin{aligned} \dim_K \operatorname{Hom}_{K\otimes\Lambda}(K\otimes\mathcal{P}_\Lambda(S), K\otimes L) &= \dim_K K\otimes\operatorname{Hom}_\Lambda(\mathcal{P}_\Lambda(S), L) \\ &= \dim_k k\otimes\operatorname{Hom}_\Lambda(\mathcal{P}_\Lambda(S), L) \\ &= \dim_k \operatorname{Hom}_{k\otimes\Lambda}(\mathcal{P}_{k\otimes\Lambda}(S), k\otimes L) \end{aligned} \tag{2.18}$$

Finally, we give a slight reformulation of Remark 2.21, which shows how decomposition matrices and Cartan matrices relate to one another.

Remark 2.24. *Let Λ be an $\mathcal{O}$-order with $\mathcal{O}$ complete and $K\otimes\Lambda$ semisimple. Then*

$$C_{k\otimes\Lambda} = D^\top \cdot C_{K\otimes\Lambda} \cdot D \tag{2.19}$$

where D is the decomposition matrix of Λ, $C_{k\otimes\Lambda}$ is the Cartan matrix of $k\otimes\Lambda$ and $C_{K\otimes\Lambda}$ is the Cartan matrix of $K\otimes\Lambda$. The matrix $C_{K\otimes\Lambda}$ will always be a diagonal, and if $K\otimes\Lambda$ is split it will be the identity matrix.

2.2 Symmetric Orders in Semisimple Algebras

An algebra A over a commutative ring R is called *symmetric* if $A \cong \operatorname{Hom}_R(A, R)$ as A-A-bimodules. The reason we are interested in symmetric algebras and orders is that for any commutative ring R and any finite group G, the group algebra RG will be symmetric. The property of being symmetric is equivalent to the existence of a bilinear form on A with certain properties, as the following proposition shows:

Proposition 2.25. *Let A be an algebra over a commutative ring R. Then A is symmetric if and only if there is an R-bilinear map*

$$T:\ A\times A \longrightarrow R \tag{2.20}$$

with the following properties:

(1) T is symmetric: $T(a,b) = T(b,a)$ for all $a,b\in A$.

(2) T is associative: $T(a\cdot b, c) = T(a, b\cdot c)$ for all $a,b,c\in A$.

(3) T is non-degenerate: If for some $a\in A$ we have $T(a,b)=0$ for all $b\in A$, then $a=0$.

(4) For any $f\in\operatorname{Hom}_R(A,R)$, there is an $a\in A$ such that $f(-) = T(a,-)$.

We call T a symmetrizing form *for A.*

Remark 2.26. *Setting as above. If T is a symmetrizing form for A, then the map*

$$A \longrightarrow \operatorname{Hom}_R(A,R):\ a\mapsto T(a,-) \tag{2.21}$$

is an isomorphism of A-A-bimodules.

If Λ is a symmetric $\mathcal{O}$-order with semisimple K-span, then a symmetrizing form for Λ has a K-linear extension to a symmetrizing form for the K-span of Λ. This is a useful fact because symmetrizing forms for semisimple algebras (or separable, if we wish to treat fields of positive characteristic as well) can be described explicitly as “traces weighted by a central unit”, as the following proposition shows:

Proposition 2.27. *Let F be a field and let*

$$A = \bigoplus_{i=1}^{n} D_i^{d_i \times d_i} \tag{2.22}$$

be a finite-dimensional separable F-algebra, where the D_i are certain (separable) skew-fields over F (Note that in the case $\mathrm{char}(F) = 0$ *"separable" is the same as "semisimple"). Denote by $\varepsilon_1, \ldots, \varepsilon_n$ the central primitive idempotents in A*

Then A is symmetric, and any symmetrizing form T for A is of the form

$$T = T_u : A \times A \longrightarrow K : \ (a, b) \mapsto \sum_{i=1}^{n} \mathrm{tr}_{Z(D_i)/F}\, \mathrm{tr.red.}_{D_i^{d_i \times d_i}/Z(D_i)}(\varepsilon_i \cdot u \cdot a \cdot b) \tag{2.23}$$

for some central unit $u \in Z(A)^\times$. Here $\mathrm{tr}_{Z(D_i)/F}$ *denotes the usual trace for field extensions, and* $\mathrm{tr.red.}_{D_i^{d_i \times d_i}/Z(D_i)}$ *denotes the reduced trace (we give an alternative definition in the remark below which does not rely on reduced traces).*

Remark 2.28. *Setting as above. An equivalent definition for T_u is the following: Let $\bar{F}$ be the algebraic closure of F. Then*

$$\bar{F} \otimes_F A \cong \bigoplus_{i=1}^{n'} \bar{F}^{d'_i \times d'_i} \tag{2.24}$$

for some $n' \geqslant n$ and certain natural numbers d'_i. The value of $T_u(a, b)$ can now be obtained by taking the image of $u \cdot a \cdot b$ in the right hand side of (2.24), taking the trace in each of the $\bar{F}^{d'_i \times d'_i}$, and finally summing up all those traces.

Remark 2.29. *We sometimes write $T_u(a)$ instead of $T_u(a, 1)$. This is justified since T_u is associative, and hence $T_u(a, b) = T_u(ab, 1)$. So in fact the two-argument version of T_u can be reconstructed from the single-argument version.*

Now suppose we are given an $\mathcal{O}$-order Λ with semisimple K-span A together with an element $u \in Z(A)^\times$. We would like to have a criterion for when the restriction of T_u to $\Lambda \times \Lambda$ is a symmetrizing form for Λ (in which case Λ would be symmetric).

Definition 2.30. *Let R be a PID with field of fractions F. Let V be a finite-dimensional F-vector space equipped with a symmetric non-degenerate F-bilinear form*

$$(-, =) : \ V \times V \longrightarrow F \tag{2.25}$$

and let L be a full R-lattice in V.

The set

$$L^\sharp := \{v \in V \mid (v, L) \subseteq R\} \tag{2.26}$$

is called the dual *of the lattice L, and is again a full R-lattice in V. The lattice L is called* self-dual *(or* unimodular*) if $L = L^\sharp$.*

Proposition 2.31. *Let R be a PID with field of fractions F. Let A be a finite-dimensional symmetric F-algebra and let Λ be an R-order in A. Then a symmetrizing form $T : \ A \times A \longrightarrow F$*

for A restricts to a symmetrizing form $T|_{\Lambda\times\Lambda} : \Lambda \times \Lambda \longrightarrow R$ if and only if Λ is a self-dual lattice with respect to T.

Rather than exposing properties of given orders, the main part of this thesis will be concerned with the "converse" of this problem: Given some properties, what does an order look like which fulfills those properties? We consider symmetry in this context because group rings happen to be symmetric orders, and there even is an explicit description of the symmetrizing element u. Furthermore, the property of being self-dual behaves well under Morita and even derived equivalences, meaning that not only the property itself transfers via such equivalences, but also information about the symmetrizing element.

We now turn to the question of how to find self-dual lattices respectively orders. The following shows how to check whether a given lattice is self-dual:

Proposition 2.32. *Let R be a PID with field of fractions F, and let V be a finite dimensional F-vector space equipped with a non-degenerate symmetric bilinear form $T : V \times V \longrightarrow F$. A full R-lattice $L \subset V$ is self-dual if and only if the determinant of the Gram matrix of T with respect to an R-basis of L (which will also be an F-basis of V) is a unit in R.*

The above proposition shows that we can usually pass to smaller coefficient rings without loosing self-duality:

Proposition 2.33. *Let R be a PID with field of fractions F and let $R' \subset R$ be a PID with field of fractions F'. Assume that the field extension F/F' is algebraic. Moreover let V' be a finite-dimensional F'-vector space equipped with a non-degenerate symmetric bilinear form $T' : V' \times V' \longrightarrow F'$. Set $V := F \otimes_{F'} V'$ and denote the F-bilinear continuation of T' to $V \times V$ by T.*

If L' is a full R'-lattice in V', then L' is self-dual in V (with respect to T') if and only if the R-lattice $L := R \otimes_{R'} L'$ is self-dual in V (with respect to T).

When looking for self-dual orders in a symmetric finite-dimensional K-algebra, the following "inclusion reversing" property of taking duals is essential:

Remark 2.34. *Let R be a PID with field of fractions F, and let V be a finite-dimensional F-vector space equipped with a non-degenerate bilinear form.*

(1) If L is a full R-lattice in V, then $L^{\sharp\sharp} = L$.

(2) Let L_1 and L_2 be two full R-lattices in V such that $L_1 \subseteq L_2$. Then $L_2^\sharp \subseteq L_1^\sharp$. Moreover, the R-modules L_2/L_1 and $L_1^\sharp/L_2^\sharp$ are isomorphic.

The last remark shows that if A is a symmetric K-algebra (with some fixed symmetrizing from), Λ is a self-dual full $\mathcal{O}$-order in A and $\Gamma \supseteq \Lambda$ is some overorder, then

$$\Gamma^\sharp \subseteq \Lambda = \Lambda^\sharp \subseteq \Gamma \tag{2.27}$$

From this one can see the rough idea of how one goes about finding self-dual orders: Given an overorder, one knows an upper and lower bound for the self-dual order. Making additional assumptions on the self-dual order will usually lead to smaller overorders, until, hopefully, one arrives at an overorder which is self-dual itself, and thus equal to the self-dual order in question. Note that the whole concept of "approximation from above" has no apparent analogue in the context of finite-dimensional k-algebras.

An important tool when looking for self-dual orders is the so-called *discriminant*. If Λ is a self-dual $\mathcal{O}$-order and $\Gamma \supseteq \Lambda$ is an overorder, the discriminant can be used to compute the length of the quotient Γ/Λ as an $\mathcal{O}$-module (which one could call the "index" of Λ in Γ).

Definition 2.35 (Discriminant). *Let R be a PID with field of fractions F, and let V be a finite-dimensional F-vector space equipped with a non-degenerate symmetric bilinear form $T:\ V \times V \longrightarrow F$. Let $\varphi_L:\ \mathrm{Hom}_R(L,R) \longrightarrow L$ be an arbitrary R-lattice isomorphism and define $\tilde{T}:\ V \longrightarrow \mathrm{Hom}_K(V,F):\ v \mapsto T(v,-)$. Then we call the fractional ideal in F generated by*

$$\det\left((\mathrm{id}_K \otimes \varphi_L) \circ \tilde{T}\right) \tag{2.28}$$

the discriminant *of L. We denote the discriminant by* $\mathrm{discrim}(L)$.

Remark 2.36. *Situation as in the above definition.*

(1) φ_L is unique only up to composition with some R-lattice automorphism of L. But since the determinant of such an automorphism lies in $R^\times$, the fractional ideal generated by the determinant in (2.28) is independent of the choice of φ_L. Thus the discriminant of a lattice is well-defined.

(2) By choosing bases, we may identify L with $R^{1\times n}$ (for $n = \dim_F(V)$), $\mathrm{Hom}_R(L,R)$ with $R^{n\times 1}$, the map φ_L with the transposition map $-^\top:\ R^{n\times 1} \longrightarrow R^{1\times n}$ and $\tilde{T}$ with the map $R^{1\times n} \longrightarrow R^{n\times 1}:\ v \mapsto G \cdot v^\top$, where G is the Gram matrix of T (with respect to the standard basis of $R^{1\times n}$). With this in mind it is clear that the discriminant of L is simply the fractional ideal generated by the determinant of the Gram matrix of T taken with respect to an R-basis of L

Definition 2.37 (Discriminants of Number Fields). *Let R be a PID with field of fractions F and let E be a finite separable field extension of F. Denote by S the integral closure of R in E (which is a full R-lattice in E). Equip E with the trace bilinear form*

$$E \times E \longrightarrow F:\ (a,b) \mapsto \mathrm{tr}_{E/F}(a \cdot b) \tag{2.29}$$

The discriminant (as defined in Definition 2.35) of S with respect to this bilinear form is called the discriminant of the field extension *E/F, denoted by* $\mathrm{discrim}_F E$.

Proposition 2.38 (Discriminants & Ring Extensions). *Let $R \subset R'$ be two PID's with fields of fractions $F \subset F'$. Let V be a finite-dimensional F-vector space equipped with a non-degenerate symmetric bilinear form. Equip $F' \otimes V$ with the F'-bilinear continuation of said form. Then for any full R-lattice L in V we have*

$$\mathrm{discrim}(R' \otimes_R L) = R' \cdot \mathrm{discrim}(L) \tag{2.30}$$

For the formulation of the proposition below it is convenient to introduce the category $\mathbf{fgtormod}_R$ of finitely-generated torsion R-modules. Passing to $K_0(\mathbf{fgtormod}_R)$ amounts to counting composition factors, which in case of a PID is equivalent to factorizing the product of all elementary divisors of a given module.

Proposition 2.39. *Situation again as in Definition 2.35. If $L \subseteq L^\sharp$, then* $\mathrm{discrim}(L)$ *is an ideal in R (not just a fractional one in F), and*

$$[R/\mathrm{discrim}(L)] = \left[L^\sharp/L\right] \in K_0(\mathbf{fgtormod}_R) \tag{2.31}$$

In particular, if L is contained in some self-dual lattice L', then

$$[R/\operatorname{discrim}(L)] = 2 \cdot [L'/L] \in K_0(\mathbf{fgtormod}_R) \tag{2.32}$$

Proposition 2.40. *Let R be a PID with field of fractions F. Let*

$$A := \bigoplus_{i=1}^{n} F^{d_i \times d_i} \quad \text{for some } n \in \mathbb{N} \text{ and certain } d_1, \ldots, d_n \in \mathbb{N} \tag{2.33}$$

Further let $u = (u_1, \ldots, u_n) \in Z(A)^\times = F^\times \times \ldots \times F^\times$. Assume there is some order Λ which is self-dual with respect to T_u. Then for any maximal order Γ which contains Λ we have

$$[\Gamma/\Lambda] = \frac{1}{2} \cdot \sum_{i=1}^{n} d_i^2 \cdot [R/u_i^{-1}R] \quad (\text{in } K_0(\mathbf{fgtormod}_R)) \tag{2.34}$$

This proposition follows fairly easily from the following remark and Proposition 2.39. It should be noted that (2.34) makes sense only if all u_i^{-1} lie in R. It is not hard to see that this is necessary for there to be a self-dual order in the first place.

Remark 2.41. *A maximal R-order in (2.33) is conjugate by an element of $A^\times$ to the order*

$$\Gamma := \bigoplus_{i=1}^{n} R^{d_i \times d_i} \tag{2.35}$$

What we have just treated are self-dual orders in *split* semisimple algebras. However, it is always possible to reduce to the split case. For that assume that Λ is a self-dual $\mathcal{O}$-order with semisimple K-span, that E is a splitting field for $K \otimes \Lambda$ (with E/K finite) and that $\mathcal{O}'$ is the integral closure of $\mathcal{O}$ in E. Then $\mathcal{O}'$ is a (local) PID as well, and there is an embedding $K_0(\mathbf{fgtormod}_{\mathcal{O}}) \hookrightarrow K_0(\mathbf{fgtormod}_{\mathcal{O}'})$ (both being isomorphic to $\mathbb{Z}$, the index of the image of the embedding being precisely the so-called *ramification index* of E/K). Now Proposition 2.40 allows as to calculate the "index" of $\mathcal{O}' \otimes_{\mathcal{O}} \Lambda$ in a maximal $\mathcal{O}'$-order containing it. According to Remark 2.7 the order Λ is contained in some maximal $\mathcal{O}$-order $\Gamma \subset K \otimes \Lambda$ (even unique up to conjugation if $\mathcal{O}$ is complete). The only real added difficulty in the non-split case is that $\mathcal{O}' \otimes_{\mathcal{O}} \Gamma$ is usually not a maximal $\mathcal{O}'$-order. To calculate the index of $\mathcal{O}' \otimes_{\mathcal{O}} \Gamma$ in a maximal $\mathcal{O}'$-order, once would usually calculate the discriminant of Γ with respect to the trace bilinear form T_1, since by Proposition 2.38 those are well-behaved under ring extensions, and a maximal order in $E \otimes_K \Lambda$ is self-dual with respect to T_1 (since it is of the shape (2.35)).

2.3 Group Algebras, Blocks and Characters

Let G be a finite group. In this section we recall some definitions and theorems from the representation theory of finite groups, with a particular focus on order-theoretic properties of the group algebra $\mathcal{O}G$.

Remark 2.42. *Let R be a PID. The R-bilinear continuation of the map*

$$G \times G \longrightarrow R : \ (g,h) \mapsto \begin{cases} 1 & \text{if } g = h^{-1} \\ 0 & \text{otherwise} \end{cases} \tag{2.36}$$

defines a symmetrizing form for RG. In particular, RG is a symmetric order.

In light of Remark 2.29, we can equivalently consider the following single-argument version of the symmetrizing form for RG:

$$T: \ RG \longrightarrow R: \ \sum_{g \in G} r_g \cdot g \mapsto r_{1_G} \tag{2.37}$$

The restriction of this to G is constant on conjugacy classes (since it is non-zero precisely on the unit element of G). Such functions are called *class functions.* If R is an algebraically closed field of characteristic zero then such functions can be written as linear combinations of (ordinary) characters.

Definition 2.43. *If V is a KG-module, with pertinent representation*

$$\Delta_V: \ KG \longrightarrow \operatorname{End}_K(V) \cong K^{\dim_K V \times \dim_K V} \tag{2.38}$$

we associate to it the character

$$\chi_V: \ KG \longrightarrow K: \ a \mapsto \operatorname{tr}(\Delta_V(a)) \tag{2.39}$$

*So χ maps g to the trace of the K-vector space endomorphism of V induced by g. Note that a character is determined by its values on G, and is constant of conjugacy classes. We denote the set of all functions from G to K which are constant on conjugacy classes (*class functions from G to K*) by* $\operatorname{CF}(G, K)$.

We denote the set of characters associated to simple KG-modules by $\operatorname{Irr}_K(G)$.

Proposition 2.44. *(1) If KG is split, then the K-span of $\operatorname{Irr}_K(G)$, denoted by $K \operatorname{Irr}_K(G)$, is equal to* $\operatorname{CF}(G, K)$.

(2) If V and W are two KG-modules, then $V \cong W$ if and only if $\chi_V = \chi_W$.

(3) Denote by $\mathbb{Z} \operatorname{Irr}_K(G)$ the $\mathbb{Z}$-span of $\operatorname{Irr}_K(G)$ in $\operatorname{CF}(G, K)$. The map

$$K_0(\mathbf{mod}_{KG}) \longrightarrow \mathbb{Z} \operatorname{Irr}_K(G): \ [V] \mapsto \chi_V \tag{2.40}$$

is an isomorphism of abelian groups. If we equip $\mathbb{Z} \operatorname{Irr}_K(G)$ with restriction of the scalar product

$$\operatorname{CF}(G, K) \times \operatorname{CF}(G, K) \longrightarrow K: \frac{1}{|G|} \sum_{g \in G} \chi(g) \eta(g^{-1}) \tag{2.41}$$

this isomorphism becomes an isometry (where $K_0(\mathbf{mod}_{KG})$ is given a scalar product as in Remark 2.11).

We want to write the map T defined in (2.37) as a sum of irreducible characters, in order to find a $u \in Z(KG)$ with $T = T_u$. It is convenient to work with absolutely irreducible characters, that is, with $\operatorname{Irr}_{\bar{K}}(G)$, instead of $\operatorname{Irr}_K(G)$. One can elementarily verify that $|G| \cdot T$ is the character of the regular KG-module. If

$$\bar{K}G \cong \bigoplus_{\chi \in \operatorname{Irr}_{\bar{K}}(G)} \bar{K}^{\chi(1) \times \chi(1)} \tag{2.42}$$

is the Wedderburn-decomposition of $\bar{K}G$, then it is clear that the simple module associated to χ, which corresponds to $\bar{K}^{1\times\chi(1)}$ on the right hand side, occurs $\chi(1)$ times in $\bar{K}G$. This leads to the following proposition, which gives us the symmetrizing form for RG explicitly:

Proposition 2.45. *With T as in (2.37) we have*

$$T(-) = \frac{1}{|G|} \cdot \sum_{\chi \in \mathrm{Irr}_{\bar{K}}(G)} \chi(1) \cdot \chi(-) \tag{2.43}$$

Furthermore, $T = T_u$ with

$$u = \frac{1}{|G|} \cdot \sum_{\chi \in \mathrm{Irr}_{\bar{K}}(G)} \chi(1) \cdot \varepsilon_\chi \in Z(\mathbb{Q}G) \subseteq Z(\bar{K}G) \tag{2.44}$$

where the ε_χ denote the central primitive idempotents in $\bar{K}G$ (the indexing being chosen such that $\chi(\varepsilon_\chi) \neq 0$).

Remark 2.46 (Characters & the Center)**.** *Let R be a PID with field of fractions F of characteristic zero. For each $\chi \in \mathrm{Irr}_F(G)$ let $\bar{\chi} \in \mathrm{Irr}_{\bar{F}}(G)$ be some absolutely irreducible constituent of χ. Let $F(\bar{\chi})$ denote the field obtained by adjoining to F all values $\bar{\chi}(g)$ for $g \in G$. The map*

$$Z(RG) \longrightarrow \bigoplus_{\chi \in \mathrm{Irr}_F(G)} F(\bar{\chi}) : \ z \mapsto \left(\frac{\bar{\chi}(z)}{\bar{\chi}(1)} \right)_{\chi \in \mathrm{Irr}_F(G)} \tag{2.45}$$

is an injective R-algebra homomorphism that maps $Z(RG)$ to a full R-lattice in the right hand side (which means, in particular, that $\bigoplus_{\chi \in \mathrm{Irr}_F(G)} F(\bar{\chi}) \cong Z(FG)$). Note that the sums over conjugacy classes in G form an R-basis of $Z(RG)$, and therefore an embedding of the center of RG into the Wedderburn-decomposition of $Z(FG)$ can be calculated from the absolutely irreducible characters of G. It should be noted though that the "generic" algorithm to calculate character tables works exactly the other way around, that is, it calculates the Wedderburn-decomposition of $Z(FG)$ first, and then from that the characters.

Definition 2.47 (Blocks)**.** *Let*

$$1_{\mathcal{O}G} = \sum_{i=1}^{n} b_i \tag{2.46}$$

be the (unique) decomposition of $1_{\mathcal{O}G} \in Z(\mathcal{O}G)$ as a sum of pairwise orthogonal primitive idempotents in $Z(\mathcal{O}G)$. The b_i are called block idempotents *of $\mathcal{O}G$ and the algebras*

$$B_i := \mathcal{O}G \cdot b_i \tag{2.47}$$

are known as blocks *of $\mathcal{O}G$. We say a module (over $\mathcal{O}G$ or KG)* lies in the block B_i *if b_i acts on it as the identity. We say a character χ of G* lies in the block B_i *if $\chi(b_i) \neq 0$ and $\chi(b_j) = 0$ for all $j \neq i$ (Or, equivalently, if the corresponding simple KG-module lies in B_i). We denote the set of all irreducible characters of G that lie in B_i by $\mathrm{Irr}_K(B_i)$. The block containing the trivial character of G is called the* principal block *of $\mathcal{O}G$.*

In the above definition, blocks need not be absolutely indecomposable. That is, there might well be a p-modular system $(K', \mathcal{O}', k')$ extending $(K, \mathcal{O}, k)$ such that the number of blocks of $\mathcal{O}'G$ is strictly greater than that of $\mathcal{O}G$. A sufficient condition for a block B of $\mathcal{O}G$ not to split up under extensions of the p-modular system is that the algebra $k \otimes_{\mathcal{O}} B$ is split. When we want to ascertain that this is the case, we usually make the assumption that k is algebraically closed.

Definition 2.48 (Defect Group)**.** *Assume k is algebraically closed. If B is a block of $\mathcal{O}G$, we call a subgroup D of G a defect group of B if the following conditions hold:*

(1) If V is an indecomposable $k \otimes B$-module, then there is some kD-module W such that V is a direct summand of $W \otimes_{kD} kG$ and W is a direct summand of $V|_{kD}$.

(2) No proper subgroup of D fulfills the first condition.

It can be shown that any defect group of B is a p-group, and that all defect groups of B are conjugate to each other in G. We call $\nu_p(|D|)$ the defect *of B.*

Remark 2.49. *Assume k is algebraically closed and let B be a block of $\mathcal{O}G$ of defect d. If $\chi \in \mathrm{Irr}_{\bar{K}}(B)$ then*

$$\mathrm{ht}(\chi) := d - \nu_p\left(\frac{\chi(1)}{|G|}\right) \tag{2.48}$$

is called the height *of χ. If then $0 \leqslant \mathrm{ht}(\chi) \leqslant d$. Furthermore, in each block lies at least one character of height zero. Moreover, it was conjectured by Brauer that the defect group of a block is abelian if and only if all characters that lie within it are of height zero. This conjecture is known as* "Brauer's height zero conjecture".

Remark 2.50 (Brauer Correspondence)**.** *Let k be algebraically closed, let G be a finite group and let $D \leqslant G$ be a p-subgroup of G. Then there is a one-to-one correspondence*

$$\{ \text{ Blocks of } kG \text{ of defect } D \ \} \leftrightarrow \{ \text{ Blocks of } kN_G(D) \text{ of defect } D \ \} \tag{2.49}$$

called the Brauer correspondence. *This correspondence is in fact given explicitly (see, for instance, [Alp86, Section 14 in Chapter IV]). For the purposes of this thesis it will suffice for us to know that there is a theorem known as "Brauer's Third Main Theorem" (also stated and proved in [Alp86]) which says (as a special case) that if D is a p-Sylow subgroup of G then the principal block of kG (which must have D as a defect group) and the principal block of $kN_G(D)$ are Brauer correspondents of each other.*

2.4 Morita Equivalences and Derived Equivalences

In this section we introduce the concepts of Morita equivalence and derived equivalence of rings. Our treatment is based mainly on [KZ98], [DV77], [Del77], [Ric89] and [Ric91a]. We start off with Morita equivalences, since they require less technical definitions than derived ones. The following is a simple but utterly unconstructive definition of the notion of Morita equivalence:

Definition 2.51 (Morita Equivalences)**.** *Let A and B be rings. We say that A and B have* equivalent module categories, *and call them* Morita equivalent, *if there are additive and exact functors*

$$\mathcal{F}: \ \mathbf{Mod}_A \longrightarrow \mathbf{Mod}_B \quad \text{and} \quad \mathcal{G}: \ \mathbf{Mod}_B \longrightarrow \mathbf{Mod}_A \tag{2.50}$$

such that $\mathcal{G} \circ \mathcal{F} \cong \mathrm{id}_{\mathbf{Mod}_A}$ and $\mathcal{F} \circ \mathcal{G} \cong \mathrm{id}_{\mathbf{Mod}_B}$.

Remark 2.52. *Even though we are using* $\mathbf{Mod}_A$ *and* $\mathbf{Mod}_B$ *instead of* $\mathbf{mod}_A$ *and* $\mathbf{mod}_B$ *in the above definition, any Morita equivalence will restrict to an equivalence between* $\mathbf{mod}_A$ *and* $\mathbf{mod}_B$ *(this is an easy consequence of the theory explained below). The reason for not working exclusively with finitely generated modules is that* $\mathbf{mod}_A$ *and* $\mathbf{mod}_B$ *need not be abelian categories (see Remark 2.67 below).*

There are two useful characterizations of Morita equivalence, one by one-sided objects (so-called progenerators) and one by two-sided objects (so-called invertible bimodules). We first give the one-sided characterization:

Theorem 2.53 (Morita)**.** *Let* A *and* B *be rings. Then* A *and* B *are Morita equivalent if and only if there is a finitely generated projective* A*-module* P *with the following two properties:*

(1) If M *is an arbitrary finitely generated* A*-module, then there is some* $n \in \mathbb{N}$ *such that there is an epimorphism*

$$\bigoplus^{n} P \twoheadrightarrow M \tag{2.51}$$

(2) $\mathrm{End}_A(P) \cong B$*.*

The projective A-module P in the preceding theorem is called a *progenerator*. We can view P as an $\mathrm{End}_A(P)$-A-bimodule, and a possible equivalence between $\mathbf{mod}_A$ and $\mathbf{mod}_{\mathrm{End}_A(P)}$ is given by

$$\mathrm{Hom}_A(P, -): \ \mathbf{Mod}_A \longrightarrow \mathbf{Mod}_{\mathrm{End}_A(P)} \tag{2.52}$$

The progenerator P gets sent to the free $\mathrm{End}_A(P)$-module of rank one by this equivalence.

There is a particularly nice way to enumerate all isomorphism classes of progenerators, and hence all algebras in a Morita equivalence class, if the ring A we start with satisfies the Krull-Schmidt theorem. Namely, in this case, assuming that $P_1, \ldots, P_n$ is a full system of representatives of isomorphism classes of projective indecomposable A-modules, the progenerators all look like

$$P = \bigoplus_{i=1}^{n} P_i^{d_i} \quad \text{for certain } d_i > 0 \tag{2.53}$$

If, more specifically, A is an artinian ring or an $\mathcal{O}$-order with $\mathcal{O}$ complete, then the P_i are of the form $\mathcal{P}(S_i)$, where $S_1, \ldots, S_n$ is a full system of representatives of isomorphism classes of simple A-modules.

Definition 2.54 (Basic Algebra)**.** *Let* A *be a ring for which the Krull-Schmidt theorem holds. Let* $P_1, \ldots, P_n$ *be a full system of representatives of projective indecomposable* A*-modules. Then we call*

$$B := \mathrm{End}_A\left(\bigoplus_{i=1}^{n} P_i\right) \tag{2.54}$$

the basic algebra of A *.*

Since the equivalence (2.52) sends P to the free module, basic algebras will be algebras such that each isomorphism class of projective indecomposable modules occurs precisely once in the regular module. Hence the following remark:

Remark 2.55. *Let A be a ring such that $A/\operatorname{Jac}(A)$ is artinian (and hence in particular semisimple) and such that there are projective indecomposable A-modules $\mathcal{P}(S)$ with $\mathcal{P}(S)/\operatorname{Rad}(\mathcal{P}(S)) \cong S$ for each simple A-module S. Then*

$$A_A \cong \bigoplus_{S \text{ simple}} \bigoplus^{\dim_{\operatorname{End}_A(S)} S} \mathcal{P}(S) \tag{2.55}$$

This follows easily by realizing that, as a ring,

$$A/\operatorname{Jac}(A) \cong \bigoplus_{S \text{ simple}} \operatorname{End}_A(S)^{\dim_{\operatorname{End}_A(S)}(S) \times \dim_{\operatorname{End}_A(S)}(S)} \tag{2.56}$$

with the simple A-module S corresponding to $\operatorname{End}_A(S)^{1\times \dim_{\operatorname{End}_A(S)}(S)}$ on the right hand side. In particular if A is an $\mathcal{O}$-order with $\mathcal{O}$ complete or a finite-dimensional k-algebra, then a basic algebra of A is distinguished among all algebras Morita equivalent to A by the fact that its simple modules are one-dimensional over their endomorphism rings (which are skew-fields).

Now we come to an alternative characterization of Morita equivalences, which has the advantage of actually specifying a particular equivalence between two module categories. The one-sided characterization of Theorem 2.53 does not do that, since in order to turn (2.52) into an equivalence between $\mathbf{Mod}_A$ and $\mathbf{Mod}_B$, one still has to choose an equivalence between $\mathbf{Mod}_{\operatorname{End}_A(P)}$ and $\mathbf{Mod}_B$. The latter can of course be done, since $\operatorname{End}_A(P)$ and B are isomorphic, but such an equivalence is not unique.

Definition 2.56. *Let A and B be rings. An A-B-bimodule X is called invertible if it is projective as a left A-module and as a right B-module and there is a B-A-bimodule Y, also projective both as a left and as a right module, with*

$$X \otimes_B Y \cong {}_A A_A \quad \text{and} \quad Y \otimes_A X \cong {}_B B_B \tag{2.57}$$

Theorem 2.57 (Morita)**.** *Let A and B be two rings. Then A and B are Morita equivalent if and only if there exists an invertible A-B-bimodule X. The functor*

$$- \otimes_A X : \ \mathbf{Mod}_A \longrightarrow \mathbf{Mod}_B \tag{2.58}$$

affords an equivalence and each equivalence between $\mathbf{Mod}_A$ and $\mathbf{Mod}_B$ is isomorphic to $-\otimes_A X'$ for some invertible A-B-bimodule X'.

The connection between invertible bimodules and progenerators is as follows: If A and B are rings and X is an invertible A-B-bimodule, then the restriction of X to B is a progenerator with endomorphism ring A. In the other direction, if P is a progenerator in $\mathbf{proj}_B$ with $\operatorname{End}_B(P) \cong A$, then we can turn P into an A-B-bimodule by fixing an isomorphism $\varphi : A \longrightarrow \operatorname{End}_B(P)$ and letting A act on P from the left via φ. The resulting A-B-bimodule will be invertible.

Now we turn to derived equivalences. In order to say what a derived equivalence is, we first need to define the *derived category* of a module (or, more generally, abelian) category and a few categories related to it. First we should introduce categories of chain complexes and their homotopy categories.

Definition 2.58 (Chain Complex). *Let $\mathcal{A}$ be an additive category. A chain complex of objects in $\mathcal{A}$ is simply a sequence in the category $\mathcal{A}$:*

$$\ldots \longrightarrow C^{i-1} \xrightarrow{d^{i-1}} C^i \xrightarrow{d^i} C^{i+1} \longrightarrow \ldots \tag{2.59}$$

However, the following way to define chain complexes simplifies notation a lot: For simplicity we assume that $\mathcal{A}$ is a full subcategory of some additive category $\mathcal{B}$ which has countably infinite direct sums (this applies to all categories we will consider). Notation-wise, we make the simplifying assumption that the objects of $\mathcal{A}$ are in particular sets and the morphisms in $\mathcal{A}$ are actually maps between those sets.

A chain complex is a $\mathbb{Z}$-graded object in $\mathcal{B}$

$$C = \bigoplus_{i\in\mathbb{Z}} C^i \quad \text{where the } C^i \text{ are objects in } \mathcal{A} \tag{2.60}$$

together with an endomorphism $d : C \longrightarrow C$ such that

(1) d is of degree 1, that is, $d(C^i) \subseteq C^{i+1}$.

(2) $d \circ d = 0$.

d is called the differential *of the complex. We denote the restriction of d to C^i by d^i.*

If (C_1, d_1) and (C_2, d_2) are two chain complexes, a homomorphism from (C_1, d_1) to (C_2, d_2) is a homomorphism $\varphi :\ C_1 \longrightarrow C_2$ such that

(1) φ is of degree zero, that is, $\varphi(C_1^i) \subseteq C_2^i$.

(2) φ commutes with the differential, that is, $\varphi \circ d_1 = d_2 \circ \varphi$.

For any complex (C, d) we define the complex "shifted by i places to the left", denoted by $(C[i], d[i])$ as follows:

$$C[i]^j := C^{i+j} \quad \text{and} \quad d[i]^j = (-1)^i \cdot d^{i+j} \tag{2.61}$$

Note that when we refer to a complex, we usually omit the differential, that is, we refer to "the complex C" instead of "the complex (C, d)".

Now we denote the category of all complexes of objects in $\mathcal{A}$ by $\mathcal{C}(\mathcal{A})$, the category of all complexes which are right-bounded (i. e., $C \in \mathcal{C}(\mathcal{A})$ with $C^i = 0$ for all $i \gg 0$) by $\mathcal{C}^-(\mathcal{A})$, the category of all complexes which are left-bounded by $\mathcal{C}^+(\mathcal{A})$ and the category of complexes which are bounded (i. e., both left and right bounded) by $\mathcal{C}^b(\mathcal{A})$.

Definition 2.59 (Zero-Homotopic Maps). *Let (C_1, d_1) and (C_2, d_2) be complexes. We define the* zero-homotopic maps *from C_1 to C_2 to be the maps of the form*

$$d_2 \circ h + h \circ d_1 \tag{2.62}$$

where $h :\ C_1 \longrightarrow C_2$ is any homomorphism of degree -1, that is, $h(C_1^i) \subseteq C_2^{i-1}$. We do not *demand that h commutes with the differential.*

The zero-homotopic maps are by construction of degree zero and commute with the differential. Hence they are homomorphisms of chain complexes. It is also easy to see that the sum of two zero-homotopic maps is again zero-homotopic. Even more, if we have three complexes C_1, C_2 and C_3, together with homomorphisms of chain complexes $f : C_1 \longrightarrow C_2$ and $g : C_2 \longrightarrow C_3$, then $g \circ f$ is zero-homotopic provided that at least one of the maps g and f is zero-homotopic. These properties show that when we factor all zero-homotopic maps out of the homomorphism sets of complexes, composition will still be well defined on the residue classes. This motivates the following definition:

Definition 2.60 (Homotopy Category). *Let $\mathcal{A}$ be an additive category. For all of the categories $\mathcal{C}(\mathcal{A})$, $\mathcal{C}^-(\mathcal{A})$, $\mathcal{C}^+(\mathcal{A})$ and $\mathcal{C}^b(\mathcal{A})$ we define corresponding* homotopy categories $\mathcal{K}(\mathcal{A})$, $\mathcal{K}^-(\mathcal{A})$, $\mathcal{K}^+(\mathcal{A})$ *and* $\mathcal{K}^b(\mathcal{A})$. *The objects of these categories are the same as in the corresponding category of complexes, but we define the set of homomorphisms between any two objects C_1 and C_2 as follows:*

$$\mathrm{Hom}_{\mathcal{K}(\mathcal{A})}(C_1, C_2) := \frac{\mathrm{Hom}_{\mathcal{C}(\mathcal{A})}(C_1, C_2)}{\{ \text{ Zero-homotopic maps from } C_1 \text{ to } C_2 \ \}} \tag{2.63}$$

Definition 2.61 (Mapping Cones & Distinguished Triangles). *Let $\mathcal{A}$ be an additive category. For any two objects $C_1, C_2 \in \mathcal{C}(\mathcal{A})$ and any homomorphism of chain complexes $f : C_1 \longrightarrow C_2$ we define the* mapping cone of f *to be the complex*

$$M(f): \quad \ldots \longrightarrow C_1[1]^{i-1} \oplus C_2^{i-1} \xrightarrow{\begin{bmatrix} d_1[1]^{i-1} & f^i \\ 0 & d_2^{i-1} \end{bmatrix}} C_1[1]^i \oplus C_2^i \longrightarrow \ldots \tag{2.64}$$

So, as a $\mathbb{Z}$-graded object $M(f) = C_1 \oplus C_2[1]$, but the differential is different from the direct sum of d_1 and $d_2[1]$. Namely, it looks as follows (note that applying the shifting functor"$-[1]$", apart from shifting one to the left, multiplies the differential by -1; this is essential to translate (2.64) from above into (2.65) below):

$$d_{M(f)}^{i-1} : \ C_1[1]^{i-1} \oplus C_2^{i-1} \longrightarrow C_1[1]^i \oplus C_2^i : \ (c_1, c_2) \mapsto (-d_1^i(c_1), f^i(c_1) + d_2^{i-1}(c_2)) \tag{2.65}$$

Moreover, the natural embedding $\iota : \ C_2 \hookrightarrow C_1[1] \oplus C_2$ and the projection $\tau : \ C_1[1] \oplus C_2 \longrightarrow C_1[1]$ define homomorphisms of chain complexes when $C_2 \oplus C_1[1]$ is equipped with the differential $d_{M(f)}$. The resulting triple of morphisms

$$C_1 \xrightarrow{f} C_2 \xrightarrow{\iota} M(f) \xrightarrow{\tau} C_1[1] \tag{2.66}$$

is called a distinguished triangle *in $\mathcal{K}(\mathcal{A})$.*

If $\mathcal{A}$ is an additive category, then the categories $\mathcal{K}(\mathcal{A})$, $\mathcal{K}^-(\mathcal{A})$, $\mathcal{K}^+(\mathcal{A})$ and $\mathcal{K}^b(\mathcal{A})$ form so-called *triangulated categories*. The formal definition is fairly tedious; we include it nonetheless for completeness' sake. We should note that for our purposes, most of the axioms listed in Definition 2.63 below should be understood as lemmata on mapping cones (which, technically, would require a proof).

Definition 2.62 (Triangles). *Let $\mathcal{A}$ be an additive category equipped with an auto-equivalence $T : \ \mathcal{A} \longrightarrow \mathcal{A}$. A* triangle *in $\mathcal{A}$ is given by the following data:*

(1) Objects C_1, C_2 and C_3 in $\mathcal{A}$.

(2) Morphisms $f:\ C_1 \longrightarrow C_2$*,* $g:\ C_2 \longrightarrow C_3$ *and* $h:\ C_3 \longrightarrow T(C_1)$*.*

We also write a triangle as follows:

$$C_1 \xrightarrow{f} C_2 \xrightarrow{g} C_3 \xrightarrow{h} T(C_1) \tag{2.67}$$

Given two triangles (C_1, C_2, C_3, f, g, h) *and* $(C_1', C_2', C_3', f', g', h')$*, a homomorphism between the two is given by three homomorphisms* $\varphi_i :\ C_i \longrightarrow C_i'$ *(*$i \in \{1,2,3\}$*) such that the following diagram commutes:*

$$\begin{array}{ccccccc} C_1 & \xrightarrow{f} & C_2 & \xrightarrow{g} & C_3 & \xrightarrow{h} & T(C_1) \\ \downarrow{\scriptstyle \varphi_1} & & \downarrow{\scriptstyle \varphi_2} & & \downarrow{\scriptstyle \varphi_3} & & \downarrow{\scriptstyle T(\varphi_1)} \\ C_1' & \xrightarrow{f'} & C_2' & \xrightarrow{g'} & C_3' & \xrightarrow{h'} & T(C_1') \end{array} \tag{2.68}$$

The triangles over $\mathcal{A}$ *then form an (additive) category themselves, so in particular there is a notion of isomorphism of triangles.*

Definition 2.63 (Triangulated Categories)**.** *A* triangulated category *is an additive category* $\mathcal{D}$ *together with the following data:*

(1) An auto-equivalence $T:\ \mathcal{D} \longrightarrow \mathcal{D}$ *of the category, in case of the homotopy categories given by the shift functor "*$-[1]$*".*

(2) A collection of triangles over $\mathcal{D}$ *called* distinguished triangles*. In the homotopy categories the distinguished triangles are given by the triangles isomorphic to mapping cones as defined above.*

The collection of distinguished triangles needs to satisfy the following axioms:

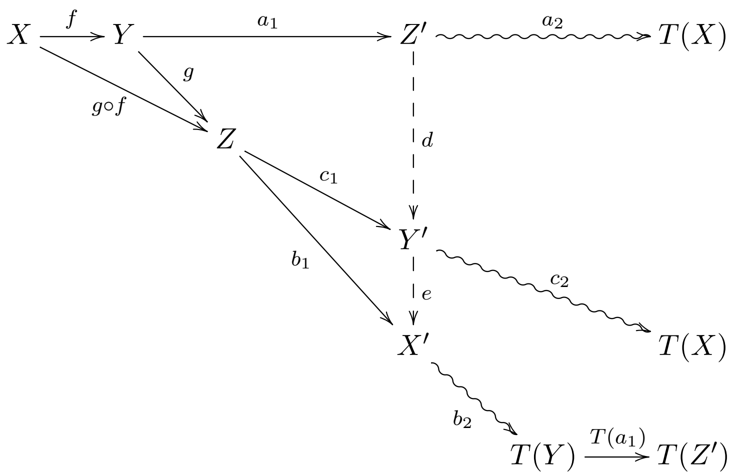
(1) The collection of distinguished triangles is closed under triangle isomorphisms. For each object X *in* $\mathcal{D}$*, the triangle* $(X, X, 0, \mathrm{id}_X, 0, 0)$ *is distinguished. For any two objects* X *and* Y *in* $\mathcal{D}$ *and any morphism* $f:\ X \longrightarrow Y$*, there is an object* Z *as well as morphisms* $g:\ Y \longrightarrow Z$ *and* $h:\ Z \longrightarrow T(X)$ *such that the triangle* (X, Y, Z, f, g, h) *is distinguished.*

(2) A triangle (X, Y, Z, f, g, h) *is distinguished if and only if the triangle* $(Y, Z, T(X), g, h, -T(f))$ *is distinguished.*

(3) If (X, Y, Z, f, g, h) *and* (X', Y', Z', f', g', h') *are two distinguished triangles and* $\varphi :\ X \longrightarrow X'$ *and* $\psi :\ Y \longrightarrow Y'$ *are two morphisms satisfying* $\psi \circ f = f' \circ \varphi$*, then there exists a morphism* $\chi :\ Z \longrightarrow Z'$ *such that the triple* (φ, ψ, χ) *defines a triangle homomorphism.*

(4) If (X, Y, Z', f, a_1, a_2) *and* (Y, Z, X', g, b_1, b_2) *and* $(X, Z, Y', g \circ f, c_1, c_2)$ *are distinguished triangles, then there are morphisms* $d:\ Z' \longrightarrow Y'$ *and* $e:\ Y' \longrightarrow X'$ *such that* (id_X, g, d) *and* (f, id_Z, e) *define triangle homomorphisms and the triangle* $(Z', Y', X', d, e, T(a_1) \circ b_2)$

is distinguished. The following diagram should help understand what this axiom says:



(2.69)

Definition 2.64 (Exact Functors & Equivalences)**.** *Let $\mathcal{C}$ and $\mathcal{D}$ be two triangulated categories. A functor $\mathcal{F}: \mathcal{C} \longrightarrow D$ is called* exact *if it is additive and sends distinguished triangles to distinguished triangles.*

The triangulated categories $\mathcal{C}$ and $\mathcal{D}$ are called equivalent *if there are exact functors $\mathcal{F}: \mathcal{C} \longrightarrow \mathcal{D}$ and $\mathcal{G}: \mathcal{D} \longrightarrow \mathcal{C}$ such that $\mathcal{G} \circ \mathcal{F} \cong \mathrm{id}_{\mathcal{C}}$ and $\mathcal{F} \circ \mathcal{G} \cong \mathrm{id}_{\mathcal{D}}$.*

Technically, we could now call two rings A and B *derived equivalent* if $\mathcal{K}^b(\mathbf{proj}_A)$ and $\mathcal{K}^b(\mathbf{proj}_B)$ are equivalent as triangulated categories. However, derived equivalences are usually defined as being equivalences between the bigger *bounded derived categories* $\mathcal{D}^b(A)$ and $\mathcal{D}^b(B)$. We may then think of $\mathcal{K}^b(\mathbf{proj}_A)$ respectively $\mathcal{K}^b(\mathbf{proj}_B)$ as subcategories of $\mathcal{D}^b(A)$ respectively $\mathcal{D}^b(B)$, which are characteristic in the sense that they are mapped into each other by any equivalence between $\mathcal{D}^b(A)$ and $\mathcal{D}^b(B)$.

Remark 2.65 (Quotients by Null-Systems)**.** *If $\mathcal{C}$ is a triangulated category with translation T and $\mathcal{N}$ is a collection of objects in $\mathcal{C}$ such that*

(1) $0_{\mathcal{C}}$ lies is $\mathcal{N}$.

(2) $\mathcal{N}$ is stable under T and T^{-1}.

(3) If any two terms in a distinguished triangle lie in $\mathcal{N}$, so does the third.

then we call $\mathcal{N}$ a null system *in $\mathcal{C}$. Then there is a category $\mathcal{C}/\mathcal{N}$, unique up to equivalence, and an exact functor $\mathcal{Q}: \mathcal{C} \longrightarrow \mathcal{C}/\mathcal{N}$ with the following universal property:*

(1) $\mathcal{Q}$ maps the objects in $\mathcal{N}$ to $0_{\mathcal{C}/\mathcal{N}}$.

(2) If $\mathcal{G}: \mathcal{C} \longrightarrow \mathcal{D}$ is an exact functor which maps all objects in $\mathcal{N}$ to $0_{\mathcal{D}}$, then $\mathcal{G}$ factors through $\mathcal{Q}$ (up to an isomorphism of functors).

It can be shown that if $\mathcal{A}$ is an abelian category, then the acyclic (that is, exact) complexes form null systems in $\mathcal{K}(\mathcal{A})$, $\mathcal{K}^-(\mathcal{A})$, $\mathcal{K}^+(\mathcal{A})$ and $\mathcal{K}^b(\mathcal{A})$.

Definition 2.66. *Let $\mathcal{A}$ be an abelian category. We define the derived category, right-bounded derived category, left-bounded derived category and bounded derived category of $\mathcal{A}$ as follows:*

$$\begin{array}{rclrcl} \mathcal{D}(A) & := & \mathcal{K}(\mathcal{A})/\mathcal{N} & \mathcal{D}^-(A) & := & \mathcal{K}^-(\mathcal{A})/\mathcal{N}^- \\ \mathcal{D}^+(A) & := & \mathcal{K}^+(\mathcal{A})/\mathcal{N}^+ & \mathcal{D}^b(A) & := & \mathcal{K}^b(\mathcal{A})/\mathcal{N}^b \end{array} \tag{2.70}$$

where $\mathcal{N}, \mathcal{N}^-, \mathcal{N}^+$ and $\mathcal{N}^b$ denote the null-system of acyclic complexes in the respective homotopy categories. If A is a ring, we write $\mathcal{D}(A)$ instead of $\mathcal{D}(\mathbf{Mod}_A)$ (of course we also adopt the analogous notation for the left/right bounded and bounded derived category of a ring).

We call two rings A and B derived equivalent *if $\mathcal{D}^b(A)$ and $\mathcal{D}^b(B)$ are equivalent as triangulated categories.*

Remark 2.67. *(1) The reason why one usually works with the derived category of $\mathbf{Mod}_A$ instead of the one of $\mathbf{mod}_A$ is that for an arbitrary ring A, $\mathbf{mod}_A$ is not necessarily an abelian category. For instance, if F is a field and*

$$A = F\left[x_i | i \in \mathbb{N}\right] / (x_i x_j | i, j \in \mathbb{N}) \quad \text{and} \quad I := (x_i | i \in \mathbb{N})_A \trianglelefteq A \tag{2.71}$$

then, even though A and A/I are finitely generated A-modules, the kernel of the natural epimorphism $A \twoheadrightarrow A/I$ is equal to I which is not finitely generated. So, in this example, the kernel of a homomorphism does not exist in $\mathbf{mod}_A$. Since the ability of taking kernels is a necessary prerequisite for taking homology, we cannot apply the construction of the derived category to $\mathbf{mod}_A$ (in this case).

If, on the other hand, A is noetherian then $\mathbf{mod}_A$ is abelian and hence has a well-defined derived category (and essentially everything we discuss below would work for this so-defined derived category of A as well).

(2) There are full and faithful embeddings of $\mathcal{D}^+(A)$ and $\mathcal{D}^-(A)$ into $\mathcal{D}(A)$ and of $D^b(A)$ into $\mathcal{D}^-(A)$ and $\mathcal{D}^+(A)$. We hence think of $\mathcal{D}^b(A)$, $\mathcal{D}^-(A)$ and $\mathcal{D}^+(A)$ as (full) subcategories of $\mathcal{D}(A)$.

(3) The restriction of the functor $\mathcal{Q} : \mathcal{K}^-(\mathbf{Mod}_A) \longrightarrow \mathcal{D}^-(A)$ to complexes with projective terms:

$$\mathcal{Q}_{\text{res}} : \mathcal{K}^-(\mathbf{Proj}_A) \longrightarrow \mathcal{D}^-(A) \tag{2.72}$$

is an equivalence of triangulated categories. On the left hand side, $\mathcal{D}^b(A)$ corresponds to the full subcategory of complexes with non-trivial homology in only finitely many degrees (which is not *the same as $\mathcal{K}^b(\mathbf{Proj}_A)$). This equivalence is very useful for us since it allows us to compute in the (right bounded) derived category without having to deal with an explicit construction of the derived category.*

The definition of the derived category we gave above is not particularly handy. With this definition it is, for instance, extremely hard to compute homomorphisms between objects (this is however possible, at least for the right bounded derived category, using the third part of the preceding remark). The notion of a quasi-isomorphism we are about to introduce gives us an at least sufficient criterion for when two complexes become isomorphic in the derived category.

Definition 2.68 (Quasi-Isomorphism)**.** *Let A be a ring and let $X, Y \in \mathcal{C}(\mathbf{Mod}_A)$ be two complexes. A homomorphism $f : X \longrightarrow Y$ is called a* quasi-isomorphism *if the induced maps* *on the homology groups of X and Y:*

$$H^i(f):\ H^i(X) \longrightarrow H^i(Y) \tag{2.73}$$

are isomorphisms for all $i \in \mathbb{Z}$. If there is a quasi-isomorphism from X to Y then we call X and Y quasi-isomorphic.

Proposition 2.69. *Let A be a ring and let $X, Y \in \mathcal{C}(\mathbf{Mod}_A)$ be two quasi-isomorphic complexes. Then X and Y become isomorphic when passing to the derived category, that is, $\mathcal{Q}(X) \cong \mathcal{Q}(Y)$.*

A note on the proof. The first axiom for triangulated categories tells us that each homomorphism $f:\ X \longrightarrow Y$ between two objects in the triangulated category is contained in some distinguished triangle. One can show that this triangle is actually unique up to isomorphism (this is not entirely trivial; see [DV77, Proposition 1-2]). Therefore if any term in a triangle is zero and Z denotes another term occurring in that triangle, then the triangle is isomorphic to a shifted version of the triangle constructed from the morphism $0 \longrightarrow Z$, namely $(0, Z, Z, 0, \mathrm{id}_Z, 0)$, by the first axiom. The two (potentially) non-zero terms in the original triangle must therefore be isomorphic. Hence all one has to show is that the mapping cone of a quasi-isomorphism is acyclic (because the acyclic complexes are isomorphic to zero in the derived category), which is just a technical verification. □

Remark 2.70. *(1) Isomorphisms in the homotopy categories are in fact quasi-isomorphisms. The third part of Remark 2.67 can therefore be used to see that we can take homology of objects in the derived category, i. e. there are functors*

$$H^i(-):\ \mathcal{D}(A) \longrightarrow \mathbf{Mod}_A \tag{2.74}$$

such that $H^i(\mathcal{Q}(-))$ is the i-th homology functor on the homotopy category of $\mathbf{Mod}_A$. We could also define $H^i(-)$ directly as $\mathrm{Hom}_{\mathcal{D}(A)}((0 \to A_A \to 0)[i], -)$.

(2) Let M be an A-module with projective resolution $P \in \mathcal{K}^-(\mathbf{Proj}_A)$. This resolution comes together with an epimorphism $P^0 \twoheadrightarrow M$ (if we choose indices in the usual way), and this epimorphism may be construed as a homomorphism from P to the complex $0 \longrightarrow M \longrightarrow 0$. One can easily check that this homomorphism is indeed a quasi-isomorphism. Therefore, a module (viewed as a stalk complex) and its projective resolution are isomorphic in the derived category. This idea can be generalized to make the third part of Remark 2.67 constructive: For an arbitrary right bounded complex we can construct a quasi-isomorphic complex with projective terms by successively taking projective covers of pullbacks (this process is also called "taking a projective resolution").

Theorem 2.71 (Rickard)**.** *Let A and B be rings. The following are equivalent:*

(1) $\mathcal{D}^b(A)$ and $\mathcal{D}^b(B)$ are equivalent as triangulated categories.

(2) $\mathcal{K}^-(\mathbf{Proj}_A)$ and $\mathcal{K}^-(\mathbf{Proj}_B)$ are equivalent as triangulated categories (and thus, of course, also $\mathcal{D}^-(A)$ and $\mathcal{D}^-(B)$).

(3) $\mathcal{K}^b(\mathbf{proj}_A)$ and $\mathcal{K}^b(\mathbf{proj}_B)$ are equivalent as triangulated categories.

Definition 2.72 (Tilting Complex). *Let A be a ring. A complex $T \in \mathcal{K}^b(\mathbf{proj}_A)$ is called a* tilting complex *(or, more precisely, a* one-sided tilting complex*) if*

(1) $\mathrm{Hom}_{\mathcal{K}^b(\mathbf{proj}_A)}(T[i], T) = 0$ *for all* $i \neq 0$.

(2) The smallest (full) triangulated subcategory of $\mathcal{K}^b(\mathbf{proj}_A)$ which contains T and is closed under taking direct summands and direct sums, denoted by $\mathrm{add}(T)$*, is equal to $\mathcal{K}^b(\mathbf{proj}_A)$ itself.*

If T only fulfills the first condition, then it is called a partial tilting complex.

Remark 2.73. *The property of being a tilting complex is preserved under derived equivalences. That is, if A and B are rings, $\mathcal{G} : \mathcal{D}^b(A) \longrightarrow \mathcal{D}^b(B)$ induces an equivalence, and if $T \in \mathcal{K}^b(\mathbf{proj}_A)$ is a tilting complex, then $\mathcal{G}(T)$ is a tilting complex as well. In fact, $\mathcal{G}$ will induce an equivalence $\mathcal{K}^b(\mathbf{proj}_A) \longrightarrow \mathcal{K}^b(\mathbf{proj}_B)$ by restriction.*

Theorem 2.74 (Rickard). *Two rings A and B are derived equivalent if and only if there is a tilting complex $T \in \mathcal{K}^b(\mathbf{proj}_A)$ with $\mathrm{End}_{\mathcal{D}^b(A)}(T) \cong B$.*

Theorem 2.75 (Rickard). *If A is a ring and T is a tilting complex defined over A, then there is an equivalence of triangulated categories*

$$\mathcal{G}_T : \mathcal{D}^b(A) \longrightarrow \mathcal{D}^b(\mathrm{End}_{\mathcal{D}^b(A)}(T)) \tag{2.75}$$

which sends T to $0 \longrightarrow \mathrm{End}_{\mathcal{D}^b(A)}(T) \longrightarrow 0$.

Essentially, one-sided tilting complexes are a generalization of progenerators. If $P \in \mathbf{proj}_A$ is a progenerator, then the stalk complex $0 \to P \to 0$ is a one-sided tilting complex. Also in complete analogy with the Morita equivalence case, a one-sided tilting complex will not fix a unique derived equivalence. To fix this, [Ric91a] introduced *two-sided tilting complexes.* In order to understand those, we need the concept of a *derived tensor product.*

Definition 2.76 (Tensor Product of Complexes). *Let A, B and C be rings. If $X \in \mathcal{C}^-({}_B\,\mathbf{Mod}_A)$ and $Y \in \mathcal{C}^-({}_A\,\mathbf{Mod}_C)$ are two complexes, we define the tensor product of them to be the complex in $\mathcal{C}^-({}_B\,\mathbf{Mod}_C)$ with the following terms:*

$$(X \otimes_A Y)^i := \bigoplus_{z\in\mathbb{Z}} X^{i-z} \otimes_A Y^z \tag{2.76}$$

and the following differential:

$$d^i_{X\otimes_A Y} := \sum_{z\in\mathbb{Z}} d_X^{i-z} \otimes \mathrm{id}_{Y^z} + (-1)^{i-z} \cdot \mathrm{id}_{X^{i-z}} \otimes d^z_{Y^z} \tag{2.77}$$

Given additional complexes X' and Y' as well as morphisms of chain complexes $f : X \longrightarrow X'$ and $g : Y \longrightarrow Y'$, the morphism $f \otimes g$ is simply defined to be the following:

$$(f \otimes g)^i := \sum_{z\in\mathbb{Z}} f^{i-z} \otimes g^z \tag{2.78}$$

Remark 2.77. *The tensor product for complexes defined above distributes over direct sums and is associative (this is simply a technical verification).*

Proposition 2.78. *Let R be a commutative ring and let A, B and C be R-algebras. The above definition of a tensor product of complexes gives rise to a well defined tensor product on the homotopy categories (note that we identify $\mathbf{Mod}_{B^{\mathrm{op}}\otimes_R A}$ with the category of R-linear B-A-bimodules):*

$$-\otimes_A = :\ \mathcal{K}^-(\mathbf{Mod}_{B^{\mathrm{op}}\otimes_R A}) \times \mathcal{K}^-(\mathbf{Mod}_{A^{\mathrm{op}}\otimes_R C}) \longrightarrow \mathcal{K}^-(\mathbf{Mod}_{B^{\mathrm{op}}\otimes_R C}) \tag{2.79}$$

Some notes on the proof. It is easy to verify that if we replace the homotopy categories by the corresponding categories of chain complexes, (2.79) does indeed define an R-bilinear functor. It only remains to show that it is well defined on morphisms (that it is well defined on objects is clear, since the objects of the homotopy category and the objects of the category of chain complexes are the same). This means: If f is a morphism between two objects in $\mathcal{C}^-(\mathbf{Mod}_{B^{\mathrm{op}}\otimes_R A})$ and f' is another morphism homotopic to f, and similarly g is a morphism between two objects in $\mathcal{C}^-(\mathbf{Mod}_{A^{\mathrm{op}}\otimes_R C})$ and g' is another morphism homotopic to g, then $f\otimes g$ is homotopic to $f'\otimes g'$. By R-bilinearity this can be reduced to showing:

(1) If $f:\ X\longrightarrow X'$ is homotopic to zero and $g:\ Y\longrightarrow Y'$ is any morphism, then $f\otimes g:\ X\otimes_A Y\longrightarrow X'\otimes_A Y'$ is homotopic to zero. This can be seen as follows: If $f^i = h^{i-1}\circ d^i_X + d^{i-1}_{X'}\circ h^i$, then $(f\otimes g)^i = \tilde{h}^{i-1}\circ d^i_{X\otimes_A Y} + d^{i-1}_{X'\otimes_A Y'}\circ \tilde{h}^i$ if we put

$$\tilde{h}^i = \sum_{z\in\mathbb{Z}} h^{i-z}\otimes g^z \tag{2.80}$$

(2) If $f:\ X\longrightarrow X'$ is any morphism and $g:\ Y\longrightarrow Y'$ is homotopic to zero, then $f\otimes g:\ X\otimes_A Y\longrightarrow X'\otimes_A Y'$ is homotopic to zero. This can be seen as follows: If $g^i = h^{i-1}\circ d^i_Y + d^{i-1}_{Y'}\circ h^i$, then $(f\otimes g)^i = \tilde{h}^{i-1}\circ d^i_{X\otimes_A Y} + d^{i-1}_{X'\otimes_A Y'}\circ \tilde{h}^i$ if we put

$$\tilde{h}^i = \sum_{z\in\mathbb{Z}} (-1)^{i+1-z}\cdot f^{i-z}\otimes h^z \tag{2.81}$$

Of course all of the assertions we made here require a (very technical) verification. □

Definition 2.79 (Left Derived Functor)**.** *Let A and B be a rings. Recall that there is an equivalence $\mathcal{Q}_{\mathrm{res}}:\ \mathcal{K}^-(\mathbf{Proj}_A)\longrightarrow\mathcal{D}^-(A)$ (see (2.72)), and we denote by $\mathcal{Q}_{\mathrm{res}}^{-1}:\mathcal{D}^-(A)\longrightarrow\mathcal{K}^-(\mathbf{Proj}_A)$ a quasi-inverse, that is, an exact functor such that $\mathcal{Q}_{\mathrm{res}}\circ\mathcal{Q}_{\mathrm{res}}^{-1}\cong\mathrm{id}_{\mathcal{D}^-(A)}$ and $\mathcal{Q}_{\mathrm{res}}^{-1}\circ\mathcal{Q}_{\mathrm{res}}\cong\mathrm{id}_{\mathcal{K}^-(\mathbf{Proj}_A)}$.*

(1) Now let $\mathcal{F}:\ \mathcal{K}^-(\mathbf{Mod}_A)\longrightarrow\mathcal{K}^-(\mathbf{Mod}_B)$ be an exact functor. We call the functor

$$\mathrm{L}\mathcal{F} := \mathcal{Q}\circ\mathcal{F}\circ\mathcal{Q}_{\mathrm{res}}^{-1} \tag{2.82}$$

a (or "the") left derived functor *of $\mathcal{F}$ (unique up to isomorphism of functors). Note that the $\mathcal{Q}$ on the left refers to the functor $\mathcal{K}^-(\mathbf{Mod}_B)\longrightarrow\mathcal{D}^-(B)$, while the $\mathcal{Q}$ on the right refers to the (restriction of) the corresponding functor with B replaced by A.*

(2) If $\mathcal{G}:\ \mathbf{Mod}_A\longrightarrow\mathbf{Mod}_B$ is an additive functor, then it induces an exact functor $\mathcal{K}^-(\mathcal{G}):\ \mathcal{K}^-(\mathbf{Mod}_A)\longrightarrow\mathcal{K}^-(\mathbf{Mod}_B)$. We define

$$\mathrm{L}\mathcal{G} := \mathrm{L}\mathcal{K}^-(\mathcal{G}) \tag{2.83}$$

and call if the left derived functor *of* $\mathcal{G}$.

Remark 2.80. *(1) We can define the right derived functor of an exact functor* $\mathcal{F} : \mathcal{K}^+(\mathbf{Mod}_A) \longrightarrow \mathcal{K}^+(\mathbf{Mod}_B)$ *using a construction analogous to the one outlined above: The functor* $\mathcal{Q} : \mathcal{K}^+(\mathbf{Mod}_A) \longrightarrow \mathcal{D}^+(A)$ *restricts to an equivalence* $\mathcal{Q}_{\mathrm{res}} : \mathcal{K}^+(\mathbf{Inj}_A) \longrightarrow \mathcal{D}^+(A)$ *which has a quasi-inverse* $\mathcal{Q}_{\mathrm{res}}^{-1}$. *The right derived functor of* $\mathcal{F}$ *will be* $\mathcal{Q} \circ \mathcal{F} \circ \mathcal{Q}_{\mathrm{res}}^{-1}$.

(2) The definition of derived functors used here goes back to Verdier (in [DV77]). While the process we outlined to construct them by taking projective or injective resolutions and then applying the original functor is constructive, it may seem to be a somewhat arbitrary definition. Therefore we should note that Verdier also gives a definition of derived functors in terms of a universal property that characterizes them. However, this property is too technical for us to include it here.

Definition 2.81. *Let* A, B *and* C *be* R*-algebras and let (as before)* $\mathcal{Q}_{\mathrm{res}}^{-1}$ *denote the "projective resolution functors". Then we define the* left derived tensor product

$$-\otimes_A^{\mathbb{L}} = : \mathcal{D}^-(B^{\mathrm{op}} \otimes_R A) \times \mathcal{D}^-(A^{\mathrm{op}} \otimes_R C) \longrightarrow \mathcal{D}^-(B^{\mathrm{op}} \otimes_R C) \tag{2.84}$$

as follows:

$$-\otimes_A^{\mathbb{L}} = \quad := \quad \mathcal{Q}(\mathcal{Q}_{\mathrm{res}}^{-1}(-) \otimes_A \mathcal{Q}_{\mathrm{res}}^{-1}(=)) \tag{2.85}$$

With this definition it is clear that, for a complex $Y \in \mathcal{K}^-(\mathbf{Mod}_{A^{\mathrm{op}} \otimes_R C})$, the functor $- \otimes_A^{\mathbb{L}} \mathcal{Q}(Y)$ is actually the left derived functor of the functor $- \otimes_A Y$ (and similarly for the other argument). It is also true that the derived tensor product distributes over direct sums. Many other properties which one would reasonably expect to be true do however require additional assumptions on A, B and C.

Proposition 2.82. *Let* A, A', A'' *and* A''' *be* R*-algebras which are projective as* R*-modules. Then:*

(1) The left derived tensor product is associative:

$$(-\otimes_{A'}^{\mathbb{L}} =)\otimes_{A''}^{\mathbb{L}} \equiv \quad \cong \quad - \otimes_{A'}^{\mathbb{L}} (= \otimes_{A''}^{\mathbb{L}} \equiv) \tag{2.86}$$

as functors from $\mathcal{D}^-(A^{\mathrm{op}} \otimes_R A') \times \mathcal{D}^-(A'^{\mathrm{op}} \otimes_R A'') \times \mathcal{D}^-(A''^{\mathrm{op}} \otimes_R A''')$ *to* $\mathcal{D}^-(A^{\mathrm{op}} \otimes_R A''')$.

(2) Assume furthermore that $B \subseteq A$ *and* $B'' \subseteq A''$ *are* R*-subalgebras such that* A *is projective as a left* B*-module and* A'' *is projective as a right* B''*-module (the most important case is when* $B = B'' = R$*). Then the following diagram commutes:*

$$\begin{array}{ccc} \mathcal{D}^-(A^{\mathrm{op}} \otimes_R A') \times \mathcal{D}^-(A'^{\mathrm{op}} \otimes_R A'') & \xrightarrow{-\otimes_{A'}^{\mathbb{L}} =} & \mathcal{D}^-(A^{\mathrm{op}} \otimes_R A'') \\ \downarrow {\scriptstyle \mathrm{res} \times \mathrm{res}} & & \downarrow {\scriptstyle \mathrm{res}} \\ \mathcal{D}^-(B^{\mathrm{op}} \otimes_R A') \times \mathcal{D}^-(A'^{\mathrm{op}} \otimes_R B'') & \xrightarrow[-\otimes_{A'}^{\mathbb{L}} =]{} & \mathcal{D}^-(B^{\mathrm{op}} \otimes_R B'') \end{array} \tag{2.87}$$

(3) Let $X \in \mathcal{K}^-(A^{\mathrm{op}} \otimes_R A')$ *and* $Y \in \mathcal{K}^-(A'^{\mathrm{op}} \otimes A'')$. *Assume that either the restriction of* X *to a complex of right* A'*-modules lies in* $\mathcal{K}^-(\mathbf{Proj}_{A'})$ *or the restriction of* Y *to left*

A'-modules lies in $\mathcal{K}^-({}_{A'}\,\mathbf{Proj})$. Then

$$\mathcal{Q}(X) \otimes^{\mathbb{L}}_{A'} \mathcal{Q}(Y) \cong \mathcal{Q}(X \otimes_{A'} Y) \tag{2.88}$$

This simplifies the computation of derived tensor products in many cases.

Definition 2.83 (Invertible Complexes). *Let A and B be R-algebras. We call $X \in \mathcal{D}^b(A^{\mathrm{op}} \otimes_R B)$* invertible *if there is a $Y \in \mathcal{D}^b(B^{\mathrm{op}} \otimes_R A)$ such that*

$$X \otimes^{\mathbb{L}}_B Y \cong 0 \longrightarrow {}_AA_A \longrightarrow 0 \quad \textit{and} \quad Y \otimes^{\mathbb{L}}_A X \cong 0 \longrightarrow {}_BB_B \longrightarrow 0 \tag{2.89}$$

Theorem 2.84 (Rickard). *Let A and B be R-algebras which are projective as R-modules. Then A and B are derived equivalent if and only if there is an invertible object X in $\mathcal{D}^b(A^{\mathrm{op}} \otimes_R B)$. Such an X is called a* two-sided tilting complex*, and the functor*

$$- \otimes^{\mathbb{L}}_A X : \ \mathcal{D}^-(A) \longrightarrow \mathcal{D}^-(B) \tag{2.90}$$

affords an equivalence.

Definition 2.85 (Standard Equivalences). *Derived equivalences afforded by tensoring with a two-sided tilting complex are called* standard.

Theorem 2.86 (Rickard). *Let A and B be R-algebras which are projective as R-modules. If $\mathcal{G} : \ \mathcal{D}^-(A) \longrightarrow \mathcal{D}^-(B)$ is an equivalence of triangulated categories, then there is a standard equivalence $- \otimes^{\mathbb{L}}_A X$ such that*

$$\mathcal{G}(C) \cong C \otimes^{\mathbb{L}}_A X \quad \textit{for all objects } C \in \mathcal{D}^-(A) \tag{2.91}$$

We say that $\mathcal{G}$ and $- \otimes^{\mathbb{L}}_A X$ "agree on objects".

Corollary 2.87. *(1) If $X \in \mathcal{D}^b(A^{\mathrm{op}} \otimes_R B)$ is a two-sided tilting complex, then the restriction of X to right B-modules is a one-sided tilting complex (more precisely: there is a one-sided tilting complex $T \in \mathcal{K}^b(\mathbf{proj}_B)$ such that $T \cong X|_B$ in $\mathcal{D}^b(B)$).*

(2) If $T \in \mathcal{K}^b(\mathbf{proj}_B)$ is a one-sided tilting complex with endomorphism ring isomorphic to A, then there is some two-sided tilting complex $X \in \mathcal{D}^b(A^{\mathrm{op}} \otimes_R B)$ such that $X|_B$ is isomorphic to T in $\mathcal{D}^b(B)$.

2.5 Automorphism Groups and Derived Equivalences

In this section we are going to discuss a few more recent results on derived equivalences. Most of these are generalizations of theorems on Morita equivalences, but the proofs are usually much harder than in the Morita case.

Definition 2.88. *Let $R \in \{\mathcal{O}, k\}$, and let A be any R-algebra that is free and finitely generated as an R-module (i. e. an order if $R = \mathcal{O}$). Then define the* Picard group *of A as follows:*

$$\mathrm{Pic}_R(A) := \{ \textit{ Isomorphism classes of invertible } A^{\mathrm{op}} \otimes_R A\textit{-modules } \} \tag{2.92}$$

$\mathrm{Pic}_R(A)$ *is a group with "$-\otimes_A =$" as its product.*

Similarly define the derived Picard group *of A as follows:*

$$\mathrm{TrPic}_R(A) := \{ \text{ Isomorphism classes of invertible objects in } \mathcal{D}^b(A^{\mathrm{op}} \otimes_R A) \ \} \tag{2.93}$$

$\mathrm{TrPic}_R(A)$ is a group with "$-\otimes^{\mathbb{L}}_A =$" as its product.

Notation 2.89. *Let A, B and B' be rings. Let $\alpha : B \to A, \beta : B' \to A$ be ring homomorphisms. Then define ${}_\alpha A_\beta$ to be the B-B'-bimodule which is (as a set) equal to A with the action*

$$B \times A \times B' \longrightarrow A : \ (b, x, b') \mapsto \alpha(b) \cdot x \cdot \beta(b') \tag{2.94}$$

Remark 2.90. *Situation as in Definition 2.88. There is a group homomorphism*

$$(\mathrm{Aut}_R(A), \circ) \to (\mathrm{Pic}_R(A), \otimes_A) : \ \alpha \mapsto {}_{\mathrm{id}}A_\alpha \cong {}_{\alpha^{-1}}A_{\mathrm{id}} \tag{2.95}$$

The kernel of this homomorphism consists of all inner automorphisms of A, and we denote its image by $\mathrm{Out}_R(A)$. However, by abuse of notation, we will not always distinguish between elements of $\mathrm{Out}_R(A)$ and arbitrarily chosen preimages in $\mathrm{Aut}_R(A)$. In case $R = k$ is an algebraically closed field, $\mathrm{Out}_k(A)$ is a linear algebraic group defined over k, and we denote its connected component by $\mathrm{Out}^0_k(A)$.

Remark 2.91. *Situation as above. If $X \in \mathrm{Pic}_R(A)$, then X is projective and finitely generated as a left A-module and as a right A-module. Hence if $P \in \mathbf{proj}_A$, then $P \otimes_A X$ is again in $\mathbf{proj}_A$. If P is indecomposable, then so is $P \otimes_A X$, since X is invertible. This implies that there is a group homomorphism from the Picard group of A into the symmetric group $\Sigma_{\mathcal{P}}$ on $\mathcal{P}$*

$$\mathrm{Pic}_R(A) \longrightarrow \Sigma_{\mathcal{P}} : \ X \mapsto [P \mapsto P \otimes_A X] \tag{2.96}$$

where $\mathcal{P}$ is the set of all isomorphism classes of finitely generated projective indecomposable A-modules. Define $\mathrm{Pic}^s_R(A)$ to be the kernel of this group homomorphism, and $\mathrm{Out}^s_R(A)$ to be the intersection of $\mathrm{Pic}^s_R(A)$ with $\mathrm{Out}_R(A)$. Define $\mathrm{Aut}^s_R(A)$ to be the preimage of $\mathrm{Out}^s_R(A)$ under the canonical epimorphism $\mathrm{Aut}_R(A) \twoheadrightarrow \mathrm{Out}_R(A)$.

Remark 2.92. *Situation as above. Then we get a series of embeddings*

$$\mathrm{Out}_R(A) \hookrightarrow \mathrm{Pic}_R(A) \hookrightarrow \mathrm{TrPic}_R(A) \tag{2.97}$$

Notation 2.93. *If A is a ring and $T \in \mathcal{K}^b(\mathbf{proj}_A)$ is a tilting complex with endomorphism ring B, then we denote by*

$$\mathcal{G}_T : \ \mathcal{D}^b(A) \xrightarrow{\sim} D^b(B) \tag{2.98}$$

an equivalence which agrees on objects with taking T-resolutions. For proper definitions and proof of the existence of such an equivalence we refer the reader to [Ric89]. A particular property of $\mathcal{G}_T$ is that is sends T to $0 \to B \to 0$.

Remark 2.94. *Let Λ be an $\mathcal{O}$-order. The functor $k \otimes_{\mathcal{O}} - : \mathbf{Mod}_\Lambda \longrightarrow \mathbf{Mod}_{k\otimes\Lambda}$ has a (unique) left-derived functor $k \otimes^{\mathbb{L}}_{\mathcal{O}} - : \mathcal{D}^-(\Lambda) \longrightarrow \mathcal{D}^-(k \otimes \Lambda)$, which restricts to a functor from $\mathcal{K}^b(\mathbf{proj}_\Lambda)$ to $\mathcal{K}^b(\mathbf{proj}_{k\otimes\Lambda})$. For a $C \in \mathcal{K}^b(\mathbf{proj}_\Lambda)$, $k \otimes^{\mathbb{L}}_{\mathcal{O}} C$ is obtained by simply applying $k \otimes_{\mathcal{O}} -$ to this complex viewed as a sequence of modules. Hence, for objects $C \in \mathcal{K}^b(\mathbf{proj}_\Lambda)$, there is no harm in writing $k \otimes_{\mathcal{O}} C$ instead of $k \otimes^{\mathbb{L}}_{\mathcal{O}} C$.*

Remark 2.95. *Let A and B be R-algebras and let $\mathcal{F}: \mathcal{D}^b(A) \longrightarrow \mathcal{D}^b(B)$ be an equivalence that sends the stalk complex $0 \to A \to 0$ to $0 \to B \to 0$. Then there is an $\alpha: A \xrightarrow{\sim} B$ such that $\mathcal{F}(X) \cong X \otimes^{\mathbb{L}}_A {}_\alpha B_{\mathrm{id}}$ for all objects $X \in \mathcal{D}^b(A)$. This follows from [Ric89, Proposition 7.1].*

Proposition 2.96. *Let A be a finite-dimensional k-algebra and $T \in \mathcal{K}^b(\mathbf{proj}_A)$ a tilting complex with endomorphism ring B. Then there exists a two-sided tilting complex $X \in \mathcal{D}^b(B^{\mathrm{op}} \otimes_k A)$ with restriction to $\mathcal{D}^b(A)$ isomorphic to T.*

Proof. By [Ric89], there exists a functor $\mathcal{F}: \mathcal{D}^b(B) \longrightarrow \mathcal{D}^b(A)$ sending $0 \to B \to 0$ to T. By [Ric91a, Corollary 3.5] this equivalence is afforded by $\mathrm{RHom}_B(Y, -)$ for some $Y \in \mathcal{D}^b(A^{\mathrm{op}} \otimes B)$. This Y has an inverse $X \in \mathcal{D}^b(B^{\mathrm{op}} \otimes A)$ such that $\mathrm{RHom}_B(Y, -) \cong - \otimes^{\mathbb{L}}_B X$ (see [Ric91a, Definition 4.2] and the remarks following it). Since $B \otimes^{\mathbb{L}}_B X \cong \mathcal{F}(B) \cong T$, X has the desired properties. □

Proposition 2.97. *Let A be a finite-dimensional symmetric k-algebra and let*

$$T = 0 \to P_1 \to P_0 \to 0 \tag{2.99}$$

be a two-term tilting complex. Then $\mathcal{G}_T(0 \to A \to 0)$ is again a two-term tilting complex.

Proof. Set $B := \mathrm{End}_{\mathcal{D}^b(A)}(T)$. Let $X \in \mathcal{D}^b(B^{\mathrm{op}} \otimes A)$ be a two-sided tilting complex with restriction to $\mathcal{D}^b(A)$ isomorphic to T. Let $Y \in \mathcal{D}^b(A^{\mathrm{op}} \otimes B)$ be the inverse of X. By [KZ98, Lemma 9.2.6] we may assume that X is a bounded complex of A-B-bimodules that become projective upon restriction to A and restriction to B. We may then furthermore assume that $Y = \mathrm{Hom}_k(X, k)$ (see [KZ98, Corollary 9.2.5]; Note that both Lemma 9.2.6 and Corollary 9.2.5 in [KZ98] use that A is symmetric). Hence Y has non-vanishing homology in precisely two adjacent degrees, since the same can be said about X and $\mathrm{Hom}_k(-, k)$ is exact on vector spaces. Now $- \otimes^{\mathbb{L}}_A Y$ sends T to $0 \to B \to 0$, which implies that for some automorphism $\gamma: B \to B$ the functor $- \otimes^{\mathbb{L}}_A Y \otimes^{\mathbb{L}}_B {}_{\mathrm{id}}B_\gamma$ agrees with $\mathcal{G}_T(-)$ on objects. Hence the image of $0 \to A \to 0$ under $\mathcal{G}_T(-)$ is equal to the restriction $Y \otimes^{\mathbb{L}}_B {}_{\mathrm{id}}B_\gamma$ to $\mathcal{D}^b(B)$. Therefore it is a bounded complex of projective B-modules that has non-zero homology (at most) in two (adjacent) degrees. Since A and B are symmetric (so in particular self-injective), any injection of a projective module and any epimorphism onto a projective module splits. Hence $Y \otimes^{\mathbb{L}}_B {}_{\mathrm{id}}B_\gamma$ is isomorphic in $\mathcal{K}^b(\mathbf{proj}_B)$ to a two-term complex. □

Theorem 2.98 (Rouquier, Huisgen-Zimmermann, Saorín)**.** *Assume k is algebraically closed. Let A and B be finite-dimensional k-algebras and X a bounded complex of A-B-bimodules inducing an equivalence between $\mathcal{D}^b(A)$ and $\mathcal{D}^b(B)$ (i. e., a two-sided tilting complex). Then there exists a (unique) isomorphism of algebraic groups*

$$\sigma: \mathrm{Out}^0_k(A) \xrightarrow{\sim} \mathrm{Out}^0_k(B) \tag{2.100}$$

such that

$${}_{\mathrm{id}}A_\alpha \otimes^{\mathbb{L}}_A X \cong X \otimes^{\mathbb{L}}_B {}_{\mathrm{id}}B_{\sigma(\alpha)} \tag{2.101}$$

for all $\alpha \in \mathrm{Out}^0_k(A)$.

Proof. The Theorem was stated in this form in [Rou06, Theorem 3.4]. A proof can be found in [HZS01] or in [Rou]. □

Theorem 2.99 (Jensen, Su, Zimmermann). *Let A be a finite-dimensional k-algebra. Then up to isomorphism in $\mathcal{K}^b(\mathbf{proj}_A)$ there exists at most one two-term (partial) tilting complex*

$$0 \to P_1 \to P_0 \to 0 \tag{2.102}$$

with fixed homogeneous components P_0 and P_1.

Proof. See [JSZ05, Corollary 8]. □

Corollary 2.100. *Assume k is algebraically closed. Let A be a finite-dimensional k-algebra and T a tilting complex over A. Then*

(1) $T \otimes_A {}_{\mathrm{id}}A_\gamma \cong T$ for all $\gamma \in \mathrm{Out}_k^0(A)$.

(2) If T is a two-term complex, then $T \otimes_A {}_{\mathrm{id}}A_\gamma \cong T$ for all $\gamma \in \mathrm{Out}_k^s(A)$.

Proof. The first point follows from Theorem 2.98 and Proposition 2.96. The second point follows from Theorem 2.99 and the definition of $\mathrm{Out}_k^s(A)$. □

2.6 Decomposition Numbers 0 and 1

In this section we let A be a finite-dimensional semisimple K-algebra, and assume that K is complete. We give a summary of the theory of *graduated orders* in such algebras, which is a well-understood class of orders closely related to orders for which all decomposition numbers are 0 or 1. The case that a block has decomposition numbers 0 and 1 seems to arise fairly often in blocks with abelian (and other "small") defect groups. Hence the structure theory laid out below can be applied in a large number of cases. Unfortunately though, there appears to be no good group theoretical criterion to decide whether the decomposition numbers of a given block actually are bounded by one. The only sizable class of blocks for which this property is known to hold in general are blocks of cyclic defect.

Definition 2.101. *A full $\mathcal{O}$-order $\Lambda \subset A$ is called graduated if there are idempotents $e_1, \ldots, e_n$ of A such that*

(1) $e_1 + \ldots + e_n = 1$ and $e_i \cdot e_j = 0$ for $i \neq j$.

(2) The e_i are primitive idempotents in A.

(3) Each $e_i \Lambda e_i$ is a maximal order in $e_i A e_i$.

Remark 2.102. *The condition that $e_i \Lambda e_i$ should be a maximal order becomes trivial in at least one important case: when K splits A. Namely, if K splits A, then $e_i A e_i \cong K$ for all i, by virtue of the e_i's being primitive in A. Then $e_i \Lambda e_i$ must be isomorphic to $\mathcal{O}$ (which is a maximal order) due to the fact that $\mathcal{O}$ is the only $\mathcal{O}$-order in K.*

Proposition 2.103. *Let Γ be an $\mathcal{O}$-order with semisimple K-span $B := K \otimes \Gamma$. Let $\varepsilon \in Z(B)$ be a central primitive idempotent and let V be the associated simple B-module (that is, the one with $V \cdot \varepsilon \neq 0$). If $e \in \Gamma$ is a primitive idempotent, then $\varepsilon \cdot e$ is primitive in εB if and only if V occurs with multiplicity 1 in $K \otimes e\Gamma$.*

The preceding proposition sheds some light on what exactly the connection between graduated orders and decomposition numbers 0 and 1 is. The following is an easily applicable corollary to this proposition:

Corollary 2.104. *Let Γ be an $\mathcal{O}$-order with semisimple and split K-span $B := K \otimes \Gamma$. Define $\varepsilon_1, \ldots, \varepsilon_l$ to be the central primitive idempotents of B. If all decomposition numbers of Γ are $\leqslant 1$, then*

$$\bigoplus_{i=1}^{l} \varepsilon_i \Gamma \tag{2.103}$$

is a graduated order (which is called the "graduated hull" of Γ).

Proposition 2.105. *Let $\varepsilon_1, \ldots, \varepsilon_l$ be the central primitive idempotents of A. Λ is a graduated order in A if and only if $\varepsilon_i \Lambda$ is a graduated order in $\varepsilon_i A$ for all i and*

$$\Lambda = \bigoplus_{i=1}^{l} \varepsilon_i \Lambda \tag{2.104}$$

The above proposition shows that graduated orders in semisimple algebras are simply direct sums of graduated orders in the Wedderburn components of the semisimple algebra. Therefore it suffices to understand graduated orders in (finite-dimensional) simple K-algebras.

From now on let D be a skew-field of finite dimension over K and let Θ be the a maximal order in D (unique up to conjugation by Remark 2.7). We set $\Pi := \mathrm{Jac}(\Theta)$. Then every fractional Θ-ideal in D is of the form Π^z for some $z \in \mathbb{Z}$.

Proposition 2.106. *Let $A = D^{n\times n}$ and denote by e_{ij} the (i,j)-matrix unit in A. Denote by Θ a maximal $\mathcal{O}$-order in D and by Π its maximal ideal. Then any full graduated $\mathcal{O}$-order Λ in A is conjugate to an order of the form*

$$\Lambda = \bigoplus_{i=1}^{n} \bigoplus_{j=1}^{n} \Pi^{\hat{m}_{i,j}} \cdot e_{ij} \tag{2.105}$$

for certain $\hat{m}_{i,j} \in \mathbb{Z}_{\geqslant 0}$.

By a further conjugation with a permutation matrix we may moreover achieve that all of the following holds: There is an integer v, a vector $d \in \mathbb{Z}_{>0}^{v}$ and a matrix $m \in \mathbb{Z}_{\geqslant 0}^{v\times v}$ such that $\sum_{i=1}^{v} d_i = n$ and

$$\hat{m}_{i,j} = m_{s,t} \quad \forall i,j \text{ with } \sum_{l=1}^{s-1} d_l < i \leqslant \sum_{l=1}^{s} d_l \text{ and } \sum_{l=1}^{t-1} d_l < j \leqslant \sum_{l=1}^{t} d_l \tag{2.106}$$

where the matrix m satisfies

(i) $m_{i,j} + m_{j,k} \geqslant m_{i,k} \quad \forall i,j,k \in \{1, \ldots, v\}$

(ii) $m_{i,j} + m_{j,i} > 0 \quad \forall i,j \in \{1, \ldots, v\}$ *with* $i \neq j$

(iii) $m_{i,i} = 0 \quad \forall i \in \{1, \ldots, v\}$

Definition 2.107 (Exponent Matrix & Dimension Vector)**.** *Situation as in Proposition 2.106 We call the matrix m an* exponent matrix *for Λ. We call the vector d a* dimension vector *for Λ.*

Moreover, given a $v \in \mathbb{N}$, we call a matrix $\mathbb{Z}_{\geqslant 0}^{v\times v}$ an exponent matrix *if it satisfies conditions (i), (ii) and (iii) in Proposition 2.106.*

Definition 2.108. *For any $v \in \mathbb{N}$, $d \in \mathbb{Z}_{>0}^{v}$ and any exponent matrix $m \in \mathbb{Z}_{\geqslant 0}^{v\times v}$ we define the graduated order $\Lambda(\Theta, m, d)$ as follows:*

$$\Lambda(\Theta, m, d) := ((\Pi^{m_{i,j}})^{d_i\times d_j})_{i,j} \subset D^{\sum_h d_h \times \sum_h d_h} \tag{2.107}$$

It follows immediately from the definition that m is an exponent matrix for the order $\Lambda(\Theta, m, d)$. The following remark follows immediately from Proposition 2.106 and Definition 2.108:

Remark 2.109. *Let $A = D^{n\times n}$ and let $\Lambda \subset A$ be a (full) graduated order in A. If $m \in \mathbb{Z}^{v\times v}$ is an exponent matrix for Λ and $d \in \mathbb{Z}_{\geqslant 0}^{v}$ is a dimension vector for Λ (where v is some positive number) then*

$$\Lambda \cong \Lambda(\Theta, m, d) \tag{2.108}$$

So far we have parametrized the isomorphism classes of graduated orders in simple finite dimensional K-algebras by a $v \in \mathbb{N}$, an exponent matrix $m \in \mathbb{Z}^{v\times v}$ and a dimension vector $d \in \mathbb{Z}_{>0}^{v}$. We will now have a closer look at how these parameters relate to properties of the order $\Lambda(\Theta, m, d)$. What we will see is, at least in the case $\Theta = \mathcal{O}$, that v and d specify number and k-dimension of the simple Λ-modules and m determines the Morita equivalence class of $\Lambda(\Theta, m, d)$ (though different exponent matrices may very well lead to orders which are in the same Morita equivalence class). If $\Theta \supsetneqq \mathcal{O}$ then the interpretation of the dimension vector is similar but slightly more technical.

Proposition 2.110. *Let $v \in \mathbb{N}$, let $d \in \mathbb{Z}_{>0}^{v}$ be a dimension vector and let $m \in \mathbb{Z}_{\geqslant 0}^{v\times v}$ be an exponent matrix.*

(1) $\Lambda(\Theta, m, d)$ has exactly v isomorphism classes of simple modules.

(2) Let $n = \sum_i d_i$ and let $e_1, \ldots, e_n$ be the primitive diagonal idempotents in $\Lambda(\Theta, m, d)$. Given $i, j \in \{1, \ldots, n\}$ we have

$$e_i \cdot \Lambda(\Theta, m, d) \cong e_j \cdot \Lambda(\Theta, m, d) \tag{2.109}$$

if and only if

$$\sum_{l=1}^{h-1} d_l < i, j \leqslant \sum_{l=1}^{h} d_l \text{ for some } h \in \{1, \ldots, v\} \tag{2.110}$$

Denote (for the rest of this proposition) the PIM $e_i \cdot \Lambda(\Theta, m, d)$ where i satisfies the inequalities of (2.110) for a specific $h \in \{1, \ldots, v\}$ by P_h. Also, denote the simple module $P_h/\operatorname{Rad}(P_h)$ by S_h.

(3) For all h we have

$$\dim_k(S_h) = d_h \cdot \dim_k \Theta/\Pi \tag{2.111}$$

(4) $\Lambda(\Theta, m, d)$ is basic if and only if $d = (1, \ldots, 1)$.

(5) *Let* $m' \in \mathbb{Z}_{\geqslant 0}^{v \times v}$ *be another exponent matrix and let* $d' \in \mathbb{Z}_{>0}^{v}$ *be another dimension vector. Let* $S'_1, \ldots, S'_v$ *be the simple* $\Lambda(\Theta, m', d')$*-modules (with the natural choice of indices). Then there is a Morita equivalence*

$$\mathcal{F}: \ \mathbf{mod}_{\Lambda(\Theta,m,d)} \longrightarrow \mathbf{mod}_{\Lambda(\Theta,m',d')} \quad \text{with} \quad \mathcal{F}(S_h) \cong S'_h \quad \text{for all } h \in \{1, \ldots, v\} \tag{2.112}$$

if and only if

$$m_{i,j} + m_{j,h} - m_{i,h} = m'_{i,j} + m'_{j,h} - m'_{i,h} \quad \text{for all } i, j, h \in \{1, \ldots, v\} \tag{2.113}$$

Remark 2.111. *The preceding proposition tells us in particular that by choosing an exponent matrix* $m \in \mathbb{Z}^{v \times v}$ *for a graduated order* Λ*, we also fix a bijection*

$$\{1, \ldots, v\} \leftrightarrow \{ \text{ Isomorphism classes of simple } \Lambda\text{-modules } \} \tag{2.114}$$

Remark 2.112. *From Proposition 2.110 we can easily see that if an order* Λ *is graduated, then its projective indecomposables are* irreducible lattices*, which by definition means that their* K*-span is a simple* $K \otimes \Lambda$*-module. Irreducible lattices are easily seen to have no proper (i.e. non-bijective) epimorphisms onto other lattices. Since all modules with simple top are epimorphic images of the projective cover of said simple top, it follows that the projective indecomposable lattices of* Λ *can be recognized among all lattices solely by the fact that they have simple top.*

Definition 2.113 (Zassenhaus Invariants)**.** *Given an exponent matrix* $m \in \mathbb{Z}_{\geqslant 0}^{v \times v}$ *we call the numbers*

$$m_{i,j,h} := m_{i,j} + m_{j,h} - m_{i,h} \quad \text{for } i, j, h \in \{1, \ldots, v\} \tag{2.115}$$

the Zassenhaus invariants *associated to* m*.*

Conversely we call a collection of non-negative integers

$$\{m_{i,j,h}\}_{i,j,h \in \{1,\ldots,v\}} \tag{2.116}$$

a set of Zassenhaus invariants if the following are satisfied:

(1) $m_{i,j,h} + m_{i,h,l} = m_{i,j,l} + m_{j,h,l}$ *for all* $i, j, h, l \in \{1, \ldots, v\}$

(2) $m_{i,i,i} = 0$ *for all* $i \in \{1, \ldots, v\}$

(3) $m_{i,j,i} > 0$ *for all* $i \neq j \in \{1, \ldots, v\}$

Remark 2.114. *We can interpret the Zassenhaus invariants of* $\Lambda = \Lambda(\Theta, m, d)$ *as follows:*

$$m_{i,j,l} = \frac{1}{\text{length}_{\mathcal{O}} \Theta/\Pi} \cdot \text{length}_{\mathcal{O}} \frac{e_i \Lambda e_l}{e_i \Lambda e_j \Lambda e_l} \tag{2.117}$$

where $e_1, \ldots, e_v$ *are a set of orthogonal primitive idempotents in* Λ *such that* $e_i\Lambda \cong P_i$ *(with* P_i *as defined in Proposition 2.110). We can also identify*

$$\frac{e_i \Lambda e_l}{e_i \Lambda e_j \Lambda e_l} \cong \frac{\text{Hom}_{\Lambda}(P_l, P_i)}{\text{Hom}_{\Lambda}(P_j, P_i) \circ \text{Hom}_{\Lambda}(P_l, P_j)} \tag{2.118}$$

which makes it fairly obvious that the Zassenhaus invariants are indeed invariants of the Morita equivalence class of Λ.

Proposition 2.115. *Let $\{m_{i,j,h}\}_{i,j,h}$ be a set of Zassenhaus invariants. Then*

$$m_{i,j} := m_{1,i,j} \tag{2.119}$$

defines an exponent matrix with Zassenhaus invariants $m_{i,j,h}$. It is distinguished among those exponent matrices with associated Zassenhaus invariants $\{m_{i,j,h}\}_{i,j,h}$ by the fact that its first row consists exclusively of zeros.

Proposition 2.116. *Let $v \in \mathbb{N}$, $d, d' \in \mathbb{Z}^v_{>0}$ and let $m, m' \in \mathbb{Z}^{v\times v}_{\geqslant 0}$ be exponent matrices. Denote by $\{m_{i,j,h}\}_{i,j,h}$ respectively $\{m'_{i,j,h}\}_{i,j,h}$ the Zassenhaus invariants associated to m respectively m'. Then*

(1) $\Lambda(\Theta, m, d)$ is Morita equivalent to $\Lambda(\Theta, m', d')$ if and only if there is a permutation $\sigma \in \Sigma_v$ such that

$$m_{i,j,h} = m'_{\sigma(i),\sigma(j),\sigma(h)} \quad \text{for all } i, j, h \in \{1, \ldots, v\} \tag{2.120}$$

(2) $\Lambda(\Theta, m, d)$ is isomorphic to $\Lambda(\Theta, m', d')$ if and only if there is a permutation $\sigma \in \Sigma_v$ such that

$$m_{i,j,h} = m'_{\sigma(i),\sigma(j),\sigma(h)} \quad \text{and} \quad d_i = d'_{\sigma(i)} \quad \text{for all } i, j, h \in \{1, \ldots, v\} \tag{2.121}$$

We now look at a simple example to illustrate many of the concepts we have seen so far in this section.

Example 2.117. *Let $\mathcal{O} = \Theta = \mathbb{Z}_3$ and $v = 3$. Then the graduated order $\Lambda(\mathbb{Z}_3, m, d)$ with*

$$m = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 0 & 1 \\ 2 & 1 & 0 \end{bmatrix} \qquad d = [2, 1, 1] \tag{2.122}$$

looks as follows:

$$\Lambda(\mathbb{Z}_3, m, d) = \begin{bmatrix} \mathbb{Z}_3 & \mathbb{Z}_3 & (3) & (9) \\ \mathbb{Z}_3 & \mathbb{Z}_3 & (3) & (9) \\ (3) & (3) & \mathbb{Z}_3 & (3) \\ (9) & (9) & (3) & \mathbb{Z}_3 \end{bmatrix} \tag{2.123}$$

$\operatorname{Jac}(\Lambda(\mathbb{Z}_3, m, d))$ *and* $\Lambda(\mathbb{Z}_3, m, d)/\operatorname{Jac}(\Lambda(\mathbb{Z}_3, m, d))$ *look as follows:*

$$\operatorname{Jac}(\Lambda(\mathbb{Z}_3, m, d)) = \begin{bmatrix} (3) & (3) & (3) & (9) \\ (3) & (3) & (3) & (9) \\ (3) & (3) & (3) & (3) \\ (9) & (9) & (3) & (3) \end{bmatrix} \quad \frac{\Lambda(\mathbb{Z}_3, m, d)}{\operatorname{Jac}(\Lambda(\mathbb{Z}_3, m, d))} = \begin{bmatrix} \mathbb{F}_3 & \mathbb{F}_3 & & \\ \mathbb{F}_3 & \mathbb{F}_3 & & \\ & & \mathbb{F}_3 & \\ & & & \mathbb{F}_3 \end{bmatrix} \tag{2.124}$$

The simple $\Lambda(\mathbb{Z}_3, m, d)$-modules look as follows:

$$\begin{aligned} S_1 &= [\ \mathbb{F}_3 \ \ \mathbb{F}_3 \ \ 0 \ \ 0\ ] \\ S_2 &= [\ 0 \ \ 0 \ \ \mathbb{F}_3 \ \ 0\ ] \\ S_3 &= [\ 0 \ \ 0 \ \ 0 \ \ \mathbb{F}_3\ ] \end{aligned} \tag{2.125}$$

where $\Lambda(\mathbb{Z}_3, m, d)$ acts by "matrix multiplication" from the right.

Definition 2.118 (Involution)**.** *Let Λ be an $\mathcal{O}$-order. We call a bijective $\mathcal{O}$-linear map*

$$-^\circ : \ \Lambda \longrightarrow \Lambda \tag{2.126}$$

such that

(1) $a^{\circ\circ} = a$ for all $a \in \Lambda$

(2) $(a \cdot b)^\circ = b^\circ \cdot a^\circ$ for all $a, b \in \Lambda$

an involution *on Λ.*

An involution also defines mutually inverse equivalences between the categories of left and right Λ-modules, which we also denote by $-^\circ$: If M is a left respectively right Λ-module, with action homomorphism

$$\varphi : \ \Lambda \longrightarrow \mathrm{End}_{\mathcal{O}}(M) \tag{2.127}$$

then we can turn M into a right respectively left Λ-module by taking $\varphi(-^\circ)$ as the new action homomorphism. On homomorphisms of Λ-modules, the functor simply induces the identity map.

We are interested in involutions on orders simply for the reason that group rings carry an involution:

$$-^\circ : \ \mathcal{O}G \longrightarrow \mathcal{O}G : \ \sum_{g\in G} a_g \cdot g \mapsto \sum_{g \in G} a_g \cdot g^{-1} \tag{2.128}$$

Quite surprisingly it turns out that if $\mathcal{O}$ is complete and $p \neq 2$ then involutions on $\mathcal{O}$-orders are compatible with certain Morita equivalences. In the case of an order with semisimple K-span and decomposition numbers zero and one they even constrain what an exponent matrix may look like.

Theorem 2.119 ([AN02, Corollary 24])**.** *Assume $\mathcal{O}$ is complete and k is finite with $\mathrm{char}(k) \neq 2$. Let Λ be an $\mathcal{O}$-order that carries an involution $^\circ : \Lambda \longrightarrow \Lambda$. Then there is a full set of primitive pairwise orthogonal idempotents $e_1, \ldots, e_n \in \Lambda$ and an involution $\sigma \in \Sigma_n$ (here, involution simply means that $\sigma^2 = \mathrm{id}$) such that $e_i^\circ = e_{\sigma(i)}$.*

Part of the proof. $\Lambda/\mathrm{Jac}(\Lambda)$ is a direct sum of matrix rings over finite field extensions k_i/k (since finite skew-fields are necessarily commutative):

$$\Lambda/\mathrm{Jac}(\Lambda) \cong \bigoplus_{i=1}^{v} k_i^{d_i \times d_i} \quad \text{for some } v \in \mathbb{N} \text{ and certain } d_1, \ldots, d_v \in \mathbb{N} \tag{2.129}$$

It is easily seen that $\mathrm{Jac}(\Lambda)^\circ = \mathrm{Jac}(\Lambda)$, and therefore $-^\circ$ induces an involution on $\Lambda/\mathrm{Jac}(\Lambda)$ as well. On the right hand side of (2.129) this involution has to be of the form

$$(A_1, \ldots, A_v) \mapsto (X_1^{-1} \cdot A_{\tau(1)}^\top \cdot X_1, \ldots, X_v^{-1} \cdot A_{\tau(v)}^\top \cdot X_v) \tag{2.130}$$

for some involution $\tau \in \Sigma_v$ (with $k_i = k_{\tau(i)}$ and $d_i = d_{\tau(i)}$ for all i) and certain symmetric matrices $X_i \in k_i^{d_i \times d_i}$ uniquely determined up to scalar multiples by the isomorphism chosen between left hand side and right hand side in (2.129). Conjugating the involution with the inner automorphism of $\bigoplus_i k_i^{d_i \times d_i}$ induced by $(S_1, \ldots, S_v) \in \prod_i \mathrm{GL}_{d_i}(k_i)$ replaces the X_i by $S_i^\top \cdot X_i \cdot S_i$. In particular, if $p \neq 2$, we may (and will) assume without loss that all of the X_i are diagonal matrices. Now assume for simplicity that $\tau = \mathrm{id}$ (the proof of [AN02, Lemma 23] shows us how to deal with a non-trivial τ; note that in the statement of this lemma one should add the requirement that $e'^\circ \cdot e' = 0$). Now all we need to do is lift a single (involution-invariant) idempotent $\bar{e} \in \Lambda / \operatorname{Jac}(\Lambda))$, say for instance the first diagonal idempotent in $k_1^{d_1 \times d_1}$, to an involution-invariant idempotent $e \in \Lambda$. Then we may replace Λ by $e\Lambda e$ and repeat the process. So assume that $\bar{e}$ is an involution-invariant idempotent in $\Lambda / \operatorname{Jac}(\Lambda)$ and $\hat{e}$ is some element of Λ with $\hat{e} + \operatorname{Jac}(\Lambda) = \bar{e}$. Set $e_0 := \hat{e} \cdot \hat{e}^\circ$. The element e_0 also satisfies $e_0 + \operatorname{Jac}(\Lambda) = \bar{e}$ and furthermore e_0 is involution-invariant. A constructive version of Hensel's lemma (Theorem 2.13) tells us that the series

$$e_{i+1} := 3e_i^2 - 2e_i^3 \tag{2.131}$$

converges towards a lift of $\bar{e}$. Since every element in that series is involution-invariant, so will be the limit. □

Corollary 2.120. *Assumptions as in the preceding theorem. Then there is an idempotent $e \in \Lambda$ with $e = e^\circ$ such that $e\Lambda$ is a minimal progenerator. The involution on Λ will then restrict to an involution on the basic order $e\Lambda e$ of Λ.*

Remark 2.121. *When $p = 2$ we may consider the $\mathbb{Z}_2$-order $\mathbb{Z}_2^{2\times 2}$ with the involution*

$$-^\circ : \mathbb{Z}_2^{2\times 2} \longrightarrow \mathbb{Z}_2^{2\times 2} : a \mapsto \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \cdot a^\top \cdot \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \tag{2.132}$$

In this order there is no primitive idempotent invariant under the involution. In particular, this involution does not carry over to a basic order.

Proposition 2.122. *Assume $\mathcal{O}$ is complete and k is finite with $\operatorname{char}(k) \neq 2$. Let Λ be an $\mathcal{O}$-order that carries an involution $^\circ : \Lambda \longrightarrow \Lambda$, and assume (mostly for simplicity) that Λ is basic. By $e_1, \ldots, e_v$ we denote a full set of primitive orthogonal idempotents in Λ which is stable under $-^\circ$. Let $\sigma \in \Sigma_v$ be defined by*

$$e_i^\circ = e_{\sigma(i)} \quad \text{for all } i \in \{1, \ldots, v\} \tag{2.133}$$

Define $S_i := e_i\Lambda / \operatorname{Rad}(e_i\Lambda)$. The modules $S_1, \ldots, S_v$ are representatives for the isomorphism classes of simple Λ-modules. Now the involution σ satisfies

$$\operatorname{Hom}_k(S_i^\circ, k) \cong S_{\sigma(i)} \tag{2.134}$$

Proposition 2.123. *Assume $\mathcal{O}$ is complete and k is finite with $\operatorname{char}(k) \neq 2$. Let $\Lambda \subset D^{n\times n}$ be a graduated $\mathcal{O}$-order which carries an involution $^\circ : \Lambda \to \Lambda$. Denote by $S_1, \ldots, S_v$ representatives for the isomorphism classes of simple modules. Assume $\sigma \in \Sigma_v$ is the involution such that $\operatorname{Hom}_k(S_i^\circ, k) \cong S_{\sigma(i)}$. Then the Zassenhaus invariants $\{m_{i,j,l}\}_{i,j,l}$ of Λ satisfy*

$$m_{i,j,l} = m_{\sigma(l),\sigma(j),\sigma(i)} \quad \text{for all } i, j, l \in \{1, \ldots, v\} \tag{2.135}$$

with the natural choice of indices (see Remark 2.111).

Proof. We can assume without loss that $\Lambda = \Lambda(\Theta, m, d)$ for some Θ, m and d and that the set of diagonal idempotents in Λ is stable under the involution. The assertion follows from the fact that $-^\circ$ induces an isomorphism of $\mathcal{O}$-modules

$$\frac{e_i\Lambda e_l}{e_i\Lambda e_j\Lambda e_l} \xrightarrow{\sim} \frac{e_{\sigma(l)}\Lambda e_{\sigma(i)}}{e_{\sigma(l)}\Lambda e_{\sigma(j)}\Lambda e_{\sigma(i)}} \tag{2.136}$$

□

2.7 Amalgamation Depths

In this section we have a quick look at the notion of "amalgamation depths". Roughly speaking, when dealing with a self-dual order with decomposition numbers 0 and 1, amalgamation depths are the proper concept to consider when asking what constraints self-duality imposes on the exponent matrices associated to the order.

Throughout this section we assume that $\mathcal{O}$ is complete and we consider a semisimple algebra

$$A := \bigoplus_{i=1}^{h} D_i^{n_i\times n_i} \tag{2.137}$$

where the D_i are certain finite-dimensional skew-fields over K. Furthermore we let Θ_i denote the (unique) maximal $\mathcal{O}$-orders in D_i, and Π_i their respective maximal ideals. By $\varepsilon_1, \ldots, \varepsilon_h$ we denote the central primitive idempotents in A, where ε_i is supposed to pertain to the Wedderburn component $D_i^{n_i\times n_i}$ of A. Moreover let $\Lambda \subset A$ be a full $\mathcal{O}$-order with the property that $\varepsilon_i\Lambda = \Lambda(\Theta_i, m^i, d^i)$ for some exponent matrix m^i and some dimension vector d^i. We assume that each d^i is equal to $(1, \ldots, 1)$, or, to put it another way, we assume that Λ is basic (this is ultimately just to simplify notation; we could also proceed without this assumption). By $e_1, \ldots, e_s$ we denote a full set of orthogonal primitive idempotents in Λ, of which we assume without loss that they are diagonal in each Wedderburn component.

Define sets

$$r_i := \{1 \leqslant j \leqslant s \mid \varepsilon_i \cdot e_j \neq 0\} \tag{2.138}$$

where i ranges from 1 to h. The sets r_i should be understood as the indices of those columns in the decomposition matrix which have a non-zero entry in the i-th row. We identify $D_i^{n_i\times n_i}$ with $D_i^{r_i\times r_i}$ in such a fashion that for $j \in r_i$ the idempotent $\varepsilon_i \cdot e_j$ is the (j, j)-matrix unit in $D^{r_i\times r_i}$. In the same vein, we construe the exponent matrix m^i as an element of $\mathbb{Z}_{\geqslant 0}^{r_i\times r_i}$.

Definition 2.124 (Amalgamation Depths)**.** *We call the numbers*

$$A_{a,b}^i := \mathrm{length}_{\Theta_i}\, \varepsilon_i e_a\Lambda e_b/(\varepsilon_i e_a\Lambda e_b \cap e_a\Lambda e_b) \tag{2.139}$$

for $i \in \{1, \ldots, h\}$ and $a, b \in r_i$ the amalgamation depths *of Λ.*

Note that it does not matter in the above definition whether we consider $\varepsilon_i e_a\Lambda e_b/(\varepsilon_i e_a\Lambda e_b \cap e_a\Lambda e_b)$ as a left Θ_i-module or a right Θ_i-module. This is owed to the fact that the aforementioned Θ_i-module is isomorphic to Θ_i/Π_i^l for some l, and regardless of whether we construe this a left or a right module, its length will be l.

Proposition 2.125. *Let $u = (u_1, \ldots, u_h) \in Z(A)$ and assume Λ is self-dual with respect to the bilinear form T_u. Then*

(1) $A_{a,b}^i = A_{b,a}^i$ for all $i \in \{1, \ldots, h\}$ and $a, b \in r_i$.

(2) $A^i_{a,b} = \kappa_i - m^i_{a,b} - m^i_{b,a}$ *for all* $i \in \{1, \dots, h\}$ *and* $a, b \in r_i$*, where* $\kappa_i \in \mathbb{Z}$ *is chosen so that*

$$\Pi_i^{\kappa_i} = u_i^{-1} \cdot \Theta_i^{\sharp,1} \tag{2.140}$$

Here, the dual of Θ_i *is taken with respect to the trace bilinear form* $T_1 : D_i \times D_i \longrightarrow K$*.*

Proof. The second part clearly implies the first. Thus we prove only the second part. First note that by definition

$$\varepsilon_i e_a \Lambda e_b = \Pi_i^{m^i_{a,b}} \cdot e_{a,b} \quad \text{and} \quad \varepsilon_i e_b \Lambda e_a = \Pi_i^{m^i_{b,a}} \cdot e_{b,a} \tag{2.141}$$

where $e_{a,b}$ respectively $e_{b,a}$ is the (a,b)- respectively (b,a)-matrix unit in $D_i^{r_i \times r_i}$. Now let $q \in \mathbb{Z}$ be chosen such that $\varepsilon_i e_a \Lambda e_b \cap e_a \Lambda e_b = \Pi^q \cdot e_{a,b}$. Then clearly $A^i_{a,b} = q - m^i_{a,b}$. On the other hand, since Λ is self-dual, we get that q is maximal with respect to the property that

$$T_u(\Pi_i^q \cdot e_{a,b}, \Pi_i^{m^i_{b,a}} \cdot e_{b,a}) = T_1(u_i \cdot \Pi_i^{q+m^i_{b,a}}, \Theta_i) \overset{!}{\subseteq} \mathcal{O} \tag{2.142}$$

This is equivalent to

$$u_i \cdot \Pi_i^{q+m^i_{b,a}} = \Theta_i^{\sharp,1} \tag{2.143}$$

which in turn can be rewritten as

$$\Pi^{A^i_{a,b}} = u_i^{-1} \cdot \Theta_i^{\sharp,1} \cdot \Pi^{-m^i_{a,b}-m^i_{b,a}} \tag{2.144}$$

This completes the proof. □

Remark 2.126. *(1) If* $D_i = K$ *and* K *is an unramified extension of* $\mathbb{Q}_p$*, then the integer* κ_i *in the preceding proposition is simply* $-\nu_p(u_i)$*. In particular, since the* $A^i_{a,b}$ *are by definition non-negative, we get that*

$$m^i_{a,b} + m^i_{b,a} \leqslant -\nu_p(u_i) \tag{2.145}$$

(2) Since $u_i^{-1} \cdot \varepsilon_i \in \Lambda$ *(due to the fact that* $u_i^{-1} \cdot \varepsilon_i$ *lies in* $\Lambda^{\sharp,u}$*, since* $T_u(u_i^{-1}\varepsilon, \Lambda) = T_1(1, \varepsilon_i \Lambda) \subseteq \mathcal{O}$*) it follows that*

$$A^i_{a,b} \leqslant -\nu_p(u_i) \cdot \operatorname{length}_{\Theta_i} \Theta_i / p\Theta_i \tag{2.146}$$

2.8 Representation Theory of the Symmetric Groups

In this section we give a quick summary of the representation theory of the symmetric groups that we are going to use in the last chapter. This is inevitably going to involve some combinatorics, and hence we are going to start by defining a few combinatorial objects. The significance of those objects will become clear once we turn to the actual representation theory.

Definition 2.127 (Partitions, β-Sets & Abaci)**.** *Let* $n, p \in \mathbb{N}$*.*

(1) A non-increasing sequence $\lambda \in \mathbb{Z}_{\geqslant 0}^{\mathbb{N}}$ *with* $\sum_{i=1}^{\infty} \lambda_i = n$ *is called a* partition *of* n*. The numbers* λ_i *which are non-zero are called its* parts*, and the number of parts is called the* length *of* λ*. When writing down a partition we will usually truncate the trailing zeros.*

(2) *Let λ and μ be partitions of n. We say that λ is* lexicographically greater or equal μ *(in symbols: $\lambda \geqslant \mu$) if either $\lambda = \mu$ or $\lambda_i > \mu_i$ for the smallest i such that $\lambda_i \neq \mu_i$. The so-defined relation is a total order on the set of all partitions of n.*

(3) *Let λ and μ be partitions of n. We say that λ* dominates μ *(in symbols: $\lambda \trianglerighteq \mu$) if for all $i \in \mathbb{N}$ the following inequation holds:*

$$\sum_{j=1}^{i} \lambda_i \geqslant \sum_{j=1}^{i} \mu_i \tag{2.147}$$

In particular, $\lambda \trianglerighteq \mu$ implies $\lambda \geqslant \mu$.

(4) *A partition λ of n is called* p-regular *if no p parts of λ are equal. A partition which is not p-regular is called* p-singular.

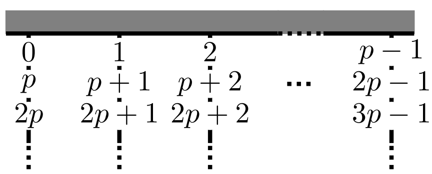
(5) *Let $\beta = \{\beta_1, \ldots, \beta_l\} \subset \mathbb{Z}_{\geqslant 0}$ be a finite set, and assume without loss that $\beta_1 < \ldots < \beta_l$. This defines a partition*

$$\lambda = (|\{0, \ldots, \beta_l\} - \beta|, |\{0, \ldots, \beta_{l-1}\} - \beta|, \ldots, |\{0, \ldots, \beta_1\} - \beta|) \tag{2.148}$$

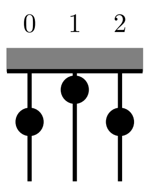
of some natural number. The set β is called a β-set *for λ. On the other hand, given a partition $\lambda = (\lambda_1, \ldots, \lambda_l)$ of n and some $l' \geqslant l$ there exists a unique β-set β for λ which has cardinality l', namely*

$$\beta = \underbrace{\{0, \ldots, l' - l - 1\}}_{= \emptyset \text{ if } l = l'} \cup \{\lambda_i + l' - i \mid i = 1, \ldots, l\} \tag{2.149}$$

(6) *Given a β-set $\beta \subset \mathbb{Z}_{\geqslant 0}$ and a natural number p (usually a prime), a customary visualization for β is a so-called* abacus diagram on p runners. *To do this, think of an abacus (as in the picture below), and for each element $b \in \beta$ place a bead at the position which is labeled by b:*



Example 2.128. $\lambda = (3, 2, 1)$ *is a partition of* 6. *A three-element β-set for λ is given by* $\{1, 3, 5\}$. *This β-set displayed on an abacus with three runners looks as follows:*



Definition 2.129 (Young Diagrams, Hooks & p-Cores)**.** *Let $n, p \in \mathbb{N}$.*

(1) *For a partition* λ *of* n *we define its* Young diagram *to be the following subset of* $\mathbb{N} \times \mathbb{N}$*:*

$$\{(i,j) \in \mathbb{N} \times \mathbb{N} \mid j \leqslant \lambda_i\} \tag{2.150}$$

The elements of the Young diagram are called nodes*. A subset* $Y \subset \mathbb{N} \times \mathbb{N}$ *of cardinality* n *is a Young diagram of some partition* λ *of* n *if and only if it has the following property: if* $(i,j) \in Y$*, then* $(i-1,j) \in Y$ *whenever* $i > 1$ *and* $(i,j-1) \in Y$ *whenever* $j > 1$.

(2) *If* λ *is a partition of* n *and* Y *is its Young diagram, then we define its* transposed $\lambda^\top$ *as follows:* $\lambda^\top$ *is the partition associated to the Young diagram* $Y^\top := \{(j,i) \mid (i,j) \in Y\}$.

(3) *Let* λ *be a partition of* n *and* Y *its Young diagram. To any node* $(i,j) \in Y$ *we associate a hook, which consists of* (i,j)*, all nodes in* Y *right of* (i,j) *as well as those nodes below* (i,j)*:*

$$(\{(i+q,j) \mid q \geqslant 0\} \cup (\{(i,j+q) \mid q \geqslant 0\}) \cap Y \tag{2.151}$$

We also define a rim hook associated to (i,j)*, which consists of the piece of the rim of* Y *enclosed by the* (i,j)*-hook:*

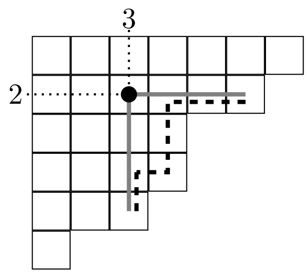
$$\{(a,b) \in Y \mid (a+1,b) \notin Y \text{ or } (a,b+1) \notin Y\} \cap \{(a,b) \in Y \mid a \geqslant i \text{ and } b \geqslant j)\} \tag{2.152}$$

Unlike the (i,j)*-hook, the* (i,j)*-rim hook will, upon removal from* Y*, always leave a Young diagram of some partition.*

It is elementary to see that the (i,j)*-hook and the* (i,j)*-rim hook contain the same number of nodes. This number is called the* length of the (i,j)-hook*. The cardinality of* $\{(i+q,j) \mid q \geqslant 1\} \cap Y$ *(that is, the number of nodes in* Y *strictly below the* (i,j)*-node) is called the* leg length *of the* (i,j)*-hook.*

(4) *A partition the Young diagram of which contains no hook of length divisible by* p *is called a* p*-core.*

Example 2.130. *We consider the partition* $\lambda = (7,6,4,4,3,1)$ *of* 25*. Its Young diagram will be visualized as follows:*



We can read off that $\lambda^\top = (6,5,5,4,2,2,1)$*. In the diagram we have also sketched the* $(2,3)$*-hook (the gray line) as well as the* $(2,3)$*-rim hook (the dashed line). We can read off that the length of the* $(2,3)$*-hook is* 7 *and that its leg length is* 3.

Remark 2.131 (Hooks & Abaci)**.** *Let* $n,p \in \mathbb{N}$*, let* λ *be a partition of* n *and let* $\beta = \{\beta_1 < \ldots < \beta_l\}$ *be a* β*-set for* λ.

(1) *The nodes in a Young diagram for λ are in bijection with pairs $(x, y) \in \mathbb{Z}^2_{\geqslant 0}$ such that $y \in \beta$, $x \notin \beta$ and $x < y$. The bijection may be chosen so that hook length of the node corresponding to the pair (x, y) is equal to $y - x$. Removing the rim hook corresponding to the pair (x, y) from the Young diagram of λ results in the Young diagram of the partition given by the β-set $(\beta - \{y\}) \cup \{x\}$.*

(2) *If we display β on an abacus with p runners, then hooks of length $q \cdot p$ for $q \in \mathbb{N}$ correspond to beads such that the spot q rows above them is not taken by another bead. Removing such a hook corresponds moving the bead q rows up.*

(3) *A p-core corresponds to an abacus diagram where all beads are as far up as they can be (this is clear by the previous point).*

The second part of the preceding remark makes the following proposition obvious:

Proposition 2.132. *Let $n, p \in \mathbb{N}$. Any partition λ of n can be turned into a p-core by successively removing rim p-hooks from λ. The so-obtained p-core is independent of the particular way in which the p-hooks were removed.*

Definition 2.133 (Core and Weight of a Partition)**.** *Let $n, p \in \mathbb{N}$ and let λ be a partition of n. The p-core obtained from λ by successively removing rim p-hooks is called the* p-core of λ*. The number of p-hooks removed in the process is called the* p-weight of λ*.*

We now turn to the actual representation theory of the symmetric group Σ_n for some henceforth fixed $n \in \mathbb{N}$. Our starting point are the so-called *Specht modules.*

Definition 2.134 (Specht Modules)**.** *For any partition λ of n there is a $\mathbb{Z}\Sigma_n$-lattice $S^\lambda_{\mathbb{Z}}$ called a* Specht lattice *(see [Jam78] for a construction). For any commutative ring R we define the Specht module defined over $R\Sigma_n$ associated to λ to be $S^\lambda_R := R \otimes S^\lambda_{\mathbb{Z}}$.*

Theorem 2.135 (Properties of Specht Modules and Simple Modules)**.** *Let p be a prime.*

(1) *If K is a field of characteristic zero, then the Specht module S^λ_K is absolutely irreducible for any partition λ of n.*

(2) *If K is a field of characteristic zero, then any simple $K\Sigma_n$-module is isomorphic to some S^λ_K, and the S^λ_K are pairwise non-isomorphic.*

(3) *If k is a field of characteristic p and λ is a p-regular partition of n, then $S^\lambda_k / \operatorname{Rad}(S^\lambda_k)$ is absolutely irreducible. We denote the simple module $S^\lambda_k / \operatorname{Rad}(S^\lambda_k)$ by D^λ_k.*

(4) *If k is a field of characteristic p, then any simple $k\Sigma_n$ is isomorphic to D^λ_k for some p-regular partition λ of n, and the D^λ_k are pairwise non-isomorphic.*

(5) *Let k be a field of characteristic p. If λ is a partition of n and μ is a p-regular partition of n such that D^μ_k is a composition factor of S^λ_k, then $\mu \trianglerighteq \lambda$. Furthermore, if λ is p-regular, then D^λ_k occurs exactly once as a composition factor of S^λ_k (or, to put it another way, D^λ_k does not occur as a composition factor of $\operatorname{Rad}(S^\lambda_k)$).*

Theorem 2.136 (Dimension Formula). *Let λ be a partition of n, and let Y be its Young diagram. The $\mathbb{Z}$-rank of the Specht lattice $S^\lambda_\mathbb{Z}$ is equal to*

$$\frac{n!}{\prod_{(a,b)\in Y} h_{a,b}} \tag{2.153}$$

where $h_{a,b}$ denotes the length of the (a,b)-hook.

Theorem 2.137 (Induction and Restriction of Specht Modules). *Let K be a field of characteristic zero and let λ be a partition of n. Then*

$$\mathrm{Res}^{\Sigma_n}_{\Sigma_{n-1}}(S^\lambda_K) \cong \bigoplus_\mu S^\mu_K \tag{2.154}$$

where the sum is taken over all partitions μ of $n-1$ which are obtained from λ by removing a node in the Young diagram.

Similarly, if μ is a partition of $n-1$, then

$$\mathrm{Ind}^{\Sigma_n}_{\Sigma_{n-1}}(S^\mu_K) \cong \bigoplus_\nu S^\nu_K \tag{2.155}$$

where the sum is taken over all partitions ν of n which are obtained from μ by adding a node in the Young diagram.

We now also fix a prime p as well as a p-modular system $(K, \mathcal{O}, k)$ and focus our attention to the group ring $\mathcal{O}\Sigma_n$.

Theorem 2.138 (Nakayama "Conjecture"). *If λ and μ are two partitions of n, then the Specht modules S^λ_K and S^μ_K lie in the same block of $\mathcal{O}\Sigma_n$ if and only if λ and μ have the same p-core and the same p-weight.*

This theorem tells us that the p-blocks of symmetric groups are parametrized by p-cores and p-weights. Using abacus diagrams we can specify a p-core by specifying the number of beads on each runner (if we assume that the first runner has no beads on it then the remaining numbers are even unique). Given a p-core κ and a p-weight w it is also easily possible to list all partitions in the corresponding block (i. e., all partitions such that the associated Specht module lies in that block). Simply represent the p-core κ in an abacus diagram with enough beads, namely make sure that there are at least w beads on each runner. Then all partitions of p-weight w with p-core κ are obtained by moving w beads up by one row (possibly moving the same bead several times).

Remark 2.139 (Weight and Defect). *It is known that blocks of weight w have defect $\nu_p((p\cdot w)!)$ and that the defect group of these blocks is conjugate in Σ_n to a p-Sylow subgroup of $\Sigma_{w\cdot p}$ (identified with the subgroup of Σ_n that fixes all but the first $w \cdot p$ letters; note that blocks of weight w can occur only if $n \geqslant w \cdot p$, so this embedding of $\Sigma_{w\cdot p}$ into Σ_n certainly exists). In particular, the defect group of a block is abelian if and only if $w < p$. In conjunction with the dimension formula (Theorem 2.136) and Remark 2.131, it follows that Brauer's height zero conjecture (see Remark 2.49) holds for blocks of symmetric groups (since one can see that there are exactly w hooks of length divisible by p in the Young diagram of a partition of weight w, and if $w < p$ then there cannot be any hooks of length divisible by p^2).*

We will now define i-induction and i-restriction functors and their divided powers. Those functors can be used to define the so-called Scopes reduction, which establishes Morita equivalences between certain blocks of symmetric groups (historically, however, Scopes reduction certainly predates and probably motivated the definition of these functors). Unfortunately, neither partitions nor β-sets and abaci are really the right combinatorial concepts to express the following definition in a natural way. This is why it may look a little contrived at first glance.

Definition 2.140 (i-Induction and i-Restriction)**.** *Let $B_1, \ldots, B_t$ be the blocks of $k\Sigma_n$, let $\kappa_1, \ldots, \kappa_t$ be their p-cores, and let $i \in \mathbb{F}_p$. Display the κ_j's on abaci with p runners and a number of beads divisible by p. We identify $\mathbb{F}_p$ with the set $\{0, \ldots, p-1\}$ which labels the runners of the abaci. For each $j \in \{1, \ldots, t\}$ we define idempotent elements $b'_j \in k\Sigma_{n-1}$ as follows: Assume that there is a block B'_j of $k\Sigma_{n-1}$ the kernel of which is obtained from κ_j by taking one bead off the i-th runner and putting it on the $(i-1)$-st runner (if $i = 0$ then the $(i-1)$-st runner is the $(p-1)$-st, since we identify the labels of the runners with the elements of $\mathbb{F}_p$). In case such a block B'_j does not exist then we define b'_j to be zero. If B'_j does exist then it is of course unique since we just specified its p-core and we define b'_j to be the block idempotent in $k\Sigma_{n-1}$ belonging to the block B'_j.*

Now we define the i-restriction functor e_i *as follows:*

$$- \otimes_{k\Sigma_n} \bigoplus_{j=1}^{t} B_j|_{k\Sigma_{n-1}} \cdot b'_j : \ \mathbf{mod}_{k\Sigma_n} \longrightarrow \mathbf{mod}_{k\Sigma_{n-1}} \tag{2.156}$$

The functor e_i is clearly exact and has a unique left and right adjoint f_i called the i-induction functor *which can be defined as follows:*

$$\mathrm{Hom}_{k\Sigma_n}\left(\bigoplus_{j=1}^{t} B_j|_{k\Sigma_{n-1}} \cdot b'_j, -\right) : \ \mathbf{mod}_{k\Sigma_{n-1}} \longrightarrow \mathbf{mod}_{k\Sigma_n} \tag{2.157}$$

Remark 2.141. *We have*

$$\bigoplus_{i\in\mathbb{F}_p} e_i(-) \cong \mathrm{Res}^{\Sigma_n}_{\Sigma_{n-1}}(-) \tag{2.158}$$

and

$$\bigoplus_{i\in\mathbb{F}_p} f_i(-) \cong \mathrm{Ind}^{\Sigma_n}_{\Sigma_{n-1}}(-) \tag{2.159}$$

Of course these functors can be lifted to the group rings defined over $\mathcal{O}$ (see Remark 2.144 below). The way we defined them we could in fact just have defined them integrally in the first place. However, in the original definition e_i applied to a $k\Sigma_n$-module V is the projection of the restriction $\mathrm{Res}^{\Sigma_n}_{\Sigma_{n-1}}(V)$ to the generalized i-eigenspace of the element $(1, n) + (2, n) + \ldots + (n-1, n) \in k\Sigma_n$ on V, and this does not immediately make sense over $\mathcal{O}$.

Definition 2.142 (Divided Powers)**.** *Setup as in Definition 2.140 and assume we are in addition given a $q \in \mathbb{N}$ with $q < n$. For each $j \in \{1, \ldots, t\}$ we define idempotent elements $b'_j \in k\Sigma_{n-q}$ as follows: Assume that there is a block B'_j of $k\Sigma_{n-q}$ the p-core of which is obtained from κ_j by taking q beads off the i-th runner and putting them on the $(i-1)$-st runner. In case such a block B'_j does not exist then we define b'_j to be zero. If B'_j does exist then it is unique and we define b'_j to be the block idempotent in $k\Sigma_{n-q}$ belonging to the block B'_j.*

*Now we define the q-*th divided power *$e_i^{(q)}$ of the i-restriction functor as follows:*

$$- \otimes_{k\Sigma_n} \bigoplus_{j=1}^{t} \left(B_j|_{k\Sigma_{n-q}\times\Sigma_q} \otimes_{k\Sigma_q} T\right) \cdot b'_j : \ \mathbf{mod}_{k\Sigma_n} \longrightarrow \mathbf{mod}_{k\Sigma_{n-q}} \tag{2.160}$$

*Here T is a one-dimensional left $k\Sigma_q$-module (that is, either the trivial or the sign representation; both choices lead to isomorphic functors). The functor $e_i^{(q)}$ is exact and has a unique left and right adjoint $f_i^{(q)}$ called the q-*th divided power *of f_i which can be defined as follows:*

$$\mathrm{Hom}_{k\Sigma_n}\left(\bigoplus_{j=1}^{t} \left(B_j|_{k\Sigma_{n-q}\times\Sigma_q} \otimes_{k\Sigma_q} T\right) \cdot b'_j, -\right) : \ \mathbf{mod}_{k\Sigma_{n-q}} \longrightarrow \mathbf{mod}_{k\Sigma_n} \tag{2.161}$$

The next theorem justifies the name "divided powers". Note that we defined functors e_i and f_i for each n, and thus we should, technically, write $e_{i,n}$ and $f_{i,n}$. However, since the proper choice of n is implied, we will usually drop the subscript n. In particular, when we write e_i^q we mean $e_{i,n-q+1} \circ \cdots \circ e_{i,n-1} \circ e_{i,n}$ and when we write f_i^q we mean $f_{i,n} \circ f_{i,n-1} \circ \cdots \circ f_{i,n-q+1}$.

Theorem 2.143. *For each $q < n$ we have isomorphisms of functors*

$$e_i^q \cong \bigoplus^{q!} e_i^{(q)} \tag{2.162}$$

and

$$f_i^q \cong \bigoplus^{q!} f_i^{(q)} \tag{2.163}$$

Remark 2.144 (Lifting to $\mathcal{O}$)**.** *We can define exact and mutually left and right adjoint functors*

$$\hat{e}_i : \ \mathbf{mod}_{\mathcal{O}\Sigma_n} \longrightarrow \mathbf{mod}_{\mathcal{O}\Sigma_{n-1}} \quad \textit{and} \quad \hat{f}_i : \ \mathbf{mod}_{\mathcal{O}\Sigma_{n-1}} \longrightarrow \mathbf{mod}_{\mathcal{O}\Sigma_n} \tag{2.164}$$

as well as (also exact and mutually left and right adjoint) functors

$$\hat{e}_i^{(q)} : \ \mathbf{mod}_{\mathcal{O}\Sigma_n} \longrightarrow \mathbf{mod}_{\mathcal{O}\Sigma_{n-q}} \quad \textit{and} \quad \hat{f}_i^{(q)} : \ \mathbf{mod}_{\mathcal{O}\Sigma_{n-q}} \longrightarrow \mathbf{mod}_{\mathcal{O}\Sigma_n} \tag{2.165}$$

This can be done since the module

$$\bigoplus_{j=1}^{t} B_j|_{k\Sigma_{n-1}} \cdot b'_j \tag{2.166}$$

from Definition 2.140 lifts to an $\mathcal{O}\Sigma_n$-$\mathcal{O}\Sigma_{n-1}$-bilattice which is projective as a left and as a right module and the module

$$\bigoplus_{j=1}^{t} \left(B_j|_{k\Sigma_{n-q}\times\Sigma_q} \otimes_{k\Sigma_q} T\right) \cdot b'_j \tag{2.167}$$

from Definition 2.142 lifts to an $\mathcal{O}\Sigma_n$-$\mathcal{O}\Sigma_{n-q}$-bilattice which is projective as a left and as a right module. The assertion that the lifts are projective is actually clear (since for a lattice

L over any $\mathcal{O}$-order, L is projective if and only if $k \otimes L$ is projective). That the module in (2.166) lifts is also trivial. The module in (2.167) can be lifted since T can be lifted, so it is possible to give an analogous construction over $\mathcal{O}$ (although it needs to be checked that the so-defined module is a lattice and in fact a lift of the module in (2.167)).

Definition 2.145 ($(w:q)$-Pairs)**.** *Let B be a block of $\mathcal{O}\Sigma_n$ and let B' be a block of $\mathcal{O}\Sigma_{n-q}$ for some $q < n$. Assume that B and B' have the same p-weight w. Represent the p-core of B on an abacus such that the leftmost runner is empty. If, for some $i \in \{1, \dots, p-1\}$, interchanging the $(i-1)$-st and i-th runner in this abacus diagram yields an abacus representation of the p-core of B', then B and B' are said to form a $(w:q)$-pair (with respect to the $(i-1)$-st and i-th runner).*

Remark 2.146. *(1) Note that if B and B' form a $(w:q)$-pair with respect to the $(i-1)$-st and i-th runner, then in the appropriate abacus representation of the p-core of B there are precisely q more beads on the i-th runner than on the $(i-1)$-st.*

(2) Any two blocks of the same defect are connected by a finite sequence of $(w:q)$-pairs.

The last theorem we cite in this section is essentially the same as the result given [Sco91], albeit slightly reformulated. It implies in particular that, if we fix a prime p, there are only finitely many Morita equivalence classes of blocks of symmetric groups of any given weight (or defect).

Theorem 2.147 (Scopes Reduction)**.** *Assume B and B' are blocks of $\mathcal{O}\Sigma_n$ respectively $\mathcal{O}\Sigma_{n-q}$ which form a $(w:q)$-pair with $q \geqslant w$. Then B and B' are Morita equivalent. A Morita equivalence between the two is afforded by the restrictions of the functors $e_j^{(q)}$ and $f_j^{(q)}$, where j is chosen so that $j \equiv i - l \mod (p)$, with l being the length of the p-core of B.*

Chapter 3

Generic Methods

In this chapter we introduce methods which work in a general setting, that is, without fixing a particular group ring or block. In the first three sections we explain how to relate lifts of two derived equivalent algebras to each other. This is an important tool in Chapter 4 and Chapter 5. The last section in this chapter contains some further results which do not fit into the framework of the first three sections.

3.1 A Correspondence of Lifts

As usual, $(K, \mathcal{O}, k)$ denotes a p-modular system and we assume that K is complete. In this section we introduce a bijection between "lifts" of derived equivalent finite-dimensional k-algebras. The idea behind this is actually very simple: Assume we are given a k-algebra $\overline{\Lambda}$ and a tilting complex $\overline{T} \in \mathcal{K}^b(\mathbf{proj}_{\overline{\Lambda}})$ with endomorphism ring $\overline{\Gamma}$. Then we can take an $\mathcal{O}$-order Λ with $k \otimes \Lambda \cong \overline{\Lambda}$ and construct from it an $\mathcal{O}$-order Γ with $k \otimes \Gamma \cong \overline{\Gamma}$. This works by lifting $\overline{T}$ to a tilting complex $T \in \mathcal{K}^b(\mathbf{proj}_{\Lambda})$ (which is possible due to Rickard's lifting theorem, given in Theorem 3.2 below) and then taking Γ to be the endomorphism ring of T. Of course, we left out some details in this short explanation which need to be taken care of to make this work properly.

Definition 3.1. *For a finite-dimensional k-algebra $\overline{\Lambda}$ define its* set of lifts *as follows:*

$$\widehat{\mathfrak{L}}(\overline{\Lambda}) := \left\{ (\Lambda, \varphi) \mid \Lambda \text{ is an } \mathcal{O}\text{-order and } \varphi : k \otimes \Lambda \xrightarrow{\sim} \overline{\Lambda} \text{ is an isomorphism} \right\} \Big/ \sim \tag{3.1}$$

where we say $(\Lambda, \varphi) \sim (\Lambda', \varphi')$ if and only if

(1) There is an isomorphism $\alpha : \Lambda \xrightarrow{\sim} \Lambda'$

(2) There is a $\beta \in \mathrm{Aut}_k(\overline{\Lambda})$ such that the functor $- \otimes^{\mathbb{L}}_{\overline{\Lambda}} {}_{\beta}\overline{\Lambda}_{\mathrm{id}}$ fixes all isomorphism classes of tilting complexes in $\mathcal{K}^b(\mathbf{proj}_{\overline{\Lambda}})$

such that $\varphi = \beta \circ \varphi' \circ (\mathrm{id}_k \otimes \alpha)$.

Our bijection will be based on the following theorem of Rickard:

Theorem 3.2 ([Ric91b, Theorem 3.3.]). *Let Λ be an $\mathcal{O}$-order and let $\overline{T} \in \mathcal{C}^b(\mathbf{proj}_{k\otimes\Lambda})$ be a tilting complex for $k \otimes \Lambda$. Then there exists a unique (up to isomorphism in $\mathcal{D}^b(\Lambda)$) tilting complex $T \in \mathcal{C}^b(\mathbf{proj}_\Lambda)$ with $k \otimes T \cong \overline{T}$. $\mathrm{End}_{\mathcal{D}^b(\Lambda)}(T)$ is torsion-free and*

$$k \otimes \mathrm{End}_{\mathcal{D}^b(\Lambda)}(T) \cong \mathrm{End}_{\mathcal{D}^b(k\otimes\Lambda)}(\overline{T}) \tag{3.2}$$

Remark 3.3. *By [Ric91b, Proposition 3.1.] it is immediately clear that we can replace the word "tilting complex" by "partial tilting complex" in the above theorem (where we understand "partial tilting complex" as defined in Definition 2.72).*

Lemma 3.4. *Let Λ be an $\mathcal{O}$-order and $T \in \mathcal{K}^b(\mathbf{proj}_\Lambda)$ a tilting complex. Define $\Gamma := \mathrm{End}_{\mathcal{D}^b(\Lambda)}(T)$, and assume that Γ is also an $\mathcal{O}$-order. Then $k \otimes \Gamma$ and $\mathrm{End}_{\mathcal{D}^b(\Lambda)}(k \otimes T)$ are (canonically) isomorphic and the diagram*

$$\begin{array}{ccc}
\mathcal{D}^-(\Lambda) & \xrightarrow{\ \mathcal{G}_T\ } & \mathcal{D}^-(\Gamma) \\
\Big\downarrow{\scriptstyle k\otimes^{\mathbb{L}}-} & & \Big\downarrow{\scriptstyle k\otimes^{\mathbb{L}}-} \\
\mathcal{D}^-(k\otimes\Lambda) & \xrightarrow{\ \mathcal{G}_{k\otimes T}\ } & \mathcal{D}^-(k\otimes\Gamma)
\end{array} \tag{3.3}$$

commutes on objects.

Proof. This follows from [Ric91a, Proposition 2.4.]. □

For the rest of the section let $\overline{\Lambda}$ and $\overline{\Gamma}$ be two derived equivalent finite-dimensional k-algebras. Furthermore let $X \in \mathcal{D}^b(\overline{\Lambda}^{\mathrm{op}} \otimes_k \overline{\Gamma})$ be a two-sided tilting complex, and let X^{-1} be its inverse. Let $\overline{T}$ be the restriction of X^{-1} to $\mathcal{D}^b(\mathbf{proj}_{\overline{\Lambda}})$ and likewise let $\overline{S}$ be the restriction of X to $\mathcal{D}^b(\mathbf{proj}_{\overline{\Gamma}})$.

Definition 3.5. *Define a map*

$$\Phi_X : \ \widehat{\mathfrak{L}}(\overline{\Lambda}) \longrightarrow \widehat{\mathfrak{L}}(\overline{\Gamma}) \tag{3.4}$$

as follows: Let T be the lift of $\overline{T} \otimes_{\overline{\Lambda}\ \mathrm{id}}\overline{\Lambda}_\varphi$ (which exists and is unique by Theorem 3.2). We put $\Phi_X(\Lambda, \varphi) = (\Gamma, \psi)$, where $\Gamma = \mathrm{End}_{\mathcal{D}^b(\Lambda)}(T)$ and $\psi : k \otimes \Gamma \xrightarrow{\sim} \overline{\Gamma}$ is an isomorphism such that the following diagram commutes on objects:

$$\begin{array}{ccccc}
\mathcal{D}^-(\Lambda) & \xrightarrow{\ \mathcal{G}_T(-)\ } & \mathcal{D}^-(\mathrm{End}(T)) & = & \mathcal{D}^-(\Gamma) \\
{\scriptstyle k\otimes^{\mathbb{L}}-}\Big\downarrow & & \Big\downarrow{\scriptstyle k\otimes^{\mathbb{L}}-} & & \Big\downarrow{\scriptstyle k\otimes^{\mathbb{L}}-} \\
\mathcal{D}^-(k\otimes\Lambda) & \xrightarrow{\ \mathcal{G}_{k\otimes T}(-)\ } & \mathcal{D}^-(\mathrm{End}(k\otimes T)) & = & \mathcal{D}^-(k\otimes\Gamma) \\
{\scriptstyle -\otimes^{\mathbb{L}}_{k\otimes\Lambda\,\varphi}\overline{\Lambda}_{\mathrm{id}}}\Big\downarrow & & \Big\downarrow{\scriptstyle \mathcal{E}} & & \Big\downarrow{\scriptstyle -\otimes^{\mathbb{L}}_{k\otimes\Gamma\,\psi}\overline{\Gamma}_{\mathrm{id}}} \\
\mathcal{D}^-(\overline{\Lambda}) & \xrightarrow[\ -\otimes^{\mathbb{L}}_{\overline{\Lambda}}X\ ]{} & \mathcal{D}^-(\overline{\Gamma}) & = & \mathcal{D}^-(\overline{\Gamma})
\end{array} \tag{3.5}$$

Here, $\mathcal{E}$ is defined so that the bottom left square commutes.

Proof of well-definedness. First note that the top left square commutes on objects by Lemma 3.4. Thus the left half of the diagram will commute on objects. Note furthermore that $\mathcal{E}$ sends $0 \to k \otimes \Gamma \to 0$ to $0 \to \overline{\Gamma} \to 0$, and hence a ψ making the diagram commutative on objects can be chosen due to Remark 2.95. This ψ is unique up to an automorphism β of $\overline{\Gamma}$ such that $-\otimes^{\mathbb{L}}_{\overline{\Gamma}}\,{}_{\mathrm{id}}\overline{\Gamma}_{\beta}$ fixes all objects in $\mathcal{D}^{-}(\overline{\Gamma})$, and hence in particular fixes all tilting complexes. Therefore the equivalence class of (Γ, ψ) is certainly independent of the particular choice of ψ.

Now assume $(\Lambda, \varphi) \sim (\Lambda', \varphi')$, that is, there are α and β as in Definition 3.1 such that $\varphi = \beta \circ \varphi' \circ (\mathrm{id}_k \otimes \alpha)$. We need to show that $(\Gamma, \psi) := \Phi_X(\Lambda, \varphi) \sim \Phi_X(\Lambda', \varphi') =: (\Gamma', \psi')$, Φ_X being given by the construction above. We get the following diagram (where we define T' analogous to T):

$$\begin{array}{ccccccc}
\mathcal{D}^{-}(\Gamma') & \xleftarrow{\mathcal{G}_{T'}(-)} & \mathcal{D}^{-}(\Lambda') & \xrightarrow{-\otimes^{\mathbb{L}}_{\Lambda'}\,{}_{\mathrm{id}}\Lambda'_{\alpha}} & \mathcal{D}^{-}(\Lambda) & \xrightarrow{\mathcal{G}_{T}(-)} & \mathcal{D}^{-}(\Gamma) \\
\downarrow {\scriptstyle k\otimes^{\mathbb{L}}-} & & \downarrow {\scriptstyle k\otimes^{\mathbb{L}}-} & & \downarrow {\scriptstyle k\otimes^{\mathbb{L}}-} & & \downarrow {\scriptstyle k\otimes^{\mathbb{L}}-} \\
\mathcal{D}^{-}(k\otimes\Gamma') & \xleftarrow{\mathcal{G}_{k\otimes T'}(-)} & \mathcal{D}^{-}(k\otimes\Lambda') & \xrightarrow[-\otimes^{\mathbb{L}}_{k\otimes\Lambda'}\,{}_{\mathrm{id}}k\otimes\Lambda'_{\mathrm{id}_k\otimes\alpha}]{} & \mathcal{D}^{-}(k\otimes\Lambda) & \xrightarrow{\mathcal{G}_{k\otimes T}(-)} & \mathcal{D}^{-}(k\otimes\Gamma) \\
\downarrow {\scriptstyle -\otimes^{\mathbb{L}}_{k\otimes\Gamma'}\,{}_{\psi'}\overline{\Gamma}_{\mathrm{id}}} & & \downarrow {\scriptstyle -\otimes^{\mathbb{L}}_{k\otimes\Lambda'}\,{}_{\varphi'}\overline{\Lambda}_{\mathrm{id}}} & & \downarrow {\scriptstyle -\otimes^{\mathbb{L}}_{k\otimes\Lambda}\,{}_{\varphi}\overline{\Lambda}_{\mathrm{id}}} & & \downarrow {\scriptstyle -\otimes^{\mathbb{L}}_{k\otimes\Gamma}\,{}_{\psi}\overline{\Gamma}_{\mathrm{id}}} \\
\mathcal{D}^{-}(\overline{\Gamma}) & \xleftarrow[-\otimes^{\mathbb{L}}_{\overline{\Lambda}}X]{} & \mathcal{D}^{-}(\overline{\Lambda}) & = & \mathcal{D}^{-}(\overline{\Lambda}) & \xrightarrow[-\otimes^{\mathbb{L}}_{\overline{\Lambda}}X]{} & \mathcal{D}^{-}(\overline{\Gamma})
\end{array} \tag{3.6}$$

This diagram will commute at the very least on tilting complexes (that is, if we take a tilting complex in any of those categories and take its image under a series of arrows in the above diagram, the isomorphism class of the outcome will not depend on the path we have chosen). Note that all horizontal arrows are equivalences, and so we get a diagram (again commutative on tilting complexes)

$$\begin{array}{ccc}
\mathcal{D}^{-}(\Gamma') & \xrightarrow{\mathcal{F}_1(-)} & \mathcal{D}^{-}(\Gamma) \\
\downarrow {\scriptstyle k\otimes^{\mathbb{L}}-} & & \downarrow {\scriptstyle k\otimes^{\mathbb{L}}-} \\
\mathcal{D}^{-}(k\otimes\Gamma') & \xrightarrow{\mathcal{F}_2(-)} & \mathcal{D}^{-}(k\otimes\Gamma) \\
\downarrow {\scriptstyle -\otimes^{\mathbb{L}}_{k\otimes\Gamma'}\,{}_{\psi'}\overline{\Gamma}_{\mathrm{id}}} & & \downarrow {\scriptstyle -\otimes^{\mathbb{L}}_{k\otimes\Gamma}\,{}_{\psi}\overline{\Gamma}_{\mathrm{id}}} \\
\mathcal{D}^{-}(\overline{\Gamma}) & = & \mathcal{D}^{-}(\overline{\Gamma})
\end{array} \tag{3.7}$$

where $\mathcal{F}_1$ and $\mathcal{F}_2$ are two equivalences. Due to commutativity on tilting complexes, $\mathcal{F}_2$ needs to send $0 \to k\otimes\Gamma' \to 0$ to $0 \to k\otimes\Gamma \to 0$. Due to unique lifting (and again commutativity), $\mathcal{F}_1$ needs to send $0 \to \Gamma' \to 0$ to $0 \to \Gamma \to 0$. Hence there is an isomorphism $\alpha : \Gamma' \to \Gamma$ such that $\mathcal{F}_1(-)$ agrees on objects with $-\otimes^{\mathbb{L}}_{\Gamma'}\,{}_{\alpha}\Gamma_{\mathrm{id}}$. Due to commutativity, $\mathcal{F}_2(-)$ then needs to agree on tilting complexes with $-\otimes^{\mathbb{L}}_{k\otimes\Gamma'}\,{}_{\mathrm{id}_k\otimes\alpha}k\otimes\Gamma_{\mathrm{id}}$ (this is owed to the fact that every tilting complex lies in the image of $k\otimes^{\mathbb{L}}-$ due to Theorem 3.2). Commutativity on tilting complexes of the lower square then implies that $\psi' = \beta\circ\psi\circ(\mathrm{id}_k\otimes\alpha)$ for some $\beta \in \mathrm{Aut}_k(\overline{\Gamma})$ so that $-\otimes^{\mathbb{L}}_{\overline{\Gamma}}\,{}_{\beta}\overline{\Gamma}_{\mathrm{id}}$ fixes all tilting complexes in $\mathcal{K}^b(\mathbf{proj}_{\overline{\Gamma}})$. By definition this means $(\Gamma, \psi) \sim (\Gamma', \psi')$. □

Proposition 3.6. *The maps Φ_X and $\Phi_{X^{-1}}$ are mutually inverse. In particular, they induce a bijection*

$$\widehat{\mathfrak{L}}(\overline{\Lambda}) \longleftrightarrow \widehat{\mathfrak{L}}(\overline{\Gamma}) \tag{3.8}$$

Proof. We keep the notation of Definition 3.5. Set $(\Gamma, \psi) := \Phi_X(\Lambda, \varphi)$ and $(\Lambda', \tilde{\varphi}) := \Phi_{X^{-1}}(\Gamma, \psi)$. Furthermore, let S be the lift of $\overline{S}\otimes_{\overline{\Gamma}}\,{}_{\mathrm{id}}\overline{\Gamma}_{\psi}$ Consider the following diagram (obtained from composing (3.5) with itself)

$$\begin{array}{ccccc}
\mathcal{D}^-(\Lambda) & \xrightarrow{\mathcal{G}_T(-)} & \mathcal{D}^-(\Gamma) & \xrightarrow{\mathcal{G}_S(-)} & \mathcal{D}^-(\Lambda') \\
\downarrow{\scriptstyle k\otimes^{\mathbb{L}}-} & & \downarrow{\scriptstyle k\otimes^{\mathbb{L}}-} & & \downarrow{\scriptstyle k\otimes^{\mathbb{L}}-} \\
\mathcal{D}^-(k\otimes\Lambda) & \xrightarrow{\mathcal{G}_{k\otimes T}(-)} & \mathcal{D}^-(k\otimes\Gamma) & \xrightarrow{\mathcal{G}_{k\otimes S}(-)} & \mathcal{D}^-(k\otimes\Lambda') \\
\downarrow{\scriptstyle -\otimes^{\mathbb{L}}_{k\otimes\Lambda\ \varphi}\overline{\Lambda}_{\mathrm{id}}} & & \downarrow{\scriptstyle \otimes^{\mathbb{L}}_{k\otimes\Gamma\ \psi}\overline{\Gamma}_{\mathrm{id}}} & & \downarrow{\scriptstyle -\otimes^{\mathbb{L}}_{k\otimes\Lambda'\ \tilde{\varphi}}\overline{\Lambda}_{\mathrm{id}}} \\
\mathcal{D}^-(\overline{\Lambda}) & \xrightarrow[-\otimes^{\mathbb{L}}_{\overline{\Lambda}}X]{} & \mathcal{D}^-(\overline{\Gamma}) & \xrightarrow[-\otimes^{\mathbb{L}}_{\overline{\Gamma}}X^{-1}]{} & \mathcal{D}^-(\overline{\Gamma})
\end{array} \tag{3.9}$$

Commutativity on objects implies that $\mathcal{G}_{k\otimes S}\circ\mathcal{G}_{k\otimes T}$ sends the stalk complex $0\to k\otimes\Lambda\to 0$ to the stalk complex $0\to k\otimes\Lambda'\to 0$. Hence, due to unique lifting, $\mathcal{G}_S\circ\mathcal{G}_T$ sends $0\to\Lambda\to 0$ to $0\to\Lambda'\to 0$. $\mathcal{G}_S\circ\mathcal{G}_T$ hence agrees on objects with $-\otimes^{\mathbb{L}}_{\Lambda\,\mathrm{id}}\Lambda_\alpha$ for some isomorphism $\alpha:\Lambda'\xrightarrow{\sim}\Lambda$. Thus, $\mathcal{G}_{k\otimes S}\circ\mathcal{G}_{k\otimes T}$ will agree on tilting complexes with $-\otimes^{\mathbb{L}}_{k\otimes\Lambda\ \mathrm{id}}k\otimes\Lambda_{\mathrm{id}_k\otimes\alpha}$. Hence, due to commutativity on objects, we must have that $-\otimes^{\mathbb{L}}_{k\otimes\Lambda\ \tilde{\varphi}}\overline{\Lambda}_{\mathrm{id}}$ and $-\otimes^{\mathbb{L}}_{k\otimes\Lambda\ \mathrm{id}_k\otimes\alpha}k\otimes\Lambda_{\mathrm{id}}\otimes^{\mathbb{L}}_{k\otimes\Lambda\ \varphi}\overline{\Lambda}_{\mathrm{id}} = -\otimes^{\mathbb{L}}_{k\otimes\Lambda\ \varphi\circ(\mathrm{id}_k\otimes\alpha)}\overline{\Lambda}_{\mathrm{id}}$ agree on tilting complexes. This however is the same as saying that $\tilde{\varphi}=\beta\circ\varphi\circ(\mathrm{id}_k\otimes\alpha)$, where $\beta\in\mathrm{Aut}_k(\overline{\Lambda})$ is an automorphism such that $-\otimes^{\mathbb{L}}_{\overline{\Lambda}\ \beta}\overline{\Lambda}_{\mathrm{id}}$ fixes all tilting complexes. This means, by definition, that $(\Lambda,\varphi)\sim(\Lambda',\tilde{\varphi})$. So we proved that $\Phi_{X^{-1}}\circ\Phi_X=\mathrm{id}$, and $\Phi_X\circ\Phi_{X^{-1}}=\mathrm{id}$ follows by swapping the roles of X and X^{-1}. □

Proposition 3.7. $\mathrm{Out}_k(\overline{\Lambda})$ *acts on* $\widehat{\mathfrak{L}}(\overline{\Lambda})$ *from the left via*

$$\alpha\cdot(\Lambda,\varphi):=(\Lambda,\alpha\circ\varphi) \tag{3.10}$$

Proof. The above formula clearly defines an action of $\mathrm{Aut}_k(\overline{\Lambda})$. In order to verify that it defines an action of $\mathrm{Out}_k(\overline{\Lambda})$, we just need to check that for any inner automorphism α of $\overline{\Lambda}$ we have $(\Lambda,\varphi)\sim(\Lambda,\alpha\circ\varphi)$. But an inner automorphism α of $\overline{\Lambda}$ gives us an inner automorphism $\varphi^{-1}\circ\alpha\circ\varphi$ of $k\otimes\Lambda$, which lifts to an inner automorphism $\hat{\alpha}$ of Λ (since the natural map of unit groups $\Lambda^\times\to(k\otimes\Lambda)^\times$ is surjective). (Λ,φ) and $(\Lambda,\alpha\circ\varphi)=(\Lambda,\varphi\circ(\mathrm{id}_k\otimes\hat{\alpha}))$ are then clearly equivalent in the sense of Definition 3.1. □

Proposition 3.8. *If k is algebraically closed, then* $\mathrm{Out}^0_k(\overline{\Lambda})$ *lies in the kernel of the action of* $\mathrm{Out}_k(\overline{\Lambda})$ *on* $\widehat{\mathfrak{L}}(\overline{\Lambda})$.

Proof. This follows directly from Theorem 2.98. □

Proposition 3.9. *Let* $\mathrm{Out}_k(\overline{\Lambda})_{\overline{T}}$ *respectively* $\mathrm{Out}_k(\overline{\Gamma})_{\overline{S}}$ *denote the stabilizers of the isomorphism classes of* $\overline{T}$ *respectively* $\overline{S}$. *There is an isomorphism*

$$-^X:\mathrm{Out}_k(\overline{\Lambda})_{\overline{T}}\xrightarrow{\sim}\mathrm{Out}_k(\overline{\Gamma})_{\overline{S}} \tag{3.11}$$

such that for all $\alpha\in\mathrm{Out}_k(\overline{\Lambda})_{\overline{T}}$ *we have*

$$\Phi_X(\alpha\cdot(\Lambda,\varphi))=\alpha^X\cdot\Phi_X(\Lambda,\varphi) \tag{3.12}$$

In particular, Φ_X *induces a bijection*

$$\mathrm{Out}_k(\overline{\Lambda})_{\overline{T}}\setminus\widehat{\mathfrak{L}}(\overline{\Lambda})\longleftrightarrow\mathrm{Out}_k(\overline{\Gamma})_{\overline{S}}\setminus\widehat{\mathfrak{L}}(\overline{\Gamma}) \tag{3.13}$$

Proof. Set

$$-^X : \mathrm{Out}_k(\overline{\Lambda})_{\overline{T}} \longrightarrow \mathrm{TrPic}(\overline{\Gamma}) : \alpha \mapsto X^{-1} \otimes^{\mathbb{L}}_{\overline{\Lambda}} {}_{\mathrm{id}}\overline{\Lambda}_\alpha \otimes^{\mathbb{L}}_{\overline{\Lambda}} X \tag{3.14}$$

First note that the restriction of X^{-1} to $\mathcal{D}^b(\Lambda)$ is isomorphic to $\overline{T}$ by definition of $\overline{T}$. Since α stabilizes the isomorphism class of $\overline{T}$, the restriction of $X^{-1} \otimes^{\mathbb{L}}_{\overline{\Lambda}} {}_{\mathrm{id}}\overline{\Lambda}_\alpha \otimes^{\mathbb{L}}_{\overline{\Lambda}} X$ to $\mathcal{D}^b(\overline{\Gamma})$ is isomorphic to $0 \to \overline{\Gamma} \to 0$. Thus $X^{-1} \otimes^{\mathbb{L}}_{\overline{\Lambda}} {}_{\mathrm{id}}\overline{\Lambda}_\alpha \otimes^{\mathbb{L}}_{\overline{\Lambda}} X$ is isomorphic to $0 \to {}_{\mathrm{id}}\overline{\Gamma}_\beta \to 0$ for some $\beta \in \mathrm{Aut}_k(\overline{\Gamma})$. That is, the image of $-^X$ as defined above is indeed contained in $\mathrm{Out}_k(\overline{\Gamma}) \leqslant \mathrm{TrPic}(\overline{\Gamma})$. Now $\overline{S}$ is by definition just the restriction of X to $\mathcal{D}^b(\overline{\Gamma})$, and hence $\overline{S} \otimes^{\mathbb{L}}_{\overline{\Gamma}} X^{-1} \otimes^{\mathbb{L}}_{\overline{\Lambda}} {}_{\mathrm{id}}\overline{\Lambda}_\alpha \otimes^{\mathbb{L}}_{\overline{\Lambda}} X$ is isomorphic to the restriction of ${}_{\mathrm{id}}\overline{\Lambda}_\alpha \otimes^{\mathbb{L}}_{\overline{\Lambda}} X$ to $\mathcal{D}^b(\overline{\Gamma})$ which is again isomorphic to $\overline{S}$. So $-^X$ does indeed define a map with image in $\mathrm{Out}_k(\overline{\Gamma})_{\overline{S}}$. It is also easy to see that $-^X$ is a group homomorphism, and that $-^{X^{-1}}$ is a two-sided inverse for $-^X$.

Now the claim of (3.12) follows from the commutativity of the following diagram

$$\begin{array}{ccc} \mathcal{D}^-(\overline{\Lambda}) & \xrightarrow{-\otimes^{\mathbb{L}}_{\overline{\Lambda}} X} & \mathcal{D}^-(\overline{\Gamma}) \\ {\scriptstyle -\otimes^{\mathbb{L}}_{\overline{\Lambda}} {}_{\mathrm{id}}\overline{\Lambda}_\alpha}\downarrow & & \downarrow{\scriptstyle -\otimes^{\mathbb{L}}_{\overline{\Gamma}} {}_{\mathrm{id}}\overline{\Gamma}_{\alpha^X}} \\ \mathcal{D}^-(\overline{\Lambda}) & \xrightarrow[-\otimes^{\mathbb{L}}_{\overline{\Lambda}} X]{} & \mathcal{D}^-(\overline{\Gamma}) \end{array} \tag{3.15}$$

by gluing it below diagram (3.5). □

Definition 3.10. *Define the set $\mathfrak{L}(\overline{\Lambda})$ to be the set of all isomorphism classes of $\mathcal{O}$-orders Λ such that $k \otimes \Lambda \cong \overline{\Lambda}$. Clearly, $\mathfrak{L}(\overline{\Lambda})$ is in bijection with $\mathrm{Out}_k(\overline{\Lambda}) \setminus \widehat{\mathfrak{L}}(\overline{\Lambda})$. Furthermore, we define the projection map*

$$\Pi : \widehat{\mathfrak{L}}(\overline{\Lambda}) \longrightarrow \mathfrak{L}(\overline{\Lambda}) : (\Lambda, \varphi) \mapsto \Lambda \tag{3.16}$$

Corollary 3.11. *Assume k is algebraically closed, $\overline{\Lambda}$ is symmetric, and $\overline{T}$ is a two-term complex. Assume furthermore that $\mathrm{Out}_k^s(\overline{\Lambda}) = \mathrm{Out}_k(\overline{\Lambda})$ and $\mathrm{Out}_k^s(\overline{\Gamma}) = \mathrm{Out}_k(\overline{\Gamma})$ (a sufficient criterion for this is for instance that the Cartan matrices of $\overline{\Lambda}$ and $\overline{\Gamma}$ have no non-trivial permutation symmetries). Then there is a bijection*

$$\mathfrak{L}(\overline{\Lambda}) \longleftrightarrow \mathfrak{L}(\overline{\Gamma}) \tag{3.17}$$

Proof. Proposition 2.97 implies that $\overline{S}$ may be assumed to be a two-term complex as well. The assertion now follows from (3.13) together with Theorem 2.99, since the latter implies that $\mathrm{Out}_k(\overline{\Lambda})_{\overline{T}} = \mathrm{Out}_k(\overline{\Lambda})$ and $\mathrm{Out}_k(\overline{\Gamma})_{\overline{S}} = \mathrm{Out}_k(\overline{\Gamma})$. □

The following proposition is useful to prove a "unique lifting property" for the group ring of $\mathrm{SL}_2(p^f)$ in defining characteristic, which we will do in Chapter 5.

Proposition 3.12. *Assume k is algebraically closed. Let $\Lambda \in \mathfrak{L}(\overline{\Lambda})$, and let $\gamma : k \otimes \Lambda \xrightarrow{\sim} \overline{\Lambda}$. be an isomorphism. Now assume*

$$\overline{\mathrm{Aut}_{\mathcal{O}}(\Lambda)} \cdot \mathrm{Out}_k^0(\overline{\Lambda}) = \mathrm{Out}_k(\overline{\Lambda}) \tag{3.18}$$

where $\overline{\mathrm{Aut}_{\mathcal{O}}(\Lambda)}$ is the image of $\mathrm{Aut}_{\mathcal{O}}(\Lambda)$ in $\mathrm{Out}_k(\overline{\Lambda})$ (here we identify $k \otimes \Lambda$ with $\overline{\Lambda}$ via γ). Then the fiber $\Pi^{-1}(\{\Lambda\})$ has cardinality one.

Proof. Let $(\Lambda, \varphi) \in \widehat{\mathfrak{L}}(\overline{\Lambda})$ for some $\varphi : k \otimes \Lambda \xrightarrow{\sim} \overline{\Lambda}$. Now if (3.18) holds, we can write $\gamma \circ \varphi^{-1} = \gamma \circ (\mathrm{id}_k \otimes \hat{\alpha}) \circ \gamma^{-1} \circ \beta$ for some $\hat{\alpha} \in \mathrm{Aut}_{\mathcal{O}}(\Lambda)$ and $\beta \in \mathrm{Aut}_k(\overline{\Lambda})$ such that the image of β in $\mathrm{Out}_k(\overline{\Lambda})$ lies in $\mathrm{Out}_k^0(\overline{\Lambda})$. Hence $\gamma \circ (\mathrm{id}_k \otimes \hat{\alpha}^{-1}) = \beta \circ \varphi$. Proposition 3.8 (together with the definition of "$\sim$") implies $(\Lambda, \gamma) \sim (\Lambda, \beta^{-1} \circ \gamma \circ (\mathrm{id}_k \otimes \hat{\alpha}^{-1})) = (\Lambda, \varphi)$. □

We can also state a version of this proposition that does not rely on an algebraically closed field. To do so, we just need one little preparatory fact concerning extension of scalars.

Proposition 3.13. *Let A be a k-algebra and let S and T be two tilting complexes over A. Then $S \cong T$ (in $\mathcal{D}^b(A)$) if and only if $\bar{k} \otimes S \cong \bar{k} \otimes T$ in $\mathcal{D}^b(\bar{k} \otimes A)$.*

Proof. If $\bar{k} \otimes S \cong \bar{k} \otimes T$, then there has to be some finite extension k' of k such that $k' \otimes S \cong k' \otimes T$. By restriction we will also have $S^{[k':k]} \cong T^{[k':k]}$. There is a k-algebra B and an invertible complex X of A-B-bimodules such that $S \otimes_A^{\mathbb{L}} X$ is the stalk complex of a module. But then $T \otimes_A^{\mathbb{L}} X$ will be the stalk complex of a module as well, since it becomes isomorphic to $S \otimes_A^{\mathbb{L}} X$ upon tensoring with k' (note that the functors $- \otimes_A^{\mathbb{L}} X$ and $k' \otimes_k -$ commute with each other; also, tilting complexes which are stalk complexes of modules are distinguished by the fact that they have non-trivial homology in only a single degree). Now we can simply apply Krull-Schmidt theorem. So $S^{[k':k]} \otimes_A^{\mathbb{L}} X \cong T^{[k':k]} \otimes_A^{\mathbb{L}} X$ implies that $S \otimes_A^{\mathbb{L}} X \cong T \otimes_A^{\mathbb{L}} X$ and therefore $S \cong T$. □

Proposition 3.14. *Let $\Lambda \in \mathfrak{L}(\overline{\Lambda})$, and let $\gamma : k \otimes \Lambda \xrightarrow{\sim} \overline{\Lambda}$. be an isomorphism. Now assume*

$$\overline{\mathrm{Aut}_{\mathcal{O}}(\Lambda)} \cdot G = \mathrm{Out}_k(\overline{\Lambda}) \tag{3.19}$$

where $\overline{\mathrm{Aut}_{\mathcal{O}}(\Lambda)}$ is as in Proposition 3.12 and and $G \leqslant \mathrm{Out}_k(\overline{\Lambda})$ is a subgroup such that the $\bar{k}$-linear extensions of all elements of G lie in $\mathrm{Out}_{\bar{k}}^0(\bar{k} \otimes_k \overline{\Lambda})$. Then the fiber $\Pi^{-1}(\{\Lambda\})$ has cardinality one.

Proof. The proof is the same as that of Proposition 3.12, except that we need to know that G acts trivially on $\widehat{\mathfrak{L}}(\overline{\Lambda})$ (since we cannot apply Proposition 3.8 directly anymore). In order to see that G acts trivially on $\widehat{\mathfrak{L}}(\overline{\Lambda})$ it suffices to show that G fixes all tilting objects in $\mathcal{D}^b(\overline{\Lambda})$. Here we can use Proposition 3.13: For any tilting complex T over $\overline{\Lambda}$ and any $\alpha \in G$, the complexes T and $T \otimes_{\overline{\Lambda}}^{\mathbb{L}} {}_{\alpha}\overline{\Lambda}_{\mathrm{id}}$ are isomorphic since they become isomorphic over $\bar{k} \otimes_k \overline{\Lambda}$. □

3.2 Tilting Orders in Semisimple Algebras

What we would want to do now is to partition the set $\widehat{\mathfrak{L}}(\overline{\Lambda})$ into manageable pieces, so that the map Φ_X defined in the previous section restricts to a bijection between corresponding pieces of $\widehat{\mathfrak{L}}(\overline{\Lambda})$ and $\widehat{\mathfrak{L}}(\overline{\Gamma})$. In order to do that in the next section, we first need to study how those properties of orders that we are interested in behave under derived equivalences. The results in this section are elementary and therefore certainly known, but we were unable to find explicit references for most of them, which is why we include proofs. We assume throughout this section that Λ is an $\mathcal{O}$-order in a finite-dimensional semisimple K-algebra A.

Lemma 3.15. *If $T \in \mathcal{K}^b(\mathbf{proj}_{\Lambda})$ is a tilting complex for Λ and $\mathrm{End}_{\mathcal{D}^b(\Lambda)}(T)$ is torsion-free as an $\mathcal{O}$-module, then $K \otimes T$ is a tilting complex for A. Furthermore, $\mathrm{End}_{\mathcal{D}^b(\Lambda)}(T)$ is a full $\mathcal{O}$-order in $\mathrm{End}_{\mathcal{D}^b(A)}(K \otimes T)$.*

Proof. First we show that $\mathrm{Hom}_{\mathcal{D}^b(A)}(K \otimes T, K \otimes T[i]) = 0$ for $i \neq 0$. Assume that $\varphi \in \mathrm{Hom}_{\mathcal{D}^b(A)}(K \otimes T, K \otimes T[i])$ for some i. Then we may view φ (or rather a representative of it) as a morphism of graded modules $K \otimes T \to K \otimes T[i]$ commuting with the differential. As such we may restrict it to T, and for a large enough $n \in \mathbb{N}$, we will have $\mathrm{Im}(\pi^n \cdot \varphi) \subseteq T[i]$. Hence $\pi^n \cdot \varphi$ defines an element in $\mathrm{Hom}_{\mathcal{D}^b(\Lambda)}(T, T[i])$. For $i = 0$ this implies that $\mathrm{End}_{\mathcal{D}^b(\Lambda)}(T)$ is a full $\mathcal{O}$-lattice in $\mathrm{End}_{\mathcal{D}^b(A)}(K \otimes T)$. For $i \neq 0$ this implies that $\pi^n \cdot \varphi$ is homotopic to zero, and hence so is φ (by dividing the homotopy by π^n).

Since $K \otimes^{\mathbb{L}} - : \mathcal{D}^-(\Lambda) \to \mathcal{D}^-(A)$ is a functor of triangulated categories that maps T to $K \otimes T$, it is clear that $\mathrm{add}(K \otimes T)$ contains the image of $\mathrm{add}(T)$. But $\mathrm{add}(T)$ is equal to $\mathcal{K}^b(\mathbf{proj}_\Lambda)$ by definition, and so in particular contains $0 \to \Lambda \to 0$, which maps to $0 \to A \to 0$, which in turn clearly generates $\mathcal{K}^b(\mathbf{proj}_A)$. Hence $\mathrm{add}(K \otimes T) = \mathcal{K}^b(\mathbf{proj}_A)$. □

Lemma 3.16. *Situation as above. Let $V_1, \ldots, V_n$ be representatives for the isomorphism classes of simple A-modules. Then there are sets Ω_i for $i \in \mathbb{Z}$ with $\biguplus_i \Omega_i = \{1, \ldots, n\}$ and numbers $\delta_j \in \mathbb{Z}_{>0}$ for $j \in \{1, \ldots, n\}$ such that*

$$K \otimes T \cong_{\mathcal{D}^b(A)} \ldots \xrightarrow{0} \underbrace{\bigoplus_{j \in \Omega_i} V_j^{\delta_j}}_{\text{degree } i} \xrightarrow{0} \bigoplus_{j \in \Omega_{i+1}} V_j^{\delta_j} \xrightarrow{0} \ldots \tag{3.20}$$

In particular, each V_j occurs as a direct summand of precisely one of the $H^i(K \otimes T)$. Also, it follows that $H^i(K \otimes T) \cong \bigoplus_{j \in \Omega_i} V_j^{\delta_j}$, and the map below is an isomorphism:

$$\bigoplus_i H^i : \ \mathrm{End}_{\mathcal{D}^b(A)}(K \otimes T) \xrightarrow{\sim} \bigoplus_i \mathrm{End}_A(H^i(K \otimes T)) \cong \bigoplus_i \bigoplus_{j \in \Omega_i} \mathrm{End}_A(V_j)^{\delta_j \times \delta_j} \tag{3.21}$$

Proof. $K \otimes T$ is, as a complex over A, certainly split, and hence isomorphic in the homotopy category to a complex C with zero differentials. Clearly $H^i(C) = C^i$ and $\mathrm{End}_{\mathcal{D}^b(A)}(C) = \bigoplus_i \mathrm{End}_A(C^i)$. So all that remains to show is that any V_j occurs in precisely on C^i. But $\mathrm{Hom}_{\mathcal{D}^b(A)}(C, C[l]) = 0$ for $l \neq 0$ implies that $\mathrm{Hom}_A(C^i, C^{i+l}) = 0$ for all $l \neq 0$ and hence that V_j occurs in at most one C^i. In the other direction, the fact that $\mathrm{add}(C) = \mathcal{D}^b(A)$ implies that each V_j has to occur in some C^i. □

Definition 3.17. *The above lemma contains a definition of sets Ω_i and numbers δ_j associated to the tilting complex T. We keep this notation. In addition to those, define $\varepsilon : \{1, \ldots, n\} \to \mathbb{Z}$ to map j to the unique i such that $j \in \Omega_i$.*

Note that in the context of perfect isometries, the numbers $(-1)^{\varepsilon(i)}$ are known as the "signs" in the "bijection with signs" induced by a perfect isometry.

Theorem 3.18 ([Zim99, Theorem 1])**.** *Assume Λ is a symmetric order. Then any $\mathcal{O}$-algebra Γ which is derived equivalent to Λ is again an $\mathcal{O}$-order, and symmetric.*

We close this section by making Theorem 3.18 constructive. We wish to give an explicit symmetrizing form (as defined in Section 2.2) for Γ, provided we know one for Λ (which we usually do, for instance in the case when Λ is a block of a group ring).

In the statement of the following theorem we assume that A is of the shape

$$A \cong \bigoplus_{i=1}^{n} D_i^{d_i \times d_i} \tag{3.22}$$

for certain skew-fields D_i (finite-dimensional over K) and certain numbers d_i. Moreover, we identify

$$Z(A) \cong \bigoplus_{i=1}^{n} Z(D_i) \tag{3.23}$$

Theorem 3.19 (Transfer of the Symmetrizing Form). *Let Λ be symmetric, and let $T \in \mathcal{K}^b(\mathbf{proj}_\Lambda)$ be a tilting complex. Set $\Gamma = \mathrm{End}_{\mathcal{D}^b(\Lambda)}(T)$, and $B = \mathrm{End}_{\mathcal{D}^b(A)}(K \otimes T)$. Identify*

$$Z(A) = \bigoplus_{j=1}^{n} Z(\mathrm{End}_A(V_j)) = Z(B) \tag{3.24}$$

Let $u = (u_1, \ldots, u_n) \in Z(A)^\times$ such that Λ is self-dual with respect to the trace bilinear form $T_u : A \times A \to K$ induced by u. Then Γ is self-dual with respect to the trace bilinear form $T_{\tilde{u}} : B \times B \to K$, where

$$\tilde{u} = ((-1)^{\varepsilon(1)} \cdot u_1, \ldots, (-1)^{\varepsilon(n)} \cdot u_n) \in Z(B)^\times \tag{3.25}$$

where ε is as defined in Definition 3.17.

Proof. Let $\hat{u} = (\hat{u}_1, \ldots, \hat{u}_n) \in Z(B)$ be an element such that Γ is actually self-dual with respect to $T_{\hat{u}}$. Then the p-valuations of the $\hat{u}_i$ are in fact independent of the particular choice of $\hat{u}$, since the coset $\hat{u} \cdot Z(\Gamma)^\times \in Z(B)^\times / Z(\Gamma)^\times$ is. Furthermore, the $u' \in Z(B)^\times$ such that Γ is integral with respect to $T_{u'}$ are precisely the elements of $\hat{u} \cdot (Z(\Gamma) \cap Z(B)^\times)$. An element of $\hat{u} \cdot Z(\Gamma) \cap Z(B)^\times$ lies in $\hat{u} \cdot Z(\Gamma)^\times$ if and only in $\nu_p(u_i') = \nu_p(\hat{u}_i)$ for all i (all of those assertions are elementary). Now assume we had shown that Γ is integral with respect to $T_{\tilde{u}}$. Then we have $\tilde{u} \in \hat{u} \cdot (Z(\Gamma) \cap Z(B)^\times)$. Thus $\nu_p(\tilde{u}_i) \geqslant \nu_p(\hat{u}_i)$ for all i, and equality for all i holds if and only if $\tilde{u} \in \hat{u} \cdot Z(\Gamma)^\times$, that is, if Γ is self-dual with respect to $T_{\tilde{u}}$. So we have seen (up to the assumption above that we have yet to prove) that if Λ is self-dual with respect T_u and Γ is self-dual with respect to $T_{\hat{u}}$, then $\nu_p(u_i) \geqslant \nu_p(\hat{u}_i)$, and, by swapping the roles of Λ and Γ, also $\nu_p(\hat{u}_i) \geqslant \nu_p(u_i)$. In conclusion, we have $\nu_p(\tilde{u}_i) = \nu_p(u_i) = \nu_p(\hat{u}_i)$ for all i, which, by the above considerations, implies that Γ is self-dual with respect to $T_{\tilde{u}}$.

So far we have reduced the problem to showing that Γ is integral with respect to $T_{\tilde{u}}$, which we will do now. So let $\varphi \in \mathrm{End}_{\mathcal{D}^b(\Lambda)}(T)$ (and fix a representative in $\mathrm{End}_{\mathcal{C}^b(\mathbf{proj}_\Lambda)}(T)$). Then φ^i induces an endomorphism of T^i, and we can decompose the A-module $K \otimes T^i$ as follows

$$K \otimes T^i = \underbrace{H^i(K \otimes T)}_{=:H^i} \oplus \underbrace{\mathrm{Im}(\mathrm{id}_K \otimes d^{i-1})}_{=:Z_-^i} \oplus \underbrace{K \otimes T^i / \mathrm{Ker}(\mathrm{id}_K \otimes d^i)}_{=:Z_+^i} \tag{3.26}$$

Define π_{H^i}, $\pi_{Z_-^i}$ and $\pi_{Z_+^i}$ to be the corresponding projections. Define $B^i := \mathrm{End}_A(K \otimes T^i)$, $B_H^i := \pi_{H^i} B^i \pi_{H^i} = \mathrm{End}_A(H^i)$, $B_+^i := \pi_{Z_+^i} B^i \pi_{Z_+^i} = \mathrm{End}_A(Z_+^i)$ and $B_-^i := \pi_{Z_-^i} B^i \pi_{Z_-^i} =$

$\mathrm{End}_A(Z^i_-)$. Now we have

$$\begin{aligned} & \sum_i (-1)^i \cdot T_{1_{B^i}\cdot u}(\varphi^i) \\ = & \sum_i T_{\pi_{H^i}\cdot\tilde{u}}(\pi_{H^i}\varphi^i\pi_{H^i}) + (-1)^i \cdot T_{\pi_{Z^i_+}\cdot u}(\pi_{Z^i_+}\varphi^i\pi_{Z^i_+}) + (-1)^i \cdot T_{\pi_{Z^i_-}\cdot u}(\pi_{Z^i_-}\varphi^i\pi_{Z^i_-}) \\ \overset{(*)}{=} & \sum_i T_{\pi_{H^i}\cdot\tilde{u}}(\pi_{H^i}\varphi^i\pi_{H^i}) \overset{(**)}{=} T_{\tilde{u}}(\varphi) \end{aligned} \tag{3.27}$$

Here $(*)$ holds because

$$T_{\pi_{Z^i_+}\cdot u}(\pi_{Z^i_+}\varphi^i\pi_{Z^i_+}) = T_{\pi_{Z^{i+1}_-}\cdot u}(\pi_{Z^{i+1}_-}\varphi^{i+1}\pi_{Z^{i+1}_-}) \tag{3.28}$$

as φ is a map of chain complexes. The equality $(**)$ holds in fact just by definition, as we have identified $\bigoplus \mathrm{End}_A(H^i) = B$. The left side is trivially integral, as $\varphi^i \in \mathrm{End}_\Lambda(T^i)$, and $\mathrm{End}_\Lambda(T^i)$ is a self-dual (and so in particular integral) lattice in B^i with respect to $T_{1_{B^i}\cdot u}$. Hence the right side is also integral. So Γ is indeed integral with respect to $T_{\tilde{u}}$. This concludes the proof. □

3.3 Partitioning the Set of Lifts by Rational Conditions

Now we continue with what we started in Section 3.1. We want to define "rational conditions" on lifts that behave well under the map Φ_X, that is, conditions such that Φ_X restricts to a bijective map between the lifts of $\overline{\Lambda}$ that fulfill the given conditions and the lifts of $\overline{\Gamma}$ that fulfill certain corresponding conditions. Probably the simplest of those conditions is to demand that the K-span of Λ shall be Morita-equivalent to a certain semisimple K-algebra A. It follows from the previous section that Φ_X sends lifts of $\overline{\Lambda}$ with K-span Morita equivalent to A to lifts of $\overline{\Gamma}$ with the same property (and Φ_X^{-1} does it the other way round).

Theorem 3.20. *Let $\overline{\Lambda}$ and $\overline{\Gamma}$ be finite-dimensional k-algebras that are derived equivalent. Let the derived equivalence be afforded by the (one-sided) tilting complex $\overline{T}$, and let X be a two-sided tilting complex such that its inverse has restriction to $\mathcal{D}^b(\mathbf{proj}_{\overline{\Lambda}})$ isomorphic to $\overline{T}$. Set $\Phi := \Pi \circ \Phi_X$. Define*

$$\widehat{\mathfrak{L}}_s(\overline{\Lambda}) := \{(\Lambda, \varphi) \in \widehat{\mathfrak{L}}(\overline{\Lambda}) \mid K \otimes \Lambda \text{ is semisimple } \} \tag{3.29}$$

Then Φ_X induces a bijection

$$\widehat{\mathfrak{L}}_s(\overline{\Lambda}) \longleftrightarrow \widehat{\mathfrak{L}}_s(\overline{\Gamma}) \tag{3.30}$$

The following holds:

(i) If $(\Lambda, \varphi), (\Lambda', \varphi') \in \widehat{\mathfrak{L}}(\overline{\Lambda})$ are two lifts with $Z(K \otimes \Lambda) \cong Z(K \otimes \Lambda')$, then

$$Z(K \otimes \Phi(\Lambda, \varphi)) \cong Z(K \otimes \Phi(\Lambda', \varphi')) \tag{3.31}$$

and every choice of an isomorphism $\gamma : Z(K\otimes\Lambda) \to Z(K\otimes\Lambda')$ gives rise to a (canonically defined) isomorphism $\Phi(\gamma) : Z(K \otimes \Phi(\Lambda, \varphi)) \to Z(K \otimes \Phi(\Lambda', \varphi'))$.

(ii) If $(\Lambda, \varphi), (\Lambda', \varphi') \in \widehat{\mathfrak{L}}(\overline{\Lambda})$ are two lifts and $\gamma : Z(\Lambda) \xrightarrow{\sim} Z(\Lambda')$ is an isomorphism, then $\Phi(\gamma) : Z(\Phi(\Lambda, \varphi)) \to Z(\Phi(\Lambda', \varphi'))$ is well defined and an isomorphism as well.

(iii) *If* $(\Lambda, \varphi), (\Lambda', \varphi') \in \widehat{\mathfrak{L}}_s(\overline{\Lambda})$ *are two lifts, and* $\gamma : Z(K \otimes \Lambda) \xrightarrow{\sim} Z(K \otimes \Lambda')$ *is an isomorphism such that* $D^{\Lambda} = D^{\Lambda'}$ *up to permutation of columns (where rows are identified via* γ*), then* $D^{\Phi(\Lambda,\varphi)} = D^{\Phi(\Lambda',\varphi')}$ *up to permutation of columns (where rows are identified via* $\Phi(\gamma)$*).*

(iv) *If* $(\Lambda, \varphi), (\Lambda', \varphi') \in \widehat{\mathfrak{L}}_s(\overline{\Lambda})$ *are two lifts with* $D^{\Lambda} = D^{\Lambda'}$ *up to permutation of rows and columns then* $D^{\Phi(\Lambda,\varphi)} = D^{\Phi(\Lambda',\varphi')}$ *up to permutation of rows and columns.*

Proof. The fact that Φ_X induces a bijection between $\widehat{\mathfrak{L}}_s(\overline{\Lambda})$ and $\widehat{\mathfrak{L}}_s(\overline{\Gamma})$ follows from the last section.

Let $(\Lambda, \varphi) \in \widehat{\mathfrak{L}}(\overline{\Lambda})$. Then $Z(K \otimes \Phi(\Lambda))$ is naturally isomorphic to $K \otimes Z(\Phi(\Lambda))$. But there is an isomorphism between $Z(\Lambda)$ and $Z(\Phi(\Lambda, \varphi))$ (letting $c \in Z(\Lambda)$ correspond to the endomorphism of the tilting complex that is given by multiplication with c in every degree). That proves (i), and shows how $\Phi(\gamma)$ should be defined. The claim of (ii) also follows.

To the proof of (iii): Let $T \in \mathcal{C}^b(\mathbf{proj}_{\Lambda})$ be the lift of $\overline{T}$ (we identify $k \otimes \Lambda$ and $\overline{\Lambda}$ via φ). Write $\overline{T} = \overline{T}_0 \oplus \overline{T}_1$ such that $\mathcal{G}_{\overline{T}}(\overline{T}_0) \cong 0 \to \overline{P} \to 0$ for a projective indecomposable $\overline{\Gamma}$-module $\overline{P}$. By Remark 3.3 there is a corresponding direct sum decomposition $T = T_0 \oplus T_1$ and we will have $\mathcal{G}_T(T_0) \cong 0 \to P \to 0$, where P is the unique projective indecomposable $\Phi(\Lambda, \varphi)$-module with $k \otimes P \cong \overline{P}$. Then take e_P to be the endomorphism of T inducing the identity on T_0 and the zero map on T_1. Clearly this is a primitive idempotent in $\Phi(\Lambda, \varphi)$ (which is just $\mathrm{End}_{\mathcal{D}^b(\Lambda)}(T)$, so this statement makes sense) with $e_P\Phi(\Lambda, \varphi) \cong P$. So the decomposition number associated to P and the simple $K \otimes \Phi(\Lambda, \varphi)$-module corresponding to the simple $K \otimes \Lambda$-module V_j (under the isomorphism of the centers) is just the $\mathrm{End}_{K\otimes\Lambda}(V_j)$-rank of the image of e_P in $\mathrm{End}_{K\otimes\Lambda}(V_j)^{\delta_j \times \delta_j}$ under the map given in Lemma 3.16. On the other hand (due the way Lemma 3.16 was obtained) this is just the absolute value of the coefficient of $[V_j] \in K_0(\mathbf{mod}_{K\otimes\Lambda})$ in the image under the decomposition map of $\sum_i (-1)^i \cdot [T_0^i] \in K_0(\mathbf{proj}_{\Lambda})$. But due to the isomorphism $K_0(\mathbf{proj}_{\Lambda}) \cong K_0(\mathbf{proj}_{\overline{\Lambda}})$ we can compute this coefficient, and hence the decomposition matrix of $\Phi(\Lambda, \varphi)$, from the knowledge of a direct sum decomposition of $\overline{T}$ and the knowledge of the decomposition matrix of Λ (since the latter determines the map $K_0(\mathbf{proj}_{\overline{\Lambda}}) \longrightarrow K_0(\mathbf{mod}_{K\otimes\Lambda})$). Therefore, if the decomposition matrices of Λ and Λ' coincide, then so do the decomposition matrices of $\Phi(\Lambda, \varphi)$ and $\Phi(\Lambda', \varphi')$. This concludes the proof of (iii). The explicit formula for the decomposition matrix of $\Phi(\Lambda, \varphi)$ we obtained above is in fact independent of the knowledge of $Z(K \otimes \Lambda)$. This implies (iv). □

Remark 3.21. *The last theorem shows that the lifts* $(\Lambda, \varphi) \in \widehat{\mathfrak{L}}(\overline{\Lambda})$ *that satisfy certain conditions (as listed in the theorem) correspond via* Φ_X *to lifts* $(\Gamma, \psi) \in \widehat{\mathfrak{L}}(\overline{\Gamma})$ *that satisfy a corresponding set of conditions. We shall call these kinds of conditions on* Λ *"rational conditions".*

3.4 Additional Results

By $(K, \mathcal{O}, k)$ we denote a p-modular system. Assume K to be complete, and by π denote a uniformizer for $\mathcal{O}$. The following theorem is related to the concept of amalgamation depths. Assume we are given an $\mathcal{O}$-order Λ such that $K \otimes \Lambda$ is split semisimple, $k \otimes \Lambda$ is split, and all decomposition numbers are $\leqslant 1$. When trying to determine a basic order for Λ, we are usually faced with the following problem: Given two orthogonal primitive idempotents $e, f \in \Lambda$, determine the isomorphism type of $e\Lambda f$ as an $e\Lambda e$-module (or, equivalently, as an $f\Lambda f$-module or a $Z(\Lambda)$-module). In some cases it turns out that this module is isomorphic to

$\varepsilon eZ(\Lambda)$ where $\varepsilon \in Z(K \otimes \Lambda)$ is the sum of all central primitive idempotents which annihilate neither e nor f (note that $\varepsilon eZ(\Lambda)$ needs not necessarily be an $e\Lambda e$-module at all). The following theorem gives a method of verifying whether this is the case, by showing that if $\varepsilon eZ(\Lambda)$ meets the conditions of the theorem, then we just have to check that a single amalgamation depth of $\varepsilon eZ(\Lambda)$ coincides with the corresponding amalgamation depth of $e\Lambda f$. This may be done for instance by determining one particular exponent matrix. Note that the condition of the theorem that $\varepsilon eZ(\Lambda)$ should be a symmetric algebra will be met in case that $\varepsilon eZ(\Lambda) \cong_{e\Lambda e} e\Lambda f$ holds in the first place and if one additionally assumes that Λ is symmetric and carries an involution that fixes the center of Λ as well as e and f.

Theorem 3.22. *Let $\Lambda \subseteq \mathcal{O}^n$ be a local symmetric order. Fix a K^n-equivariant symmetric bilinear form $K^n \times K^n \to K$ such that $\Lambda = \Lambda^\sharp$ with respect to that form. By ε_i we denote the i-th standard basis vector in K^n. Then we claim: If $L \leqslant K^n$ is a full Λ-lattice with*

$$\frac{L \cdot \varepsilon_i}{L \cdot \varepsilon_i \cap L} \cong_{\mathcal{O}} \frac{\Lambda \cdot \varepsilon_i}{\Lambda \cdot \varepsilon_i \cap \Lambda} \quad \textit{for } \text{some } i \in \{1, \ldots, n\} \tag{3.32}$$

then $L \cong_\Lambda \Lambda$.

Proof. Let L be a full Λ-lattice in K^n not isomorphic to Λ. Without loss we may assume that $L \subseteq \mathcal{O}^n$ and $L \cdot \varepsilon_i = \mathcal{O} \cdot \varepsilon_i$ for all $i \in \{1, \ldots, n\}$.

Assume $\Lambda \subsetneqq L \subseteq \mathcal{O}^n$. We then have $L^\sharp \subseteq \mathrm{Jac}(\Lambda)$, and hence $L \supseteq \mathrm{Jac}(\Lambda)^\sharp$. Now, since Λ is local and symmetric, the following holds:

$$\frac{\mathrm{Jac}(\Lambda)^\sharp \cdot \pi}{\Lambda \cdot \pi} = \mathrm{Soc}\left(\frac{\Lambda}{\Lambda \cdot \pi}\right) \tag{3.33}$$

Therefore: If $l \in \Lambda$ such that $l + \Lambda \cdot \pi \in \mathrm{Soc}(\Lambda / \Lambda \cdot \pi)$ then $l \cdot \pi^{-1} \in L$.

Now let $l \in \Lambda \cdot \varepsilon_i \cap \Lambda$ (where i is arbitrary) such that $l \notin \Lambda \cdot \pi$. Then $l \cdot \Lambda + \Lambda \cdot \pi / \Lambda \cdot \pi \cong_\Lambda k$ (where k is viewed as the simple Λ-module). This implies that $l + \Lambda \cdot \pi \in \mathrm{Soc}(\Lambda / \Lambda \cdot \pi)$, and thus according to the above $l \cdot \pi^{-1} \in L$. Since $L \cdot \varepsilon_i = \Lambda \cdot \varepsilon_i = \mathcal{O} \cdot \varepsilon_i$, we conclude

$$\mathrm{length}_{\mathcal{O}} \frac{L \cdot \varepsilon_i}{L \cdot \varepsilon_i \cap L} \leqslant \mathrm{length}_{\mathcal{O}} \frac{\Lambda \cdot \varepsilon_i}{\Lambda \cdot \varepsilon_i \cap \Lambda} - 1 \tag{3.34}$$

and this holds for each i, as i was chosen arbitrary. One implication of the above is that for any idempotent $1 \neq \varepsilon \in K^n$ and any i with $\varepsilon \cdot \varepsilon_i \neq 0$ the epimorphism

$$\Lambda \cdot \varepsilon_i / \Lambda \cdot \varepsilon_i \cap \Lambda \twoheadrightarrow \Lambda \cdot \varepsilon_i / (\Lambda \cdot \varepsilon_i) \cap (\Lambda \cdot \varepsilon) \tag{3.35}$$

is proper (to see this choose $L = \Lambda \cdot \varepsilon \oplus \Lambda \cdot (1 - \varepsilon)$). Also we have shown at this point that a Λ-lattice L with $\Lambda \subseteq L \subseteq \mathcal{O}^n$ cannot possibly be a counterexample to the statement of the Theorem.

Now we consider an arbitrary Λ-lattice $L \subseteq \mathcal{O}^n$ with $L \cdot \varepsilon_i = \mathcal{O} \cdot \varepsilon_i$ for all i. We pick an element $v \in L$ with $v \cdot \varepsilon_1 = \varepsilon_1$. If $v \cdot \varepsilon_i \in \mathcal{O}^\times \cdot \varepsilon_i$ for all i then clearly $\Lambda \subseteq v^{-1} \cdot L \subseteq \mathcal{O}^n$, and what we have shown above implies that L is not a counterexample to the Theorem. So assume that there is a $j \in \{2, \ldots, n\}$ such that $v \cdot \varepsilon_j = r \cdot \varepsilon_j$ with $r \in (\pi)_{\mathcal{O}}$. Then we pick a $w \in L$ with $w \cdot \varepsilon_j = \varepsilon_j$ and look at $v' = v - r \cdot w$. By construction $v' \cdot \varepsilon_1 \in \mathcal{O}^\times \cdot \varepsilon_1$ and

$v' \cdot \varepsilon_j = 0$. Now we have a series of epimorphisms

$$\frac{\Lambda \cdot \varepsilon_1}{\Lambda \cdot \varepsilon_1 \cap \Lambda} \twoheadrightarrow \frac{\Lambda \cdot \varepsilon_1}{\Lambda \cdot \varepsilon_1 \cap \Lambda \cdot (1-\varepsilon_j)} \twoheadrightarrow \frac{L \cdot \varepsilon_1}{L \cdot \varepsilon_1 \cap \Lambda \cdot v'} \twoheadrightarrow \frac{L \cdot \varepsilon_1}{L \cdot \varepsilon_1 \cap L} \tag{3.36}$$

of which at least the first one is proper, since it is a special case of the epimorphism in (3.35). Hence the first and the last term cannot possibly be isomorphic. Repetition of this argument with ε_1 replaced by $\varepsilon_2, \varepsilon_3, \ldots, \varepsilon_n$ yields that the Theorem holds for L. □

The following theorem gives a statement analogous to the theorem above, now for group rings of arbitrary finite groups (it would be interesting to know if the corresponding theorem for arbitrary symmetric orders with semisimple K-span holds as well). It is a generalization of a proposition found in [But73], where the assertion is proved for the group V_4 in characteristic two.

Theorem 3.23 (cf. [But73, Proposition 1.5]). *Let G be a finite group and $\chi \in \mathrm{Irr}_K(G)$ a character of that group. Let V be the simple KG-module associated to that character. Assume that V is absolutely irreducible and that for a (or, equivalently, any) full $\mathcal{O}G$-lattice $L \subset V$, $L/\pi L$ is absolutely irreducible as well. Let*

$$\varepsilon = \frac{\chi(1)}{|G|} \sum_{g \in G} \chi(g^{-1}) \cdot g \in Z(KG)$$

be the central primitive idempotent belonging to the character χ and set $d(\chi) := \nu_p(\chi(1)) - \nu_p(|G|)$. Furthermore, let M be an $\mathcal{O}G$-lattice with $M \cdot \varepsilon \neq 0$.

Assume there is an $m \in M \cap M \cdot \varepsilon$ such that $m \notin M \cdot \pi$ and $m \in M \cdot \varepsilon \cdot p^{d(\chi)}$. Then M has a non-zero projective direct summand.

Proof. First we argue that we can assume without loss that $m \cdot \mathcal{O}G$ is an irreducible lattice. To see this let $e_1, \ldots, e_l$ be a full set of orthogonal primitive idempotents in $\mathcal{O}G$. Then, for some i, the element $m \cdot e_i$ is not contained in $M \cdot \pi$ (since if all of them were contained in $M \cdot \pi$, then so would be m). The $\mathcal{O}G$-module $m \cdot e_i \cdot \mathcal{O}G$ is an epimorphic image of $e_i \cdot \mathcal{O}G$, and by Brauer reciprocity (see Proposition 2.23) the multiplicity of V in $K \otimes e_i \cdot \mathcal{O}G$ is equal to one. Therefore $K \otimes m \cdot e_i \cdot \mathcal{O}G \cong V$, and hence we may simply replace m by $m \cdot e_i$ (and we will assume that we have done this from here on out).

Due to our assumptions on V, all $\mathcal{O}G$-lattices in $K \otimes m \cdot \mathcal{O}G \cong V$ are of the form $m \cdot \mathcal{O}G \cdot \pi^z$ for some $z \in \mathbb{Z}$, and therefore we must have $M \cap m \cdot KG = m \cdot \mathcal{O}G$. This means that $m \cdot \mathcal{O}G$ is a pure sublattice of M.

Now we may choose an $\mathcal{O}$-basis of $m \cdot \mathcal{O}G$ from the set $\{m \cdot g \mid g \in G\}$. By virtue of $m \cdot \mathcal{O}G$ being a pure sublattice of M we may complete such a basis to an $\mathcal{O}$-basis of M. So we get a subset $\{g_1, \ldots, g_h\} \subseteq G$ such that

$$M \cong_{\mathcal{O}} \bigoplus_{i=1}^{h} m \cdot g_i \cdot \mathcal{O} \oplus (\text{other summands}) \tag{3.37}$$

For $i \in \{1, \ldots, h\}$ define an $\mathcal{O}$-homomorphism $\Psi_i : M \to \mathcal{O}$ which is the projection onto the $m \cdot g_i \cdot \mathcal{O}$-summand in the above decomposition (sending $m \cdot g_i$ to 1). Note that there are maps

$\Delta_{i,j}: G \to \mathcal{O}$ for $j \in \{1, \ldots, h\}$ such that

$$\Psi_j(n \cdot g) = \sum_{i=1}^{h} \Delta_{i,j}(g) \cdot \Psi_i(n) \quad \forall g \in G, n \in M \tag{3.38}$$

Also note that $\hat{\Delta} : g \mapsto (\Delta_{i,j}(g))_{i,j=1}^{h}$ defines a representation of G that makes $\mathcal{O}^{1\times h}$ isomorphic to $m \cdot \mathcal{O}G$ as an $\mathcal{O}G$-module. Now define $\mathcal{O}G$-homomorphisms $\Phi_i : M \to \mathcal{O}G$ via

$$\Phi_i(n) := \widehat{\mathrm{Tr}}_1^G(\Psi_i)(n) = \sum_{g \in G} \Psi_i(n \cdot g^{-1}) \cdot g \tag{3.39}$$

If we choose $\hat{m} \in M$ such that $\hat{m} \cdot \varepsilon \cdot p^{d(\chi)} = m$, then

$$\begin{aligned}
\varepsilon \cdot \sum_{i=1}^{h} \Phi_i(\hat{m} \cdot g_i) &= \sum_{i=1}^{h} \Phi_i(\hat{m} \cdot \varepsilon \cdot g_i) \\
&= p^{-d(\chi)} \cdot \sum_{i=1}^{h} \sum_{g \in G} \Psi_i(m \cdot g_i \cdot g^{-1}) \cdot g \\
&= p^{-d(\chi)} \cdot \sum_{i=1}^{h} \sum_{g \in G} \sum_{j=1}^{h} \Psi_j(m \cdot g_i) \cdot \Delta_{j,i}(g^{-1}) \cdot g \\
&= p^{-d(\chi)} \cdot \sum_{g \in G} \left(\sum_{i=1}^{h} \Delta_{i,i}(g^{-1}) \right) \cdot g \\
&= p^{-d(\chi)} \cdot \sum_{g \in G} \chi(g^{-1}) \cdot g \;=\; \varepsilon \cdot \alpha \quad \text{for some } \alpha \in \mathcal{O}^{\times}
\end{aligned}$$

We conclude from this that the element $\sum_i \Phi_i(\hat{m} \cdot g_i) \in \mathcal{O}G$ is not nilpotent modulo p and therefore that it is not contained in the radical of $\mathcal{O}G$. So there will be a projective indecomposable P and and epimorphism $\Pi : \mathcal{O}G \to P$ such that $\Pi \circ \xi$ is onto, where we take ξ to be

$$\xi : \bigoplus^{h} M \to \mathcal{O}G : (m_1, \ldots, m_h) \mapsto \sum_{i=1}^{h} \Phi_i(m_i)$$

Now it follows from the Krull-Schmidt theorem that P is a direct summand of M. □

The following proposition shows that the equality $\dim_k Z(kG) = \mathrm{rank}_{\mathcal{O}} Z(\mathcal{O}G) = \dim_K Z(KG)$ holds not only for group rings but (appropriately formulated) for symmetric orders in general. On the one hand this shows that this equality cannot be used to distinguish group rings among arbitrary symmetric orders (in some sense a negative result). On the other hand it justifies why we more or less implicitly assume that (3.40) is an isomorphism in the chapters on dihedral blocks and $\mathrm{SL}_2(p^f)$ (although, in the context of those chapters, comparing dimensions instead of applying the proposition below would also do the job).

Proposition 3.24. *Let R be a PID and let Λ be a symmetric R-order. Let π be a prime element in R and set $k := R/(\pi)_R$. Then the natural homomorphism*

$$k \otimes_R Z(\Lambda) \longrightarrow Z(k \otimes_R \Lambda) \tag{3.40}$$

is an isomorphism.

Proof. By definition, Λ is symmetric if and only if

$$\Lambda \cong_{\Lambda^{\mathrm{op}}\otimes_R\Lambda} \mathrm{Hom}_R(\Lambda, R) \tag{3.41}$$

We have

$$Z(\Lambda) \cong \mathrm{End}_{\Lambda^{\mathrm{op}}\otimes_R\Lambda}(\Lambda) \cong \mathrm{Hom}_{\Lambda^{\mathrm{op}}\otimes_R\Lambda}(\Lambda, \mathrm{Hom}_R(\Lambda, R)) \cong \mathrm{Hom}_R(\Lambda \otimes_{\Lambda^{\mathrm{op}}\otimes_R\Lambda} \Lambda, R) \tag{3.42}$$

Hence

$$\mathrm{rank}_R\, Z(\Lambda) = \mathrm{rank}_R\, \Lambda \otimes_{\Lambda^{\mathrm{op}}\otimes_R\Lambda} \Lambda \tag{3.43}$$

Similarly, with $\overline{\Lambda} := k \otimes_R \Lambda$,

$$\dim_k Z(\overline{\Lambda}) = \dim_k \overline{\Lambda} \otimes_{\overline{\Lambda}^{\mathrm{op}}\otimes_k\overline{\Lambda}} \overline{\Lambda} \tag{3.44}$$

But

$$\begin{aligned} k \otimes_R (\Lambda \otimes_{\Lambda^{\mathrm{op}}\otimes_R\Lambda} \Lambda) &\cong (k \otimes \Lambda) \otimes_{\Lambda^{\mathrm{op}}\otimes_R\Lambda} \Lambda \\ &\cong (k \otimes \Lambda) \otimes_{\Lambda^{\mathrm{op}}\otimes_R\Lambda} (k \otimes \Lambda) \cong \overline{\Lambda} \otimes_{\overline{\Lambda}^{\mathrm{op}}\otimes_k\overline{\Lambda}} \overline{\Lambda} \end{aligned} \tag{3.45}$$

Now (3.43) and (3.44) imply

$$\mathrm{rank}_R\, Z(\Lambda) = \dim_k Z(\overline{\Lambda}) \tag{3.46}$$

As for the injectivity, notice that $\Lambda/Z(\Lambda)$ is a torsion-free R-module (as $r \cdot m \in Z(\Lambda)$ for some $r \in R \setminus \{0\}$ implies $m \in Z(\Lambda)$), and hence we can write

$$\Lambda \cong_R Z(\Lambda) \oplus \text{ (Rest)} \tag{3.47}$$

which implies that $k \otimes_R Z(\Lambda)$ embeds into $k \otimes_R \Lambda$. □

Chapter 4

Blocks of Dihedral Defect

In this chapter we look at blocks of dihedral defect, that is, blocks with defect group D_{2^n}. In particular our ground ring will be an extension of the 2-adic integers. Over algebraically closed fields of characteristic two, blocks of dihedral defect have essentially been classified by Erdmann in [Erd90], leaving only finitely many possible Morita equivalence classes for any given n. Those Morita equivalence classes are given explicitly in terms of basic algebras, and our obvious aim is to lift these basic algebras to basic orders over an appropriate discrete valuation ring. Nevertheless, there is another motivation to look at dihedral blocks which has to do with a shortcoming of the classification in [Erd90]. Namely, what is actually classified in [Erd90] are "algebras of dihedral type". Those are algebras (over an algebraically closed field in arbitrary characteristic) which are symmetric, indecomposable, have non-degenerate Cartan matrix and fulfill a special condition concerning the shape of their stable Auslander-Reiten quiver (which is a notion we do not need in this thesis and therefore have not introduced). The problem at hand is that not all of these "algebras of dihedral type" are necessarily Morita equivalent to some dihedral defect block of some group ring, and in fact it was believed (and indeed turns out to be true) that only two out of the four possible Morita equivalence classes with two isomorphism classes of simple modules given in the classification can actually occur in group rings (for algebras of dihedral type with a different number of isomorphism classes of simple modules the question whether or not they may occur in group rings was not an open problem). Most recently, it has been shown in [BLS11] that in principal blocks of dihedral defect (with two simple modules), only two Morita equivalence classes can occur. In this thesis we show that this is indeed true for arbitrary such blocks. Note that this a result on blocks defined over an algebraically closed field. Integral methods show up only in its proof.

The main technical tools of this chapter are the methods relating lifts of derived equivalent algebras to one another described in Chapter 3. This allows a reduction of the lifting problem for dihedral blocks to algebras with decomposition numbers 0 and 1, which are accessible by linear algebra. It then turns out that, when appropriate rational conditions are imposed, there either is a unique lift or no lift at all. Of course, a block of a group ring defined over a field of positive characteristic always lifts to a block of the corresponding group ring defined over an appropriate discrete valuation ring. Hence, as long as we stick to rational conditions which are known to hold for the integral block, non-existence of a lift satisfying those rational conditions implies that the algebra in question cannot be a block of a group ring. This principle is used to establish the above mentioned result that certain Morita equivalence classes of algebras of dihedral type do not occur in group rings.

A quick note may be in place at this point as to why we do not treat dihedral blocks with just one isomorphism class of simple modules, or quaternion and semidihedral blocks (which are also classified in [Erd90]). In the case of the dihedral blocks with one simple module, the answer is that since these are local algebras, our methods using derived equivalences are useless (as finite-dimensional local algebras are derived equivalent if and only if they are Morita equivalent). However, such blocks are known to be nilpotent (a concept we did not introduce), and the theory of nilpotent blocks implies that they have to be Morita equivalent to the group ring of D_{2^n} over a sufficiently large discrete valuation ring. Quaternion and semidihedral blocks are, in principle, accessible by our methods, but in the "interesting" cases (in particular the quaternion case with two simple modules where Donovan's conjecture is still open) a reduction to decomposition numbers 0 and 1 is not possible.

4.1 Generalities

For this entire chapter we specialize K to be the 2-adic completion of the maximal unramified extension of $\mathbb{Q}_2$ (so, in particular, k will be algebraically closed). By $(K, \mathcal{O}, k)$ we denote the corresponding 2-modular system. We fix a finite group G and a block Λ of $\mathcal{O}G$ with dihedral defect group D_{2^n} for some fixed $n \geqslant 3$ (we use the convention where $|D_{2^n}| = 2^n$). Set $A := K \otimes \Lambda$ and $\overline{\Lambda} := k \otimes \Lambda$. For any $i \geqslant 2$ we denote by ζ_i a primitive 2^i-th root of unity in $\bar{K}$ (that is, we fix a choice for each i). In what follows, by a "character" we always mean an absolutely irreducible ordinary character with values in $\bar{K}$.

Lemma 4.1 (Facts from Number Theory). *(i) Define $K_i := K(\zeta_i + \zeta_i^{-1})$. K_i/K is a field extension of degree 2^{i-2}. Its Galois group is cyclic, and we denote by γ_i one of its generators. Hence the subfield lattice of K_i is just a chain, and in fact equal to*

$$K = K_2 \subset K_3 \subset \ldots \subset K_i \tag{4.1}$$

We denote by $\mathcal{O}_i$ the integral closure of $\mathcal{O}$ in K_i.

(ii) The field extension K_i/K is totally ramified and the 2-valuation of its discriminant is equal to $(i-1) \cdot 2^{i-2} - 1$. See [Lia76, Theorem 1] (the result from that paper carries over to our situation without change, as the 2-valuation of the discriminant of $K(\zeta_i + \zeta_i^{-1})/K$ equals the 2-valuation of the discriminant of $\mathbb{Q}(\zeta_i + \zeta_i^{-1})/\mathbb{Q}$ since both extensions have the same degree).

(iii) If G is any finite group, then KG is isomorphic to a direct sum of matrix rings over fields (i. e., no non-commutative division algebras occur in the Wedderburn decomposition of KG). To see this let D be a skew-field that occurs in the Wedderburn decomposition of $\mathbb{Q}_2G$, and denote the center of D by E. Then by [Rei75, Corollary 31.10] the unique unramified extension E' of E of degree equal to the index of D will split D. We may write $E' = E \cdot F$ for some unramified extension F of $\mathbb{Q}_2$. Then $F \otimes_{\mathbb{Q}_2} D$ will be isomorphic to a direct sum of matrix rings over E'. Since F is contained in K, this proves the assertion.

Theorem 4.2 (Brauer). *(i) There are precisely $2^{n-2} + 3$ characters in Λ. Four of these characters have height zero, the rest have height one. See [Bra74, Theorem 1].*

(ii) *All characters of Λ take values in K_{n-1} (see [Bra74, Proposition (5A)]). There are exactly 5 characters in Λ with values in K. The remaining characters lie in families F_r for $r = 1, \dots, n-3$, where each F_r is a single $\mathrm{Gal}(K_{n-1}/K)$-conjugacy class of characters. Each F_r consists of 2^r elements (see [Bra74, Theorem 3]). Together with Lemma 4.1 (i) and elementary Galois theory the latter implies that a character in F_r takes values in K_{r+2}.*

(iii) *The four characters of height zero in Λ take values in K. See [Bra74, Theorem 4].*

Note that we may as well denote the one-element set containing the unique K-rational character of height one by F_0, and use indices $r = 0, \dots, n-3$. The grouping of the characters into four height zero characters and $n-2$ families F_r of height one characters seems more natural in what follows.

Corollary 4.3. *From the above it follows immediately that Λ is an $\mathcal{O}$-order in*

$$A = \bigoplus_{i=1}^{4} K^{\delta_i \times \delta_i} \oplus \bigoplus_{r=0}^{n-3} K_{r+2}^{\delta'_r \times \delta'_r} \quad \text{for certain } \delta_i, \delta'_i \in \mathbb{Z}_{>0} \tag{4.2}$$

that is self-dual with respect to T_u, where $u = (u_1, u_2, u_3, u_4, \dots, u_{n+2}) \in Z(A)$ with $\nu_2(u_i) = -n$ for $i = 1, \dots, 4$ and $\nu_2(u_i) = -n+1$ for $i > 4$. Of course the analogous statement will hold for a basic order of Λ.

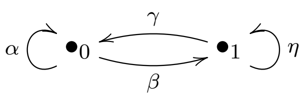
Theorem 4.4 (Erdmann). *The basic algebra of $\overline{\Lambda}$ is isomorphic to one of the algebras of dihedral type in the list given in the appendix of [Erd90].* (Technically, this follows from [Erd90, Lemma IX.2.2] together with the fact that $\overline{\Lambda}$ is known to be of tame representation type and thus has to occur in the list.)

Theorem 4.5 (Holm and Linckelmann). *(i) In Erdmann's classification, the algebras $\mathcal{D}(2A)^{\kappa,c}$ and $\mathcal{D}(2B)^{\kappa,c}$, for any combination $\kappa = 2^{n-2} \geqslant 1$ and $c \in \{0,1\}$, are derived equivalent. In particular, for fixed n, there are at most two derived equivalence classes of 2-blocks over k with defect group D_{2^n} and two simple modules. See [Hol97].*

(ii) *There is precisely one derived equivalence class of 2-blocks over k with defect group D_{2^n} and three simple modules. See [Lin94, Theorem 1].*

4.2 Blocks with Two Simple Modules

Assume in this section that Λ has precisely two isomorphism classes of simple modules. We first assume that $\overline{\Lambda}$ is Morita equivalent to $\mathcal{D}(2B)^{\kappa,c}$ for some $c \in \{0,1\}$ and $\kappa = 2^{n-2}$ (the latter is implied by $\kappa + 3 = \dim_k Z(\mathcal{D}(2B)^{\kappa,c}) = \dim_K Z(A) = 2^{n-2}+3$). Now let Λ_0 be a basic algebra of Λ. From [Erd90] we know that $k \otimes \Lambda_0 \cong kQ/I$, where

$Q =$  (4.3)

and

$$I = \langle \beta\eta,\ \eta\gamma,\ \gamma\beta,\ \alpha^2 - c\cdot\alpha\beta\gamma,\ \alpha\beta\gamma - \beta\gamma\alpha,\ \gamma\alpha\beta - \eta^\kappa \rangle \tag{4.4}$$

We may assume the following rational structure on Λ_0

$$\begin{array}{c|ccc}
\mathbf{Z(A)} & \mathbf{u} & \mathbf{0} & \mathbf{1} \\
\hline
K & u_1 & 1 & 0 \\
K & u_1 & 1 & 0 \\
K & u_2 & 1 & 1 \\
K & u_2 & 1 & 1 \\
\hline
K_{r+2} & u_3 & 0 & 1
\end{array} \quad [\text{ exactly once for each } r = 0, \ldots, n-3 \,] \tag{4.5}$$

where $u_1, u_2 \in K$ have 2-valuation $-n$ and $u_3 \in K$ has 2-valuation $-n+1$.

Remark 4.6. *We say that a lift Γ of $k \otimes \Lambda_0$ satisfies the rational conditions given above if all of the following conditions hold:*

(1) $K \otimes \Gamma$ is Morita equivalent to $K \oplus K \oplus K \oplus K \oplus \bigoplus_{r=0}^{n-3} K_{r+2}$ (so in particular $K \otimes \Gamma$ will be semisimple).

(2) The decomposition matrix of Γ is as in (4.5), where the individual rows pertain to the summand of the center that is given on the left of the table.

(3) There exists some $u = (u_1, u_1, u_2, u_2, u_3, \ldots, u_3) \in K^{n+2} \subseteq K \oplus K \oplus K \oplus K \oplus \bigoplus_{r=0}^{n-3} K_{r+2}$ with $\nu_2(u_1) = \nu_2(u_2) = -n$ and $\nu_2(u_3) = -n+1$ such that Γ is self-dual with respect to T_u.

We should probably also explain what we mean when we say that two lifts Γ and Γ' of $k \otimes \Lambda_0$ subject to the above rational conditions have equal *center. The point is that the rows of the decomposition matrix of Γ are canonically in bijection with the Wedderburn components of $Z(K \otimes \Gamma)$ (or, equivalently, central primitive idempotents in $K \otimes \Gamma$). Naturally we demand that there should be an isomorphism $\gamma : Z(\Gamma) \xrightarrow{\sim} Z(\Gamma')$ such that if $\varepsilon \in Z(K \otimes \Gamma)$ is a central primitive idempotent, then the rows in the respective decomposition matrices pertaining to ε respectively $(\mathrm{id}_K \otimes \gamma)(\varepsilon)$ are equal (up to some fixed permutation of the columns).*

Lemma 4.7. *Let*

$$\Gamma \subseteq \mathcal{O} \oplus \mathcal{O} \oplus \bigoplus_{r=0}^{n-3} \mathcal{O}_{r+2} \tag{4.6}$$

be a local $\mathcal{O}$-order such that $k \otimes \Gamma$ is generated by a single nilpotent element η (so, in particular, $k \otimes \Gamma = k[\eta]$). Furthermore assume that Γ is symmetric with respect to T_u, where $u = (u_1, u_2, u_3 \ldots, u_n) \in K \oplus K \oplus \bigoplus_{r=0}^{n-3} K_{r+2}$ with $\nu_2(u_1) = \nu_2(u_2) = -n$ and $\nu_2(u_i) = -n+1$ for all $i > 2$. Then for some $x \in k^\times$ there exists a preimage $\hat{\eta}$ of $x \cdot \eta$ in Γ of the form

$$(0, 4, \pi_0, \ldots, \pi_{n-3}) \tag{4.7}$$

where the π_r are prime elements in the ring $\mathcal{O}_{r+2}$.

Proof. If $\hat{\eta} = (a, b, d_0 \ldots, d_{n-3})$ is a preimage of η, then $a \in (2)_{\mathcal{O}}$, and hence $\hat{\eta} - a \cdot (1, \ldots, 1)$ is a preimage of η as well. So we may assume without loss that $a = 0$. Hence some non-zero scalar multiple of η will have a preimage in Γ of the following shape:

$$\hat{\eta} = (0, 2^l, \pi_0, \ldots, \pi_{n-3}) \quad \text{with } \pi_r \in \mathrm{Jac}(\mathcal{O}_{r+2}) \tag{4.8}$$

Note that we do not know yet that the π_r are prime elements in $\mathcal{O}_{r+2}$. All we can say at this point is $\nu_2(\pi_r) \geqslant 2^{-r}$ (because we know the ramification indices of the extensions K_{r+2}/K to be 2^r). The fact that Γ is self-dual with respect to u implies that

$$\nu_2\left(\left[\mathcal{O}_{n-1}^{2^{n-2}+1} : \mathcal{O}_{n-1}\otimes_{\mathcal{O}}\Gamma\right]\right) = \frac{1}{2}\left(2n + (2^{n-2}-1)(n-1)\right) \tag{4.9}$$

Here, for two $\mathcal{O}_{n-1}$-lattices $N \subseteq M$ such that M/N is a torsion module, we denote by $[M : N]$ the product of all elementary divisors of M/N (of course, this is only well-defied up to units). The left-hand side of the above equation is equal to the 2-valuation of the determinant of the $(2^{n-2}+1)\times(2^{n-2}+1)$ Vandermonde matrix $M(S)$ associated to the values

$$S = \{s_1,\dots,s_{2^{n-2}+1}\} := \{0, 2^l, \pi_r^{\alpha_r} \mid r = 0,\dots,n-3,\ \alpha_r \in \mathrm{Gal}(K_{r+2}/K)\} \tag{4.10}$$

But the factorization (note that we fix an arbitrary total ordering on the Galois groups $\mathrm{Gal}(K_i/K)$)

$$\begin{aligned}\prod_{i>j}(s_i - s_j) = \ & \pm 2^l\cdot\left(\prod_{r=0}^{n-3}\prod_{\alpha\in\mathrm{Gal}(K_{r+2}/K)}\pi_r^\alpha\right)\cdot\left(\prod_{r=0}^{n-3}\prod_{\alpha\in\mathrm{Gal}(K_{r+2}/K)}(2^l-\pi_r^\alpha)\right)\\ &\cdot\prod_{r=0}^{n-3}\left(\left(\prod_{q=r+1}^{n-3}\prod_{\alpha\in\mathrm{Gal}(K_{r+2}/K)}\prod_{\beta\in\mathrm{Gal}(K_{q+2}/K)}(\pi_r^\alpha-\pi_q^\beta)\right)\right.\\ &\left.\cdot\prod_{\alpha>\beta\in\mathrm{Gal}(K_{r+2}/K)}(\pi_r^\alpha-\pi_r^\beta)\right)\end{aligned} \tag{4.11}$$

of $\det M(S)$ yields the following estimate of its 2-valuation:

$$\begin{aligned}&\nu_2(\det M(S))\\ \geqslant\ & l + \sum_{r=0}^{n-3}\frac{1}{2^r}\cdot 2^r + \sum_{r=0}^{n-3}\frac{1}{2^r}\cdot 2^r + \sum_{r=0}^{n-3}\left(\sum_{q=r+1}^{n-3}(2^{r+q}\cdot\frac{1}{2^q}) + \frac{1}{2}\nu_2\,\mathrm{discrim}_K(K_{r+2})\right)\\ =\ & l + 2(n-2) + \sum_{r=0}^{n-3}\left((n-3-r)\cdot 2^r + \frac{1}{2}\left((r+1)\cdot 2^r - 1\right)\right)\\ =\ & \frac{1}{2}n + \frac{1}{8}2^n n + l - \frac{1}{8}2^n - \frac{3}{2}\end{aligned} \tag{4.12}$$

Here we used that for any $x \in \mathrm{Jac}(\mathcal{O}_i)$ we have

$$\begin{aligned}\nu_2\prod_{\alpha>\beta\in\mathrm{Gal}(K_i/K)}(x^\alpha - x^\beta) \ &=\ \nu_2\left(\left[\mathcal{O}_{n-1}^{2^{i-2}} : \mathcal{O}_{n-1}\otimes_{\mathcal{O}}\mathcal{O}[x]\right]\right)\\ &\geqslant\ \nu_2\left(\left[\mathcal{O}_{n-1}^{2^{i-2}} : \mathcal{O}_{n-1}\otimes_{\mathcal{O}}\mathcal{O}_i\right]\right) = \frac{1}{2}\nu_2(\mathrm{discrim}_K(K_i))\end{aligned} \tag{4.13}$$

Now the right hand side of (4.9) has to be greater than or equal to the right hand side of (4.12). This implies $l \leqslant 2$. On the other hand, the assumptions on u would imply that $\nu_2(T_u(\hat\eta)) < 0$ if $l \leqslant 1$, which is of course impossible. Hence $l = 2$, and in particular the "$\geqslant$" in (4.12) is really an equality, which is easily seen to be equivalent to $\nu_2(\pi_r) = 2^{-r}$ for all

$r = 0, \dots, n-3$. □

Theorem 4.8. *If $\Gamma, \Gamma' \in \mathfrak{L}(\mathcal{D}(2B)^{\kappa,c})$ (where $\kappa = 2^{n-2}$) satisfy the rational conditions stated in (4.5) and $Z(\Gamma) = Z(\Gamma')$, then $\Gamma \cong \Gamma'$. Furthermore, the existence of such a lift implies $c = 0$.*

Proof. Our general approach is to determine the structure of Γ up to some parameters, and then conclude that these parameters are determined by the knowledge of $Z(\Gamma)$. We assume (without loss) that

$$\Gamma \subseteq \mathcal{O} \oplus \mathcal{O} \oplus \mathcal{O}^{2\times 2} \oplus \mathcal{O}^{2\times 2} \oplus \bigoplus_{r=0}^{n-3} \mathcal{O}_{r+2} \tag{4.14}$$

Choose lifts $\hat{e}_0$ and $\hat{e}_1$ in Γ of the idempotents e_0 and e_1 in $\mathcal{D}(2B)^{\kappa,c}$. Assume without loss that these idempotents are diagonal, and identify in the obvious way

$$\begin{array}{ll} \Gamma_{00} := \hat{e}_0\Gamma\hat{e}_0 \subseteq \mathcal{O}\oplus\mathcal{O}\oplus\mathcal{O}\oplus\mathcal{O} & \Gamma_{11} := \hat{e}_1\Gamma\hat{e}_1 \subseteq \mathcal{O}\oplus\mathcal{O}\oplus\bigoplus_{r=0}^{n-3}\mathcal{O}_{r+2} \\ \Gamma_{10} := \hat{e}_1\Gamma\hat{e}_0 \subseteq \mathcal{O}\oplus\mathcal{O} & \Gamma_{01} := \hat{e}_0\Gamma\hat{e}_1 \subseteq \mathcal{O}\oplus\mathcal{O} \end{array} \tag{4.15}$$

We first look at Γ_{11}. Note that $e_1\mathcal{D}(2B)^{\kappa,c}e_1 \cong k[\eta]$, and therefore Lemma 4.7 tells us that there is a lift $\hat{\eta} \in \Gamma_{11}$ of some non-zero scalar multiple of η of the form $(0, 4, \pi_0, \dots, \pi_{n-3})$.

Now we consider Γ_{00}. We may assume without loss that Γ_{00} is equal to the row space of

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 2^a & x & y \\ 0 & 0 & 2^b & z \\ 0 & 0 & 0 & 2^n \end{bmatrix} \quad \text{for certain } a, b \in \mathbb{Z}_{>0} \text{ and } x, y, z \in (2)_{\mathcal{O}} \tag{4.16}$$

We may furthermore assume without loss that $\hat{\alpha} := [0, 2^a, x, y]$ is a lift of a (non-zero) scalar multiple of α. To see this first note that $\hat{\alpha} \notin \Gamma_{01} \cdot \Gamma_{10} + 2 \cdot \Gamma_{00}$, and therefore the image of $\hat{\alpha}$ in $\mathcal{D}(2B)^{\kappa,c}$ will be of the form $c_1 \cdot \alpha + c_2 \cdot \beta\gamma + c_3 \cdot \alpha\beta\gamma$ with $c_1, c_2, c_3 \in k$ and $c_1 \neq 0$. For all $c_1, c_2 \in k$ there is an automorphism of $\mathcal{D}(2B)^{\kappa,c}$ with $\alpha \mapsto \alpha + c_1 \cdot \beta\gamma + c_2 \cdot \alpha\beta\gamma$, $\beta \mapsto \beta$, $\gamma \mapsto \gamma$ and $\eta \mapsto \eta$ (to verify this just plug the right hand sides into the defining relations of $\mathcal{D}(2B)^{\kappa,c}$). Thus we may replace α by an appropriate multiple of the image of $\hat{\alpha}$ in $\mathcal{D}(2B)^{\kappa,c}$.

Next we look at the trace form T_u to get some restrictions on the parameters (by "$\sim$" we mean "equal up to units in $\mathcal{O}$"):

$$\begin{array}{rcl} T_u([1,1,1,1]) \sim 2^{-n} \cdot (2 + 2 \cdot \frac{u_2}{u_1}) \overset{!}{\in} \mathcal{O} & \Longrightarrow & \frac{u_1}{u_2} \equiv -1 \mod (2^{n-1}) \\ T_u([0,0,2^b,z]) \sim 2^{-n} \cdot (2^b + z) \overset{!}{\in} \mathcal{O} & \Longrightarrow & z \equiv -2^b \mod (2^n) \\ & \overset{\text{w.l.o.g.}}{\Longrightarrow} & z = -2^b \\ T_u(\hat{\alpha}) = 2^{-n} \cdot (2^a + (x+y) \cdot \frac{u_2}{u_1}) \overset{!}{\in} \mathcal{O} & \Longrightarrow & x + y \equiv -\frac{u_1}{u_2} \cdot 2^a \mod (2^n) \\ & \overset{\text{w.l.o.g.}}{\Longrightarrow} & x = 2^a - y \end{array} \tag{4.17}$$

Now let $\hat{\gamma} \in \Gamma_{10}$ and $\hat{\beta} \in \Gamma_{01}$ be lifts of non-zero scalar multiples of γ and β such that

$$\hat{\beta} \cdot \hat{\gamma} = [0, 0, 2^b, -2^b] + \xi \cdot [0, 0, 0, 2^n] \quad \text{for some } \xi \in \mathcal{O} \tag{4.18}$$

Then we have

$$\hat{\gamma} \cdot \hat{\beta} = [2^b, -2^b + \xi \cdot 2^n, 0, \ldots, 0] \in \Gamma_{11} \tag{4.19}$$

Since $\beta\eta = 0$ we have $\frac{1}{2} \cdot \hat{\gamma} \cdot \hat{\beta} \cdot \hat{\eta} \in \Gamma$, and thus

$$T_u\left(\frac{1}{2} \cdot \hat{\gamma} \cdot \hat{\beta} \cdot \hat{\eta}\right) = u_2 \cdot (-2^{b+1} + \xi 2^{n+1}) \sim 2^{b-n+1} \overset{!}{\in} \mathcal{O} \quad \Longrightarrow b \geqslant n-1 \tag{4.20}$$

But $a + b = n$ and a, b are both strictly greater than zero. This implies $b = n - 1$ and $a = 1$. To summarize: At this point we know that Γ_{00} is equal to the row space of

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 2 & x & 2-x \\ 0 & 0 & 2^{n-1} & 2^{n-1} \\ 0 & 0 & 0 & 2^n \end{bmatrix} \quad \text{for some } x \in (2)_{\mathcal{O}} \tag{4.21}$$

Note that (for large n) this row space will not be multiplicatively closed for all values of x. So this gives us a condition on x:

$$\hat{\alpha}^2 - 2\hat{\alpha} = [0, 0, x^2 - 2x, x^2 - 2x] \overset{!}{\in} \left\langle [0, 0, 2^{n-1}, 2^{n-1}], [0, 0, 0, 2^n] \right\rangle_{\mathcal{O}} \tag{4.22}$$

This is equivalent to $x^2 \equiv 2x \mod (2^{n-1})$, which in turn is equivalent to

$$x \equiv 0 \mod (2^{n-2}) \quad \text{or} \quad x \equiv 2 \mod (2^{n-2}) \tag{4.23}$$

For now let us assume $x \equiv 0 \mod (2^{n-2})$. Then $x = 2^{n-2} \cdot \xi$ for some $\xi \in \mathcal{O}$. But then

$$\hat{\alpha}^2 - 2\hat{\alpha} = \xi(2^{n-3}\xi - 1) \cdot [0, 0, 2^{n-1}, 2^{n-1}] \tag{4.24}$$

and

$$\hat{\alpha} \cdot \Gamma_{01} \cdot \Gamma_{10} + 2 \cdot \Gamma_{00} \subseteq \langle [0, 0, 0, 2^n] \rangle_{\mathcal{O}} + 2 \cdot \Gamma_{00} \tag{4.25}$$

Hence α^2 and $\alpha\beta\gamma$ would be linearly independent over k if $\xi(2^{n-3}\xi - 1) \in \mathcal{O}^\times$. The relation $\alpha^2 - c \cdot \alpha\beta\gamma$ prohibits this though. Therefore we must have $\xi(2^{n-3}\xi - 1) \in (2)_{\mathcal{O}}$, and thus $\alpha^2 = 0$. This implies the assertion that the existence of a lift implies $c = 0$. Furthermore, if $n > 3$, the fact that $\xi(2^{n-3}\xi - 1) \in (2)_{\mathcal{O}}$ implies $x \equiv 0 \mod (2^{n-1})$. If $n = 3$, the fact that $\xi(2^{n-3}\xi - 1) \in (2)_{\mathcal{O}}$ implies that either $x \equiv 0 \mod (2^{n-1})$ or $x \equiv 2 \mod (2^{n-1})$. Had we started with the assumption $x \equiv 2 \mod (2^{n-2})$, we would in the same fashion have arrived at $x \equiv 2 \mod (2^{n-1})$ (again with the exception of $n = 3$ where $x \equiv 0 \mod (2^{n-1})$ is also possible). Hence independent of our assumptions on x it follows that either $x \equiv 0 \mod (2^{n-1})$ or $x \equiv 2 \mod (2^{n-1})$, which means that Γ_{00} is equal to the row space of

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 2 & 0 & 2 \\ 0 & 0 & 2^{n-1} & 2^{n-1} \\ 0 & 0 & 0 & 2^n \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 2 & 2 & 0 \\ 0 & 0 & 2^{n-1} & 2^{n-1} \\ 0 & 0 & 0 & 2^n \end{bmatrix} \tag{4.26}$$

The row space of the second matrix is obtained from the row space of the first matrix by swapping the first two columns. This swapping of columns is induced by an automorphism of $K \otimes \Gamma$. Hence we may assume that we are in the case where Γ_{00} is equal to the row space of the leftmost matrix in (4.26). Note that the aforementioned automorphism which swaps the first two Wedderburn components of $Z(K \otimes \Gamma)$ might not fix $Z(\Gamma)$. This will however not matter to us since we only use that the projection of $Z(\Gamma)$ to all but the first two Wedderburn components is equal to the projection of $Z(\Gamma')$ to all but the first two Wedderburn components (instead of $Z(\Gamma) = Z(\Gamma')$; in particular, we could have made a slightly stronger assertion in the statement of the theorem).

Now if we project Γ_{00} onto its last two Wedderburn components we get an order $\Gamma'_{00} := \langle [1,1],[0,2] \rangle_{\mathcal{O}}$. Clearly Γ_{01} and Γ_{10} are both Γ'_{00}-lattices with the natural action. However, Γ'_{00} has only two non-isomorphic lattices L with $K \otimes L \cong K \otimes \Gamma'_{00}$, namely $L_1 = \mathcal{O} \oplus \mathcal{O}$ and $L_2 = \Gamma'_{00}$. Both of them are self-dual lattices in $K \otimes \Gamma'_{00}$. Assume $\Gamma_{01} = L_1$ (if we assume $\Gamma_{01} \cong L_1$, we may as well assume equality, by means of conjugation). By self-duality of Γ, we would then have $\Gamma_{10} = 2^n \cdot L_1$, and hence $\Gamma_{01}\Gamma_{10} \subset \mathrm{Jac}^2(\Gamma_{00})$. But $\beta\gamma$ certainly is not contained in $\mathrm{Jac}^2(e_0 \mathcal{D}(2B)^{\kappa,c} e_0)$. Hence we have a contradiction. This implies (without loss) $\Gamma_{01} = L_2$ and $\Gamma_{10} = [2^{n-1}, -2^{n-1}] \cdot L_2$.

All that is left to verify is that the choice of the π_i in Γ_{11} can be reconstructed from $Z(\Gamma)$. But from our knowledge of Γ_{00} and Γ_{11} we know that the following element is in $Z(\Gamma)$:

$$[0,4,0,4,\pi_0,\ldots,\pi_{n-3}] \in Z(\Gamma) \subset K \oplus K \oplus K \oplus K \oplus \bigoplus_{r=0}^{n-3} K_{r+2} \tag{4.27}$$

Hence the natural homomorphism $Z(\Gamma) \to \Gamma_{11}$ is surjective. This concludes the proof. □

Now assume that $\overline{\Lambda}$ is Morita-equivalent to $\mathcal{D}(2A)^{\kappa,c}$. Then we may assume the following rational structure of Λ_0

Z(A)	**u**	**0**	**1**	
K	u_1	1	0	
K	u_1	1	0	
K	u_2	1	1	
K	u_2	1	1	
K_{r+2}	u_3	2	1	[exactly once for each $r = 0,\ldots,n-3$]

(4.28)

where $u_1, u_2 \in K$ have 2-valuation $-n$ and $u_3 \in K$ has 2-valuation $-n+1$. We also know from [Hol97] that there is a tilting complex $\overline{T} \in \mathcal{K}^b(\mathbf{proj}_{\mathcal{D}(2A)^{\kappa,c}})$ with $\mathrm{End}_{\mathcal{D}^b(\mathcal{D}(2A)^{\kappa,c})}(\overline{T}) \cong \mathcal{D}(2B)^{\kappa,c}$ looking as follows:

$$\overline{T} = [0 \to P_1 \oplus P_1 \to P_0 \to 0] \oplus [0 \to P_1 \to 0 \to 0] \tag{4.29}$$

Let X be a two-sided tilting complex the inverse of which restricts to $\overline{T}$. Then clearly Φ_X maps a lift of $\mathcal{D}(2A)^{\kappa,c}$ satisfying the rational conditions (4.28) to a lift of $\mathcal{D}(2B)^{\kappa,c}$ satisfying the rational conditions (4.5). Hence we get the following corollary directly:

Corollary 4.9. *If there is a $\Gamma \in \mathfrak{L}(\mathcal{D}(2A)^{\kappa,c})$ subject to the rational conditions stated in (4.28), then $c = 0$. In particular, if B is a 2-block of kG with defect group D_{2^n} (where $n \geqslant 3$),* *and B has exactly two simple modules, then B is Morita-equivalent to either* $\mathcal{D}(2A)^{\kappa,0}$ *or* $\mathcal{D}(2B)^{\kappa,0}$ *with* $\kappa = 2^{n-2}$.

Corollary 4.10. *If* $\Gamma, \Gamma' \in \mathfrak{L}(\mathcal{D}(2A)^{\kappa,c})$ *(where* $\kappa = 2^{n-2}$*) satisfy the rational conditions stated in (4.28) and* $Z(\Gamma) = Z(\Gamma')$*, then* $\Gamma \cong \Gamma'$.

Proof. By Corollary 3.11 Φ_X induces a bijection between $\mathfrak{L}(\mathcal{D}(2A)^{\kappa,c})$ and $\mathfrak{L}(\mathcal{D}(2B)^{\kappa,c})$. Note that Φ_X maps the lifts of $\mathcal{D}(2A)^{\kappa,c}$ satisfying rational conditions as in (4.5) to lifts of $\mathcal{D}(2B)^{\kappa,c}$ satisfying rational conditions as in (4.28). Hence our assertion follows from Theorem 4.8. □

Remark 4.11. *For* $n = 3$*, the condition "*$Z(\Gamma) = Z(\Gamma')$*" can be dropped in Theorem 4.8 and Corollary 4.10. The reason for this is that in the proof of Theorem 4.8, we can (in the case* $n = 3$*) determine the parameter* π_0 *merely using symmetry. Namely,* $\hat{e}_1\Gamma\hat{e}_1$ *has to be equal to the row space of*

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 4 & 2 \\ 0 & 0 & 4 \end{bmatrix} \tag{4.30}$$

where the Wedderburn components are ordered as usual (that is, as in (4.5)). The line of reasoning is as follows: Lemma 4.7 tells us what the projection of $\hat{e}_1\Gamma\hat{e}_1$ *onto its first two Wedderburn components looks like. By the theory of self-dual orders we know that the intersection of* $\hat{e}_1\Gamma\hat{e}_1$ *with the first two Wedderburn components of its* K*-span is equal to the dual of the aforementioned projection with respect to* T_u*. This means that this intersection is equal to the row space of*

$$\begin{bmatrix} 2 & -2 \\ 0 & 8 \end{bmatrix} \tag{4.31}$$

and therefore $\hat{e}_1\Gamma\hat{e}_1$ *is equal to the row space of*

$$\begin{bmatrix} 1 & 1 & 1 \\ 2 & -2 & 0 \\ 0 & 8 & 0 \end{bmatrix} \tag{4.32}$$

which is the same as the one given in (4.30).

4.3 Explicit Computation of the Lifts

In this section we will compute the unique lift of $\mathcal{D}(2A)^{\kappa,c}$ explicitly (depending, of course, on a prescribed center). We know already that we may assume $c = 0$. Define a complex of $\mathcal{D}(2B)^{\kappa,0}$-modules

$$\overline{T} := \underbrace{0 \to P_1 \oplus P_1 \xrightarrow{\begin{bmatrix} \gamma \\ \gamma\alpha \end{bmatrix}} P_0 \to 0}_{=:\overline{T}_0} \oplus \underbrace{0 \to P_1 \to 0 \to 0}_{=:\overline{T}_1} \tag{4.33}$$

Here, for the sake of simplicity, we identify the generators of $\mathcal{D}(2B)^{\kappa,0}$ with homomorphisms between projective indecomposables satisfying the same relations as the original generators (as opposed to the opposite relations). We can do this since the algebra $\mathcal{D}(2B)^{\kappa,0}$ is isomorphic to its opposite algebra (it even carries an involution).

Remark 4.12. *The algebra $\mathcal{D}(2A)^{\kappa,0}$ has* Ext*-quiver*

$$Q' = \alpha' \circlearrowright \bullet_0 \underset{\beta'}{\overset{\gamma'}{\rightleftarrows}} \bullet_1 \tag{4.34}$$

with ideal of relations

$$I' = \left\langle \gamma'\beta',\ \alpha'^2,\ (\alpha'\beta'\gamma')^\kappa - (\beta'\gamma'\alpha')^\kappa \right\rangle_{kQ'} \tag{4.35}$$

where $\kappa = 2^{n-2}$. Its Cartan matrix is

$$\begin{bmatrix} 4\kappa & 2\kappa \\ 2\kappa & \kappa+1 \end{bmatrix} \tag{4.36}$$

Lemma 4.13. *$\overline{T}$ as defined in (4.33) is a tilting complex with endomorphism ring $\mathcal{D}(2A)^{\kappa,0}$.*

Proof. First note that γ and $\gamma\alpha$ form a k-basis of $\mathrm{Hom}(P_1, P_0)$. and $\beta, \alpha\beta$ form a k-basis of $\mathrm{Hom}(P_0, P_1)$. Now let $\varphi = c_1 \cdot \beta + c_2 \cdot \alpha\beta \in \mathrm{Hom}(P_0, P_1)$. Then

$$\begin{bmatrix} \gamma \\ \gamma\alpha \end{bmatrix} \cdot \varphi = 0 \Longleftrightarrow \begin{bmatrix} c_2 \cdot \gamma\alpha\beta \\ c_1 \cdot \gamma\alpha\beta \end{bmatrix} = 0 \Longleftrightarrow \varphi = 0 \tag{4.37}$$

This implies $\mathrm{Hom}(\overline{T}_0, \overline{T}[-1]) = 0$ (already in $\mathcal{C}^b(\mathbf{proj}_{\mathcal{D}(2A)^{\kappa,0}})$). $\mathrm{Hom}(\overline{T}_1, \overline{T}[-1]) = 0$ is clear since in any degree at least one of these complexes is the zero module. Now assume $\varphi = c_1 \cdot \gamma + c_2 \cdot \gamma\alpha \in \mathrm{Hom}(P_1, P_0)$. Then clearly

$$\varphi = \begin{bmatrix} c_1 & c_2 \end{bmatrix} \cdot \begin{bmatrix} \gamma \\ \gamma\alpha \end{bmatrix} \tag{4.38}$$

which implies that every chain map from $\overline{T}$ to $\overline{T}[1]$ is homotopic to zero. Furthermore $\overline{T}$ generates $\mathcal{D}^b(\mathcal{D}(2B)^{\kappa,0})$, since $P_1[1]$ is a summand of $\overline{T}$, and the mapping cone of the projection map $\overline{T}_0 \to \overline{T}_1 \oplus \overline{T}_1$ is isomorphic to $P_0[0]$. So we have seen that $\overline{T}$ is a tilting complex.

Now we claim that the endomorphisms

$$\begin{array}{ccc} P_1 \oplus P_1 & \xrightarrow{\begin{bmatrix} \gamma \\ \gamma\alpha \end{bmatrix}} & P_0 \\ {\scriptstyle \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}} \downarrow & & \downarrow {\scriptstyle \alpha} \\ P_1 \oplus P_1 & \longrightarrow & P_0 \end{array} \tag{4.39}$$

(which we denote by α') and

$$\begin{array}{ccc} P_1 \oplus P_1 & \longrightarrow & P_0 \\ {\scriptstyle \begin{bmatrix} 0 \\ 1 \end{bmatrix}} \downarrow & & \downarrow \\ P_1 & \longrightarrow & 0 \end{array} \qquad \begin{array}{ccc} P_1 & \longrightarrow & 0 \\ {\scriptstyle \begin{bmatrix} \eta & 0 \end{bmatrix}} \downarrow & & \downarrow \\ P_1 \oplus P_1 & \longrightarrow & P_0 \end{array} \tag{4.40}$$

(which we denote by β' and γ') together with the idempotent endomorphisms coming from the decomposition $\overline{T} = \overline{T}_0 \oplus \overline{T}_1$ (which we denote by e_0' and e_1') generate the endomorphism ring of $\overline{T}$. To prove this, we determine the dimension of the subalgebra of $\mathrm{End}(\overline{T})$ they generate. It should be noted that one can deduce from the shape of $\overline{T}$ and the Cartan matrix of $\mathcal{D}(2B)^{\kappa,0}$ that the Cartan matrix of $\mathrm{End}(\overline{T})$ is equal to that of $\mathcal{D}(2A)^{\kappa,0}$. First look at the endomorphism ring of $\overline{T}_0$ in the category $\mathcal{C}^b(\mathcal{D}(2B)^{\kappa,0})$ (which we identify as a subring of $\mathrm{End}(P_1 \oplus P_1) \oplus \mathrm{End}(P_0)$). Here α' and $\beta' \cdot \gamma'$ generate the subalgebra

$$\left\langle \left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, 1 \right) \right\rangle_k \oplus \left\langle \left(\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \alpha \right) \right\rangle_k \oplus \left(\begin{bmatrix} \eta k[\eta] & \eta k[\eta] \\ \eta k[\eta] & \eta k[\eta] \end{bmatrix}, 0 \right) \tag{4.41}$$

which has dimension $2 + 4 \cdot 2^{n-2}$. The zero-homotopic endomorphisms generate the subspace

$$\left\langle \left(\begin{bmatrix} \gamma\alpha\beta & 0 \\ 0 & 0 \end{bmatrix}, \alpha\beta\gamma \right), \left(\begin{bmatrix} 0 & \gamma\alpha\beta \\ 0 & 0 \end{bmatrix}, 0 \right), \left(\begin{bmatrix} 0 & 0 \\ \gamma\alpha\beta & 0 \end{bmatrix}, \beta\gamma \right), \left(\begin{bmatrix} 0 & 0 \\ 0 & \gamma\alpha\beta \end{bmatrix}, \alpha\beta\gamma \right) \right\rangle_k \tag{4.42}$$

which has two-dimensional intersection with the vector space in (4.41). Hence the subalgebra of the endomorphism ring (in $\mathcal{D}^b(\mathcal{D}(2B)^{\kappa,0})$) of $\overline{T}_0$ generated by α' and $\beta' \cdot \gamma'$ is 2^n-dimensional. Since we know the dimension of $\mathrm{End}(\overline{T}_0)$ to be 2^n, it follows that α' and $\beta' \cdot \gamma'$ generate $\mathrm{End}(\overline{T}_0)$.

With much less effort one can see that (in the category $\mathcal{C}^b(\mathcal{D}(2B)^{\kappa,0})$) we have $\mathrm{Hom}(\overline{T}_0, \overline{T}_1) \cong k[\eta] \oplus k[\eta]$, and β' generates this space as an $\mathrm{End}(\overline{T}_0)$-module. Similarly $\mathrm{Hom}(\overline{T}_1, \overline{T}_0) \cong \eta k[\eta] \oplus \eta k[\eta]$ and γ' generates this space as an $\mathrm{End}(\overline{T}_0)$-module. Furthermore $\gamma' \cdot \alpha' \cdot \beta' = \eta$ generates $\mathrm{End}(\overline{T}_1) = \mathrm{End}(P_1)$ as a k-algebra. The above considerations imply that $e_0', e_1', \alpha', \beta'$ and γ' generate the endomorphism ring (in $\mathcal{D}^b(\mathcal{D}(2B)^{\kappa,0})$) of $\overline{T}$ as a k-algebra.

Now one can easily verify that α', β' and γ' satisfy the relations given in (4.34), and this is all we have to check, since we know that the endomorphism ring of $\overline{T}$ has the same dimension as $\mathcal{D}(2A)^{\kappa,0}$. □

Theorem 4.14. *Define K-algebras A and B as follows:*

$$A := K \oplus K \oplus K^{2\times 2} \oplus K^{2\times 2} \oplus \bigoplus_{r=0}^{n-3} K_{r+2} \quad B := K \oplus K \oplus K^{2\times 2} \oplus K^{2\times 2} \oplus \bigoplus_{r=0}^{n-3} K_{r+2}^{3\times 3} \tag{4.43}$$

Define idempotents $\hat{e}_0, \hat{e}_1 \in A$:

$$\hat{e}_0 := \left(1, 1, \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, 0, \ldots, 0\right) \quad \hat{e}_1 := 1_A - \hat{e}_0 \tag{4.44}$$

and define idempotents $\hat{e}_0', \hat{e}_1' \in B$:

$$\hat{e}_0' := \left(1, 1, \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \ldots, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}\right) \quad \hat{e}_1' := 1_B - \hat{e}_0' \tag{4.45}$$

Any lift Λ of $\mathcal{D}(2B)^{\kappa,0}$ subject to the rational conditions in (4.5) is isomorphic to the

$\mathcal{O}$-order in A generated by the idempotents $\hat{e}_0$, $\hat{e}_1$ and

$$\begin{array}{rcl} \hat{e}_0A\hat{e}_0 \ni \hat{\alpha} & = & (0,\ 2,\ 0,\ 2) \\ \hat{e}_1A\hat{e}_1 \ni \hat{\eta} & = & (0,\ 4,\ \pi_0,\ \ldots,\ \pi_{n-3}) \\ \hat{e}_0A\hat{e}_1 \ni \hat{\beta} & = & (1,\ 1) \\ \hat{e}_1A\hat{e}_0 \ni \hat{\gamma} & = & (2^{n-1},\ 2^{n-1}) \end{array} \tag{4.46}$$

for certain prime elements $\pi_i \in K_{i+2}$. Any lift Γ of $\mathcal{D}(2A)^{\kappa,0}$ subject to the rational conditions in (4.28) is isomorphic to the $\mathcal{O}$-order in B generated by the idempotents $\hat{e}'_0$, $\hat{e}'_1$ and

$$\begin{array}{rcl} \hat{e}'_0B\hat{e}'_0 \ni \hat{\alpha}' & = & \left(0,\ 2,\ 2,\ 0,\ \begin{bmatrix} 0 & 1 \\ 0 & 2 \end{bmatrix},\ \ldots,\ \begin{bmatrix} 0 & 1 \\ 0 & 2 \end{bmatrix}\right) \\ \hat{e}'_0B\hat{e}'_1 \ni \hat{\beta}' & = & \left(1,\ 1,\ \begin{bmatrix} 0 \\ 1 \end{bmatrix},\ \ldots,\ \begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) \\ \hat{e}'_1B\hat{e}'_0 \ni \hat{\gamma}' & = & \left(-2,\ -2,\ \begin{bmatrix} \pi_0 & -2 \end{bmatrix},\ \ldots,\ \begin{bmatrix} \pi_{n-3} & -2 \end{bmatrix}\right) \end{array} \tag{4.47}$$

for certain prime elements $\pi_i \in K_{i+2}$. In particular, any block with dihedral defect group D_{2^n} and two simple modules is isomorphic to an order of one of the above shapes.

Furthermore, if X is a two-sided tilting complex the inverse of which restricts to $\overline{T}$, the lifts of (4.46) and (4.47) with equal π_i correspond to each other under the bijection Φ_X.

Proof. We have already seen in the proof of Theorem 4.8 that Λ has to be as in (4.46). We did however not see (and in general it is not true) that $\hat{\alpha}, \hat{\beta}, \hat{\gamma}$ and $\hat{\eta}$ may be assumed to be lifts of the elements α, β, γ and η. What we did see is that $\hat{\alpha}$ and $\hat{\eta}$ may be assumed to reduce to scalar multiples of α and η. Since we will need it below we now show that we may in fact assume that $\hat{\alpha}, \hat{\gamma}$ and $\hat{\eta}$ reduce to α, γ and η. To see that one simply verifies that for all $c_1, c_2, c_3, c_4 \in k$ with $c_1, c_2, c_4 \neq 0$ the following

$$\mathcal{D}(2B)^{\kappa,0} \longrightarrow \mathcal{D}(2B)^{\kappa,0}:\ \alpha \mapsto c_1\alpha,\ \beta \mapsto \frac{c_4^\kappa}{c_1c_2}\beta + \frac{c_3c_4^\kappa}{c_1c_2^2}\alpha\beta,\ \gamma \mapsto c_2\gamma + c_3\gamma\alpha,\ \eta \mapsto c_4\eta \tag{4.48}$$

defines an automorphism of $\mathcal{D}(2B)^{\kappa,0}$.

Now we show that Γ as given in (4.47) equals $\Phi_X(\Lambda)$. We choose

$$T := \underbrace{0 \to \hat{P}_1 \oplus \hat{P}_1 \xrightarrow{\begin{bmatrix} \hat{\gamma} \\ \hat{\gamma}\hat{\alpha} \end{bmatrix}} \hat{P}_0 \to 0}_{=:T_0} \oplus \underbrace{0 \to \hat{P}_1 \to 0 \to 0}_{=:T_1} \tag{4.49}$$

as a lift of $\overline{T}$ (where the $\hat{P}_i$ are the projective indecomposable Λ-modules). Now

$$\begin{array}{ccc} \hat{P}_1 \oplus \hat{P}_1 & \longrightarrow & \hat{P}_0 \\ {\scriptstyle \begin{bmatrix} 0 & 1 \\ 0 & 2 \end{bmatrix}} \downarrow & & \downarrow {\scriptstyle \hat{\alpha}} \\ \hat{P}_1 \oplus \hat{P}_1 & \longrightarrow & \hat{P}_0 \end{array} \tag{4.50}$$

is a lift of α' (which we denote by $\hat{\alpha}'$), and

$$\begin{array}{ccc} \hat{P}_1 \oplus \hat{P}_1 & \longrightarrow & \hat{P}_0 \\ {\scriptstyle \begin{bmatrix} 0 \\ 1 \end{bmatrix}} \downarrow & & \downarrow \\ \hat{P}_1 & \longrightarrow & 0 \end{array} \qquad \begin{array}{ccc} \hat{P}_1 & \longrightarrow & 0 \\ {\scriptstyle \begin{bmatrix} \hat{\eta} & -2 \end{bmatrix}} \downarrow & & \downarrow \\ \hat{P}_1 \oplus \hat{P}_1 & \longrightarrow & \hat{P}_0 \end{array} \tag{4.51}$$

are lifts of β' and γ' (which we denote by $\hat{\beta}'$ and $\hat{\gamma}'$). We now have to calculate the action of those endomorphisms on homology. For that identify $K \otimes P_0 \cong K \oplus K \oplus K \oplus K$ and $K \otimes P_1 \cong K \oplus K \oplus \bigoplus_{r=0}^{n-3} K_{r+2}$. Only for the third and fourth Wedderburn-component we need to do any actual work. Choose $\begin{bmatrix} 0 & 1 \end{bmatrix}$ as a basis for the projection of the kernel of the differential to the third Wedderburn-component, and $\begin{bmatrix} -2 & 1 \end{bmatrix}$ as a basis of the projection to the fourth Wedderburn-component. Now, for instance,

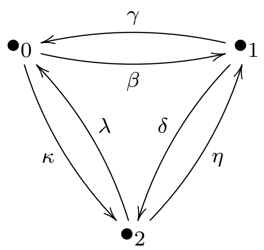
$$\begin{bmatrix} 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 0 & 1 \\ 0 & 2 \end{bmatrix} = 2 \cdot \begin{bmatrix} 0 & 1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} -2 & 1 \end{bmatrix} \cdot \begin{bmatrix} 0 & 1 \\ 0 & 2 \end{bmatrix} = 0 \cdot \begin{bmatrix} -2 & 1 \end{bmatrix} \tag{4.52}$$

which leads to the corresponding entries of $\hat{\alpha}'$ in the third and fourth Wedderburn-component.

□

4.4 Blocks with Three Simple Modules

Assume in this section that Λ has precisely three isomorphism classes of simple modules. We first assume that $\overline{\Lambda}$ is Morita-equivalent to $\mathcal{D}(3K)^c$ for $c = 2^{n-2}$ (we leave out the parameters a and b in [Erd90], which are both known to be equal to one in blocks of group rings). From [Erd90] we know that a basic algebra of $\overline{\Lambda}$ is isomorphic to kQ/I, where

$Q =$ 

(4.53)

and

$$I = \langle \beta\delta,\ \delta\lambda,\ \lambda\beta,\ \gamma\kappa,\ \kappa\eta,\ \eta\gamma,\ \beta\gamma - \kappa\lambda,\ (\eta\delta)^c - \lambda\kappa,\ (\delta\eta)^c - \gamma\beta \rangle \tag{4.54}$$

We may assume the following rational structure on Λ

Z(A)	**u**	**0**	**1**	**2**	
K	u_1	1	0	0	
K	u_2	0	1	0	
K	u_3	0	0	1	
K	u_4	1	1	1	
K_{r+2}	u_5	0	1	1	[exactly once for each $r = 0, \ldots, n-3$]

(4.55)

where $u_1, u_2, u_3, u_4 \in K$ have 2-valuation $-n$ and $u_5 \in K$ has 2-valuation $-n+1$. Note that we could be more restrictive regarding the values of the u_i if we were just interested in blocks of group rings Morita equivalent to $\mathcal{D}(3K)^c$ (namely we could demand that $u_4 = u_1+u_2+u_3$ and $u_5 = u_2 + u_3$). But since we want to transfer our unique lifting result via derived equivalences we have to admit more potential values for the u_i.

Theorem 4.15. *If $\Gamma, \Gamma' \in \mathfrak{L}(\mathcal{D}(3K)^c)$ (where $c = 2^{n-2}$) are subject to the rational conditions stated above and $Z(\Gamma) = Z(\Gamma')$, then $\Gamma \cong \Gamma'$.*

Proof. First note that $e_0\mathcal{D}(3K)^c e_0 = k[\kappa\lambda] \cong k[T]/(T^2)$, $e_1\mathcal{D}(3K)^c e_1 = k[\delta\eta] \cong k[T]/(T^{c+1})$ and $e_2\mathcal{D}(3K)^c e_2 = k[\eta\delta] \cong k[T]/(T^{c+1})$. Now we have $\delta\eta + \eta\delta + \kappa\lambda \in Z(\mathcal{D}(3K)^c)$ (which is easily verified using the quiver relations), and hence

$$e_i Z(\mathcal{D}(3K)^c) e_i = e_i \mathcal{D}(3K)^c e_i \tag{4.56}$$

for $i \in \{0, 1, 2\}$.

Now we have a look at the $e_i\mathcal{D}(3K)^c e_j$. First observe that any path not contained in I which connects two distinct vertices must involve the edge connecting its source with its target. The reason for this is that if a path involves a subpath starting and ending at the same vertex e_i, that subpath lies in $e_i Z(\mathcal{D}(3K)^c) e_i$, and the path can hence be rewritten as the product of a shorter path connecting source and target of the original path and an element of the center. Hence we may assume that any path is congruent modulo I to a path involving a subpath connecting source and target of the original path such that this subpath traverses no vertex twice. But all paths of length $\geqslant 2$ that connect two different vertices that do not pass through any vertex twice lie in I. So we have seen that for $i \neq j$

$$e_i\mathcal{D}(3K)^c e_j = e_i\mathcal{D}(3K)^c e_i \cdot \varphi \cdot e_j\mathcal{D}(3K)^c e_j \tag{4.57}$$

where $\varphi \in \{\beta, \gamma, \kappa, \lambda, \delta, \eta\}$ is the unique edge from e_i to e_j. Using (4.56) to pull elements across the φ we even get

$$e_i\mathcal{D}(3K)^c e_j = \varphi \cdot e_j\mathcal{D}(3K)^c e_j = e_i\mathcal{D}(3K)^c e_i \cdot \varphi \tag{4.58}$$

Now identify $K \otimes \Gamma$ and $K \otimes \Gamma'$ (denote this K-algebra by A), and assume without loss (by replacing Γ and Γ' by a conjugate order) that the $\hat{e}_i$ for $i \in \{0, 1, 2\}$ are lifts of e_i in *both* Γ and Γ' simultaneously and that they are diagonal (when identifying A with a direct sum of matrix rings). Now $k \otimes Z(\Gamma) = k \otimes Z(\Gamma') = Z(\mathcal{D}(3K)^c)$, since clearly $k \otimes Z(\Gamma) \subseteq Z(k \otimes \Gamma)$, and equality holds since the dimensions are equal (see Proposition 3.24). There is a commutative diagram

$$\begin{array}{ccc} \hat{e}_i Z(\Gamma) \hat{e}_i & \longrightarrow & \hat{e}_i \Gamma \hat{e}_i \\ \downarrow & & \downarrow \\ e_i Z(k \otimes \Gamma) e_i & = & e_i \cdot k \otimes \Gamma \cdot e_i \end{array} \tag{4.59}$$

where the maps are the canonical ones. Clearly the map in the top row must be onto (since otherwise not both of the vertical arrows could be surjective), and hence

$$\hat{e}_i \Gamma \hat{e}_i = \hat{e}_i Z(\Gamma) \hat{e}_i = \hat{e}_i Z(\Gamma') \hat{e}_i = \hat{e}_i \Gamma' \hat{e}_i \tag{4.60}$$

Now we start looking at the $e_i\Gamma e_j$ for $i \neq j$. First note that for $i \neq j$ we have

$$\hat{e}_j\Gamma\hat{e}_i = \{x \in \hat{e}_jA\hat{e}_i \mid x \cdot \hat{e}_i\Gamma\hat{e}_j \subseteq \hat{e}_j\Gamma\hat{e}_j\} \tag{4.61}$$

and the analogous equation for Γ'. This follows easily from the fact that Γ and Γ' are self-dual with respect to some trace bilinear form (not necessarily the same one for Γ and Γ'). Hence it follows that if we had $\hat{e}_i\Gamma\hat{e}_j = \hat{e}_i\Gamma'\hat{e}_j$, then we would automatically also have $\hat{e}_j\Gamma\hat{e}_i = \hat{e}_j\Gamma'\hat{e}_i$ (due to the fact that $\hat{e}_j\Gamma\hat{e}_j = \hat{e}_j\Gamma'\hat{e}_j$). Now identify in the obvious way

$$\begin{array}{lll} \hat{e}_0\Gamma\hat{e}_1 \subseteq \mathcal{O} & \hat{e}_0\Gamma\hat{e}_2 \subseteq \mathcal{O} & \hat{e}_1\Gamma\hat{e}_2 \subseteq \mathcal{O} \oplus \bigoplus_{r=0}^{n-3} \mathcal{O}_{r+2} \\ \hat{e}_1\Gamma\hat{e}_0 \subseteq \mathcal{O} & \hat{e}_2\Gamma\hat{e}_0 \subseteq \mathcal{O} & \hat{e}_2\Gamma\hat{e}_1 \subseteq \mathcal{O} \oplus \bigoplus_{r=0}^{n-3} \mathcal{O}_{r+2} \end{array} \tag{4.62}$$

By conjugation we may assume that $\hat{e}_0\Gamma\hat{e}_1 = \mathcal{O}$ and $\hat{e}_0\Gamma\hat{e}_2 = \mathcal{O}$ and still have that the other inclusions hold (see Proposition 2.115). Formula (4.61) then implies that $\hat{e}_1\Gamma\hat{e}_0 = (2^n)_\mathcal{O}$ and $\hat{e}_2\Gamma\hat{e}_0 = (2^n)_\mathcal{O}$. In the same vein we may assume that the projection of $\hat{e}_1\Gamma\hat{e}_2$ into the various summands $\mathcal{O}_i$ is surjective. Now, again by conjugation, we may replace $\hat{e}_1\Gamma\hat{e}_2$ by $(\alpha, \beta_0, \ldots, \beta_{n-3}) \cdot \hat{e}_1\Gamma\hat{e}_2$. where the $\beta_i \in \mathcal{O}_{i+2}^\times$ and $\alpha \in \mathcal{O}^\times$ are units. Hence all that is left to prove is that the valuation of the projection of $\hat{e}_1\Gamma\hat{e}_2$ to $\mathcal{O}$ (the first summand in the above embedding) is uniquely determined, and, more precisely, equal to $(2)_\mathcal{O}$.

Now note that Lemma 4.7 says that $\hat{e}_1\Gamma\hat{e}_1$ (and $\hat{e}_2\Gamma\hat{e}_2$, for that matter) is generated by a single element of the form $(0, 4, \pi_0, \ldots, \pi_{n-3})$. By general theory of self-dual orders, the intersection of $\hat{e}_1\Gamma\hat{e}_1$ with $K \oplus K$ (i. e., the first two Wedderburn components) is the dual of the projection to the first two Wedderburn components, hence

$$\hat{e}_1\Gamma\hat{e}_1 \cap K \oplus K \oplus \bigoplus_{r=0}^{n-3}\{0\} = \left\langle [2^{n-2}, c \cdot 2^{n-2}], [0, 2^n] \right\rangle_\mathcal{O} \quad \text{for some } c \in \mathcal{O}^\times \tag{4.63}$$

Note that as an $\hat{e}_1\Gamma\hat{e}_1$-module $\hat{e}_1\Gamma\hat{e}_2$ is just the projection of the regular module to all but the first Wedderburn component, since $e_1\mathcal{D}(3K)^c e_2$ is generated by a single element as an $e_1\mathcal{D}(3K)^c e_2$-module. Hence (4.63) implies that the amalgamation depth of $\hat{e}_1\Gamma\hat{e}_2$ with respect to the second Wedderburn component of $\hat{e}_1\Gamma\hat{e}_1$ is equal to $n-2$. By Proposition 2.125 this implies that the 2-valuation of the projection of $\hat{e}_1\Gamma\hat{e}_2 \cdot \hat{e}_2\Gamma\hat{e}_1$ to $\mathcal{O}$ (i. e. the first occurring Wedderburn component) is equal to 2. On the other hand $\hat{e}_0\Gamma\hat{e}_1 \cdot \hat{e}_1\Gamma\hat{e}_2 \subseteq 2 \cdot \hat{e}_0\Gamma\hat{e}_2$ implies that the 2-valuation of the projection of $\hat{e}_1\Gamma\hat{e}_2$ to $\mathcal{O}$ is at least one, and, in the same fashion, $\hat{e}_0\Gamma\hat{e}_2 \cdot \hat{e}_2\Gamma\hat{e}_1 \subseteq 2 \cdot \hat{e}_0\Gamma\hat{e}_1$ implies that the 2-valuation of the projection of $\hat{e}_2\Gamma\hat{e}_1$ to $\mathcal{O}$ is at least one. Hence both these projections must have 2-valuation precisely equal to one, which concludes the proof. □

There are two other possible basic algebras for dihedral blocks with three simple modules.

Namely $\mathcal{D}(3A)_1^c$, which has decomposition matrix

Z(A)	**u**	**0**	**1**	**2**	
K	u_1	1	0	0	
K	u_2	1	1	0	
K	u_3	1	0	1	
K	u_4	1	1	1	
K_{r+2}	u_5	2	1	1	[exactly once for each $r = 0, \dots, n-3$]

(4.64)

and $\mathcal{D}(3B)_1^c$, which has decomposition matrix

Z(A)	**u**	**0**	**1**	**2**	
K	u_1	1	0	0	
K	u_2	1	1	0	
K	u_3	1	0	1	
K	u_4	1	1	1	
K_{r+2}	u_5	0	1	0	[exactly once for each $r = 0, \dots, n-3$]

(4.65)

The restriction placed on the symmetrizing element u is again that $\nu_2(u_i) = -n$ for $i = 1, \dots, 4$, and $\nu_2(u_5) = -n + 1$.

Corollary 4.16. *If $\Gamma, \Gamma' \in \mathfrak{L}(\mathcal{D}(3A)_1^c)$ or $\Gamma, \Gamma' \in \mathfrak{L}(\mathcal{D}(3B)_1^c)$ (where $c = 2^{n-2}$) are subject to the rational conditions stated above and $Z(\Gamma) = Z(\Gamma')$, then $\Gamma \cong \Gamma'$.*

Proof. In [Lin94] it is shown that there is a two term tilting complex

$$[P_1 \to 0] \oplus [P_2 \to 0] \oplus [P_1 \oplus P_2 \to P_0] \tag{4.66}$$

is a tilting complex in $\mathcal{K}^b(\mathbf{proj}_{\mathcal{D}(3A)_1^c})$ with endomorphism ring $\mathcal{D}(3K)^c$, and

$$[P_2 \to 0] \oplus [P_1 \oplus P_2 \to P_0] \oplus [P_1 \to 0] \tag{4.67}$$

is a tilting complex in $\mathcal{K}^b(\mathbf{proj}_{\mathcal{D}(3B)_1^c})$ with endomorphism ring $\mathcal{D}(3K)^c$. Now we can argue just as in Corollary 4.10. □

Chapter 5

The Group Ring of $\mathbf{SL_2(p^f)}$

Let, as always, $(K, \mathcal{O}, k)$ be a p-modular system with K complete. Before explaining what we do in this chapter, let us fix some notation: We set $\mathrm{SL}_2(p^f) := \mathrm{SL}_2(\mathbb{F}_{p^f})$ and

$$\Delta_2(p^f) := \left\{ \begin{bmatrix} a & b \\ 0 & a^{-1} \end{bmatrix} \;\middle|\; a, b \in \mathbb{F}_{p^f},\ a \neq 0 \right\} \cong C_p^f \rtimes C_{p^f-1} \tag{5.1}$$

Note that $\Delta_2(p^f)$ is the normalizer of a p-Sylow subgroup of $\mathrm{SL}_2(p^f)$, namely of the group of unipotent upper triangular 2×2-matrices. Also note that k is a splitting field for either one of $\Delta_2(p^f)$ or $\mathrm{SL}_2(p^f)$ if and only if $\mathbb{F}_{p^f} \subseteq k$.

What we want to do in this chapter is describe the integral group ring $\mathcal{O}\,\mathrm{SL}_2(p^f)$, at least in the case when k is a splitting field for $\mathrm{SL}_2(p^f)$. However, at first sight, it may not quite seem like that is what we are actually doing, so let us start by giving some context. The basic algebra for the group ring of $k\,\mathrm{SL}_2(p^f)$ (where k is algebraically closed or at least a splitting field) was described (in terms of quivers with relations) in [Kos94] and [Kos98] (the first one being based on a fairly explicit description of the projective indecomposable $\bar{\mathbb{F}}_2\,\mathrm{SL}_2(2^f)$-modules in [Alp79]). In [Neb00a] and [Neb00b], $\mathcal{O}$-orders Λ were constructed such that $k \otimes \Lambda$ is isomorphic to the algebras given in [Kos94] and [Kos98]. Furthermore it was shown that the K-span of Λ is Morita equivalent to $K\,\mathrm{SL}_2(p^f)$ and that Λ is symmetric with respect to a trace bilinear form T_u such that $u \in Z(K \otimes \Lambda)$ has p-valuation $-f$ in every Wedderburn component of $Z(\bar{K} \otimes \Lambda)$. Based on this it was conjectured that this Λ is in fact a basic order of $\mathcal{O}\,\mathrm{SL}_2(p^f)$, though the evidence given was of course merely circumstantial. We show that this conjecture is indeed true, by virtue of Λ being the only lift of the basic algebra of $k\,\mathrm{SL}_2(p^f)$ with the given properties. We do this by transferring the problem to the group ring $k\Delta_2(p^f)$ via the derived equivalences proved in [Oku00] and [Yos09] (which prove Broué's conjecture in this case). The group ring $k\Delta_2(p^f)$ is then simple enough to be treated directly.

5.1 The Algebra $k\Delta_2(p^f)$ and Unique Lifting

Assume that $K/\mathbb{Q}_p$ is an unramified algebraic extension (possibly an infinite one). By $\bar{k}$ we denote the algebraic closure of k. In this section we will write $k\Delta_2(p^f)$ explicitly as a quotient of a quiver algebra (at least in the case when k splits $\Delta_2(p^f)$), and use this presentation to show that it lifts uniquely to an $\mathcal{O}$-order satisfying certain properties.

Definition 5.1. *Assume that A is an abelian p'-group such that kA is split. Denote by $\hat{A}$ the character group of A, that is,* $\mathrm{Hom}(A, k^\times)$ *(abstractly we will have $A \cong \hat{A}$). Assume moreover that A is acting on a p-group P by automorphisms. Let*

$$\mathrm{Jac}(kP)/\,\mathrm{Jac}^2(kP) \cong \bigoplus_{i=1}^{l} S_i \tag{5.2}$$

be a decomposition of $\mathrm{Jac}(kP)/\,\mathrm{Jac}^2(kP)$ *as a direct sum of simple kA-modules $S_1, \ldots, S_l$. We define the set $X(P, A)$ to be the disjoint union*

$$\biguplus_{i=1}^{l} \{\chi_{S_i}\} \tag{5.3}$$

where $\chi_{S_i} \in \hat{A}$ denotes the character of A associated to S_i.

Lemma 5.2. *Let $P = C_p^f$ and A be a group acting on P by automorphisms. View P as an $\mathbb{F}_p$-vector space by identifying C_p^f with $(\mathbb{F}_p^f, +)$. Under this identification, P becomes an $\mathbb{F}_pA$ module. Then*

$$\mathrm{Jac}(kP)/\,\mathrm{Jac}^2(kP) \cong_{kA} k \otimes_{\mathbb{F}_p} P \tag{5.4}$$

Proof. First note that after identifying P with $\mathbb{F}_p^f$, the fact that A acts on P by automorphisms translates into A acting linearly on $\mathbb{F}_p^f$, as each automorphism of $(\mathbb{F}_p^f, +)$ is automatically $\mathbb{F}_p$-linear. This turns P into an $\mathbb{F}_pA$-module (in fact, the isomorphism type of this module is independent of the choice of the identification of P with $\mathbb{F}_p^f$). Let $x_1, \ldots, x_f$ be a minimal generating system for $P = C_p^f$. Then $1 \otimes x_1, \ldots, 1 \otimes x_f$ is a k-basis for $k \otimes_{\mathbb{F}_p} P$. Now define a k-linear map

$$\Phi:\ k \otimes_{\mathbb{F}_p} P \to \mathrm{Jac}(kP)/\,\mathrm{Jac}^2(kP):\ 1 \otimes x_i \mapsto x_i - 1 \tag{5.5}$$

Since the $x_i - 1$ lie in $\mathrm{Jac}(kP)$ and they are a minimal (with respect to inclusion) generating set for kP as a k-algebra, they form a $k = kP/\,\mathrm{Jac}(kP)$ basis of $\mathrm{Jac}(kP)/\,\mathrm{Jac}^2(kP)$. Hence Φ is an isomorphism of vector spaces. We only need to check that Φ is A-equivariant (or, more generally, $\mathrm{Aut}(P)$-equivariant). This amounts to showing that for all $n_1, \ldots, n_f \in \mathbb{Z}_{\geqslant 0}$ the following holds:

$$x_1^{n_1} \cdots x_f^{n_f} - 1 \equiv \sum_{i=1}^{f} n_i \cdot (x_i - 1) \mod \mathrm{Jac}^2(kP) \tag{5.6}$$

Let $x, y \in P$. Then clearly $(x-1)(y-1) \in \mathrm{Jac}^2(P)$, and hence $xy - x - y + 1 \equiv 0 \mod \mathrm{Jac}^2(kP)$. This can be rewritten as $xy - 1 \equiv (x-1) + (y-1) \mod \mathrm{Jac}^2(kP)$. Applying this equality iteratedly clearly implies (5.6). ☐

Proposition 5.3. *Let $G = P \rtimes A$ with $P \cong C_p^f$ and A an abelian p'-group. If k splits G then*

$$kG \cong kQ/I \tag{5.7}$$

where Q is the quiver which has vertices e_χ in bijection with the elements $\chi \in \hat{A}$, and an arrow $e_\chi \xrightarrow{s_{\chi,\psi}} e_{\chi\cdot\psi}$ for each $\chi \in \hat{A}$ and $\psi \in X(P, A)$. I is the ideal generated by the relations

$$s_{\chi,\psi} \cdot s_{\chi\cdot\psi,\varphi} = s_{\chi,\varphi} \cdot s_{\chi\cdot\varphi,\psi} \quad \textit{for all } \chi \in \hat{A} \textit{ and } \psi, \varphi \in X(P, A) \tag{5.8}$$

and

$$\prod_{i=0}^{p-1} s_{\chi\cdot\psi^i,\psi} = 0 \quad \textit{for all } \chi \in \hat{A} \textit{ and } \psi \in X(P,A) \tag{5.9}$$

Proof. We first look at kP. We have $kC_p \cong k[T]/\langle T^p\rangle$, and

$$kP \cong \bigotimes^{f} kC_p \cong k[T_1,\dots,T_f]/(T_1^p,\dots,T_f^p) \tag{5.10}$$

Given any minimal generating set $t_1,\dots,t_f$ of kP contained in $\mathrm{Jac}(kP)$, the epimorphism $k[T_1,\dots,T_f] \twoheadrightarrow kP$ sending T_i to t_i has the same kernel $(T_1^p,\dots,T_f^p)$. This is simply because any automorphism of $k[T_1,\dots,T_f]$ mapping the ideal $(T_1,\dots,T_f)$ into itself will map the ideal $(T_1^p,\dots,T_f^p)$ into itself as well.

Now consider the action of A on $\mathrm{Jac}(kP)$ by conjugation. Since kA is abelian and semisimple, there is a basis $t_1,\dots,t_{p^f-1}$ of $\mathrm{Jac}(kP)$ such that for each i the conjugates $u^{-1}t_iu$ are a multiple of t_i for all $u \in A$. We may choose a minimal generating set for kP from said t_i's, say (after reindexing) $t_1,\dots,t_f$. As the images of $t_1,\dots,t_f$ in $\mathrm{Jac}(kP)/\mathrm{Jac}^2(kP)$ form a basis, there is a bijective map

$$X(P,A) \longrightarrow \{t_1,\dots,t_f\}: \ \psi \mapsto s_\psi \tag{5.11}$$

such that $u^{-1}\cdot s_\psi \cdot u = \psi(u)\cdot s_\psi$ for all $u \in A$. Define furthermore for each $\chi \in \hat{A}$ the corresponding primitive idempotent $e_\chi \in kA$ via the standard formula

$$e_\chi = \frac{1}{|A|}\sum_{a\in A}\chi(a)\cdot a^{-1} \tag{5.12}$$

This is a full set of orthogonal primitive idempotents in kG. Furthermore

$$e_\chi \cdot s_\psi = \frac{1}{|A|}\sum_{a\in A}\chi(a)\cdot a^{-1}s_\psi\cdot a\cdot a^{-1} = s_\psi\cdot\frac{1}{|A|}\sum_{a\in A}\chi(a)\psi(a)\cdot a^{-1} = s_\psi \cdot e_{\chi\cdot\psi} \tag{5.13}$$

Hence define

$$s_{\chi,\psi} := e_\chi \cdot s_\psi \quad \text{for all } \chi\in\hat{A}, \psi \in X(P,A) \tag{5.14}$$

The fact that the s_ψ commute implies the relation (5.8), and the fact that $s_\psi^p = 0$ implies relation (5.9). What we have to verify though is that that the s_ψ and e_χ generate kG as a k-algebra, and that there are no further relations (i. e. $\dim_k kG = \dim_k kQ/I$).

The s_ψ generate kP as a k-algebra and the e_χ generate kA even as a k-vector space. Hence together they generate $kP\cdot kA = kG$ as a k-algebra. Now to the dimension of kQ/I. We can use relation (5.8) to rewrite a path involving the arrows $s_{\chi_1,\psi_1},\dots,s_{\chi_l,\psi_l}$ (in that order) as a path $s_{\tilde{\chi}_1,\tilde{\psi}_1}\cdots s_{\tilde{\chi}_l,\tilde{\psi}_l}$ for any chosen reordering $(\tilde{\psi}_1,\dots,\tilde{\psi}_l)$ of $(\psi_1,\dots,\psi_l)$. Notice that necessarily $\chi_1 = \tilde{\chi}_1$, and all other $\tilde{\chi}_i$ are determined by $\tilde{\chi}_1$ and the $\tilde{\psi}_i$. Also we may assume, due to relation (5.9), that no p of the ψ_i are equal. So ultimately, there are at most $|\hat{A}|\cdot p^{|X(P,A)|}$ linearly independent paths ($|\hat{A}|$ choices for the starting point χ_1, p choices for the number of occurrences of each element of $X(P,A)$ in the sequence $(\psi_1,\dots,\psi_l)$). Hence

$$\dim kQ/I \leqslant |\hat{A}|\cdot p^{|X(P,A)|} = |A|\cdot p^f = \dim_k kG \tag{5.15}$$

and thus the epimorphism $kQ/I \twoheadrightarrow kG$ is in fact an isomorphism. □

Remark 5.4. *It seems practical to keep on using the notation*

$$s_\psi = \sum_{\chi \in \hat{A}} s_{\chi,\psi} \tag{5.16}$$

With this notation we may just write

$$kG \cong kQ/\left\langle s_\psi s_\varphi - s_\varphi s_\psi,\ s_\psi^p \mid \psi, \varphi \in X(P,A)\right\rangle \tag{5.17}$$

Proposition 5.5. *Let $G = \Delta_2(p^f)$, $P = \mathbb{G}_a(\mathbb{F}_{p^f}) \cong C_p^f$ and $A = \mathbb{G}_m(\mathbb{F}_{p^f}) \cong C_{p^f-1}$ (we view P as the subgroup of G consisting of diagonal matrices and A as the subgroup of G consisting of unipotent matrices). Assume $\mathbb{F}_{p^f} \subseteq k$ and identify $\hat{A} = \mathbb{Z}/(p^f-1)\mathbb{Z}$ (where we identify i with the character that sends $a \in A$ to $a^i \in k^\times$) and write the group operation in $\hat{A}$ additively. Then*

$$X(P,A) = \{2 \cdot p^q \mid q = 0, \dots, f-1\} \tag{5.18}$$

In particular, the Ext*-quiver Q of $k\Delta_2(p^f)$ has $p^f - 1$ vertices e_i labeled by elements $i \in \mathbb{Z}/(p^f-1)\mathbb{Z}$. There are precisely f arrows $s_{i,2\cdot p^q}$ (for $q \in \{0, \dots, f-1\}$) emanating from each vertex e_i.*

Proof. $G = P \rtimes A$ is a semidirect product. The action of A on P is given by

$$P \times A \to P: \ (b,a) \mapsto b \cdot a^2 \quad \text{where we identified } A = \mathbb{F}_{p^f}^\times,\ P = \mathbb{F}_{p^f} \tag{5.19}$$

Let us denote the $\mathbb{F}_pA$ module $\mathbb{F}_{p^f}$ with the action of A specified above by M. According to Lemma 5.2 we have to determine the simple constituents of $k \otimes_{\mathbb{F}_p} M$ as a kA-module. Note that there is a (one-dimensional) $\mathbb{F}_{p^f}A$-module $\tilde{M}$ with $\tilde{M}|_{\mathbb{F}_pA} \cong M$. So clearly

$$k \otimes_{\mathbb{F}_p} M \cong \bigoplus_{\gamma \in \mathrm{Gal}(\mathbb{F}_{p^f}/\mathbb{F}_p)} k \otimes_{\mathbb{F}_{p^f}} \tilde{M}^\gamma \tag{5.20}$$

Now $\mathrm{Gal}(\mathbb{F}_{p^f}/\mathbb{F}_p) \cong C_f$ is generated by the Frobenius automorphism. So the simple constituents of $k \otimes_{\mathbb{F}_p} M$ are just copies of k on which $a \in A$ acts as $a^{2\cdot p^q}$ for $q \in \{0, \dots, f-1\}$. This shows that $X(P,A)$ is as claimed. The shape of the Ext-quiver is now immediate from Lemma 5.2. □

Notation 5.6. *We define symbols*

$$[\mathbf{q}] := 2 \cdot p^q \tag{5.21}$$

to refer to the elements of $X(P,A)$ in the situation of the above proposition.

Lemma 5.7. *Assume k splits $\Delta_2(p^f)$. $k\Delta_2(p^f)$ consists of a single block if $p = 2$, and two isomorphic blocks otherwise. In the case $p = 2$, the Cartan matrix is given by $I + J$, where I is the identity matrix, and J is the matrix that has all entries equal to one. In the case p odd, the Cartan matrix of either one of the two blocks is $I + 2 \cdot J$.*

Proof. The (i,j)-entry of the Cartan matrix is, by definition, the k-dimension of $e_i \cdot kQ/I \cdot e_j$. Let $E = \langle e_1, \dots, e_{p^f-1}\rangle_k$ be the subspace of kQ/I spanned by the idempotents. Clearly, $kQ/I = E \oplus \mathrm{Rad}(kQ/I)$. So $\dim_k e_i \cdot kQ/I \cdot e_j = \delta_{ij} + \dim_k e_i \,\mathrm{Rad}(kQ/I) e_j$. Now, using the quiver relations from Proposition 5.3, we can deduce that $\dim_k e_i \operatorname{Rad}(kQ/I)e_j$ is equal to the number of vectors $(0,\ldots,0) \neq (n_0,\ldots,n_{f-1}) \in \{0,\ldots,p-1\}^f$ such that

$$2 \cdot \sum_{q=0}^{f-1} n_q \cdot p^q \equiv i - j \mod (p^f - 1) \tag{5.22}$$

If p is odd and $i-j$ is odd as well, then (since $p^f - 1$ will be even) the congruence cannot possibly be satisfied by any sequence of n_q's. So the corresponding entries in the Cartan matrix are zero. Now assume that p is odd and $i-j$ is even. Then the above congruence is equivalent to

$$\sum_{q=0}^{f-1} n_q \cdot p^q \equiv \frac{i-j}{2} \mod \left(\frac{p^f-1}{2}\right) \tag{5.23}$$

By uniqueness of the p-adic expansion of an integer, the analogous equation modulo $p^f - 1$ has a unique solution (in the case $i - j \equiv 0 \mod (p^f - 1)$ we would have two solutions, but we said above that we only consider solutions where not all of the n_q's are zero). Hence the equation above has precisely two solutions.

Now if $p = 2$, the factor "2" in (5.22) is a unit in the ring $\mathbb{Z}/(2^f - 1)\mathbb{Z}$, and hence can be divided out. The remaining equation has a unique solution thanks to the uniqueness of the 2-adic expansion of an integer (again discounting the zero solution). □

Remark 5.8. *By counting conjugacy classes in the group $\Delta_2(2^f)$, one easily obtains that*

$$\dim_K Z(K\Delta_2(2^f)) = 2^f \tag{5.24}$$

In the same way one obtains for p odd that

$$\dim_K Z(K\Delta_2(p^f)) = p^f + 3 \tag{5.25}$$

Since $k\Delta_2(p^f)$ is the direct sum of two isomorphic blocks, the dimension of the center of either one of these blocks is $(p^f + 3)/2$.

For reasons that will become apparent in the section on descent to smaller fields, we would like to investigate a slightly larger class of algebras than the blocks of $k\Delta_2(p^f)$, namely those (split) k-algebras which become isomorphic to $k\Delta_2(p^f)$ upon extension of the ground field.

Definition 5.9. *We call a split k-algebra $\overline{\Lambda}$ with $\bar{k} \otimes \overline{\Lambda} \cong B_0(\bar{k}\Delta_2(p^f))$ a* split k-form *of the principal block $B_0(\bar{k}\Delta_2(p^f))$ of $\bar{k}\Delta_2(p^f)$.*

Remark 5.10. *If $\overline{\Lambda}$ is a split k-form of $B_0(\bar{k}\Delta_2(p^f))$, then $\overline{\Lambda}$ has the same* Ext*-quiver and the same Cartan matrix as $B_0(\bar{k}\Delta_2(p^f))$. Moreover, the k-dimension of the center of $\overline{\Lambda}$ is equal to the $\bar{k}$-dimension of the center of $B_0(\bar{k}\Delta_2(p^f))$.*

Remark 5.11. *The quiver relations given in (5.8) and (5.9) are defined over $\mathbb{F}_p$. In particular, even if k is no splitting field for $\Delta_2(p^f)$, the blocks of kQ/I are split k-forms of $B_0(\bar{k}\Delta_2(p^f))$.*

Proposition 5.12 (Shape of Split k-Forms)**.** *Let $\overline{\Lambda}$ be a split k-form of $B_0(\bar{k}\Delta_2(p^f))$. By Q we now denote the* Ext*-quiver of $B_0(\bar{k}\Delta_2(p^f))$ (as opposed to the entire group ring $\bar{k}\Delta_2(p^f)$,* *which it was before). Denote (as before) the vertices of Q by e_{2i} and the arrows by $s_{2i,q}$. Then $\overline{\Lambda}$ is isomorphic to kQ/I' for some ideal I' which contains relations*

$$\prod_{j=0}^{p-1} s_{2i+j\cdot[\mathbf{q}],q} \quad \text{for all } i \in \mathbb{Z} \text{ and } q \in \{0,\dots,f-1\} \tag{5.26}$$

and relations of the shape

$$s_{2i,q} \cdot s_{2i+[\mathbf{q}],q'} - \alpha_{2i,q,q'} \cdot s_{2i,q'} \cdot s_{2i+[\mathbf{q}'],q} \tag{5.27}$$

with i ranging over $\mathbb{Z}$, q and q' ranging over $\{0,\dots,f-1\}$ and the $\alpha_{2i,q,q'}$ being of the form

$$c_{2i,q,q'} \cdot e_{2i} + r_{2i,q,q'} \tag{5.28}$$

for some $c_{2i,q,q'} \in k^\times$ and some k-linear combination $r_{2i,q,q'}$ of closed paths of positive length starting and ending in e_{2i} (hence, by construction, the $\alpha_{2i,q,q'}$ will lie in $(e_{2i} \cdot kQ/I' \cdot e_{2i})^\times$).

The relations given in (5.26) and (5.27) together with all paths of length $|\Delta_2(p^f)|$ (or any other sufficiently large number) generate I'.

Proof. We can assume that $\overline{\Lambda} \cong kQ/I'$ for some ideal I' contained in the ideal of kQ generated by the paths of length at least two. We proceed to show that I' is of the desired form. Choose an embedding $\varphi:\ kQ/I' \hookrightarrow \bar{k}Q/I$ that maps the idempotents e_{2i} to themselves such that the $\bar{k}$-span of the image of φ is all of $\bar{k}Q/I$. Then for each i and q the image $\varphi(s_{2i,q})$ has to be equal to $x_{2i,q} \cdot s_{2i,q}$ for some $x_{2i,q} \in (e_{2i} \cdot \bar{k}Q/I \cdot e_{2i})^\times$ (since the relations in I can be used to show that $e_{2i} \cdot \bar{k}Q/I \cdot e_{2i+[\mathbf{q}]} = e_{2i} \cdot \bar{k}Q/I \cdot e_{2i} \cdot s_{2i,q}$; now if $x_{2i,q}$ were no unit in $e_{2i} \cdot \bar{k}Q/I \cdot e_{2i}$, then $\varphi(s_{2i,q})$ would be contained in $\mathrm{Jac}^2(\bar{k}Q/I)$ and therefore the $\varphi(s_{2i,q})$ together with the e_{2i} could not possibly generate $\bar{k}Q/I$ as a $\bar{k}$-algebra). Since the relations in I imply that $e_{2i} \cdot \bar{k}Q/I \cdot e_{2i} \cdot s_{2i,q} = s_{2i,q} \cdot e_{2i+[\mathbf{q}]} \cdot \bar{k}Q/I \cdot e_{2i+[\mathbf{q}]}$, the relations in (5.26) follow immediately from the corresponding relation in I by application of φ.

Analogous to the above discussion, we can also deduce that for all $i \in \mathbb{Z}$ and $q, q' \in \{0,\dots,f-1\}$

$$\varphi(s_{2i,q}) \cdot \varphi(s_{2i+[\mathbf{q}],q'}) = \beta_{2i,q,q'} \cdot \varphi(s_{2i,q'}) \cdot \varphi(s_{2i+[\mathbf{q}'],q}) \tag{5.29}$$

for some $\beta_{2i,q,q'} \in (e_{2i} \cdot \bar{k}Q/I \cdot e_{2i})^\times$. Now take $\alpha'_{2i,q,q'} := (\mathrm{id}_{\bar{k}} \otimes_k \varphi)^{-1}(\beta_{2i,q,q'}) \in \bar{k} \otimes_k kQ/I'$. Choose a k-vector space complement V of k in $\bar{k}$ and choose $\alpha_{2i,q,q'} \in e_{2i} \cdot kQ/I' \cdot e_{2i}$ such that $\alpha'_{2i,q,q'} = \alpha_{2i,q,q'} +$ (Sum of paths with coefficients in V) . Now clearly the following holds:

$$s_{2i,q} \cdot s_{2i+[\mathbf{q}],q'} = \alpha_{2i,q,q'} \cdot s_{2i,q'} \cdot s_{2i+[\mathbf{q}'],q} + \text{(Sum of paths with coefficients in } V) \tag{5.30}$$

in $\bar{k} \otimes_k kQ/I'$. Since a sum of paths with coefficients in V must be k-linearly independent from kQ/I', the relation (5.27) must hold with this choice of $\alpha_{2i,q,q'}$. To see that the coefficient of e_{2i} in $\alpha_{2i,q,q'}$ is non-zero we could simply map the relation back into $\bar{k}Q/I$ using φ and subtract it from relation (5.29). This implies $(\beta_{2i,q,q'} - \varphi(\alpha_{2i,q,q'})) \cdot s_{2i,q'} \cdot s_{2i+[\mathbf{q}'],q} = 0$, and hence $\beta_{2i,q,q'} - \varphi(\alpha_{2i,q,q'})$ is no unit in $e_{2i} \cdot \bar{k}Q/I \cdot e_{2i}$, which forces $\varphi(\alpha_{2i,q,q'})$ to be a unit.

The claim that the given relations together with all paths of some sufficiently large length generate I' can be verified by showing that they can be used to rewrite any path as a linear combination of paths of the form

$$s_{2i,q_1} \cdot s_{2i+[\mathbf{q_1}],q_2} \cdots s_{2i+[\mathbf{q_1}]+\ldots+[\mathbf{q_{l-1}}],q_l} \tag{5.31}$$

such that $q_1 \leqslant q_2 \leqslant \ldots \leqslant q_l$ and no p of the q_j's are equal. The latter requirement can be met using relation (5.26). If the q_j's are not ordered as wanted, relation (5.27) can be used to permute them. This will however produce some summands of strictly greater length. So one can apply a rewriting strategy where one starts with the paths of smallest length which are not already in the desired standard form, rewrites those (possibly altering or adding some summands of strictly greater length) and then repeats the process until the shortest paths not in standard form are bigger than the cut-off length and therefore equal to zero. □

Lemma 5.13. *Let $\overline{\Lambda}$ be a split k-form of $B_0(\bar{k}\Delta_2(p^f))$*

(1) Assume $p = 2$. Then any lift $\Lambda \in \mathfrak{L}_s(\overline{\Lambda})$ with $\dim_K Z(K \otimes \Lambda) = \dim_k Z(\overline{\Lambda})$ has the following decomposition matrix over a splitting field

$$\begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \\ 1 & 1 & \cdots & 1 \end{bmatrix} \tag{5.32}$$

up to permutation of rows.

(2) Assume $p \neq 2$. If $\Lambda \in \mathfrak{L}_s(\overline{\Lambda})$ with $\dim_K Z(K \otimes \Lambda) = \dim_k Z(\overline{\Lambda})$, then the decomposition matrix of Λ over a splitting field looks as follows:

$$\begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \\ 1 & 1 & \cdots & 1 \\ 1 & 1 & \cdots & 1 \end{bmatrix} \tag{5.33}$$

up to permutation of rows.

(3) Fix a $\Lambda \in \mathfrak{L}_s(\overline{\Lambda})$ subject to the condition on the center as above. Assume that there is some totally ramified extension of K that splits Λ.

(a) If $p = 2$, then K already splits Λ.

(b) If p is odd then all one-dimensional representations of $\bar{K} \otimes \Lambda$ are already defined over $K \otimes \Lambda$. If K does not split $K \otimes \Lambda$, then $K \otimes \Lambda$ has a unique representation of dimension greater than one, and its endomorphism ring is a totally ramified extension of K of degree two. In particular, in that case, the decomposition matrix of Λ is as in (5.33) with the last row removed.

Proof. We prove the first two parts simultaneously. Let D be the decomposition of Λ (over a splitting system). First note that all entries of D must be $\leqslant 1$, as $D^\top \cdot D$ is equal to the Cartan matrix of $\bar{k}\Delta_2(p^f)$, which has "2"'s (respectively "3"'s) on the diagonal. Using the same argument it follows that each column of D has precisely two (respectively three) non-zero entries. It also follows that for any choice of two columns, there is precisely one row (respectively two rows) in which the entry in both columns is equal to one. Denote by v_l the number of rows in which there is a non-zero entry in the first l columns. If $p = 2$ we have $v_1 = 2$, $v_{2^f-1} = 2^f$ and by the above considerations we must have $v_{l+1} - v_l \leqslant 1$ for all l. So clearly, $v_{l+1} - v_l = 1$ for all l as otherwise $v_{2^f-1} - v_1$ would be strictly smaller than $2^f - 2$. In the same vein, if p is odd we have $v_1 = 3$, $v_{(p^f-1)/2} = (p^f + 3)/2$ and again $v_{l+1} - v_l \leqslant 1$, which also implies $v_{l+1} - v_l = 1$ for all l. Now assume by induction that the first l columns of D are as in (5.32) respectively (5.33). Note that the induction hypothesis for $l = 1$ is trivially satisfied. Since $v_{l+1} - v_l = 1$, the $l + 1$-th column has to have one non-zero entry in a row in which none of the preceding l columns have a non-zero entry. By permuting the rows we may assume without loss that this row is the $l + 1$-th row. Now there has to be one more entry (respectively two more entries) in that column, which must lie in a row where all preceding columns have a non-zero entry, since otherwise there would have to be a zero (respectively an entry $\leqslant 1$) somewhere in the Cartan matrix. So the other non-zero entry (respectively entries) can only be in the last row (respectively last two rows). Note that in the first induction step we may have to permute the rows to achieve this. Hence the induction hypothesis holds for $l + 1$. This concludes the proof of the first two points.

Finally we come to the assertions on the non-splitting case. First assume that there is a simple $K \otimes \Lambda$-module V such that $\mathrm{End}_{K\otimes\Lambda}(V)$ is non-commutative. Let P be a projective indecomposable Λ-lattice (note that $k \otimes \Lambda \cong \overline{\Lambda}$ is split, so indecomposable projectives are absolutely indecomposable) such that V occurs as a composition factor of $K \otimes P$. Since the endomorphism ring of V is non-commutative, $\bar{K} \otimes V$ is not multiplicity-free, but it is still a composition factor of $\bar{K} \otimes P$. Hence there is some simple $\bar{K} \otimes \Lambda$-module which occurs in $\bar{K} \otimes P$ with multiplicity greater than one. This is the same as saying that (over a splitting system) there is a decomposition number greater than one, which, as we have seen above, is impossible. Now let V be any simple $K \otimes \Lambda$-module. As we have seen $E := \mathrm{End}_{K\otimes\Lambda}(V)$ is commutative, and therefore it is necessarily contained in any splitting field for $K \otimes \Lambda$. Since by assumption there is a splitting field that is totally ramified over K, the field E must be totally ramified over K as well. Now we look at how the decomposition matrix over K relates to the decomposition matrix over a splitting field. $\mathrm{End}_{\bar{K}\otimes\Lambda}(\bar{K} \otimes V) \cong \bar{K} \otimes_K E \cong \bigoplus^{\dim_K E} \bar{K}$. This implies that $\bar{K} \otimes V$ decomposes into $e := \dim_K E$ non-isomorphic absolutely irreducible modules $V_1, \ldots, V_e$. Whenever P is a projective indecomposable Λ-module, the multiplicity of any V_i in $\bar{K}\otimes P$ is the same as the multiplicity of V in $K\otimes P$. Hence, the decomposition matrix of Λ over a splitting field arises from the decomposition matrix over K by repeating certain rows. The shape of the decomposition matrix over a splitting field proved above then limits the simple $K \otimes \Lambda$-modules that may not be split sufficiently so that our claims follow. □

Notation 5.14. *Let Λ be an $\mathcal{O}$-order with semisimple K-span and let $\varepsilon_1, \ldots, \varepsilon_n \in Z(K \otimes \Lambda)$ be the central primitive idempotents. So, in particular, we have fixed a bijection $\{1, \ldots, n\} \leftrightarrow$ $\{$ central primitive idempotents $\}$.*

(1) Given an element $u \in Z(K \otimes \Lambda)$ we set

$$u_i := \varepsilon_i \cdot u \quad \textit{for all } i \in \{1, \ldots, n\} \tag{5.34}$$

(2) When dealing with orders Λ which have a decomposition matrix like the one in (5.32) or

(5.33), we make the following convention concerning the ordering of the central primitive idempotents: We choose indices so that the idempotents associated to rows in the decomposition matrix with more than one non-zero entry come last.

Remark 5.15. *If $\Lambda = \mathcal{O}G$ for some finite group G (or a block thereof), then the symmetrizing element u may be chosen so that*

$$u_i = \frac{\chi_i(1)}{m_i \cdot |G|} \in \mathbb{Q}^\times \tag{5.35}$$

where χ_i is the i-th irreducible K-character of G (or in the block under consideration), and m_i is the number of absolutely irreducible characters it splits up into when passing from K to its algebraic closure $\bar{K}$ (see Proposition 2.45). In particular two of the u_i are equal if (and only if) the corresponding absolutely irreducible characters have equal degree. The equality of two rows in the decomposition matrix is a sufficient criterion for the corresponding characters to have equal degree, and therefore for the corresponding u_i to be equal. Note that we potentially have two equal rows in the decomposition matrix of $\mathcal{O}\,\mathrm{SL}_2(p^f)$ if p is odd (to be precise, this happens if f is even).

Theorem 5.16. *Let A be a finite-dimensional semisimple K-algebra with $\dim_K Z(A) = \dim_{\bar{k}} Z(B_0(\bar{k}\Delta_2(p^f)))$. Assume A is split by some totally ramified extension of K. Given an element $u \in Z(A)^\times$ which has p-valuation $-f$ in every Wedderburn component of $Z(\overline{K} \otimes A)$, there is, up to conjugacy, at most one full $\mathcal{O}$-order $\Lambda_u \subset A$ satisfying the following conditions:*

(1) Λ_u is self-dual with respect to T_u.

(2) $k \otimes \Lambda_u$ is a split k-form of $B_0(\bar{k}\Delta_2(p^f))$

Addendum to the theorem (concerning the dependence on u): *Assume u and u' are two symmetrizing elements subject to the above conditions, such that Λ_u and $\Lambda_{u'}$ both exist. Then:*

(1) If $p = 2$: Λ_u and $\Lambda_{u'}$ are conjugate.

(2) If $p \neq 2$ and K splits A: Let $\kappa = \frac{p^f-1}{2}$. If $\frac{u_{\kappa+1}}{u_{\kappa+2}} = \frac{u'_{\kappa+1}}{u'_{\kappa+2}}$, then Λ_u and $\Lambda_{u'}$ are conjugate.

(3) If $p \neq 2$ and K does not split A: If $u_{\kappa+1} \cdot \mathcal{O}^\times = u'_{\kappa+1} \cdot \mathcal{O}^\times$, then Λ_u and $\Lambda_{u'}$ are conjugate.

where κ is the number of isomorphism classes of simple modules in $B_0(\bar{k}\Delta_2(p^f))$.

Proof. We assume that we are given an order $\Lambda = \Lambda_u$ satisfying the given conditions. To prove the theorem we will try to conjugate Λ into a kind of "standard form" depending on u. We let I' be an ideal in kQ as described in Proposition 5.12 such that $k \otimes_{\mathcal{O}} \Lambda \cong kQ/I'$ (we will assume that we have fixed an isomorphism and identify the two). Also, as before, we denote the idempotents in kQ by e_{2i} and the arrows by $s_{2i,q}$. We wish to treat the case where K splits A and the case where K does not split A as well as the cases p even and p odd (essentially) uniformly. So assume that

$$A = \left(\bigoplus_{i=1}^{\kappa} K \right) \oplus \tilde{K}^{\kappa \times \kappa} \quad \text{with } \kappa = \begin{cases} \dfrac{p^f - 1}{2} & \text{if } p \neq 2 \\ 2^f - 1 & \text{if } p = 2 \end{cases} \tag{5.36}$$

where $\tilde{K}$ is isomorphic to K if $p = 2$, to $K \oplus K$ if $p \neq 2$ and A is K-split, or to a fully ramified extension of K of degree two if $p \neq 2$ and A is not K-split. By $\tilde{\varepsilon}$ denote the unit element of $\tilde{K}$, construed as an idempotent in $Z(A)$. For each i let $\hat{e}_{2i} \in \Lambda$ be a lift of $e_{2i} \in kQ/I'$, and assume without loss that $\tilde{\varepsilon}\hat{e}_{2i}$ is the i-th diagonal idempotent in $\tilde{K}^{\kappa\times\kappa}$ (this may certainly be achieved by conjugating Λ by an element of $A^\times$). Assume furthermore that $(1-\tilde{\varepsilon})\cdot\hat{e}_{2i}$ has non-zero entry in the i-th direct summand of the decomposition (5.36). Hence we have fixed the elements $\hat{e}_{2i}$ as elements of the algebra A as described in (5.36). Now, using the fact that Λ is supposed to be symmetric with respect to T_u, it follows that

(1) If p is odd and K splits A:

$$\hat{e}_i\Lambda\hat{e}_i = \left\langle [1,1,1], [0, p^{\frac{f}{2}}, -c\cdot p^{\frac{f}{2}}], [0,0,p^f] \right\rangle_{\mathcal{O}} \subset \mathcal{O}\oplus\mathcal{O}\oplus\mathcal{O} \quad \text{where } c = \frac{u_{\kappa+1}}{u_{\kappa+2}} \tag{5.37}$$

This follows simply from the fact that a self-dual order (with respect to T_u) in $\mathcal{O}\oplus\mathcal{O}\oplus\mathcal{O}$ must have elementary divisors $1, p^{\frac{f}{2}}, p^f$ (as an $\mathcal{O}$ lattice in $\mathcal{O}\oplus\mathcal{O}\oplus\mathcal{O}$) and all traces with respect to T_u must be integral. Note that this also implies that f must be even (in this situation, i. e. when K splits A and p is odd).

(2) If p is odd and K does not split A:

$$\hat{e}_i\Lambda\hat{e}_i = \left\langle [1,1], [0, c\cdot\pi^f], [0, c^2\cdot\pi^{2f}] \right\rangle_{\mathcal{O}} \qquad \text{for some } c \in \mathcal{O}[\pi]^\times \tag{5.38}$$

where π is some uniformizer for the integral closure of $\mathcal{O}$ in $\tilde{K}$, which is a fully ramified extension of K in this case. Up to this point, we have used two facts: First, that the elementary divisors of $\mathcal{O}[\pi]\otimes\hat{e}_i\Lambda\hat{e}_i$ (as a lattice in $\mathcal{O}[\pi]\oplus\mathcal{O}[\pi]\oplus\mathcal{O}[\pi]$) must be $1, \pi^f, \pi^{2f}$, and second, that $\hat{e}_i\Lambda\hat{e}_i$ is generated by a single element as an $\mathcal{O}$-order (since $e_i\cdot kQ/I\cdot e_i \cong k[T]/(T^3)$ is generated by a single element as a k-algebra). In this case we need to put in some work to show that $\hat{e}_i\Lambda\hat{e}_i$ is uniquely determined (since different choices of c may give rise to different orders). Note $T_u(\{0\}\oplus p^f\mathcal{O}[\pi]) \subseteq \mathcal{O}$, and hence necessarily $\{0\}\oplus p^f\mathcal{O}[\pi] \subset (\hat{e}_i\Lambda\hat{e}_i)^\sharp = \hat{e}_i\Lambda\hat{e}_i$. Moreover an element $[0, \tilde{c}\cdot\pi^f]$ lies in $\hat{e}_i\Lambda\hat{e}_i$ if and only if $T_u([0,\tilde{c}\cdot\pi^f]) \in \mathcal{O}$. This characterizes $\hat{e}_i\Lambda\hat{e}_i$ as

$$\hat{e}_i\Lambda\hat{e}_i = \mathcal{O}\left[[0, \tilde{c}\cdot\pi^f] \;\middle|\; T_u([0,\tilde{c}\cdot\pi^f]) \in \mathcal{O} \right] \tag{5.39}$$

which is obviously uniquely determined by u and the extension $\tilde{K}/K$.

(3) If $p = 2$ then

$$\hat{e}_i\Lambda\hat{e}_i = \left\langle [1,1], [0, 2^f] \right\rangle_{\mathcal{O}} \tag{5.40}$$

by the same argument as in the first point.

In the above considerations we have used that each u_i has p-valuation $-f$. In the case $p = 2$ we have not used any further information on u. In the case $p \neq 2$ we have used the value of the quotient $u_{\kappa+1}/u_{\kappa+2}$ if K splits A and the class $u_{\kappa+1}\cdot\mathcal{O}^\times$ if it does not (since the characterization in (5.39) depends only on $u_{\kappa+1}\cdot\mathcal{O}^\times$; note that $u_{\kappa+1}$ is an element of $\tilde{K}$ in this case while in the split case $u_{\kappa+1}$ and $u_{\kappa+2}$ are both elements of K). Since we will not make any further use of the symmetrizing element u below, this will imply the addendum on the dependence on u.

Note that in either case the $\hat{e}_i\Lambda\hat{e}_i$ are equal (when we identify the unique maximal orders containing them). In particular the image in $\mathrm{End}_K(\tilde{K})$ of the action homomorphism of $\hat{e}_i\Lambda\hat{e}_i$ on $\hat{e}_i\Lambda\hat{e}_j \subset \tilde{K}$ is the same as the image of $\hat{e}_j\Lambda\hat{e}_j$ under the corresponding action homomorphism. Hence the submodule structure of $\hat{e}_i\Lambda\hat{e}_j$ is independent of whether it is construed as a left $\hat{e}_i\Lambda\hat{e}_i$-module or a right $\hat{e}_j\Lambda\hat{e}_j$-module. Now $e_i \cdot kQ/I' \cdot e_j$ is free as a $e_i \cdot kQ/I' \cdot e_i/\operatorname{Soc}(e_i \cdot kQ/I' \cdot e_i)$ left module (this is actually best seen by using the relations over $\bar{k}$ as given in Proposition 5.3 and then descending to k), and since $e_i \cdot kQ/I' \cdot e_i/\operatorname{Soc}(e_i \cdot kQ/I' \cdot e_i) \cong k \otimes \tilde{\varepsilon}\hat{e}_i\Lambda\hat{e}_i$, this implies that $\hat{e}_i\Lambda\hat{e}_j$ is free as a left $\tilde{\varepsilon}\hat{e}_i\Lambda\hat{e}_i$-module. This implies (when $\hat{e}_iA\hat{e}_j$ is identified with $\tilde{K}$ in the natural way)

$$\hat{e}_i\Lambda\hat{e}_j = x_{ij} \cdot \tilde{\varepsilon}\hat{e}_i\Lambda\hat{e}_i \quad \text{for some } x_{ij} \in \tilde{K}^\times \tag{5.41}$$

In addition, we may and will assume that the x_{ij} are integral over $\mathcal{O}$. For each i and q we have

$$\prod_{l=0}^{p-1} e_{i+l\cdot[\mathbf{q}]} \cdot kQ/I' \cdot e_{i+(l+1)\cdot[\mathbf{q}]} = 0 \tag{5.42}$$

and hence

$$\prod_{l=0}^{p-1} \hat{e}_{i+l\cdot[\mathbf{q}]} \cdot \Lambda \cdot \hat{e}_{i+(l+1)\cdot[\mathbf{q}]} \subseteq p \cdot \hat{e}_i \cdot \Lambda \cdot \hat{e}_{i+[\mathbf{q+1}]} \tag{5.43}$$

Everything from here down to (5.61) below is about showing that the inclusion in (5.43) is in fact an equality. The significance of this is that it can then be used as a formula to compute the $\hat{e}_i \cdot \Lambda \cdot \hat{e}_{i+[\mathbf{q+1}]}$ from the $\hat{e}_i \cdot \Lambda \cdot \hat{e}_{i+[\mathbf{q}]}$, showing that Λ is determined by the $\hat{e}_i \cdot \Lambda \cdot \hat{e}_{i+[\mathbf{0}]}$.

We define a "normalized index" for full $\mathcal{O}$-lattices $L_1 \supseteq L_2$ in $\tilde{K}$ as follows:

$$\mathrm{idx}(L_1, L_2) := \frac{\mathrm{length}_{\mathcal{O}}\, L_1/L_2}{\mathrm{length}_{\mathcal{O}}\, L_1/pL_1} \tag{5.44}$$

Note that the denominator is a constant independent of the choice of L_1. Note furthermore that if L is any full lattice in $\tilde{K}$, and $x_1, x_2 \in \tilde{K}^\times$ with $x_i \cdot L \subseteq L$ for $i \in \{1, 2\}$, then

$$\mathrm{idx}(L, x_1 \cdot x_2 \cdot L) = \mathrm{idx}(L, x_1 \cdot L) + \mathrm{idx}(L, x_2 \cdot L) \tag{5.45}$$

because $\mathrm{idx}(L, x_i \cdot L)$ equals a constant multiple of the p-valuation of the determinant of "multiplication with x_i" construed as a K-vector space automorphism of $\tilde{K}$. Now define

$$m_{i,q} := \mathrm{idx}\left(\tilde{\varepsilon}\hat{e}_i\Lambda\hat{e}_i, \hat{e}_i\Lambda\hat{e}_{i+[\mathbf{q}]}\right) \tag{5.46}$$

where we identify $\hat{e}_i\Lambda\hat{e}_{i+[\mathbf{q}]} \subseteq \tilde{\varepsilon}\hat{e}_i\Lambda\hat{e}_i$ as in (5.41). Define furthermore

$$a_{i,q} := \mathrm{idx}\left(\hat{e}_i \cdot \Lambda \cdot \hat{e}_{i+[\mathbf{q+1}]}, \prod_{l=0}^{p-1} \hat{e}_{i+l\cdot[\mathbf{q}]} \cdot \Lambda \cdot \hat{e}_{i+(l+1)\cdot[\mathbf{q}]}\right) = \left(\sum_{l=0}^{p-1} m_{i+l\cdot[\mathbf{q}],q}\right) - m_{i,q+1} \tag{5.47}$$

Clearly $a_{i,q} \geqslant 1$ for all i and q. We have for any $q \neq r$

$$e_i \cdot kQ/I' \cdot e_{i+[\mathbf{q}]} \cdot kQ/I' \cdot e_{i+[\mathbf{q}]+[\mathbf{r}]} = e_i \cdot kQ/I' \cdot e_{i+[\mathbf{q}]+[\mathbf{r}]} \tag{5.48}$$

and hence in particular

$$\hat{e}_i\Lambda\hat{e}_{i+[\mathbf{q}]}\Lambda\hat{e}_{i+[\mathbf{q}]+[\mathbf{q+1}]} = \hat{e}_i\Lambda\hat{e}_{i+[\mathbf{q}]+[\mathbf{q+1}]} = \hat{e}_i\Lambda\hat{e}_{i+[\mathbf{q+1}]}\Lambda\hat{e}_{i+[\mathbf{q}]+[\mathbf{q+1}]} \tag{5.49}$$

which implies for all i and q that

$$m_{i,q} + m_{i+[\mathbf{q}],q+1} = m_{i,q+1} + m_{i+[\mathbf{q+1}],q} \tag{5.50}$$

Now

$$\begin{aligned} a_{i,q} - a_{i+[\mathbf{q}],q} &= \left(\sum_{l=0}^{p-1} m_{i+l\cdot[\mathbf{q}],q}\right) - \left(\sum_{l=1}^{p} m_{i+l\cdot[\mathbf{q}],q}\right) - m_{i,q+1} + m_{i+[\mathbf{q}],q+1} \\ &= m_{i,q} - m_{i+[\mathbf{q+1}],q} - m_{i,q+1} + m_{i+[\mathbf{q}],q+1} \overset{(5.50)}{=} 0 \end{aligned} \tag{5.51}$$

Since p is relatively prime to κ, this implies that $a_{i,q} = a_q$ for some a_q independent of i. Now we sum up (5.47) over all κ values of i, and get

$$\sum_{i=1}^{\kappa} m_{2i,q+1} = p \cdot \sum_{i=1}^{\kappa} m_{2i,q} - \kappa \cdot a_q \tag{5.52}$$

Plugging this formula into itself f times yields (for all values of q)

$$\sum_{i=1}^{\kappa} m_{2i,q} = p^f \cdot \sum_{i=1}^{\kappa} m_{2i,q} - \kappa \sum_{i=1}^{f} p^{f-i} \cdot a_{q+i-1} \tag{5.53}$$

which implies

$$\sum_{i=1}^{\kappa} m_{2i,q} = \frac{\kappa}{p^f - 1} \cdot \sum_{i=1}^{f} p^{f-i} \cdot a_{q+i-1} \geqslant \frac{\kappa}{p-1} \tag{5.54}$$

with equality if and only if all a_q are equal to 1. Now we know that

$$\begin{aligned} \mathrm{Jac}(e_i \cdot kQ/I' \cdot e_i) &= \prod_{q=0}^{f-1} \prod_{j=1}^{\frac{p-1}{2}} e_{i+\frac{1}{2}\cdot([\mathbf{q}]-[\mathbf{0}])+(j-1)\cdot[\mathbf{q}]} \cdot kQ/I' \cdot e_{i+\frac{1}{2}\cdot([\mathbf{q}]-[\mathbf{0}])+j\cdot[\mathbf{q}]} \quad (p \neq 2) \\ \mathrm{Jac}(e_i \cdot kQ/I' \cdot e_i) &= \prod_{q=0}^{f-1} e_{i+[\mathbf{q}]-[\mathbf{0}]} \cdot kQ/I' \cdot e_{i+[\mathbf{q+1}]-[\mathbf{0}]} \quad (p = 2) \end{aligned} \tag{5.55}$$

In the upper equation we used that $\frac{1}{2}([\mathbf{q}] - [\mathbf{0}]) = \sum_{r=0}^{q-1} \frac{p-1}{2}[\mathbf{r}]$. Now $\tilde{\varepsilon}\hat{e}_i\Lambda\hat{e}_i \cap \hat{e}_i\Lambda\hat{e}_i$ is a pure sublattice of $\hat{e}_i\Lambda\hat{e}_i$. The k-dimension of its image in $e_i \cdot kQ/I' \cdot e_i$ must therefore be equal to its $\mathcal{O}$-rank (which is one if $p = 2$ and two otherwise), which implies that said image is equal to $\mathrm{Jac}(e_i \cdot kQ/I' \cdot e_i)$. Another ramification of $\tilde{\varepsilon}\hat{e}_i\Lambda\hat{e}_i \cap \hat{e}_i\Lambda\hat{e}_i$ being a pure sublattice of $\hat{e}_i\Lambda\hat{e}_i$ is that any proper sublattice of it maps to a proper subspace of $\mathrm{Jac}(e_i \cdot kQ/I' \cdot e_i)$. Hence

(5.55) implies the following:

$$\begin{aligned} \tilde{\varepsilon}\hat{e}_i\Lambda\hat{e}_i \cap \hat{e}_i\Lambda\hat{e}_i &= \prod_{q=0}^{f-1}\prod_{j=1}^{\frac{p-1}{2}} \hat{e}_{i+\frac{1}{2}\cdot([\mathbf{q}]-[\mathbf{0}])+(j-1)\cdot[\mathbf{q}]}\Lambda\hat{e}_{i+\frac{1}{2}\cdot([\mathbf{q}]-[\mathbf{0}])+j\cdot[\mathbf{q}]} \quad (p \neq 2) \\ \tilde{\varepsilon}\hat{e}_i\Lambda\hat{e}_i \cap \hat{e}_i\Lambda\hat{e}_i &= \prod_{q=0}^{f-1} \hat{e}_{i+[\mathbf{q}]-[\mathbf{0}]}\Lambda\hat{e}_{i+[\mathbf{q+1}]-[\mathbf{0}]} \quad (p = 2) \end{aligned} \tag{5.56}$$

This, in turn, implies that the following holds for any index i:

$$\begin{aligned} \frac{f}{2} &= \operatorname{idx}(\tilde{\varepsilon}\hat{e}_i\Lambda\hat{e}_i, \tilde{\varepsilon}\hat{e}_i\Lambda\hat{e}_i \cap \hat{e}_i\Lambda\hat{e}_i) = \sum_{q=0}^{f-1}\sum_{j=1}^{\frac{p-1}{2}} m_{i+\frac{1}{2}\cdot([\mathbf{q}]-[\mathbf{0}])+(j-1)\cdot[\mathbf{q}],q} \quad (p \neq 2) \\ f &= \operatorname{idx}(\tilde{\varepsilon}\hat{e}_i\Lambda\hat{e}_i, \tilde{\varepsilon}\hat{e}_i\Lambda\hat{e}_i \cap \hat{e}_i\Lambda\hat{e}_i) = \sum_{q=0}^{f-1} m_{i+[\mathbf{q}]-[\mathbf{0}],q} \quad (p = 2) \end{aligned} \tag{5.57}$$

Summing this up over all κ different values of i yields (regardless of whether p is even or odd)

$$\kappa \cdot \frac{f}{2} = \sum_{q=0}^{f-1} \frac{p-1}{2} \sum_{i=1}^{\kappa} m_{2i,q} \tag{5.58}$$

Now we plug in (5.54) to get

$$\kappa \cdot \frac{f}{2} = \frac{p-1}{2} \cdot \frac{\kappa}{p^f-1} \cdot \sum_{q=0}^{f-1}\sum_{i=1}^{f} p^{f-i} \cdot a_{q+i-1} = \frac{p-1}{2} \cdot \frac{\kappa}{p^f-1} \cdot \frac{p^f-1}{p-1} \cdot \sum_{q=0}^{f-1} a_q \tag{5.59}$$

We conclude

$$\sum_{q=0}^{f-1} a_q = f \tag{5.60}$$

which implies that all a_q are equal to one. This implies that the $\hat{e}_{2i}\Lambda\hat{e}_{2i+[\mathbf{0}]}$ determine Λ in the sense that the formula

$$\hat{e}_{2i}\Lambda\hat{e}_{2i+[\mathbf{q+1}]} = \frac{1}{p} \cdot \hat{e}_{2i}\Lambda\hat{e}_{2i+[\mathbf{q}]} \cdots \hat{e}_{2i+(p-1)\cdot[\mathbf{q}]}\Lambda\hat{e}_{2i+p\cdot[\mathbf{q}]} \tag{5.61}$$

shows how to calculate $\hat{e}_{2i}\Lambda\hat{e}_{2i+[\mathbf{q+1}]}$ from the knowledge of the $\hat{e}_{2j}\Lambda\hat{e}_{2j+[\mathbf{q}]}$ (for all j).

Now we may replace Λ by $y^{-1} \cdot \Lambda \cdot y$, where

$$y := \left[1, \ldots, 1, \operatorname{diag}\left(\prod_{j=0}^{i-1} x_{2j,2j+[\mathbf{0}]} \;\middle|\; i = 1, \ldots, \kappa\right)\right] \in A^{\times} \tag{5.62}$$

(the x_{ij} were defined in (5.41)) and so we may assume without loss that all $x_{2i,2i+[\mathbf{0}]}$ are equal to 1, except possibly $x_{2\kappa-[\mathbf{0}],2\kappa}$. In other words, we have fixed all but one of the $\hat{e}_{2i}\Lambda\hat{e}_{2i+[\mathbf{0}]}$. But

$$\hat{e}_{2\kappa-[\mathbf{0}]}\Lambda\hat{e}_{2\kappa} = \left\{v \in \hat{e}_{2\kappa-[\mathbf{0}]}A\hat{e}_{2\kappa} \mid \hat{e}_{2\kappa}\Lambda\hat{e}_{2\kappa-[\mathbf{0}]} \cdot v \subseteq \hat{e}_{2\kappa}\Lambda\hat{e}_{2\kappa}\right\} \tag{5.63}$$

which is either seen by formula (5.56), or from the fact that $\hat{e}_{2\kappa-[\mathbf{0}]}\Lambda\hat{e}_{2\kappa}$ is the dual of $\hat{e}_{2\kappa}\Lambda\hat{e}_{2\kappa-[\mathbf{0}]}$ with respect to the bilinear pairing induced by T_u. Now in the above formula, $\hat{e}_{2\kappa}\Lambda\hat{e}_{2\kappa}$ is explicitly known, and $\hat{e}_{2\kappa}\Lambda\hat{e}_{2\kappa-[\mathbf{0}]}$ can be calculated via (5.61) from the $\hat{e}_{2i}\Lambda e_{2i+[\mathbf{0}]}$ with $0 \leqslant i < \kappa - 1$ (which were fixed above by means of conjugation). Hence, Λ is determined in the sense that we have conjugated Λ to some fixed order determined by the data given in the statement of the theorem. This concludes the proof. □

Remark 5.17. *Situation as in the last theorem. Assume furthermore that the (unique) lift $\Lambda = \Lambda_u$ exists. Then the above proof also implies the following: If $\alpha \in \operatorname{Aut}_k(k \otimes \Lambda)$ is an automorphism of $k \otimes \Lambda$ permuting the set of idempotents $\{e_i\}_i$, then there exists an element $\hat{\alpha} \in \operatorname{Aut}_{\mathcal{O}}(\Lambda)$ inducing the corresponding permutation on the set of idempotents $\{\hat{e}_i\}_i$. This follows simply from the fact that we fixed the idempotents at the beginning of the proof of the Theorem and then only used conjugation by elements of $A^\times$ that commuted with all $\hat{e}_i$ to conjugate Λ to any potential other lift of $k\otimes\Lambda$ (also containing the same fixed set of idempotents $\{\hat{e}_i\}_i$).*

5.2 Transfer to $\mathcal{O}\mathrm{SL}_2(p^f)$

Now we will generalize the result of Theorem 5.16 to all algebras derived equivalent to $B_0(k\Delta_2(p^f))$ (for algebraically closed fields k). This includes in particular the two non-semisimple blocks of $k\,\mathrm{SL}_2(p^f)$.

Lemma 5.18. *Let k be algebraically closed and let B be the principal block of $k\Delta_2(p^f)$. There is an epimorphism of algebraic groups*

$$\prod_{i=1}^{f} Z(B)^\times \twoheadrightarrow \operatorname{Out}_k^s(B) \tag{5.64}$$

In particular, $\operatorname{Out}_k^s(B)$ is connected as an algebraic group, and hence equal to $\operatorname{Out}_k^0(B)$.

Proof. We retain the notations of the previous section, and in particular we identify B with a block of kQ/I (with Q and I as defined in Proposition 5.3). First define a homomorphisms of algebraic groups

$$\psi : \prod_{i=1}^{f} Z(B)^\times \to \operatorname{Aut}_k^s(B) \tag{5.65}$$

which sends $(z_1, \ldots, z_f)$ to the automorphism given by $s_{i,q} \mapsto z_q \cdot s_{i,q}$ (and mapping the e_i to themselves). It is clear that those are automorphisms by checking that the images satisfy the relations given in Proposition 5.3. We claim that the composition of ψ with the natural epimorphism $\operatorname{Aut}_k^s(B) \twoheadrightarrow \operatorname{Out}_k^s(B)$ is surjective. Note that $Z(B)^\times$ is an extension of $\mathbb{G}_m(k)$ by the affine plane $\operatorname{Jac}(Z(B))$, and hence is connected.

We first prove the following claim, which will be used below: If $n \in \mathbb{N}$ is relatively prime to p, then the equation $T^n - z$ for $z \in Z(B)^\times$ has a solution in $Z(B)^\times$. This follows from the fact that a full set of n orthogonal primitive idempotents can be lifted from $k[T]/(T^n - \overline{z})$ to $Z(B)[T]/(T^n - z)$ (where $\overline{z}$ is the image of z in $Z(B)/\operatorname{Jac}(Z(B)) = k$). This yields a decomposition of algebras $Z(B)[T]/(T^n - z) \cong A_1 \oplus \ldots \oplus A_n$. Since the A_i are, in particular, $Z(B)$-modules, and $Z(B)[T]/(T^n - z)$ is free of rank n as a $Z(B)$-module, we must have that each A_i is a $Z(B)$-algebra that is free of rank one as a $Z(B)$-module. Hence each A_i is canonically isomorphic (as a k-algebra) to $Z(B)$, and the image of T in any of the $A_i \cong Z(B)$ is a solution of $T^n - z = 0$.

Now we come to the actual proof of surjectivity of the composition of ψ with the natural epimorphism $\mathrm{Aut}_k^s(B) \twoheadrightarrow \mathrm{Out}_k^s(B)$. Assume that $\alpha \in \mathrm{Aut}(B)$ is an automorphism such that $P \otimes {}_{\mathrm{id}}A_\alpha \cong P$ for all projective indecomposables P. All full sets of orthogonal primitive idempotents in B are conjugate (see, for instance, [CR81, Introduction §6, Exercise 14]), and hence we may compose α with an inner automorphism of B such that the resulting automorphism fixes all idempotents. We replace α by this new automorphism (without loss of generality). Since the canonical map $Z(B) \to e_iBe_i$ is surjective, and $s_{i,q}$ is a generator for the e_iBe_i module $e_iBe_{i+[\mathbf{q}]}$, we will have $\alpha(s_{i,q}) = z_{i,q} \cdot s_{i,q}$ for certain elements $z_{i,q} \in Z(B)^\times$ (and the $z_{i,q}$ determine α). Now consider conjugation with elements v of the form $v = \sum_i c_ie_i$ for certain $c_i \in Z(B)^\times$:

$$v^{-1} \cdot \alpha(s_{i,q}) \cdot v = \underbrace{\frac{c_{i+[\mathbf{q}]}}{c_i} \cdot z_{i,q}}_{=:\tilde{z}_{i,q}} \cdot s_{i,q} \tag{5.66}$$

With $\tilde{z}_{i,q}$ defined as in the above equation we have

$$\prod_i \tilde{z}_{i,0} = \prod_i z_{i,0} \tag{5.67}$$

Furthermore we can choose the c_i in the definition of v to assign prescribed values to all but one the $\tilde{z}_{i,0}$. Choose the c_i so that all but possibly one become equal to an κ-th root of the above product (where κ is the number of simple modules in the block, which is relatively prime to p). Then by the invariance of the product given in (5.67), all $\tilde{z}_{i,0}$ will be equal. Replace (without loss) α by the composition of α with conjugation by this v, that is, assume that all $z_{i,0}$ are equal. We claim that this α (which differs from the α we started with only by an inner automorphism) lies in $\mathrm{Im}(\psi)$ (with ψ as defined in (5.65)). To show this first notice that for $q \neq r$ the product $s_{i,q} \cdot s_{i+[\mathbf{q}],r}$ is a generator for the e_iBe_i-module $e_iBe_{i+[\mathbf{q}]+[\mathbf{r}]}$, which is isomorphic to the e_iBe_i-module $e_iBe_{i+[\mathbf{q}]}$. Hence for any $c, \tilde{c} \in Z(B)^\times$ we have $c \cdot s_{i,q} = \tilde{c} \cdot s_{i,q}$ if and only if $c \cdot s_{i,q}s_{i+[\mathbf{q}],r} = \tilde{c} \cdot s_{i,q}s_{i+[\mathbf{q}],r}$. Furthermore, in order for α to be an automorphism, the following relation must hold:

$$\begin{aligned} z_{i,q} \cdot z_{i+[\mathbf{q}],q+1} \cdot s_{i,q}s_{i+[\mathbf{q}],q+1} &= z_{i,q+1} \cdot z_{i+[\mathbf{q+1}],q} \cdot s_{i,q+1}s_{i+[\mathbf{q+1}],q} \\ &= z_{i,q+1} \cdot z_{i+[\mathbf{q+1}],q} \cdot s_{i,q}s_{i+[\mathbf{q}],q+1} \end{aligned} \tag{5.68}$$

So if we assume (as an induction hypothesis) that all $z_{i,q}$ (for some fixed value of q) are equal, then this implies that $z_{i+[\mathbf{q}],q+1} \cdot s_{i,q} = z_{i,q+1} \cdot s_{i,q}$, and hence we may set $z_{i+[\mathbf{q}],q+1} = z_{i,q+1}$. Consequentially, all $z_{i,q+1}$ are equal. Therefore α agrees with an element of $\mathrm{Im}(\psi)$ on the generators $s_{i,q}$. But this implies $\alpha \in \mathrm{Im}(\psi)$. □

Remark 5.19. *By determining the kernel of the epimorphism in (5.64) one can easily deduce that*

$$\mathrm{Out}_k^s(B) \cong \prod^f k[T]/(T^2)^\times \cong (\mathbb{G}_m^f \times \mathbb{G}_a^f)(k) \quad \text{if } p \neq 2 \tag{5.69}$$

and

$$\mathrm{Out}_k^s(B) \cong \mathbb{G}_m^f(k) \quad \text{if } p = 2 \tag{5.70}$$

Lemma 5.20. *Let $\overline{\Lambda}$ be a split k-form of the principal block $\bar{k}\Delta_2(p^f)$, and assume there is a lift Λ of $\overline{\Lambda}$ subject to conditions as in Theorem 5.16 (by said theorem, this lift will be unique). Then if $\alpha \in \mathrm{Aut}_k(\overline{\Lambda})$, then there exists a $\beta \in \mathrm{Aut}_{\mathcal{O}}(\Lambda)$ such that $\alpha \circ \overline{\beta} \in \mathrm{Aut}_k^s(\overline{\Lambda})$ (where $\overline{\beta}$ denotes the image of β in $\mathrm{Aut}_k(\overline{\Lambda})$).*

Proof. This follows from the fact that (since any two full sets of orthogonal primitive idempotents are conjugate) the automorphism α can be composed with an inner automorphism (which clearly fixes all simple modules) to get an automorphism of $\overline{\Lambda}$ that induces a permutation on some full set of orthogonal primitive idempotents in $\overline{\Lambda}$. Now Remark 5.17 implies the existence of β. □

Corollary 5.21. *Let $\overline{\Gamma}$ be a k-algebra that is derived equivalent to a split k-form $\overline{\Lambda}$ of $B_0(\bar{k}\Delta_2(p^f))$. Moreover let B be a finite-dimensional semisimple K-algebra with $\dim_K Z(B) = \dim_{\bar{k}} Z(B_0(\bar{k}\Delta_2(p^f)))$ and assume B is split by some totally ramified extension of K. Given an element $u \in Z(B)^\times$ which has p-valuation $-f$ in every Wedderburn component of $Z(\overline{K} \otimes B)$, there is, up to isomorphism, at most one full $\mathcal{O}$-order $\Gamma_u \subset B$ satisfying the following conditions:*

(1) Γ_u is self-dual with respect to T_u.

(2) $k \otimes \Gamma_u$ is isomorphic to $\overline{\Gamma}$.

Proof. Recall the result of Proposition 3.14, which stated that if Λ is a lift of $\overline{\Lambda}$ for which every outer automorphism of $\overline{\Lambda}$ may be written as a composition of (the reduction of) an automorphism of Λ and an element the $\bar{k}$-linear extension of which lies in $\mathrm{Out}_{\bar{k}}^0(\bar{k} \otimes_k \overline{\Lambda})$, then Λ corresponds to a single equivalence class of lifts in $\widehat{\mathfrak{L}}(\overline{\Lambda})$. This proposition is applicable to $\overline{\Lambda}$ and the unique lift Λ of $\overline{\Lambda}$ subject to conditions as in Theorem 5.16, since we have verified in Lemma 5.18 and Lemma 5.20 above that the conditions of the proposition are met. Theorem 3.20 shows that the equivalence classes in $\widehat{\mathfrak{L}}(\overline{\Lambda})$ subject to the conditions of Theorem 5.16 (with a modified u, depending on the choice of the derived equivalence) are in bijection with the equivalence classes in $\widehat{\mathfrak{L}}(\overline{\Gamma})$ subject to the conditions given in the statement of this corollary. Therefore there is at most one equivalence class of lifts of $\overline{\Gamma}$ satisfying our assumptions. In particular there is at most one isomorphism class of orders satisfying the assumptions. □

Remark 5.22. *Broué's abelian defect conjecture states the following: Let k be an algebraically closed field, G a group, B a block of kG, P a defect group of B, and b the Brauer correspondent of B in $kN_G(P)$. Then b and B are derived equivalent.*

Broué's conjecture has been proven (in defining characteristic) for the principal block of $\mathrm{SL}_2(q)$ in [Oku00] (although this paper has unfortunately never been published). It has also been shown to hold for the unique non-principal block of maximal defect of $\mathrm{SL}_2(q)$ (which exists if q is odd) in [Yos09].

Corollary 5.23. *Assume k is algebraically closed. Then the generators for a basic order of $\mathcal{O}\,\mathrm{SL}_2(p^f)$ as conjectured in [Neb00a] (for $p = 2$) respectively in [Neb00b] (for p odd) define an $\mathcal{O}$-order which is Morita equivalent to $\mathcal{O}\,\mathrm{SL}_2(p^f)$.*

5.3 Rationality of Tilting Complexes

Our goal in this section is to perform a "Galois descent for derived equivalences" to the degree up to which this is possible. This will allow us to state a unique lifting theorem for the group ring $\mathbb{F}_{p^f}\,\mathrm{SL}_2(p^f)$, thus ridding us of the necessity to assume an algebraically closed coefficient field.

Concerning notation: In this section we often use field extensions $\tilde{K}$ and K' of K. We will always assume that $\tilde{K}$ and K' are (possibly infinite) algebraic extensions of K of finite ramification. We denote by $\tilde{O}$ respectively $\mathcal{O}'$ the corresponding discrete valuation rings and by $\tilde{k}$ respectively k' their respective residue fields.

Definition 5.24. *We call an $\mathcal{O}$-order Λ* split *if the k-algebra $k\otimes\Lambda$ is split and the K-algebra $K\otimes\Lambda$ is split.*

Lemma 5.25. *Let k be finite. Let Λ be an $\mathcal{O}$-order such that $K\otimes\Lambda$ is split semisimple. Assume that there is a field extension $\tilde{K}/K$ of finite degree such that $\tilde{\mathcal{O}}\otimes\Lambda$ is split and its decomposition matrix has full row rank (that is, its rank is equal to its number of columns). Then Λ is already split.*

Proof. Assume S is a simple Λ-module that is not absolutely irreducible. Since there are no non-commutative finite-dimensional division algebras over k, $\mathrm{End}(S)$ is commutative and hence $\mathrm{End}(\tilde{k}\otimes S)\cong\tilde{k}\otimes\mathrm{End}(S)$ is a direct sum of copies of $\tilde{k}$. Therefore $\tilde{k}\otimes S$ is a direct sum of non-isomorphic simple $\tilde{\mathcal{O}}\otimes\Lambda$-modules $\tilde{S}_1,\ldots,\tilde{S}_l$ (for some $l>1$). Each simple $\tilde{K}\otimes\Lambda$-module is of the form $\tilde{K}\otimes V$ for some simple $K\otimes\Lambda$-module V. Let L be a Λ-lattice in V. Then $\tilde{\mathcal{O}}\otimes L$ is a $\tilde{\mathcal{O}}\otimes\Lambda$-lattice in $\tilde{K}\otimes V$, and the multiplicities of $\tilde{S}_1,\ldots\tilde{S}_l$ in $\tilde{k}\otimes L$ are all equal to the multiplicity of S in $k\otimes L$. Therefore, the columns in the decomposition matrix of $\tilde{\mathcal{O}}\otimes\Lambda$ associated to the simple modules $\tilde{S}_1,\ldots,\tilde{S}_l$ are all equal, in contradiction to the assumption that the decomposition matrix of $\tilde{\mathcal{O}}\otimes\Lambda$ has full row rank. Therefore all simple Λ-modules are absolutely simple, that is, Λ is split. □

Lemma 5.26. *Assume that $\tilde{K}$ is totally ramified over K. If Λ is an $\mathcal{O}$-order such that $\tilde{k}\otimes\Lambda$ is split, then $k\otimes\Lambda$ is split.*

In particular, under the assumption that k is finite, $\tilde{K}\otimes\Lambda$ is split semisimple and the decomposition matrix of Λ over a splitting system has full row rank, $k\otimes\Lambda$ will be split.

Proof. This is clear since $\tilde{k}=k$. □

Remark 5.27. *We should note that*

(1) Full row rank of the decomposition matrix is implied if the Cartan matrix of an algebra is non-degenerate (which is a known fact in the case of group rings).

(2) Up to signs, the determinant (and therefore non-degeneracy) of the Cartan matrix is preserved under derived equivalences (even under stable equivalences of Morita type).

Definition 5.28. *Let A be a ring. We say a tilting complex $T\in\mathcal{C}^b(\mathbf{proj}_A)$ is* determined by its terms*, if any tilting complex $T'\in\mathcal{C}^b(\mathbf{proj}_A)$ with $T^i\cong T'^i$ for all $i\in\mathbb{Z}$ is isomorphic to T in $\mathcal{K}^b(\mathbf{proj}_A)$.*

Remark 5.29. *By Theorem 2.99, two-term tilting complexes defined over algebras over a field are determined by their terms. By unique lifting, the same is true for two-term tilting complexes defined over orders over complete discrete valuation rings.*

Definition 5.30. *Let $\tilde{\Lambda}$ be an $\tilde{\mathcal{O}}$-order. We call an $\mathcal{O}$-order $\Lambda \subseteq \tilde{\Lambda}$ an $\mathcal{O}$-*form *of $\tilde{\Lambda}$ if* $\operatorname{rank}_{\mathcal{O}} \Lambda = \operatorname{rank}_{\tilde{\mathcal{O}}} \tilde{\Lambda}$ *and $\tilde{\mathcal{O}} \cdot \Lambda = \tilde{\Lambda}$. We define a k-form of a finite-dimensional $\tilde{k}$-algebra is the analogous way.*

Lemma 5.31. *Let Λ be an $\mathcal{O}$-order and let $\tilde{K}$ be an unramified finite extension of K. Furthermore, let $\tilde{C} \in \mathcal{C}^b(\mathbf{mod}_{\tilde{\mathcal{O}}\otimes\Lambda})$ be a complex of $\tilde{\mathcal{O}} \otimes \Lambda$-modules and let C be the restriction of $\tilde{C}$ to Λ. Then, in the category $\mathcal{C}^b(\mathbf{mod}_{\tilde{\mathcal{O}}\otimes\Lambda})$,*

$$\tilde{\mathcal{O}} \otimes C \cong \bigoplus_{i=1}^{[\tilde{K}:K]} \tilde{C}^{\alpha_i} \tag{5.71}$$

for certain $\alpha_i \in \operatorname{Aut}_{\mathcal{O}}(\tilde{\mathcal{O}})$. Here, for an $\alpha \in \operatorname{Aut}_{\mathcal{O}}(\tilde{\mathcal{O}})$, $\tilde{C}^\alpha$ denotes the complex of $\tilde{\mathcal{O}} \otimes \Lambda$-module the terms of which are (as sets) equal to the terms of $\tilde{C}$, with differential equal to that of $\tilde{C}$, but with the following twisted action of $\tilde{\mathcal{O}} \otimes \Lambda$ on the terms:

$$\tilde{C}^i \times \tilde{\mathcal{O}} \otimes \Lambda \longrightarrow \tilde{C}^i : (m, a \otimes b) \mapsto m \cdot \alpha(a) \otimes b \tag{5.72}$$

We claim furthermore that at least one of the α_i may be chosen to be the identity automorphism of $\tilde{\mathcal{O}}$.

Proof. First note that $\tilde{\mathcal{O}} \otimes_{\mathcal{O}} \tilde{\mathcal{O}} \cong \bigoplus^{[\tilde{K}:K]} \tilde{\mathcal{O}}$, since $\tilde{K}$ is unramified over K. For $i \in \{1, \ldots, [\tilde{K} : K]\}$ denote by ε_i the epimorphism from $\tilde{\mathcal{O}} \otimes_{\mathcal{O}} \tilde{\mathcal{O}}$ to $\tilde{\mathcal{O}}$ given by projection to the i-th component of $\bigoplus^{[\tilde{K}:K]} \tilde{\mathcal{O}}$ (of course, the ordering of the ε_i is not canonical). By abuse of notation, we also denote by ε_i the unique primitive idempotent in $\tilde{\mathcal{O}} \otimes_{\mathcal{O}} \tilde{\mathcal{O}}$ that gets mapped to 1 under the projection ε_i. Now we consider the complex of $\tilde{\mathcal{O}} \otimes_{\mathcal{O}} \tilde{\mathcal{O}} \otimes_{\mathcal{O}} \Lambda$-modules $\tilde{\mathcal{O}} \otimes_{\mathcal{O}} \tilde{C}$. We can decompose this complex as follows:

$$\tilde{\mathcal{O}} \otimes_{\mathcal{O}} \tilde{C} = \bigoplus_{i=1}^{[\tilde{K}:K]} \tilde{\mathcal{O}} \otimes_{\mathcal{O}} \tilde{C} \cdot (\varepsilon_i \otimes 1_\Lambda) \tag{5.73}$$

Now consider the embedding

$$\eta : \tilde{\mathcal{O}} \hookrightarrow \tilde{\mathcal{O}} \otimes_{\mathcal{O}} \tilde{\mathcal{O}} : a \mapsto a \otimes 1 \tag{5.74}$$

If we turn $\tilde{\mathcal{O}} \otimes_{\mathcal{O}} \tilde{C}$ into a complex of $\tilde{\mathcal{O}} \otimes \Lambda$-modules via the embedding $\eta \otimes \mathrm{id}_\Lambda$ we get, by definition, $\tilde{\mathcal{O}} \otimes_{\mathcal{O}} C$. If we turn $\tilde{\mathcal{O}} \otimes_{\mathcal{O}} \tilde{C} \cdot (\varepsilon_i \otimes 1_\Lambda)$ into a complex of $\tilde{\mathcal{O}} \otimes \Lambda$-modules via the embedding $\eta \otimes \mathrm{id}_\Lambda$ we get $\tilde{C}^{\varepsilon_i \circ \eta}$. So the our first claim follows (with $\alpha_i := \varepsilon_i \circ \eta$). As for the claim that one of the α_i may be chosen equal to the identity, just note that there is an epimorphism $\tilde{\mathcal{O}} \otimes_{\mathcal{O}} \tilde{\mathcal{O}} \twoheadrightarrow \tilde{\mathcal{O}} : a \otimes b \mapsto a \cdot b$. Since the ε_i are in fact all epimorphisms from $\tilde{\mathcal{O}} \otimes_{\mathcal{O}} \tilde{\mathcal{O}}$ to $\tilde{\mathcal{O}}$, this epimorphism needs to be equal to some ε_i. But then $\alpha_i = \mathrm{id}$. □

Proposition 5.32 (Reduction to Finite Field Extensions)**.** *Let Λ and Γ be two $\mathcal{O}$-orders such that $\tilde{\mathcal{O}} \otimes \Lambda$ and $\tilde{\mathcal{O}} \otimes \Gamma$ are derived equivalent, and let $\tilde{T}$ be a tilting complex over $\tilde{\mathcal{O}} \otimes \Lambda$ with endomorphism ring $\tilde{\mathcal{O}} \otimes \Gamma$. Then there exists a finite extension K' of K which is contained* *in $\tilde{K}$ such that $\mathcal{O}' \otimes \Lambda$ is derived equivalent to an $\mathcal{O}'$-form Γ' of $\tilde{\mathcal{O}} \otimes \Gamma$, and there is a tilting complex T' over $\mathcal{O}' \otimes \Lambda$ with endomorphism ring Γ' such that $\tilde{\mathcal{O}} \otimes_{\mathcal{O}'} T' \cong \tilde{T}$ in $\mathcal{K}^b(\mathbf{proj}_{\tilde{\mathcal{O}}\otimes\Lambda})$.*

Proof. There is some invertible complex $\tilde{X} \in \mathcal{D}^b((\tilde{\mathcal{O}} \otimes \Lambda)^{\mathrm{op}} \otimes_{\tilde{\mathcal{O}}} (\tilde{\mathcal{O}} \otimes \Gamma))$ with inverse $\tilde{Y} \in \mathcal{D}^b((\tilde{\mathcal{O}} \otimes \Gamma)^{\mathrm{op}} \otimes_{\tilde{\mathcal{O}}} (\tilde{\mathcal{O}} \otimes \Lambda))$ such that the restriction of $\tilde{Y}$ to $\tilde{\mathcal{O}} \otimes \Lambda$ is isomorphic to $\tilde{T}$ in $\mathcal{D}^b(\tilde{\mathcal{O}} \otimes \Lambda)$. We can find a finite extension K' of K (contained in $\tilde{K}$) such that there are bounded complexes X' and Y' such that $\tilde{\mathcal{O}} \otimes_{\mathcal{O}'} X' \cong \tilde{X}$ and $\tilde{\mathcal{O}} \otimes_{\mathcal{O}'} Y' \cong \tilde{Y}$. This is simply because $\tilde{X}$ and $\tilde{Y}$ can be represented by bounded complexes of finitely generated modules, and so K' needs only be big enough for all terms of these complexes to be defined over $\mathcal{O}'$ and for the differentials (which are made up of finitely many homomorphisms) to be defined. Looking at the construction of the derived tensor product, it is clear that

$$\tilde{\mathcal{O}} \otimes^{\mathbb{L}}_{\mathcal{O}'} (X' \otimes^{\mathbb{L}}_{\mathcal{O}'\otimes\Gamma} Y') \cong \tilde{X} \otimes^{\mathbb{L}}_{\tilde{\mathcal{O}}\otimes\Gamma} \tilde{Y} \text{ and } \tilde{\mathcal{O}} \otimes^{\mathbb{L}}_{\mathcal{O}'} (Y' \otimes^{\mathbb{L}}_{\mathcal{O}'\otimes\Lambda} X') \cong \tilde{Y} \otimes^{\mathbb{L}}_{\tilde{\mathcal{O}}\otimes\Lambda} \tilde{X} \tag{5.75}$$

But the right hand terms in (5.75) have homology concentrated in degree zero. This means that $X' \otimes^{\mathbb{L}}_{\mathcal{O}'\otimes\Gamma} Y'$ and $Y' \otimes^{\mathbb{L}}_{\mathcal{O}'\otimes\Lambda} X'$ are isomorphic to stalk complexes in $\mathcal{D}^-((\mathcal{O}' \otimes \Lambda)^{\mathrm{op}} \otimes_{\mathcal{O}'} (\mathcal{O}' \otimes \Lambda))$ respectively $\mathcal{D}^-((\mathcal{O}' \otimes \Gamma)^{\mathrm{op}} \otimes_{\mathcal{O}'} (\mathcal{O}' \otimes \Gamma))$. Since tensoring with $\tilde{\mathcal{O}}$ renders them isomorphic to $0 \to \tilde{\mathcal{O}} \otimes \Lambda \to 0$ respectively $0 \to \tilde{\mathcal{O}} \otimes \Gamma \to 0$ it follows from the Noether-Deuring theorem for modules that they are isomorphic to $0 \to \mathcal{O}' \otimes \Lambda \to 0$ respectively $0 \to \mathcal{O}' \otimes \Gamma \to 0$. Therefore X' and Y' are invertible, and thus the restriction of Y' to $\mathcal{O}' \otimes \Lambda$ is a tilting complex T' with $\tilde{\mathcal{O}} \otimes_{\mathcal{O}'} T' \cong \tilde{T}$.

By [Ric91a, Theorem 2.1] it follows that the endomorphism ring of T' in $\mathcal{D}^b(\mathcal{O}' \otimes \Lambda)$ is an $\mathcal{O}'$-form of $\tilde{\mathcal{O}} \otimes \Lambda$. □

Remark 5.33. *We should mention the following (trivial) addendum to the above proposition: If $\tilde{\mathcal{O}}$ splits Λ and/or Γ, we may choose an $\mathcal{O}'$ which splits Λ and/or Γ. Similarly, if $\tilde{k}$ splits $k \otimes \Lambda$ and/or $k \otimes \Gamma$, we may choose an $\mathcal{O}'$ such that k' (the residue field of $\mathcal{O}'$) splits $k \otimes \Lambda$ and/or $k \otimes \Gamma$.*

Lemma 5.34. *Let Λ be an $\mathcal{O}$-order and let $T \in \mathcal{C}^b(\mathbf{mod}_\Lambda)$ be a complex with differential $d : T \longrightarrow T[-1]$. If $\tilde{\mathcal{O}} \otimes T$ is a tilting complex for $\tilde{\mathcal{O}} \otimes \Lambda$ (in particular $\tilde{\mathcal{O}} \otimes T \in \mathcal{C}^b(\mathbf{proj}_{\tilde{\mathcal{O}}\otimes\Lambda})$), then T is a tilting complex for Λ.*

Proof. First note that by Proposition 5.32 we may assume that $\tilde{K}/K$ is a field extension of finite degree. If M is a (finitely-generated) Λ-module such that $\tilde{\mathcal{O}} \otimes M$ is a projective $\tilde{\mathcal{O}} \otimes \Lambda$-module, M must itself be projective. This follows easily from the fact that $\tilde{\mathcal{O}} \otimes M$ is projective if and only if it is a direct summand of some free module, and so the restriction of $\tilde{\mathcal{O}} \otimes M$, which is just a direct sum of copies of M, is a summand of a restriction of a free module, which is again a free module. This shows that $\tilde{\mathcal{O}} \otimes T \in \mathcal{C}^b(\mathbf{proj}_{\tilde{\mathcal{O}}\otimes\Lambda})$ implies $T \in \mathcal{C}^b(\mathbf{proj}_\Lambda)$.

Now we show $\mathrm{Hom}_{\mathcal{D}^b(\Lambda)}(T, T[i]) = 0$ for all $i \neq 0$. So let $\varphi \in \mathrm{Hom}_{\mathcal{C}^b(\mathbf{proj}_\Lambda)}(T, T[i])$. Then there is a homotopy $h : \tilde{\mathcal{O}} \otimes T \longrightarrow \tilde{\mathcal{O}} \otimes T[i+1]$ such that $1_{\tilde{\mathcal{O}}} \otimes \varphi = h \circ (1_{\tilde{\mathcal{O}}} \otimes d) + (1_{\tilde{\mathcal{O}}} \otimes d) \circ h$. Since for arbitrary Λ-modules M and N we have $\mathrm{Hom}_{\tilde{\mathcal{O}}\otimes\Lambda}(\tilde{\mathcal{O}} \otimes M, \tilde{\mathcal{O}} \otimes N) \cong \tilde{\mathcal{O}} \otimes \mathrm{Hom}_\Lambda(M, N)$, we can write

$$h = \sum_{j=1}^{[\tilde{K}:K]} b_j \otimes h_j \quad \text{for certain } h_j : T \longrightarrow T[i+1] \tag{5.76}$$

where $(b_1, \ldots, b_{[\tilde{K}:K]})$ is an $\mathcal{O}$-basis of $\tilde{\mathcal{O}}$ and, without loss, $b_1 = 1_{\tilde{\mathcal{O}}}$. Hence

$$b_1 \otimes \varphi = \sum_{j=1}^{[\tilde{K}:K]} b_j \otimes (h_j \circ d + d \circ h_j) \tag{5.77}$$

This implies

$$\varphi = h_1 \circ d + d \circ h_1 \tag{5.78}$$

and therefore φ is homotopic to the zero map.

Now we show that T generates $\mathcal{K}^b(\mathbf{proj}_\Lambda)$. To see this we look at the functor

$$\mathrm{Res} : \mathcal{K}^-(\mathbf{proj}_{\tilde{\mathcal{O}}\otimes\Lambda}) \longrightarrow \mathcal{K}^-(\mathbf{proj}_\Lambda) \tag{5.79}$$

which, by definition, simply restricts the terms of the complexes from $\tilde{\mathcal{O}} \otimes \Lambda$-modules to Λ-modules. Since this is an exact functor, and $\mathrm{Res}(\tilde{\mathcal{O}} \otimes T)$ is just a direct sum of copies of T, $\mathrm{add}(T) \supseteq \mathrm{Res}(\mathrm{add}(\tilde{\mathcal{O}} \otimes T))$. But $0 \to \tilde{\mathcal{O}} \otimes \Lambda \to 0$ lies in $\mathrm{add}(\tilde{\mathcal{O}} \otimes T)$, and therefore $0 \to \Lambda \to 0$ lies in $\mathrm{add}(T)$ (since $\mathrm{Res}(0 \to \tilde{\mathcal{O}} \otimes \Lambda \to 0)$ is isomorphic to a direct sum of copies of $0 \to \Lambda \to 0$). □

Theorem 5.35. *Assume k is finite and $\tilde{K}$ is unramified over K. Let $\tilde{\Lambda}$ be an $\tilde{\mathcal{O}}$-order such that $\tilde{k} \otimes \tilde{\Lambda}$ is split, $\tilde{K} \otimes \tilde{\Lambda}$ is semisimple and the decomposition matrix of $\tilde{\Lambda}$ over a splitting system has full row rank. Let $\tilde{T} \in \mathcal{C}^b(\mathbf{proj}_{\tilde{\Lambda}})$ be a tilting complex that is determined by its terms. Set*

$$\tilde{\Gamma} := \mathrm{End}_{\mathcal{D}^b(\tilde{\Lambda})}(\tilde{T}) \tag{5.80}$$

If Λ is an $\mathcal{O}$-form of $\tilde{\Lambda}$ such that $k \otimes \Lambda$ is split and there is a totally ramified extension of K that splits $K \otimes \Lambda$, then there is an $\mathcal{O}$-form Γ of $\tilde{\Gamma}$ with the same properties that is derived equivalent to Λ.

Proof. Let T be the restriction of $\tilde{T}$ to $\mathcal{C}^b(\mathbf{proj}_\Lambda)$. By Lemma 5.31 the complex $\tilde{\mathcal{O}} \otimes T$ is isomorphic to a direct sum of complexes of the form $\tilde{T}^\alpha$ for certain $\alpha \in \mathrm{Aut}_{\mathcal{O}}(\tilde{\mathcal{O}})$. Now note that since $k \otimes \Lambda$ is split, the projective indecomposable $\tilde{\Lambda}$-modules $\tilde{P}$ are of the form $\tilde{\mathcal{O}} \otimes P$ for projective indecomposable Λ-modules P. Therefore they are isomorphic to their Galois twists. In particular, the terms of $\tilde{T}^\alpha$ and $\tilde{T}$ are isomorphic for all $\alpha \in \mathrm{Aut}_{\mathcal{O}}(\tilde{\mathcal{O}})$. Since $\tilde{T}$ is by assumption determined by its terms, we must have $\tilde{T}^\alpha \cong \tilde{T}$ for all $\alpha \in \mathrm{Aut}_{\mathcal{O}}(\tilde{\mathcal{O}})$. This shows that $\tilde{\mathcal{O}} \otimes T$ is a tilting complex, and therefore so is T (by Lemma 5.34). It is clear by [Ric91a, Theorem 2.1] (or by using linear algebra) that the endomorphism ring of T is an $\mathcal{O}$-form of the endomorphism ring of $\tilde{\mathcal{O}} \otimes T$, and of course it is derived equivalent to Λ. We have

$$\tilde{\mathcal{O}} \otimes \mathrm{End}_{\mathcal{D}^b(\Lambda)}(T) \cong \mathrm{End}_{\mathcal{D}^b(\tilde{\Lambda})}(\tilde{\mathcal{O}} \otimes T) \cong \tilde{\Gamma}^{[\tilde{K}:K]\times[\tilde{K}:K]} \tag{5.81}$$

that is, $\mathrm{End}_{\mathcal{D}^b(\Lambda)}(T)$ is on $\mathcal{O}$-form of $\tilde{\Gamma}^{[\tilde{K}:K]\times[\tilde{K}:K]}$. This will yield on $\mathcal{O}$-form of $\tilde{\Gamma}$ with the desired properties (simply by applying a Morita equivalence) once we see that $k \otimes \mathrm{End}_{\mathcal{D}^b(\Lambda)}(T)$ is split. Let K' be a totally ramified extension of K such that $K' \otimes \Lambda$ is split. Since $K' \otimes \mathrm{End}_{\mathcal{D}^b(\Lambda)}(T) \cong \mathrm{End}_{\mathcal{D}^b(K'\otimes\Lambda)}(K' \otimes T)$ is Morita equivalent to $K' \otimes \Lambda$, it follows by Lemma 5.26 that $k \otimes \mathrm{End}_{\mathcal{D}^b(\Lambda)}(T)$ is split. □

Corollary 5.36. *The assertion of the preceding Theorem remains true if $\tilde{\Lambda}$ and $\tilde{\Gamma}$ are linked by a series of derived equivalences which all are afforded by tilting complexes that are determined by their terms.*

Proof. This follows by iterated application of the preceding theorem. □

Corollary 5.37. *Let $\mathcal{O}$ be the p-adic completion of the maximal unramified extension of $\mathbb{Q}_p$. The blocks of defect C_p^f of the group ring $\mathbb{Z}_p[\zeta_{p^f-1}]\operatorname{SL}_2(p^f)$ are derived equivalent to a $\mathbb{Z}_p[\zeta_{p^f-1}]$-form (split over $\mathbb{F}_{p^f}$) of their respective Brauer correspondent in $\mathcal{O}\Delta_2(p^f)$ with $\mathbb{Q}_p[\zeta_{p^f-1}]$-span isomorphic to the $\mathbb{Q}_p[\zeta_{p^f-1}]$-span of the corresponding block of $\mathbb{Z}_p[\zeta_{p^f-1}]\Delta_2(p^f)$.*

Proof. The respective blocks of $k\operatorname{SL}_2(p^f)$ and $k\Delta_2(p^f)$ are linked by a series of two-term complexes (see [Oku00] respectively [Yos09]). Hence the first claim follows from Theorem 5.35 and Corollary 5.36. The assertion concerning the $\mathbb{Q}_p[\zeta_{p^f-1}]$-spans follows from the fact that the $\mathbb{Q}_p[\zeta_{p^f-1}]$-spans of the blocks of $\mathbb{Z}_p[\zeta_{p^f-1}]\operatorname{SL}_2(p^f)$ and $\mathbb{Z}_p[\zeta_{p^f-1}]\Delta_2(p^f)$ which are Brauer correspondents are Morita equivalent. □

Corollary 5.38. *Assume $k \supseteq \mathbb{F}_{p^f}$ and B is a block of $k\operatorname{SL}_2(p^f)$ of maximal defect. Let A be a finite-dimensional semisimple K-algebra with $\dim_K Z(A) = \dim_k Z(B)$. Assume A is split by some totally ramified extension of K. Given an element $u \in Z(A)^\times$ which has p-valuation $-f$ in every Wedderburn component of $Z(\bar{K}\otimes A)$, there is, up to conjugacy, at most one full $\mathcal{O}$-order $\Lambda_u \subset A$ satisfying the following conditions:*

(1) Λ_u is self-dual with respect to T_u.

(2) $k\otimes\Lambda_u$ is isomorphic to B.

Proof. By Corollary 5.37 the block B is derived equivalent to a split k-form $\overline{\Gamma}$ of $B_0(\bar{k}\Delta_2(p^f))$. Thus the assertion follows directly from Corollary 5.21. □

Corollary 5.39. *The generators for a basic order of $\mathbb{Z}_p[\zeta_{p^f-1}]\operatorname{SL}_2(p^f)$ as conjectured in [Neb00a] (for $p = 2$) respectively in [Neb00b] (for p odd) define a $\mathbb{Z}_p[\zeta_{p^f-1}]$-order which is Morita-equivalent to $\mathbb{Z}_p[\zeta_{p^f-1}]\operatorname{SL}_2(p^f)$.*

Corollary 5.40. *The non-semisimple blocks of $\mathbb{Z}_p[\zeta_{p^f-1}]\operatorname{SL}_2(p^f)$ are derived equivalent to their Brauer-correspondents in $\mathbb{Z}_p[\zeta_{p^f-1}]\Delta_2(p^f)$.*

Proof. As we have already seen, any non-semisimple block Γ of $\mathbb{Z}_p[\zeta_{p^f-1}]\operatorname{SL}_2(p^f)$ is derived equivalent to the unique lift $\Lambda_u \subset \mathbb{Q}_p[\zeta_{p^f-1}]\otimes B_0(\mathbb{Z}_p[\zeta_{p^f-1}]\Delta_2(p^f)) =: A$ of a split $\mathbb{F}_{p^f}$-form of $B_0(\bar{\mathbb{F}}_p\Delta_2(p^f))$ with respect to some $u \in Z(A)$ satisfying the conditions of Theorem 5.16 (this is just putting Corollary 5.37 and Theorem 5.16 together). The addendum to Theorem 5.16 tells us that if $p = 2$, then $\Lambda_u \cong B_0(\mathbb{Z}_2[\zeta_{2^f-1}]\Delta_2(2^f))$ which implies the assertion of this corollary. If $p \neq 2$ and $\mathbb{Q}_p[\zeta_{p^f-1}]$ does not split $\operatorname{SL}_2(p^f)$, then the addendum tells us (using the same notational conventions as in Theorem 5.16, including Notation 5.14; these will be used throughout this proof) that Λ_u depends only on $u_{\kappa+1}\cdot\mathcal{O}^\times$, which we may assume to be equal to $p^{-f}\cdot\mathcal{O}^\times$ by virtue of $u_{\kappa+1}$ being rational. So again, $\Lambda_u \cong B_0(\mathbb{Z}_p[\zeta_{p^f-1}]\Delta_2(p^f))$ follows and we are done. Now if p is odd and $\mathbb{Q}_p[\zeta_{p^f-1}]$ does split $\operatorname{SL}_2(p^f)$, then Λ_u depends only on the quotient $u_{\kappa+1}/u_{\kappa+2}$. Assume for the rest of the proof that we are in this case. We also fix some tilting complex T in $\mathcal{K}^b(\mathbf{proj}_{\Lambda_u})$ with endomorphism ring Γ. Furthermore let $V_{\kappa+1}$ and

$V_{\kappa+2}$ be the $(\kappa+1)$-st and $(\kappa+2)$-nd simple $\bar{\mathbb{Q}}_p \otimes A$-module. Note that the symmetrizing element u for Λ_u arises from the symmetrizing element u' we use for Γ by flipping signs in certain Wedderburn components. As mentioned in Remark 5.15, u' may be chosen so that $u'_{\kappa+1} = u'_{\kappa+2}$, since the corresponding rows in the decomposition matrix are equal (we do not make use of any particular knowledge of the decomposition matrix of $\mathrm{SL}_2(p^f)$ to establish this; the fact that the $(\kappa+1)$-st and $(\kappa+2)$-nd row of the decomposition matrix of $\Delta_2(p^f)$ over a splitting system are equal implies that the corresponding rows in the decomposition matrix of a derived equivalent order will also be equal). The sign of $u'_{\kappa+1}$ respectively $u'_{\kappa+2}$ is flipped upon passage to Λ_u depending on the sign of $[V_{\kappa+1}]$ respectively $[V_{\kappa+2}]$ as a coefficient of

$$\sum_i (-1)^i \cdot [\bar{\mathbb{Q}}_p \otimes_{\mathbb{Z}_p[\zeta_{p^f-1}]} T^i] \in K_0(\mathbf{mod}_{\bar{\mathbb{Q}}_p \Delta_2(p^f)}) \tag{5.82}$$

These signs are equal, since all of the T^i are projective modules and therefore $V_{\kappa+1}$ and $V_{\kappa+2}$ occur in their $\bar{\mathbb{Q}}_p$-span with the same multiplicities (again since the corresponding rows in the decomposition matrix are equal). We conclude that $u_{\kappa+1} = u_{\kappa+2}$, and therefore $\Lambda_u \cong B_0(\mathbb{Z}_p[\zeta_{p^f-1}]\Delta_2(p^f))$, which is what we wanted to prove. □

Chapter 6

Symmetric Groups

This chapter consists of two essentially independent results on representations of symmetric groups: The first result is a generalization of a theorem which is commonly known as "Scopes reduction" (see Theorem 2.147). We establish Morita equivalences between certain epimorphic images of the blocks involved in a $(w : q)$-pair (see Definition 2.145). This, on the one hand, generalizes the result of Scopes (see [Sco91]), where Morita equivalences between such blocks were established whenever $w \leqslant q$. On the other hand, it may be seen in the context of the result of Chuang and Rouquier (see [CR08]), which assigns a derived equivalence to any $(w : q)$-pairing. The Morita equivalence of the epimorphic images of the involved blocks could in fact be a consequence of this derived equivalence, if it could be shown that the images of the differential of the tilting complex given in [CR08] are pure sublattices.

The second result is a description of basic orders for defect two blocks of group rings of symmetric groups. This is a very much classical application of the theory of orders with decomposition numbers "0" and "1" to such blocks.

6.1 Scopes Reduction for Wedderburn Components

As usual we let $(K, \mathcal{O}, k)$ be a p-modular system and we denote by π a uniformizer for $\mathcal{O}$. We assume that Λ and Γ are two $\mathcal{O}$-orders subject to all of the following conditions:

(C1) $A := K \otimes_{\mathcal{O}} \Lambda$ and $B := K \otimes_{\mathcal{O}} \Gamma$ are semisimple K-algebras, and K is a splitting field for both.

(C2) k splits $\overline{\Lambda} := k \otimes_{\mathcal{O}} \Lambda$ and $\overline{\Gamma} := k \otimes_{\mathcal{O}} \Gamma$.

(C3) Λ and Γ have the same number of isomorphism classes of simple modules.

(C4) A and B have the same number of isomorphism classes of simple modules.

Denote by $V_1, \ldots, V_h$ the simple A-modules, by $\varepsilon_1, \ldots, \varepsilon_h$ the corresponding idempotents in $Z(A)$, by $W_1, \ldots, W_h$ the simple B-modules and by $\eta_1, \ldots, \eta_h$ the corresponding idempotents in $Z(B)$. Moreover, denote by $S_1, \ldots, S_t$ the simple $\overline{\Lambda}$-modules and by $T_1, \ldots, T_t$ the simple $\overline{\Gamma}$-modules.

Assume that there is a Λ-Γ-bimodule $\mathbf{\Omega}$ such that

(C5) $\mathbf{\Omega}$ is projective as a right Γ-module and as a left Λ-module.

(C6) There is some h_0 in $\mathbb{N}$ and a matrix $X_\otimes \in \mathbb{Z}_{\geqslant 0}^{(h-h_0)\times(h-h_0)}$ such that

$$\begin{bmatrix} [V_1 \otimes_\Lambda \mathbf{\Omega}] \\ \vdots \\ [V_h \otimes_\Lambda \mathbf{\Omega}] \end{bmatrix} = \left[\begin{array}{ccc|c} 1 & & & \\ & \ddots & & \mathbf{0} \\ & & 1 & \\ \hline & \mathbf{0} & & X_\otimes \end{array}\right] \cdot \begin{bmatrix} [W_1] \\ \vdots \\ [W_h] \end{bmatrix} \tag{6.1}$$

as an equation over $K_0(\mathbf{mod}_B)^{h\times 1}$.

(C7) $S_i \otimes_\Lambda \mathbf{\Omega} \not\cong 0$ and $\mathrm{Hom}_\Gamma(\mathbf{\Omega}, T_i) \not\cong 0$ for all $i \in \{1, \ldots, t\}$.

(C8) Let $D^{(\Lambda)}$ and $D^{(\Gamma)}$ be the decomposition matrices of Λ and Γ. We assume that the rows in $D^{(\Lambda)}$ associated to $V_1, \ldots, V_{h_0}$ are equal to the rows in $D^{(\Gamma)}$ associated to $W_1, \ldots, W_{h_0}$. Of course we are at liberty to permute the columns to meet this requirement.

Finally, we define $\varepsilon = \varepsilon_1 + \ldots + \varepsilon_{h_0} \in Z(A)$ and $\eta = \eta_1 + \ldots + \eta_{h_0} \in Z(B)$. Our objective in this section is to prove the following theorem:

Theorem 6.1. *The $\mathcal{O}$-orders $\varepsilon\Lambda$ and $\eta\Gamma$ are Morita equivalent.*

We need a couple of lemmas before we can prove this. First note that we identify $\mathbf{mod}_{\varepsilon\Lambda}$ as a full subcategory of $\mathbf{mod}_\Lambda$ and $\mathbf{mod}_{\eta\Gamma}$ as a full subcategory of $\mathbf{mod}_\Gamma$. Also note that both $- \otimes_\Lambda \mathbf{\Omega}$ and $\mathrm{Hom}_\Gamma(\mathbf{\Omega}, -)$ are exact by **(C5)**.

Lemma 6.2. *There is a matrix $X_{\mathrm{Hom}} \in \mathbb{Z}_{\geqslant 0}^{(h-h_0)\times(h-h_0)}$ such that*

$$\begin{bmatrix} [\mathrm{Hom}_\Gamma(\mathbf{\Omega}, W_1)] \\ \vdots \\ [\mathrm{Hom}_\Gamma(\mathbf{\Omega}, W_h)] \end{bmatrix} = \left[\begin{array}{ccc|c} 1 & & & \\ & \ddots & & \mathbf{0} \\ & & 1 & \\ \hline & \mathbf{0} & & X_{\mathrm{Hom}} \end{array}\right] \cdot \begin{bmatrix} [V_1] \\ \vdots \\ [V_h] \end{bmatrix} \tag{6.2}$$

as an equation over $K_0(\mathbf{mod}_A)^{h\times 1}$.

Proof. Take $X \in \mathbb{Z}_{\geqslant 0}^{h\times h}$ such that

$$\mathrm{Hom}_\Gamma(\mathbf{\Omega}, W_i) \cong_A \bigoplus_{j=1}^{h} \bigoplus^{X_{i,j}} V_j \quad \forall i \in \{1, \ldots, h\} \tag{6.3}$$

Then

$$X_{i,j} = \dim_K \mathrm{Hom}_A(V_j, \mathrm{Hom}_\Gamma(\mathbf{\Omega}, W_i)) = \dim_K \mathrm{Hom}_B(V_j \otimes_\Lambda \mathbf{\Omega}, W_i) \tag{6.4}$$

and since $V_j \otimes_\Lambda \mathbf{\Omega} \cong W_j$ for $j \leqslant h_0$, it follows that $X_{i,j} = \delta_{ij}$ for $j \leqslant h_0$. Furthermore, since W_i with $i \leqslant h_0$ does not occur as a summand of $V_j \otimes_\Lambda \mathbf{\Omega}$ for $j > h_0$, it also follows that $X_{i,j} = 0$ for $1 \leqslant i \leqslant h_0 < j \leqslant h$. This proves the claim (it is also easy to see that $X_{\mathrm{Hom}} = X_\otimes^\top$). □

We now choose $t_0 \leqslant t$ such that (after reordering) $S_1, \ldots, S_{t_0}$ are precisely those simple $\overline{\Lambda}$-modules which have non-zero decomposition number with at least one of $V_1, \ldots, V_{h_0}$, and $T_1, \ldots, T_{t_0}$ are precisely those simple $\overline{\Gamma}$-modules which have non-zero decomposition number with at least one of $W_1, \ldots, W_{h_0}$.

Lemma 6.3. *After reordering $T_1, \ldots, T_{t_0}$ appropriately we have*

$$S_i \otimes_\Lambda \mathbf{\Omega} \cong T_i \quad \text{and} \quad \operatorname{Hom}_\Gamma(\mathbf{\Omega}, T_i) \cong S_i \tag{6.5}$$

for all $i \in \{1, \ldots, t_0\}$.

Proof. Fix some $i \in \{1, \ldots, t_0\}$ and choose $j \in \{1, \ldots, h_0\}$ such that the decomposition number $D^{(\Lambda)}_{V_j, S_i}$ is non-zero. Let L be a Λ-lattice in V_j. The Γ-module $L \otimes_\Lambda \mathbf{\Omega}$ is torsion-free as an $\mathcal{O}$-module (since $\mathbf{\Omega}$ is projective as a left Λ-module), and therefore $L \otimes_\Lambda \mathbf{\Omega}$ is a Γ-lattice in $V_j \otimes_\Lambda \mathbf{\Omega} \cong W_j$. By **(C8)** the row in $D^{(\Lambda)}$ belonging to V_j and the row in $D^{(\Gamma)}$ belonging to W_j are equal, and therefore $\operatorname{length}_\Lambda L/\pi L = \operatorname{length}_\Gamma L \otimes_\Lambda \mathbf{\Omega}/\pi L \otimes_\Lambda \mathbf{\Omega}$. On the other hand, due to **(C7)** and the fact that $- \otimes_\Lambda \mathbf{\Omega}$ is exact, the length of $L \otimes_\Lambda \mathbf{\Omega}/\pi L \otimes_\Lambda \mathbf{\Omega} \cong (L/\pi L) \otimes_\Lambda \mathbf{\Omega}$ is greater than or equal to the length of $L/\pi L$. Equality can only hold if for every simple subquotient V of $L/\pi L$ the analogous equality holds, that is, $\operatorname{length}_\Lambda V = \operatorname{length}_\Gamma V \otimes_\Lambda \mathbf{\Omega}$. Hence, we must in particular have that $S_i \otimes_\Lambda \mathbf{\Omega}$ is simple. Using the exact same reasoning we can also deduce that $\operatorname{Hom}_\Gamma(\mathbf{\Omega}, T_i)$ is simple. Hence also $\operatorname{Hom}_\Gamma(\mathbf{\Omega}, S_i \otimes_\Lambda \mathbf{\Omega})$ is simple, and our claim will follow once we have established that it is isomorphic to S_i. But

$$\operatorname{Hom}_\Lambda(S_i, \operatorname{Hom}_\Gamma(\mathbf{\Omega}, S_i \otimes_\Lambda \mathbf{\Omega})) \cong_{\mathcal{O}} \operatorname{Hom}_\Gamma(S_i \otimes_\Lambda \mathbf{\Omega}, S_i \otimes_\Lambda \mathbf{\Omega}) \neq 0 \tag{6.6}$$

and therefore we are done. □

Lemma 6.4. $\varepsilon\mathbf{\Omega} = \mathbf{\Omega}\eta = \varepsilon\mathbf{\Omega}\eta$. *Hence we may view $\mathbf{\Omega}\eta$ as an $\varepsilon\Lambda$-$\eta\Gamma$-bimodule.*

Proof. $V_i^* \otimes_K W_j$ for $i, j \in \{1, \ldots, h\}$ are representatives for the isomorphism classes of simple (K-linear) A-B-bimodules (here "$-^*$" is short for "$\operatorname{Hom}_K(-, K)$"). Thus we may write

$$K \otimes_{\mathcal{O}} \mathbf{\Omega} \cong \bigoplus_{i,j=1}^{h} \bigoplus^{z_{ij}} V_i^* \otimes_K W_j \tag{6.7}$$

where the z_{ij} are certain non-negative integers that are yet to be determined. Note that $V_i \otimes_A V_i^* \cong K$ for any i and $V_j \otimes_A V_i^* \cong 0$ for $i \neq j$. Hence we get for $1 \leqslant l \leqslant h_0$

$$W_l \overset{(6.1)}{\cong} V_l \otimes_\Lambda \mathbf{\Omega} \cong \bigoplus_{j=1}^{h} \bigoplus^{z_{lj}} W_j \tag{6.8}$$

and therefore $z_{ij} = \delta_{ij}$ for $1 \leqslant i \leqslant h_0$. In the same manner we get

$$V_l \overset{(6.2)}{\cong} \operatorname{Hom}_\Gamma(\mathbf{\Omega}, W_l) \cong \bigoplus_{i,j=1}^{h} \bigoplus^{z_{ij}} \operatorname{Hom}_B(V_i^* \otimes_K W_j, W_l) \cong \bigoplus_{i=1}^{h} \bigoplus^{z_{il}} V_i \tag{6.9}$$

which implies $z_{ij} = \delta_{ij}$ for $1 \leqslant j \leqslant h_0$. So

$$\varepsilon \cdot K \otimes_{\mathcal{O}} \mathbf{\Omega} \cong \bigoplus_{i=1}^{h_0} V_i^* \otimes_K W_i \cong K \otimes_{\mathcal{O}} \mathbf{\Omega} \cdot \eta \tag{6.10}$$

Our claim follows. □

Lemma 6.5. *We have*

$$\mathbf{\Omega}\eta/\operatorname{Jac}(\Lambda)\mathbf{\Omega}\eta \cong \bigoplus_{i=1}^{t_0}\bigoplus^{\dim_k T_i} S_i^* \tag{6.11}$$

and

$$\mathbf{\Omega}\eta/\mathbf{\Omega}\eta\operatorname{Jac}(\Gamma) \cong \bigoplus_{i=1}^{t_0}\bigoplus^{\dim_k S_i} T_i \tag{6.12}$$

Proof. We just prove the first isomorphism, the second one can be proven in the same fashion. First note that if S is in $\mathbf{mod}_\Lambda$ and N in ${}_\Lambda\mathbf{mod}$, and S is simple, then $S \otimes_\Lambda N \cong S \otimes_\Lambda N/\operatorname{Rad} N$. So $S_i \otimes_\Lambda \mathbf{\Omega} \cong T_i$ for $1 \leqslant i \leqslant t_0$ implies

$$\mathbf{\Omega}/\operatorname{Jac}(\Lambda)\mathbf{\Omega} \cong \bigoplus_{i=1}^{t_0}\bigoplus^{\dim_k T_i} S_i^* \oplus (\text{some } S_j^* \text{ with } j > t_0) \tag{6.13}$$

Now the epimorphism $\mathbf{\Omega} \twoheadrightarrow \mathbf{\Omega}\eta$ clearly induces an epimorphism $\mathbf{\Omega}/\operatorname{Jac}(\Lambda)\mathbf{\Omega} \twoheadrightarrow \mathbf{\Omega}\eta/\operatorname{Jac}(\Lambda)\mathbf{\Omega}\eta$. The only possible summands of $\mathbf{\Omega}\eta/\operatorname{Jac}(\Lambda)\mathbf{\Omega}\eta$ are $S_1^*, \ldots, S_{t_0}^*$, as it may be viewed as a semisimple left $\varepsilon\Lambda$-module. So we have an epimorphism

$$\mathcal{P}_{\varepsilon\Lambda}\left(\bigoplus_{i=1}^{t_0}\bigoplus^{\dim_k T_i} S_i^*\right) \twoheadrightarrow \mathbf{\Omega}\eta \tag{6.14}$$

and we will know this is an isomorphism once we have seen that the $\mathcal{O}$-rank of its source and target are equal:

$$\operatorname{rank}_{\mathcal{O}} \mathbf{\Omega}\eta = \sum_{i=1}^{h_0} \dim_K V_i \cdot \dim_K W_i \tag{6.15}$$

$$\begin{aligned}
\operatorname{rank}_{\mathcal{O}} \mathcal{P}_{\varepsilon\Lambda}\left(\bigoplus_{i=1}^{t_0}\bigoplus^{\dim_k T_i} S_i^*\right) &= \sum_{j=1}^{t_0}\sum_{i=1}^{h_0} \dim_K V_i^* \cdot D^{(\Lambda)}_{V_i^*, S_j^*} \cdot \dim_k T_j \\
&= \sum_{i=1}^{h_0} \dim_K V_i \cdot \left(\sum_{j=1}^{t_0} D^{(\Gamma)}_{W_i, T_j} \dim_k T_j\right) \\
&= \sum_{i=1}^{h_0} \dim_K V_i \cdot \dim_K W_i
\end{aligned}$$

Hence, as a left $\varepsilon\Lambda$-module, $\mathbf{\Omega}\eta$ is isomorphic to the projective cover of the right hand side of (6.11), and therefore its radical quotient is as claimed. □

Proof of Theorem 6.1. We can now simply mimic the reasoning of [Sco91, Theorem 4.2]. We claim that $\operatorname{Hom}_{\eta\Gamma}(\mathbf{\Omega}\eta, - \otimes_{\varepsilon\Lambda} \mathbf{\Omega}\eta)$ is isomorphic to the identity functor on $\mathbf{mod}_{\varepsilon\Lambda}$, and that $\operatorname{Hom}_{\eta\Gamma}(\mathbf{\Omega}\eta, -) \otimes_{\varepsilon\Lambda} \mathbf{\Omega}\eta$ is isomorphic to the identity functor on $\mathbf{mod}_{\eta\Gamma}$. In order to show that, we must specify a natural transformation. For $M \in \mathbf{mod}_{\varepsilon\Lambda}$ let

$$\psi_M : \operatorname{Hom}_{\eta\Gamma}(M \otimes_{\varepsilon\Lambda} \mathbf{\Omega}\eta, M \otimes_{\varepsilon\Lambda} \mathbf{\Omega}\eta) \xrightarrow{\sim} \operatorname{Hom}_{\varepsilon\Lambda}(M, \operatorname{Hom}_{\eta\Gamma}(\mathbf{\Omega}\eta, M \otimes_{\varepsilon\Lambda} \mathbf{\Omega}\eta)) \tag{6.16}$$

denote the natural isomorphism. Then sending M to the morphism $t_M = \psi_M(\mathrm{id}_{M\otimes_{\varepsilon\Lambda}\mathbf{\Omega}\eta})$ gives a natural transformation from the identity functor on $\mathbf{mod}_{\varepsilon\Lambda}$ to $\mathcal{F} = \mathrm{Hom}_\Gamma(\mathbf{\Omega}\eta, - \otimes_{\varepsilon\Lambda} \mathbf{\Omega}\eta)$. We have to show that each t_M is an isomorphism. Clearly, t_M is an isomorphism if M is simple, because $S_i \cong \mathrm{Hom}_{\eta\Gamma}(\mathbf{\Omega}\eta, S_i \otimes_{\varepsilon\Lambda} \mathbf{\Omega}\eta)$ for all $i \in \{1, \ldots, t_0\}$ and each t_M is nonzero. It follows easily by induction that t_M is an isomorphism for all $\varepsilon\Lambda$-modules M of finite length. So assume that M is of infinite length (albeit still finitely generated). Then there is a commutative diagram with exact rows and columns:

$$\begin{array}{ccccccccc} & & & & & & 0 & & \\ & & & & & & \downarrow & & \\ 0 & \longrightarrow & M & \xrightarrow{\cdot\pi} & M & \longrightarrow & M/M\pi & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle t_M} & & \downarrow{\scriptstyle t_M} & & \downarrow{\scriptstyle t_{M/M\pi}} & & \\ 0 & \longrightarrow & \mathcal{F}(M) & \xrightarrow{\cdot\pi} & \mathcal{F}(M) & \longrightarrow & \mathcal{F}(M/M\pi) & \longrightarrow & 0 \\ & & & & & & \downarrow & & \\ & & & & & & 0 & & \end{array} \tag{6.17}$$

That implies that there is an exact sequence

$$0 \to \mathrm{Ker}(t_M) \xrightarrow{\cdot\pi} \mathrm{Ker}(t_M) \to 0 \to \mathrm{Coker}(t_M) \xrightarrow{\cdot\pi} \mathrm{Coker}(t_M) \to 0 \to 0 \tag{6.18}$$

The Nakayama-lemma now implies that t_M is an isomorphism.

To prove that $\mathcal{G} = \mathrm{Hom}_{\eta\Gamma}(\mathbf{\Omega}\eta, -) \otimes_{\varepsilon\Lambda} \mathbf{\Omega}\eta$ is isomorphic to the identity functor on $\mathbf{mod}_{\eta\Gamma}$ we may proceed in the exact same manner. Namely, we look at the natural isomorphism

$$\phi_N : \mathrm{Hom}_{\varepsilon\Lambda}(\mathrm{Hom}_{\eta\Gamma}(\mathbf{\Omega}\eta, N), \mathrm{Hom}_{\eta\Gamma}(\mathbf{\Omega}\eta, N)) \xrightarrow{\sim} \mathrm{Hom}_{\eta\Gamma}(\mathrm{Hom}_{\eta\Gamma}(\mathbf{\Omega}\eta, N) \otimes_{\varepsilon\Lambda} \mathbf{\Omega}\eta, N) \tag{6.19}$$

and then take the natural transformation (now *from* $\mathcal{G}$ *to* the identity functor) that sends N to $t_N = \phi_N(\mathrm{id}_{\mathrm{Hom}_{\eta\Gamma}(\mathbf{\Omega}, N)})$. It then follows by induction as above that each t_N is an isomorphism. □

6.2 Application to Symmetric Groups

Let B be a block of $\mathcal{O}\Sigma_n$ for some $n \in \mathbb{N}$, let w denote its weight, and let B' be a block of $\mathcal{O}\Sigma_{n-q}$ for some $q < n$ such that B and B' form a $(w : q)$-pair (see Definition 2.145). Let κ be the p-core of B and let l be its length. Choose an abacus representation of κ (on p runners; the same goes for all abaci representations in this section) such that the first runner is empty, and then add w rows filled with beads on top of that (the resulting diagram will be the unique abacus representation of κ with $p \cdot w + l$ beads). We assume that with respect to this abacus representation, B and B' form a $(w : q)$-pair with respect to the $(i-1)$-st and i-th runner. By b and b' we denote the block idempotents in $\mathcal{O}\Sigma_n$ respectively $\mathcal{O}\Sigma_{n-q}$ that belong to B respectively B'.

Definition 6.6 (Non-Exceptionality). *(1) We call a partition λ in B* non-exceptional *if, when represented on an abacus with $p \cdot w + l$ beads, there are exactly q beads on the i-th runner which can be moved to the place next to them on the $(i-1)$-st runner (that is, that place is not taken by another bead).*

(2) *We call a partition η in B'* non-exceptional *if, when represented on an abacus with $p \cdot w + l$ beads, there are exactly q beads on the $(i-1)$-st runner which can be moved to the place next to them on the i-th runner.*

Remark 6.7. *Note that all partitions in B are obtained by moving w beads (possibly the same several times) down by one row in the abacus diagram of κ with $p \cdot w + l$ beads. In the beginning of that process there are q beads on the i-th runner which can be moved left to the $(i-1)$-st (lets call this number N_1) and no beads on the $(i-1)$-st runner can be moved right to the i-th (lets call this number N_2). Whenever we move a bead up, it either increases both N_1 and N_2 by one, decreases them both by one, or leaves them both unchanged. Either way, the difference $N_1 - N_2$ does not change. If one picks q beads on the i-th runner and moves them left to the $(i-1)$-st, then N_1 is replaced by $N_1 - q$ and N_2 is replaced by $N_2 + q$. Since non-exceptional partitions correspond to the situation where $N_1 = 0$ respectively $N_2 = 0$, one can hence deduce that by moving q beads from the i-th to the $(i-1)$-st runner (or the other way round), one obtains a non-exceptional partition if and only if one started with a non-exceptional partition. Using Theorem 2.137 (concerning induction and restriction of Specht modules) we get the following two corollaries:*

(1) *If λ is a non-exceptional partition in B, then $S_K^\lambda|_{K\Sigma_{n-q}} \cdot b'$ is a direct sum of $q!$ times S_K^μ, where μ is obtained from λ by moving the q beads on the i-th runner which can be moved to the $(i-1)$-st runner to the $(i-1)$-st runner.*

(2) *If λ is an exceptional partition in B, then $S_K^\lambda|_{K\Sigma_{n-q}} \cdot b'$ is a direct sum of certain S_K^μ (with multiplicities always divisible by $q!$), where the partitions μ are all exceptional.*

Definition 6.8. *We define a B-B'-bimodule*

$$\mathbf{\Omega} = B|_{\mathcal{O}\Sigma_{n-q}\times\Sigma_q} \cdot b' \otimes_{\mathcal{O}\Sigma_q} T \tag{6.20}$$

where T is the trivial left $\mathcal{O}\Sigma_q$-module.

Remark 6.9. *On $\mathbf{mod}_B$ respectively $\mathbf{mod}_{B'}$ the functors $- \otimes_B \mathbf{\Omega}$ respectively $\operatorname{Hom}_{B'}(\mathbf{\Omega}, -)$ coincide (by definition) with the divided power functors $\hat{e}_{i-l}^{(q)}$ respectively $\hat{f}_{i-l}^{(q)}$.*

Definition 6.10. *We define a matrix X_{res} with rows indexed by the partitions belonging to B and columns indexed by the partitions belonging to B'*

$$(X_{\text{res}})_{\lambda,\mu} = \textit{Multiplicity of } S_K^\mu \textit{ as a direct summand of } S_K^\lambda \otimes_B \mathbf{\Omega} \tag{6.21}$$

We define a matrix X_{ind} with rows indexed by the partitions belonging to B' and columns indexed by the partitions belonging to B

$$(X_{\text{ind}})_{\mu,\lambda} = \textit{Multiplicity of } S_K^\lambda \textit{ as a direct summand of } \operatorname{Hom}_{B'}(\mathbf{\Omega}, S_K^\mu) \tag{6.22}$$

Remark 6.11. *Note that this are square matrices, and if we order the partitions of B and those of B' in such a way that the non-exceptional partitions (with respect to the $(w : q)$-pairing we consider) come first, X_{res} and X_{ind} are diagonal joins of a $h_0 \times h_0$-permutation matrix (without loss, by reordering the non-exceptional partitions, the identity matrix) and some $(h - h_0) \times (h - h_0)$-matrix, where h denotes the number of partitions in B and h_0 is the number of non-exceptional partitions.*

Lemma 6.12 ([Sco90, Lemma 7.1.1 and Appendix 15])**.** *X_{res} and X_{ind} are invertible over $\mathbb{Q}$.*

Lemma 6.13. *Let D^λ be a simple B-module that occurs as a composition factor of some non-exceptional S_k^μ. Then*

$$\left[D^\lambda \otimes_B \mathbf{\Omega}\right] = \left[D^{\lambda'}\right] \qquad \textit{for some composition factor } D^{\lambda'} \textit{ of } S_k^\mu \otimes_B \mathbf{\Omega}$$

Similarly, if $D^{\lambda'}$ is a simple B'-module that occurs as a composition factor of some non-exceptional $S_k^{\mu'}$ then

$$\left[\mathrm{Hom}_{B'}(\mathbf{\Omega}, D^{\lambda'})\right] = \left[D^\lambda\right] \qquad \textit{for some composition factor } D^\lambda \textit{ of } \mathrm{Hom}_{B'}(\mathbf{\Omega}, S_k^{\mu'})$$

Proof. Define matrices

$$(\overline{X}_{\text{res}})_{\lambda,\mu} = \text{Multiplicity of } D_k^\mu \text{ as a composition factor of } D_k^\lambda \otimes_B \mathbf{\Omega} \tag{6.23}$$

and

$$(\overline{X}_{\text{ind}})_{\mu,\lambda} = \text{Multiplicity of } D_k^\lambda \text{ as a composition factor of } \mathrm{Hom}_{B'}(\mathbf{\Omega}, D_k^\mu) \tag{6.24}$$

where λ runs over all p-regular partitions in B and μ runs over all p-regular partitions in B'. Denote by $D^{(B)}$ and $D^{(B')}$ the decomposition matrices of B respectively B'. Then we have, due to the exactness and $\mathcal{O}$-linearity of the functors $- \otimes_B \mathbf{\Omega}$ and $\mathrm{Hom}_{B'}(\mathbf{\Omega}, -)$, equalities $X_{\text{res}} \cdot D^{(B')} = D^{(B)} \cdot \overline{X}_{\text{res}}$ and $X_{\text{ind}} \cdot D^{(B)} = D^{(B')} \cdot \overline{X}_{\text{ind}}$.

We prove the first claim (the second one is analogous). $\overline{X}_{\text{res}}$ and $\overline{X}_{\text{ind}}$ need to have full row rank, since $D^{(B)}$ and $D^{(B')}$ have full row rank and X_{ind} and X_{res} are both invertible over $\mathbb{Q}$, and thus $X_{\text{res}} \cdot D^{(B')}$ and $X_{\text{ind}} \cdot D^{(B)}$ both have full row rank. So no row of $\overline{X}_{\text{res}}$ and $\overline{X}_{\text{ind}}$ can be zero, and this means that any simple module of B respectively B' restricts respectively induces to a module of composition length at least 1 of B' respectively B. Assume that a row of $\overline{X}_{\text{res}}$ has more than one nonzero entry. Then the corresponding simple module D^μ restricts to a module of composition length *at least* 2. Therefore if M is a B-module of composition length l such that D^μ occurs as a composition factor of M then $\mathrm{Hom}_{B'}(\mathbf{\Omega}, M \otimes_B \mathbf{\Omega})$ has composition length strictly greater than l. But $[\mathrm{Hom}_{B'}(\mathbf{\Omega}, S_k^\lambda \otimes_B \mathbf{\Omega})] = [S_k^\lambda]$ for each non-exceptional λ (simply by applying Theorem 2.137 and then dividing by $(q!)^2$), which means that D^μ may not occur as a composition factor of any non-exceptional Specht module. So any D^λ that satisfies the assumptions of this lemma will restrict to a simple module, as claimed. □

Remark 6.14. *Note that if D^λ and D^μ both occur as composition factors of non-exceptional Specht modules in B, and $D^\lambda \ncong D^\mu$, then $D^\lambda \otimes_B \mathbf{\Omega} \ncong D^\mu \otimes_B \mathbf{\Omega}$. This is due to the fact that*

$$\mathrm{Hom}_B(D^\lambda, \mathrm{Hom}_{B'}(\mathbf{\Omega}, D^\lambda \otimes_B \mathbf{\Omega})) \cong \mathrm{Hom}_{B'}(D^\lambda \otimes_B \mathbf{\Omega}, D^\lambda \otimes_B \mathbf{\Omega}) \ncong \{0\} \tag{6.25}$$

and therefore $\mathrm{Hom}_{B'}(\mathbf{\Omega}, D^\lambda \otimes_B \mathbf{\Omega}) \cong D^\lambda$. In the same way we see that $\mathrm{Hom}_{B'}(\mathbf{\Omega}, D^\mu \otimes_B \mathbf{\Omega}) \cong D^\mu$, which clearly proves the claim.

Lemma 6.15. *The rows of the decomposition matrix of B pertaining to non-exceptional partitions in B are equal to the rows in the decomposition matrix of B' pertaining to the non-exceptional in B'. More precisely, the decomposition number associated to S_K^λ and D^μ (for a* *non-exceptional λ in B) is equal to that associated to $S_K^\lambda \otimes_B \mathbf{\Omega}$ and $D^\mu \otimes_B \mathbf{\Omega}$ if D^μ occurs as a composition factor of some non-exceptional Specht module.*

Proof. First note that the following diagram is commutative:

$$\begin{array}{ccc} K_0(\mathbf{mod}_{K\otimes B}) & \xrightarrow{-\otimes_B \mathbf{\Omega}} & K_0(\mathbf{mod}_{K\otimes B'}) \\ \big\downarrow D & & \big\downarrow D \\ K_0(\mathbf{mod}_{k\otimes B}) & \xrightarrow{-\otimes_B \mathbf{\Omega}} & K_0(\mathbf{mod}_{k\otimes B'}) \end{array} \tag{6.26}$$

Furthermore the upper horizontal arrow induces a bijection between non-exceptional Specht modules in B and non-exceptional Specht modules in B'. Also, the bottom horizontal arrow induces a bijection between simple modules in B and B' which have non-zero decomposition number with such a non-exceptional Specht module. Now the assertion follows directly. □

Corollary 6.16. *The $\mathcal{O}$-orders εB and $\eta B'$ are Morita-equivalent, where ε is the sum of all non-exceptional central primitive idempotents in $K \otimes B$ and η is the sum of all non-exceptional central primitive idempotents in $K \otimes B'$.*

Proof. This is simply an application of Theorem 6.1, with $\Lambda = B$, $\Gamma = B'$ and the bimodule $\mathbf{\Omega}$ defined in Definition 6.8. Conditions **(C1)-(C5)** are immediately seen to be satisfied. Condition **(C6)** is treated in Remark 6.11. Condition **(C7)** is a consequence of Lemma 6.13 and Condition **(C8)** is verified in Lemma 6.15. □

6.3 Basic Algebras for Blocks of Defect Two

In this section we determine basic orders of defect two blocks of symmetric groups. Blocks of symmetric groups of small defect have been subject to extensive study in the past, mostly due to the fact that the structure of such blocks seems to become more complicated the bigger the defect gets. For defect two blocks, it has been shown in [Sco95] that (among other things) the decomposition numbers are all $\leqslant 1$. Hence the projections of such a defect two block B to the individual Wedderburn components of $\mathbb{Q}_p \otimes B$ are graduated orders. These can be described easily in terms of exponent matrices. Therefore describing these projections is what we do first. This yields an overorder of a basic order of B. We then go on to determine how a basic order of B is embedded in this graduated overorder. Here the shape of the Ext-quiver and the decomposition matrix play an important role, as these control to what extent we can modify generators of the basic order by conjugation. We find that those considerations determine a basic order of B up to conjugacy. The basic order is then given explicitly by generators in a direct product of matrix rings over $\mathbb{Q}_p$. It should not be too hard, in any particular case, to derive from this a presentation as a quiver algebra of a basic algebra of $\mathbb{F}_p \otimes_{\mathbb{Z}_p} B$.

For the principal block of $\mathbb{Z}_p\Sigma_{2p}$, the basic orders have been determined in [Neb02], and our approach is a generalization of that. The aforementioned paper relies, however, heavily on the explicit knowledge of (among other things) the decomposition matrix of the blocks treated in it. We, on the other hand, get by without such explicit information, which allows us to treat *all* defect two blocks of symmetric groups (and makes it possible for the reader to verify the proofs without inspecting any large tables). In fact, the following is a list of the known properties of defect two blocks of symmetric groups that we are going to use:

Remark 6.17 (Known facts). *(i) The decomposition numbers of a defect two block of a symmetric group are all $\leqslant 1$ and the off-diagonal Cartan numbers are all $\leqslant 2$. The diagonal Cartan numbers are all $\geqslant 3$. (see [Sco95]).*

(ii) The decomposition matrices of defect two blocks of symmetric groups can be computed by combinatorial means, for instance using the Jantzen-Schaper formula (see [Ben87]).

(iii) The dimension of the Ext^1*-spaces between simple modules in defect two blocks is $\leqslant 1$ (see [Sco95]).*

(iv) The Ext*-quiver of a defect two block is a bipartite graph according to [CT99, Theorem 3.2]. Bipartite means, in this context, that the set of vertices of the quiver can be partitioned in two parts such that any edge connects vertices coming from different parts.*

Remark 6.18. *In what follows, p will always be an odd prime and K is assumed to be an unramified extension of $\mathbb{Q}_p$. When we say that a partition is in a defect two block, that simply means that it is of p-weight two.*

Definition 6.19 (Jantzen-Schaper-Filtration). *Let λ be a partition of some $n \in \mathbb{N}$, and let*

$$(-,=):\ S^\lambda_\mathcal{O} \times S^\lambda_\mathcal{O} \to \mathcal{O} \tag{6.27}$$

be the natural bilinear form on $S^\lambda_\mathcal{O}$ inherited from the permutation module $M^\lambda_\mathcal{O}$ (see [Jam78] for details). Then we define for $i \in \mathbb{Z}_{\geqslant 0}$

$$S^\lambda_\mathcal{O}(i) := \left\{ m \in S^\lambda_\mathcal{O} \mid (m, S^\lambda_\mathcal{O}) \subseteq p^i \cdot \mathcal{O} \right\} \tag{6.28}$$

and

$$S^\lambda_k(i) := \frac{S^\lambda_\mathcal{O}(i) + p \cdot S^\lambda_\mathcal{O}}{p \cdot S^\lambda_\mathcal{O}} \leqslant S^\lambda_k \tag{6.29}$$

The filtration $S^\lambda_k = S^\lambda_k(0) \geqslant S^\lambda_k(1) \geqslant S^\lambda_k(2) \geqslant \dots$ is called the Jantzen-Schaper filtration *of S^λ_k.*

Remark 6.20. *If S^λ_k is multiplicity-free, then all layers $S^\lambda_k(i)/S^\lambda_k(i{+}1)$ of the Jantzen-Schaper filtration are semisimple. So, in particular, this holds for an S^λ_k in a defect two block.*

Proof. Consider the restriction of the standard bilinear form $(-,=)$ on $S^\lambda_\mathcal{O}$ to $S^\lambda_\mathcal{O}(i)$ for some i. By definition of $S^\lambda_\mathcal{O}(i)$, this takes values in $(p^i)_\mathcal{O}$. Thus we may look at $p^{-i} \cdot (-,=)$, which defines a bilinear form on $S^\lambda_\mathcal{O}(i)$ with values in $\mathcal{O}$. We reduce this modulo p to get a bilinear form on $k \otimes_\mathcal{O} S^\lambda_\mathcal{O}(i)$. Clearly

$$X := \frac{k \otimes_\mathcal{O} S^\lambda_\mathcal{O}(i)}{k \otimes_\mathcal{O} S^\lambda_\mathcal{O}(i) \cap k \otimes_\mathcal{O} S^\lambda_\mathcal{O}(i)^\perp} \tag{6.30}$$

is a self-dual $k\Sigma_n$-module. Since S^λ_k (and therefore also $k \otimes_\mathcal{O} S^\lambda_\mathcal{O}(i)$) is multiplicity-free, and all simple $k\Sigma_n$-modules are self-dual, we must hence have that X is semisimple (as any simple module occurring in the radical would otherwise turn up again in the socle, giving it a multiplicity of at least two). Now we have the natural epimorphism $S^\lambda_\mathcal{O}(i) \twoheadrightarrow S^\lambda_k(i)$, giving rise to an epimorphism $k \otimes_\mathcal{O} S^\lambda_\mathcal{O}(i) \twoheadrightarrow S^\lambda_k(i)$ which maps $k \otimes_\mathcal{O} S^\lambda_\mathcal{O}(i) \cap k \otimes_\mathcal{O} S^\lambda_\mathcal{O}(i)^\perp$ into

$S_k^\lambda(i+1)$. Thus we get an epimorphism $X \twoheadrightarrow S_k^\lambda(i)/S_k^\lambda(i+1)$, implying that the latter is also semisimple. □

Definition 6.21. *Define for a p-regular partition μ of n the set*

$$c_\mu := \left\{\lambda \text{ a partition of } n \mid D^\mu \text{ is a composition factor of } S_k^\lambda\right\} \tag{6.31}$$

Define for any partition λ of n the set

$$r_\lambda := \left\{\mu \text{ a p-regular partition of } n \mid D^\mu \text{ is a composition factor of } S_k^\lambda\right\} \tag{6.32}$$

Definition 6.22 (Mullineux Map)**.** *The map*

$$-^M : \{ \text{ p-regular partitions of } n \} \longrightarrow \{ \text{ p-regular partitions of } n \} \tag{6.33}$$

determined by the condition

$$D^\mu \otimes_k \mathrm{sign} \cong D^{\mu^M} \qquad \text{for all p-regular partitions } \mu \text{ of } n \tag{6.34}$$

is called the Mullineux map*.*

Theorem 6.23. *Let λ be a partition in a defect two block. Then the Jantzen-Schaper quotients of S_k^λ and $S_k^{\lambda^\top}$ may be described as follows:*

(i) If λ and $\lambda^\top$ are both p-regular, then

$$\begin{array}{lcl lcl} S_k^\lambda(0)/S_k^\lambda(1) & \cong & D^\lambda & S_k^{\lambda^\top}(0)/S_k^{\lambda^\top}(1) & \cong & D^{\lambda^\top} \\ S_k^\lambda(1)/S_k^\lambda(2) & \cong & \bigoplus_{\mu\in r_\lambda\setminus\{\lambda,\lambda^{\top M}\}} D^\mu & S_k^{\lambda^\top}(1)/S_k^{\lambda^\top}(2) & \cong & \bigoplus_{\mu\in r_\lambda\setminus\{\lambda,\lambda^{\top M}\}} D^{\mu^M} \\ S_k^\lambda(2)/S_k^\lambda(3) & \cong & D^{\lambda^{\top M}} & S_k^{\lambda^\top}(2)/S_k^{\lambda^\top}(3) & \cong & D^{\lambda^M} \end{array}$$

and all further layers are zero. Furthermore, $r_\lambda \setminus \{\lambda, \lambda^{\top M}\} \neq \emptyset$, meaning all of the above layers are non-trivial.

(ii) If λ is p-regular and $\lambda^\top$ is p-singular, then

$$\begin{array}{lcl lcl} S_k^\lambda(0)/S_k^\lambda(1) & \cong & D^\lambda & S_k^{\lambda^\top}(0)/S_k^{\lambda^\top}(1)] & \cong & 0 \\ S_k^\lambda(1)/S_k^\lambda(2) & \cong & \bigoplus_{\mu\in r_\lambda\setminus\{\lambda\}} D^\mu & S_k^{\lambda^\top}(1)/S_k^{\lambda^\top}(2) & \cong & \bigoplus_{\mu\in r_\lambda\setminus\{\lambda\}} D^{\mu^M} \\ S_k^\lambda(2)/S_k^\lambda(3) & \cong & 0 & S_k^{\lambda^\top}(2)/S_k^{\lambda^\top}(3) & \cong & D^{\lambda^M} \end{array}$$

and all further layers are zero.

(iii) If λ and $\lambda^\top$ are both p-singular, then there is a p-regular partition μ of n (which will necessarily be the p-regularization of λ) such that

$$S_k^\lambda(1)/S_k^\lambda(2) \;\cong\; D^\mu \qquad S_k^{\lambda^\top}(1)/S_k^{\lambda^\top}(2) \;\cong\; D^{\mu^M}$$

and all other layers are zero.

Proof. By [Fay03, Theorem 4.8] (specialized to the defect two case) we have

$$S_k^\lambda(i)/S_k^\lambda(i+1) \cong \left(S_k^{\lambda^\top}(2-i)/S_k^{\lambda^\top}(3-i)\right) \otimes_{\mathcal{O}} \mathrm{sign} \tag{6.35}$$

This clearly implies that the first and the third layer of the filtration are always as claimed. Our claim on the middle layer in case (i) and (ii) simply follows from the fact that all decomposition numbers are zero or one (that is, any simple module that occurs as a composition factor of S_k^λ does so with multiplicity one), and Remark 6.20.

Now we show that when λ and $\lambda^\top$ are both p-regular, the set $r_\lambda \setminus \{\lambda, \lambda^{\top M}\}$ is non-empty. Assume otherwise. By [Jam78, Corollary 13.18] S_k^λ is indecomposable, and thus $\mathrm{Ext}^1_{k\Sigma_n}(D^\lambda, D^{\lambda^{\top M}})$ must be non-zero. Now, as mentioned in Remark 6.17, the Ext-quiver of a defect two block of a symmetric group is bipartite. The bipartition is given by the so-called *relative p-sign* (see [FT07, Proposition 2.2.]). Given any partition η, define its relative p-sign $\sigma_p(\eta)$ to be $(-1)^{\sum l_i}$, where l_i are the leg lengths of a sequence of p-hooks that may be removed from η to leave a p-core. Since for p odd the leg length and the arm length of a p-hook always leave the same residue modulo two, we have $\sigma_p(\eta) = \sigma_p(\eta^\top)$ for any partition λ. By [Tan06, Proposition 2.5.], for odd p and p-regular η of even weight, $\sigma_p(\eta) = \sigma_p(\eta^M)$ will hold. Therefore, $\sigma_p(\lambda^{\top M}) = \sigma_p(\lambda)$, that is, D^λ and $D^{\lambda^{\top M}}$ are in the same part of the bipartition. But then, Ext^1 between the two cannot be non-zero, giving us the desired contradiction.

The only part of our claim left to prove is that whenever λ and $\lambda^\top$ are both p-singular, S_k^λ will be simple. By (6.35) it is clear that in this case, $S_k^\lambda \cong S_k^\lambda(1)/S_k^\lambda(2)$. According to Remark 6.20 the module S_k^λ is hence semisimple. But by [Jam78, Corollary 13.18], S_k^λ is also indecomposable. It follows that S_k^λ is simple, as claimed. □

Remark 6.24. *Note that the last remark and theorem determine the submodule structure of S_k^λ for each partition λ in a defect two block.*

Lemma 6.25. *Let λ be a partition of some n, and let S_K^λ be equipped with the natural bilinear form inherited from M_K^λ. If $L \subset S_K^\lambda$ is an $\mathcal{O}\Sigma_n$-lattice, we denote its dual with respect to this form by $L^\sharp$. Define $\hat{S}(j) := (p^{-j} \cdot S_{\mathcal{O}}^\lambda) \cap S_{\mathcal{O}}^{\lambda\sharp}$. Then there is an ascending filtration*

$$S_{\mathcal{O}}^\lambda = \hat{S}(0) \leqslant \hat{S}(1) \leqslant \ldots \leqslant \hat{S}(l) = S_k^{\lambda\sharp} \quad \text{for some } l \in \mathbb{N} \tag{6.36}$$

and the quotients $\hat{S}(j)/\hat{S}(j+1)$ are isomorphic to $S_k^\lambda(j)$.

Proof.

$$\hat{S}(j)/\hat{S}(j+1) \cong \frac{p^{-j}S_{\mathcal{O}}^\lambda \cap S_{\mathcal{O}}^{\lambda\sharp} + p^{-j+1}S_{\mathcal{O}}^\lambda}{p^{-j+1}S_{\mathcal{O}}^\lambda} \cong \frac{S_{\mathcal{O}}^\lambda \cap p^j S_{\mathcal{O}}^{\lambda\sharp} + pS_{\mathcal{O}}^\lambda}{pS_{\mathcal{O}}^\lambda} = S_k^\lambda(j)$$

□

Theorem 6.26. *Let λ be a p-regular partition in a defect two block. Let J_0, J_1 and J_2 be the sets of p-regular partitions μ such that D^μ occurs in $S_k^\lambda(0)/S_k^\lambda(1)$, $S_k^\lambda(1)/S_k^\lambda(2)$ and $S_k^\lambda(2)/S_k^\lambda(3)$ respectively. By ε^λ denote the primitive idempotent in $Z(K\Sigma_n)$ belonging to λ.*

Then the $\mathcal{O}$-order $\varepsilon^\lambda\mathcal{O}\Sigma_n$ is Morita-equivalent to the graduated order $\Lambda = \Lambda(\mathcal{O}, m, (1,\ldots,1))$ for an exponent matrix $m \in \mathbb{Z}_{\geqslant 0}^{J_0\cup J_1\cup J_2\times J_0\cup J_1\cup J_2}$ subject to the conditions:

$$\begin{aligned} m_{\alpha\lambda} &= 0 \quad \forall\alpha\in J_0\cup J_1\cup J_2 && (6.37)\\ m_{\lambda\alpha} &= i \quad \forall\alpha\in J_i \quad \textit{for } i\in\{0,1,2\} && (6.38)\\ m_{\alpha\beta}-m_{\beta\alpha} &= m_{\lambda\beta}-m_{\lambda\alpha} \quad \forall\alpha,\beta && (6.39)\\ 0<m_{\alpha\beta}+m_{\beta\alpha} &\leqslant 2 \quad \forall\alpha\neq\beta && (6.40)\end{aligned}$$

These conditions completely determine the matrix m.

Denote for each $\gamma\in r_\lambda$ by e^γ the diagonal matrix unit in Λ belonging to γ. Then we may choose a Morita equivalence $\mathcal{F}$ between $\mathbf{mod}_{\varepsilon^\lambda\mathcal{O}\Sigma_n}$ and $\mathbf{mod}_\Lambda$ such that $\mathcal{F}(D^\gamma)\cong e^\gamma\Lambda/\operatorname{Rad}e^\gamma\Lambda$.

Proof. Since K and k split Σ_n and all decomposition numbers are known to be 0 or 1 it follows that $\varepsilon^\lambda\mathcal{O}\Sigma_n$ (as well as, of course, its basic algebra) is a graduated order (see Corollary 2.104). Thus $\Lambda = \Lambda(\mathcal{O}, m, (1,\ldots,1))$ for some exponent matrix m. We may assume without loss that $\mathcal{F}(S_\mathcal{O}^{\lambda\sharp})\cong\mathcal{O}^{1\times J_0\cup J_1\cup J_2}$. We may also assume that $\mathcal{F}(D^\gamma)\cong e^\gamma\Lambda/\operatorname{Rad}e^\gamma\Lambda$. At this point we have fixed an exponent matrix m, and we need to show that it satisfies (6.37)-(6.40).

The order $\varepsilon^\lambda\mathcal{O}\Sigma_n$ carries an involution, as $K\Sigma_n$ carries the standard involution $g\mapsto g^{-1}$, and this involution fixes ε^λ and maps $\mathcal{O}\Sigma_n$ into itself. Dualizing followed by the standard involution also fixes all simple modules (this fact is usually stated as "The simple $k\Sigma_n$-modules are self-dual"). Hence the order Λ may also be equipped with an involution $^\circ:\Lambda\to\Lambda$ that fixes all the e^γ (this is by Corollary 2.120 and Proposition 2.122). By Proposition 2.123 this implies (6.39). The upper bound in inequation (6.40) is just formula (2.145) of Remark 2.126. The lower bound in inequation (6.40) is part of the definition of an exponent matrix.

Since $S_\mathcal{O}^\lambda$ has simple top D^λ, so does $\mathcal{F}(S_\mathcal{O}^\lambda)$. The uniqueness explained in Remark 2.112 thus implies $\mathcal{F}(S_\mathcal{O}^\lambda)\cong e^\lambda\Lambda$. But $e^\lambda\Lambda\cong\left[(p)_\mathcal{O}^{m_{\lambda\alpha}}\right]_\alpha$, and therefore $m_{\lambda\alpha}$ equals (for each α) the multiplicity of $\mathcal{F}(D^\alpha)$ in $\mathcal{F}(S_\mathcal{O}^{\lambda\sharp})/\mathcal{F}(S_\mathcal{O}^\lambda)$, which has been determined in Lemma 6.25. This implies (6.38). Note that in principle $\mathcal{F}$ applied to an irreducible lattice is, as a lattice in $K^{1\times J_0\cup J_1\cup J_2}$, only determined up to multiplication by powers of p. For the above quotient we choose however the maximal representative of $\mathcal{F}(S_\mathcal{O}^\lambda)$ that is contained in $\mathcal{O}^{1\times J_0\cup J_1\cup J_2}=\mathcal{F}(S_\mathcal{O}^{\lambda\sharp})$.

We may equip the vector space $K^{1\times J_0\cup J_1\cup J_2}$ with a Λ-equivariant non-degenerate symmetric bilinear form (which one we choose is irrelevant for our purposes). Then for each $\mathcal{O}\Sigma_n$-lattice $L\leqslant S_K^\lambda$ we have $\mathcal{F}(L^\sharp)\cong\mathcal{F}(L)^\sharp$. This is best seen by choosing an involution-invariant idempotent in $e\in\varepsilon^\lambda\mathcal{O}\Sigma_n$ that affords the Morita equivalence, since

$$L^\sharp\cdot e\cong\operatorname{Hom}_\mathcal{O}(L,\mathcal{O})^\circ\cdot e\cong\operatorname{Hom}_\mathcal{O}(L\cdot e^\circ,\mathcal{O})^\circ=\operatorname{Hom}_\mathcal{O}(L\cdot e,\mathcal{O})^\circ\cong(L\cdot e)^\sharp \tag{6.41}$$

By looking the standard bilinear pairing of $K^{1\times J_0\cup J_1\cup J_2}$ and $K^{J_0\cup J_1\cup J_2\times 1}$ we see that

$$\operatorname{Hom}_\mathcal{O}(\mathcal{F}(S_\mathcal{O}^{\lambda\sharp}),\mathcal{O})\cong_\Lambda\mathcal{O}^{J_0\cup J_1\cup J_2\times 1} \tag{6.42}$$

On the other hand, as was just seen,

$$\begin{aligned}\operatorname{Hom}_\mathcal{O}(\mathcal{F}(S_\mathcal{O}^{\lambda\sharp}),\mathcal{O}) &\cong \operatorname{Hom}_\mathcal{O}(\operatorname{Hom}_\mathcal{O}(\mathcal{F}(S_\mathcal{O}^\lambda),\mathcal{O})^\circ,\mathcal{O})\\ &\cong \mathcal{F}(S_\mathcal{O}^\lambda)^\circ\cong(e^\lambda\Lambda)^\circ\cong\Lambda e^\lambda\end{aligned} \tag{6.43}$$

This clearly implies (6.37).

The conditions (6.37)-(6.40) determine m, since (6.39) determines for all α, β the difference $m_{\alpha\beta} - m_{\beta\alpha}$, and hence determines $m_{\alpha\beta} + m_{\beta\alpha}$ modulo 2. As (6.40) states that $m_{\alpha\beta} + m_{\beta\alpha} \in \{1, 2\}$, this is already enough to determine the sum $m_{\alpha\beta} + m_{\beta\alpha}$. This clearly determines the values of the $m_{\alpha\beta}$. □

Remark 6.27. *The preceding theorem determines the exponent matrices for every $\varepsilon^\lambda \mathcal{O}\Sigma_n$ with λ in a defect two block, even when λ is p-singular. Namely, if λ is p-singular, we have the following two cases:*

(1) λ is p-singular and $\lambda^\top$ is p-regular:
We have an isomorphism

$$\varphi: \ \varepsilon^\lambda \mathcal{O}\Sigma_n \to \varepsilon^{\lambda^\top} \mathcal{O}\Sigma_n : \ \varepsilon^\lambda \cdot g \mapsto \operatorname{sign}(g) \cdot \varepsilon^{\lambda^\top} \cdot g \tag{6.44}$$

and when we retract the simple $\varepsilon^{\lambda^\top}\mathcal{O}\Sigma_n$-modules with φ, we get $\varphi_*(D^{\mu^M}) \cong D^\mu$. Hence

$$m^\lambda_{\mu\nu} = m^{\lambda^\top}_{\mu^M \nu^M} \quad \textit{for all } \mu, \nu \in r_\lambda \tag{6.45}$$

where m^λ and $m^{\lambda^\top}$ denote the exponent matrices of $\varepsilon^\lambda \mathcal{O}\Sigma_n$ and $\varepsilon^{\lambda^\top}\mathcal{O}\Sigma_n$. $m^{\lambda^\top}$ has of course been determined by the last theorem.

(2) λ is p-singular and $\lambda^\top$ is p-singular:
According to Theorem 6.23 the set r_λ will contain just one element, and hence the exponent matrix will be the 1×1 zero matrix.

Corollary 6.28. *Let λ be a partition in a defect two block and let J_0, J_1, J_2 be defined as in Theorem 6.26. Then the* Ext*-quiver of $k \otimes_\mathcal{O} \varepsilon^\lambda \mathcal{O}\Sigma_n$ is maximally bipartite. More precisely: If $J_1 \neq \emptyset$, then there is an edge from every partition in J_1 to every partition if $J_0 \cup J_2$ and there are no further edges. If $J_1 = \emptyset$, then the* Ext*-quiver consists of a single vertex with no edges.*

Proof. Due to the last remark it is clear that the claim can be reduced to the case where λ is p-regular. Σ_{J_1} acts naturally via automorphisms on the basic order of $\varepsilon^\lambda \mathcal{O}\Sigma_n$. In particular it acts via quiver automorphisms on the Ext-quiver of $k \otimes_\mathcal{O} \varepsilon^\lambda \mathcal{O}\Sigma_n$ by permuting the vertices labeled by elements of J_1. This can easily be derived from the fact that the set of equations (6.37)-(6.40) is invariant under the operation of Σ_{J_1} on the indices, and those equations determine the exponent matrix m completely.

J_0 and J_2 each consist of at most one partition. First suppose that $J_1 \neq \emptyset$. Then by Theorem 6.23 the top of S^λ_k has a single constituent labeled by the partition in J_0, and the socle of S^λ_k has constituents labeled by the partitions in J_2 (also, of course, at most one). Therefore $S^\lambda_k / \operatorname{Soc} S^\lambda_k$ has at least one non-semisimple quotient of length two, implying the existence of an edge from the partition in J_0 to one partition in J_1. Provided $J_2 \neq \emptyset$, the module $\operatorname{Rad} S^\lambda_k$ has a non-semisimple submodule of length two, implying the existence of an edge from the element of J_2 to one element of J_1. Now using the action of Σ_{J_1} we conclude that the Ext-quiver has at least the postulated edges. The case $J_1 = \emptyset$ is trivial, since then the Ext-quiver consists of a single vertex (see Theorem 6.23).

As we already mentioned in Remark 6.17, the Ext-quiver of any defect two block of a symmetric group is known to be bipartite by [CT99]. We can use the epimorphism $k\Sigma_n \twoheadrightarrow k \otimes_\mathcal{O} \varepsilon^\lambda \mathcal{O}\Sigma_n$ to retract modules and sequences of modules, in particular simple modules and extensions of simple modules. Hence the Ext-quiver of $k \otimes_{\mathcal{O}} \varepsilon^\lambda \mathcal{O}\Sigma_n$ is a subquiver of the bipartite Ext-quiver of the defect two block. It will therefore be bipartite as well. But if any further edges were to be added to the quiver constructed above, it would cease to be bipartite (for then there would be a closed path of length three). Hence we have constructed the full Ext-quiver of $k \otimes_{\mathcal{O}} \varepsilon^\lambda \mathcal{O}\Sigma_n$. □

Lemma 6.29. *Let λ be a p-regular partition in a defect two block, and let $i \in \mathbb{Z}_{\geqslant 0}$. If D^γ is a simple module occurring in $S_k^\lambda(i)/S_k^\lambda(i+1)$ and D^ω is a simple module occurring in $S_k^\lambda(i+1)/S_k^\lambda(i+2)$, then $\mathrm{Ext}^1_{k\Sigma_n}(D^\gamma, D^\omega) \neq \{0\}$. On the other hand, whenever two simple modules D^γ and D^ω occur in the same layer of the Jantzen-Schaper-filtration, then $\mathrm{Ext}^1_{k\Sigma_n}(D^\gamma, D^\omega) = \{0\}$.*

Proof. This follows directly from Corollary 6.28. □

Theorem 6.30. *Let λ and μ be two distinct p-regular partitions in some defect two block. If $\mathrm{Ext}^1_{k\Sigma_n}(D^\lambda, D^\mu) \neq \{0\}$, then*

(i) $|c_\lambda \cap c_\mu| = 2$

(ii) $\lambda \in c_\lambda \cap c_\mu$ *or* $\mu \in c_\lambda \cap c_\mu$

Proof. To prove the claim of (i), we argue by contradiction. So let $c_\lambda \cap c_\mu$ consist of just one element, say η. Then by Theorem 6.23 and Lemma 6.29, D^λ and D^μ occur in successive Jantzen-Schaper layers of S_k^η. Hence, by Theorem 6.26, $m_{\mu\lambda} + m_{\lambda\mu} = 1$ (where m is the exponent matrix of $\varepsilon^\eta \mathcal{O}\Sigma_n$). By Proposition 2.125 this implies that the amalgamation depth $A^\eta_{\lambda,\mu}$ is equal to $1 = 2 - m_{\lambda\mu} - m_{\mu\lambda}$. But if e_λ and e_μ are primitive idempotents in $\mathcal{O}\Sigma_n$ corresponding to D^λ and D^μ, then $e_\lambda \mathcal{O}\Sigma_n e_\mu = \varepsilon^\eta \cdot e_\lambda \mathcal{O}\Sigma_n e_\mu$ (since for all partitions $\varphi \neq \eta$, either $\varepsilon^\varphi e_\mu$ or $\varepsilon^\varphi e_\lambda$ is zero, by definition of c_μ and c_λ). Since by definition $A^\eta_{\lambda,\mu} = \mathrm{length}_{\mathcal{O}}\, e_\lambda \mathcal{O}\Sigma_n e_\mu / \varepsilon^\eta \cdot e_\lambda \mathcal{O}\Sigma_n e_\mu$, it follows that $A^\eta_{\lambda,\mu}$ is zero, which is a contradiction.

Now we prove (ii). Let ν be an element of $c_\lambda \cap c_\mu$. By Lemma 6.29, either λ of μ must occur in one of $S_k^\nu(0)/S_k^\nu(1)$ or $S_k^\nu(2)/S_k^\nu(3)$. By Theorem 6.23 it follows that $\nu \in \{\lambda, \mu, \lambda^{M\top}, \mu^{M\top}\}$. Since we know already that $|c_\lambda \cap c_\mu| = 2$, we only have to check that $c_\lambda \cap c_\mu \neq \{\lambda^{M\top}, \mu^{M\top}\}$. Suppose the contrary. Then $S_k^{\lambda^{M\top}}$ has D^μ as a composition factor, and therefore $S_k^{\lambda^M}$ has D^{μ^M} as a composition factor. It follows $\mu^M \rhd \lambda^M$. But in the same way the fact that $S_k^{\mu^{M\top}}$ has D^λ as a composition factor implies that $\lambda^M \rhd \mu^M$, which yields the desired contradiction. □

At this point we fix a defect two block B of some $\mathcal{O}\Sigma_n$, and we wish to describe its basic order, which we shall denote by Λ. We describe Λ as an order in the K-algebra A which we are about to define.

Definition 6.31. *The exponent matrices for B determined in Theorem 6.26 shall be denoted by $m^\lambda_{\mu\nu}$. By d_λ we denote the dimension of the Specht module S_K^λ.*

Definition 6.32. *Define a K-algebra A spanned by a K-basis*

$$\varepsilon^\lambda_{\mu\nu} \quad \text{for } \lambda \text{ a partition in } B \text{ and } \mu, \nu \in r_\lambda \tag{6.46}$$

equipped with the following multiplication law:

$$\varepsilon^\lambda_{\mu\nu} \cdot \varepsilon^{\tilde\lambda}_{\tilde\mu\tilde\nu} = \delta_{\lambda\tilde\lambda} \cdot \delta_{\nu\tilde\mu} \cdot \varepsilon^\lambda_{\mu\tilde\nu} \tag{6.47}$$

Note that this A is isomorphic to a direct sum of full matrix algebras over K, and the $\varepsilon_{\mu\nu}^{\lambda}$ are just the matrix units. Note also that $A \cong K \otimes_{\mathcal{O}} \Lambda$. We will henceforth assume that Λ is embedded in A.

The central primitive idempotents in A are given by

$$\varepsilon^{\lambda} := \sum_{\mu \in r_{\lambda}} \varepsilon_{\mu\mu}^{\lambda} \tag{6.48}$$

and we shall assume without loss that for each λ

$$\varepsilon^{\lambda}\Lambda = \bigoplus_{\mu,\nu \in r_{\lambda}} \langle p^{m_{\mu\nu}^{\lambda}} \cdot \varepsilon_{\mu\nu}^{\lambda} \rangle_{\mathcal{O}} \tag{6.49}$$

and that for each p-regular μ the idempotent $\sum_{\lambda \in c_{\mu}} \varepsilon_{\mu\mu}^{\lambda}$ is a primitive idempotent in Λ (corresponding to D^{μ}). This can all be achieved by conjugation within A.

Theorem 6.33. *The order Λ is conjugate in A to the $\mathcal{O}$-algebra generated by the following elements of A: For each p-regular μ in B the idempotent*

$$e_{\mu} := \sum_{\lambda \in c_{\mu}} \varepsilon_{\mu\mu}^{\lambda} \tag{6.50}$$

and for each (ordered) pair (μ, ν) of (distinct) p-regular partitions in B with $\mathrm{Ext}^1_{k\otimes_{\mathcal{O}}B}(D^{\mu}, D^{\nu}) \neq \{0\}$ an element $x_{\mu\nu}$, which is defined as

$$x_{\mu\nu} := p^{m_{\mu\nu}^{\lambda}} \cdot \varepsilon_{\mu\nu}^{\lambda} + p^{m_{\mu\nu}^{\eta}} \cdot \varepsilon_{\mu\nu}^{\eta} \quad \text{where } c_{\mu} \cap c_{\nu} = \{\lambda, \eta\} \text{ and } \mu > \nu \tag{6.51}$$

respectively

$$x_{\mu\nu} := p^{m_{\mu\nu}^{\lambda}} \cdot \varepsilon_{\mu\nu}^{\lambda} - \frac{d_{\lambda}}{d_{\eta}} \cdot p^{m_{\mu\nu}^{\eta}} \cdot \varepsilon_{\mu\nu}^{\eta} \quad \text{where } c_{\mu} \cap c_{\nu} = \{\lambda, \eta\} \text{ and } \mu < \nu \tag{6.52}$$

Proof. It follows easily from Nakayama's lemma (for $\mathcal{O}$-modules) that a set of elements of Λ generate Λ as an $\mathcal{O}$-algebra if and only if their images in $k \otimes_{\mathcal{O}} \Lambda$ generate $k \otimes_{\mathcal{O}} \Lambda$ as a k-algebra. It is well known (see for instance [Ben91, Proposition 4.1.7], and be aware that the condition "k algebraically closed" may be replaced by "k is a splitting field") that a full set of primitive idempotents $\overline{e}_{\mu}$ (with the natural choice of indices) and any basis of $\overline{e}_{\mu} \operatorname{Jac}(k \otimes_{\mathcal{O}} \Lambda)/\operatorname{Jac}(k \otimes_{\mathcal{O}} \Lambda)^2 \overline{e}_{\nu}$ (where μ, ν run over all p-regular partitions) generates $k \otimes_{\mathcal{O}} \Lambda$. By [Ben91, Proposition 2.4.3] we have

$$\dim_k \overline{e}_{\mu} \cdot \left(\operatorname{Jac}(k \otimes_{\mathcal{O}} \Lambda)/\operatorname{Jac}(k \otimes_{\mathcal{O}} \Lambda)^2\right) \cdot \overline{e}_{\nu} = \dim_k \mathrm{Ext}^1_{k\otimes_{\mathcal{O}}B}(D^{\mu}, D^{\nu}) \tag{6.53}$$

Moreover we know (as mentioned in Remark 6.17) that all $\mathrm{Ext}^1_{k\otimes_{\mathcal{O}}B}(D^{\mu}, D^{\nu})$ are at most one-dimensional. Now pick a specific pair μ, ν of p-regular partitions with $\mu > \nu$ such that $\mathrm{Ext}^1_{B}(D^{\mu}, D^{\nu}) \neq \{0\}$. Our goal is to pick some element in $e_{\mu}\Lambda e_{\nu}$ that is suitable as a generator due to the above considerations.

First we should note that since Λ is a self-dual order with respect to the symmetric Λ-

equivariant bilinear form

$$A \times A \to K : (a,b) \mapsto \frac{1}{n!} \sum_{\lambda} d_\lambda \cdot \mathrm{Tr}(\varepsilon^\lambda \cdot a \cdot b) \tag{6.54}$$

we can define the bilinear pairing

$$\begin{array}{cccc} T: & e_\nu A e_\mu \times e_\mu A e_\nu & \to & K \\ & \left(\sum_{\lambda \in c_\mu \cap c_\nu} f_\lambda \cdot \varepsilon^\lambda_{\nu\mu},\ \sum_{\lambda \in c_\mu \cap c_\nu} g_\lambda \cdot \varepsilon^\lambda_{\mu\nu}\right) & \mapsto & \frac{1}{n!} \cdot \sum_{\lambda \in c_\mu \cap c_\nu} d_\lambda \cdot f_\lambda \cdot g_\lambda \end{array} \tag{6.55}$$

to get $e_\nu \Lambda e_\mu = \{v \in e_\nu A e_\mu \mid T(v, e_\mu \Lambda e_\nu) \subseteq \mathcal{O}\}$ (and the analogous equation for $e_\mu \Lambda e_\nu$). We will use this together with the fact that $\nu_p\left(\frac{d_\lambda}{n!}\right) = -2$ for all λ. We distinguish the following cases:

(i) $c_\mu \cap c_\nu = \{\lambda, \eta\}$, $m^\lambda_{\mu\nu} + m^\lambda_{\nu\mu} = 2$ and $m^\eta_{\mu\nu} + m^\eta_{\nu\mu} = 2$. By Theorem 6.23 and Theorem 6.26 this can only happen if $\{\lambda, \eta\} = \{\mu, \nu\}$. But then D^μ is a composition factor of S^ν, so $\mu \rhd \nu$, and D^ν is a composition factor of S^μ, so $\nu \rhd \mu$. Clearly this is a contradiction, so this case does not occur at all.

(ii) $c_\mu \cap c_\nu = \{\lambda, \eta\}$ and $m^\lambda_{\mu\nu} + m^\lambda_{\nu\mu} = 1$. In this case $T(e_\nu \Lambda e_\mu, p^{m^\lambda_{\mu\nu}} \cdot \varepsilon^\lambda_{\mu\nu}) = p^{-1} \cdot \mathcal{O}$, which implies $p^{m^\lambda_{\mu\nu}} \cdot \varepsilon^\lambda_{\mu\nu} \notin e_\mu \Lambda e_\nu$ (note however that $p^{m^\lambda_{\mu\nu}+1} \cdot \varepsilon^\lambda_{\mu\nu}$ is in $e_\mu \Lambda e_\nu$ by the same argument), and thus

$$e_\mu \Lambda e_\nu \subsetneq \langle p^{m^\lambda_{\mu\nu}} \cdot \varepsilon^\lambda_{\mu\nu} \rangle_{\mathcal{O}} \oplus \langle p^{m^\eta_{\mu\nu}} \cdot \varepsilon^\eta_{\mu\nu} \rangle_{\mathcal{O}} \tag{6.56}$$

However the projection onto each summand (that is, multiplication by ε^λ) has to be surjective by (6.49), and so there is an element in $e_\mu \Lambda e_\nu$ of the form $p^{m^\lambda_{\mu\nu}} \cdot \varepsilon^\lambda_{\mu\nu} + \alpha^\eta_{\mu\nu} \cdot p^{m^\eta_{\mu\nu}} \cdot \varepsilon^\eta_{\mu\nu}$ for some $\alpha^\eta_{\mu\nu} \in \mathcal{O}^\times$. So we can state that

$$e_\mu \Lambda e_\nu = \left\langle p^{m^\lambda_{\mu\nu}} \cdot \varepsilon^\lambda_{\mu\nu} + \alpha^\eta_{\mu\nu} \cdot p^{m^\eta_{\mu\nu}} \cdot \varepsilon^\eta_{\mu\nu},\ p^{m^\lambda_{\mu\nu}+1} \cdot \varepsilon^\lambda_{\mu\nu} \right\rangle_{\mathcal{O}} \tag{6.57}$$

and by dualizing it follows that

$$e_\nu \Lambda e_\mu = \left\langle p^{m^\lambda_{\nu\mu}} \cdot \varepsilon^\lambda_{\nu\mu} - \frac{d_\lambda}{d_\eta} \cdot (\alpha^\eta_{\mu\nu})^{-1} \cdot p^{m^\eta_{\nu\mu}} \cdot \varepsilon^\eta_{\nu\mu},\ p^{m^\lambda_{\nu\mu}+1} \cdot \varepsilon^\lambda_{\nu\mu} \right\rangle_{\mathcal{O}} \tag{6.58}$$

Using Theorem 3.22 we conclude that

$$(\varepsilon^\lambda + \varepsilon^\eta) \cdot e_\mu \Lambda e_\mu = \langle \varepsilon^\lambda_{\nu\mu} + \varepsilon^\eta_{\nu\mu}, p \cdot \varepsilon^\lambda_{\nu\mu} \rangle_{\mathcal{O}} \tag{6.59}$$

and therefore $\langle p^{m^\lambda_{\nu\mu}+1} \cdot \varepsilon^\lambda_{\nu\mu}, p^{m^\eta_{\nu\mu}+1} \cdot \varepsilon^\eta_{\nu\mu} \rangle_{\mathcal{O}}$ is the unique maximal $e_\mu \Lambda e_\mu$-submodule of $e_\nu \Lambda e_\mu$. It is therefore equal to $e_\mu \operatorname{Jac}(\Lambda)^2 e_\nu$. Hence we may take

$$x_{\mu\nu} = p^{m^\lambda_{\mu\nu}} \cdot \varepsilon^\lambda_{\mu\nu} + \alpha^\eta_{\mu\nu} \cdot p^{m^\eta_{\mu\nu}} \cdot \varepsilon^\eta_{\mu\nu} \tag{6.60}$$

as a generator (since it is not contained in $e_\mu \operatorname{Jac}(\Lambda)^2 e_\nu$), and by the same argument we may pick

$$x_{\nu\mu} = p^{m^\lambda_{\nu\mu}} \cdot \varepsilon^\lambda_{\nu\mu} - \frac{d_\lambda}{d_\eta} \cdot (\alpha^\eta_{\mu\nu})^{-1} \cdot p^{m^\eta_{\nu\mu}} \cdot \varepsilon^\eta_{\nu\mu} \tag{6.61}$$

Now we have to show that all the $\alpha_{\mu\nu}^{\lambda}$ may be chosen to be equal to one. To do that, first note that due to Theorem 6.30 we may assume that all parameters are of the form $\alpha_{\mu\nu}^{\nu}$ (for p-regular partitions $\mu > \nu$). Of course, in this case, it may also be assumed that ν is the lexicographically greatest element in $c_\mu \cap c_\nu$.

Assume ν_0 is a p-regular partition in B such that all $\alpha_{\mu\nu}^{\nu}$ with $\nu < \nu_0$ are equal to one and $\alpha_{\mu\nu_0}^{\nu_0} \neq 1$ for some μ. Assume moreover that ν_0 is lexicographically maximal with respect to this property. Our goal is to show that after conjugation by an appropriate unit in A and renormalization of the generators by multiplying with elements of $\mathcal{O}^\times$ afterwards, we can make all $\alpha_{\mu\nu}^{\nu}$ with $\nu \leqslant \nu_0$ equal to one, which yields that without loss, all $\alpha_{\mu\nu}^{\nu}$ may be chosen equal to one. To do this, we conjugate with a u (i. e., replace each $x_{\mu\nu}$ by $u \cdot x_{\mu\nu} \cdot u^{-1}$), where

$$u := \sum_{\mu \in r_{\nu_0}} \alpha_{\mu\nu_0}^{\nu_0} \cdot \varepsilon_{\mu\nu_0}^{\nu_0} + \sum_{\lambda \neq \nu_0} \varepsilon^{\lambda} \quad \in A^\times \tag{6.62}$$

Note that in this formula we take all $\alpha_{\mu\nu_0}^{\nu_0}$ that are not defined to equal one. The conjugation with this unit will obviously make all $\alpha_{\mu\nu_0}^{\nu_0}$ equal to one, and not affect any $\alpha_{\mu\nu}^{\nu}$ with $\nu < \nu_0$ (since all elements of $c_\mu \cap c_\nu$ will be lexicographically smaller than ν_0). After renormalizing the other generators that were altered by the conjugation, we have $\alpha_{\mu\nu}^{\nu} = 1$ for all $\nu \leqslant \nu_0$. That concludes the proof. □

It has been proved in [Pea04, Corollary 5.4.5.] that defect two blocks of symmetric groups over k are tightly graded (that is, there is a grading such that all arrows in the Ext-quiver are homogeneous of degree one). The following reproves that result (in a very simple fashion), and additionally shows that the images of the $x_{\mu\nu}$ in the basic algebra over k are homogeneous generators. That should simplify the calculation of the quiver relations from our description of the block.

Corollary 6.34. *Let $\Lambda = \mathcal{O}\langle \{e_\mu\}_\mu, \{x_{\mu\nu}\}_{\mu,\nu}\rangle$ as in Theorem 6.33. Let Q be the* Ext*-quiver, denote by E_μ the vertices and denote by $X_{\mu\nu}$ an edge from E_μ to E_ν. By kQ we denote the quiver algebra (with multiplication convention $X_{\mu\nu} \cdot X_{\nu\tau} \neq 0$). Then the kernel of the epimorphism*

$$\Phi: \ kQ \twoheadrightarrow \Lambda/p\Lambda : \begin{cases} X_{\mu\nu} \mapsto x_{\mu\nu} + p\Lambda \\ E_\mu \mapsto e_\mu + p\Lambda \end{cases} \tag{6.63}$$

is a homogeneous ideal, where we define the vertices of Q to be homogeneous of degree zero and the arrows to be homogeneous of degree one.

Proof. Since $\operatorname{Ker}\Phi = \bigoplus_{\mu,\nu} E_\mu \cdot \operatorname{Ker}\Phi \cdot E_\nu$, and a path connecting E_μ with E_ν has even respectively odd length if and only if μ and ν lie in the same part respectively in different parts of the bipartition, we may assume that $\operatorname{Ker}\Phi$ is generated by elements that involve only paths of even length and such elements that only involve paths of odd length. By general theory we may assume that all paths involved in any element of $\operatorname{Ker}\Phi$ have at least length two. By [Sco95, Theorem I], the projective indecomposable of $\Lambda/p\Lambda$ have common Loewy length five, i. e. every path of length $\geqslant 5$ is zero. A path of length four will correspond to a top onto socle endomorphism of a projective indecomposable. Thus all paths of length four that start and end at a separate vertex are sent to zero under Φ, and all paths of length four that start and end at the same fixed vertex of Q will be mapped under Φ into a one-dimensional subspace of $\Lambda/p\Lambda$.

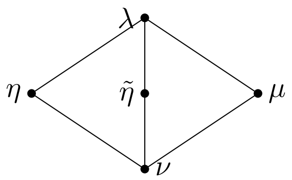
So, considering all of this, all we need to show is that if $Y_\mu := X_{\mu\alpha}X_{\alpha\beta}X_{\beta\gamma}X_{\gamma\mu}$ is not in $\operatorname{Ker}\Phi$, then neither is $Y_\mu + \sum_\nu q_\nu \cdot X_{\mu\nu}X_{\nu\mu}$ for any choice of $q_\nu \in k$. It follows easily from Theorem 6.30 that $|c_\mu \cap c_\alpha \cap c_\beta \cap c_\gamma| = 1$, and lets denote the single element of this set by λ. Hence $y_\mu := x_{\mu\alpha}x_{\alpha\beta}x_{\beta\gamma}x_{\gamma\mu}$ is equal to $v \cdot \varepsilon_{\mu\mu}^{\lambda}$ for some $v \in \mathcal{O}$. The fact that $y_\mu \in \Lambda \setminus p\Lambda$ implies $\nu_p(v) = 2$. Let $T : e_\mu\Lambda e_\mu \times e_\mu\Lambda e_\mu \to \mathcal{O}$ the symmetric bilinear form on $e_\mu\Lambda e_\mu$ as given in (6.55). Then $T(y_\mu, 1) \in \mathcal{O}^\times$. On the other hand $T(x_{\mu\nu}x_{\nu\mu}, 1) = 0$ for any ν by definition of the $x_{\mu\nu}$. Hence $T(y_\mu + \sum_\nu \hat{q}_\nu \cdot x_{\mu\nu}x_{\nu\mu}, 1) \in \mathcal{O}^\times$ for any choice of $\hat{q}_\nu \in \mathcal{O}$, which implies $y_\mu + \sum_\nu \hat{q}_\nu \cdot x_{\mu\nu}x_{\nu\mu} \notin p\Lambda$. Thus $\Phi(Y_\mu + \sum_\nu q_\nu \cdot X_{\mu\nu}X_{\nu\mu}) \neq 0$. □

Example 6.35. *We look at the principal block of $\mathbb{Z}_3\Sigma_7$. The decomposition matrix is given as follows (we assign arbitrary names to the partitions in order to unclutter notation a bit):*

Dim.	*Name*		(7)	$(5,2)$	$(4,3)$	$(4,2,1)$	$(3,2,1^2)$
1	λ	(7)	1	.	.	.	.
14	μ	$(5,2)$	1	1	.	.	.
14	ν	$(4,3)$	.	1	1	.	.
35	η	$(4,2,1)$	1	1	1	1	.
20	φ	$(4,1^3)$	.	.	.	1	.
35	$\tilde{\eta}$	$(3,2,1^2)$	1	.	1	1	1
14	$\tilde{\nu}$	$(2^3,1)$	1	.	.	.	1
14	$\tilde{\mu}$	$(2^2,1^3)$	.	.	1	.	1
1	$\tilde{\lambda}$	(1^7)	.	.	1	.	.

(6.64)

and the Ext*-quiver is given by*



The 4×4-exponent matrices are given as follows

$$m^\eta = m^{\tilde{\eta}} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 2 & 1 & 1 & 0 \end{pmatrix} \tag{6.65}$$

with row/column indexing $(\mu, \lambda, \nu, \eta)$ and $(\eta, \lambda, \nu, \tilde{\eta})$. The 2×2-exponent matrices are

$$m^\mu = m^\nu = m^{\tilde{\nu}} = m^{\tilde{\mu}} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \tag{6.66}$$

with row/column indexing (λ, μ), (μ, ν), $(\lambda, \tilde{\eta})$ and $(\nu, \tilde{\eta})$. The 1×1-exponent matrices are of course trivial.

By Theorem 6.33 we now get the following generators for the basic order:

$$\begin{array}{rclrcl} x_{\lambda\eta} & = & \varepsilon^{\eta}_{\lambda\eta} + 3\cdot\varepsilon^{\tilde{\eta}}_{\lambda\eta} & x_{\eta\lambda} & = & 3\cdot\varepsilon^{\eta}_{\eta\lambda} - \varepsilon^{\tilde{\eta}}_{\eta\lambda} \\ x_{\lambda\tilde{\eta}} & = & \varepsilon^{\tilde{\eta}}_{\lambda\tilde{\eta}} + 3\cdot\varepsilon^{\tilde{\nu}}_{\lambda\tilde{\eta}} & x_{\tilde{\eta}\lambda} & = & 3\cdot\varepsilon^{\tilde{\eta}}_{\tilde{\eta}\lambda} - \varepsilon^{\tilde{\nu}}_{\tilde{\eta}\lambda} \\ x_{\lambda\mu} & = & 3\cdot\varepsilon^{\mu}_{\lambda\mu} + 3\cdot\varepsilon^{\eta}_{\lambda\mu} & x_{\mu\lambda} & = & \varepsilon^{\mu}_{\mu\lambda} - \varepsilon^{\eta}_{\mu\lambda} \\ x_{\nu\eta} & = & \varepsilon^{\eta}_{\nu\eta} + 3\cdot\varepsilon^{\tilde{\eta}}_{\nu\eta} & x_{\eta\nu} & = & 3\cdot\varepsilon^{\eta}_{\eta\nu} - \varepsilon^{\tilde{\eta}}_{\eta\nu} \\ x_{\nu\tilde{\eta}} & = & \varepsilon^{\tilde{\eta}}_{\nu\tilde{\eta}} + 3\cdot\varepsilon^{\tilde{\mu}}_{\nu\tilde{\eta}} & x_{\tilde{\eta}\nu} & = & 3\cdot\varepsilon^{\tilde{\eta}}_{\tilde{\eta}\nu} - \varepsilon^{\tilde{\mu}}_{\tilde{\eta}\nu} \\ x_{\mu\nu} & = & 3\cdot\varepsilon^{\nu}_{\mu\nu} + \varepsilon^{\eta}_{\mu\nu} & x_{\nu\mu} & = & \varepsilon^{\nu}_{\nu\mu} - 3\cdot\varepsilon^{\eta}_{\nu\mu} \end{array} \tag{6.67}$$

and of course the following idempotents:

$$\begin{array}{rcl} e_{\lambda} & = & \varepsilon^{\lambda}_{\lambda\lambda} + \varepsilon^{\mu}_{\lambda\lambda} + \varepsilon^{\eta}_{\lambda\lambda} + \varepsilon^{\tilde{\eta}}_{\lambda\lambda} + \varepsilon^{\tilde{\nu}}_{\lambda\lambda} \\ e_{\mu} & = & \varepsilon^{\mu}_{\mu\mu} + \varepsilon^{\nu}_{\mu\mu} + \varepsilon^{\eta}_{\mu\mu} \\ e_{\nu} & = & \varepsilon^{\nu}_{\nu\nu} + \varepsilon^{\eta}_{\nu\nu} + \varepsilon^{\tilde{\eta}}_{\nu\nu} + \varepsilon^{\tilde{\mu}}_{\nu\nu} + \varepsilon^{\tilde{\lambda}}_{\nu\nu} \\ e_{\eta} & = & \varepsilon^{\eta}_{\eta\eta} + \varepsilon^{\varphi}_{\eta\eta} + \varepsilon^{\tilde{\eta}}_{\eta\eta} \\ e_{\tilde{\eta}} & = & \varepsilon^{\tilde{\eta}}_{\tilde{\eta}\tilde{\eta}} + \varepsilon^{\tilde{\nu}}_{\tilde{\eta}\tilde{\eta}} + \varepsilon^{\tilde{\mu}}_{\tilde{\eta}\tilde{\eta}} \end{array} \tag{6.68}$$

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