What Does Rebalancing Really Achieve?

Keith Cuthbertson* (Cass Business School)
Simon Hayley (Cass Business School)
Nick Motson (Cass Business School)
Dirk Nitzsche (Cass Business School)

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Abstract

There is now a substantial literature on the effects of rebalancing on portfolio performance. However, this literature contains frequent misattribution between 'rebalancing returns' which are specific to the act of rebalancing, and 'diversification returns' which can be earned by both rebalanced and unrebalanced strategies. Confusion on this issue can encourage investors to follow strategies which involve insufficient diversification and excessive transactions costs. This paper identifies the misleading claims that are made for rebalanced strategies and demonstrates theoretically and by simulation that the apparent advantages of rebalanced strategies over infinite horizons give an inaccurate impression of their performance over finite horizons.

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* Cass Business School, 106 Bunhill Row, London EC1Y 8TZ, UK. E-mail: K.Cuthbertson@city.ac.uk. Tel +44 20 7040 5170. The authors are grateful to Max Bruche, Ales Cerny, Michael Dempster, Aneel Keswani, Ian Marsh, Frank McGroarty, Anthony Neuberger, Richard Payne, Scott Willenbrock, participants at the EFMA 2015 Annual Meeting, seminars at the ISMA Centre, University of Reading, Cass Business School and Smurfit Business School (University College Dublin) for useful comments. The usual disclaimer applies.

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1. Introduction

Rebalancing is a vital part of many investment strategies. Passive strategies usually involve choosing portfolio weights according to some predetermined rule and a key practical element of such strategies is the frequency and hence cost of rebalancing. Only if portfolio weights are allowed to evolve over time according to the relative returns on the component assets (a buy and hold strategy, B&H) will there be no rebalancing. Other passive strategies require at least periodic rebalancing, and in the extreme some passive strategies assume continuous rebalancing to keep asset weights constant (an assumption that is often required to produce tractable closed form results). Rebalancing is so widespread that a good understanding of its effects is vital.

The well-established literature on optimal growth considers portfolio weights chosen to maximise the expected geometric growth rate of the portfolio value, consistent with maximising the expected logarithm of final wealth (Kelly 1956, Luenberger 1997, Thorpe 2010, Platen and Rendek 2010). Rebalancing is a vital part of these strategies. Another strand of the literature considers arbitrary fixed weight portfolios which are compared with a B&H strategy (e.g. Fernholz and Shay 1982, Booth and Fama 1992, Luenberger 1997, Mulvey et al. 2007, Qian 2012, Willenbrock, 2012). A key result from this part of the literature is that a rebalanced portfolio of independently and identically distributed (IID) assets has a higher expected portfolio growth rate than the corresponding B&H strategy even when assets follow a random walk. “Volatility pumping" is a strategy which seeks to boost this effect by deliberately choosing high volatility assets and rebalancing each period to fixed weights.

In this paper we focus on the choice between constant portfolio weight rebalanced strategies and the corresponding B&H strategies. We examine under what circumstances and for what reasons one strategy outperforms the other. The purpose of this paper is to correct misleading claims that are widely made about the benefits of rebalancing.

It is widely noted that even when there is no predictable time structure to asset returns rebalanced portfolios generate "excess growth" (defined in this literature as expected geometric portfolio growth which is greater than the average geometric growth of the underlying assets). We demonstrate mathematically that unrebalanced portfolios also generate "excess growth", and that growth rates of unrebalanced and rebalanced portfolios only diverge to the extent that drift in portfolio weights gradually leaves the unrebalanced portfolio less well diversified.

Confusion on this issue appears to have arisen in part because of the difficulty in making meaningful comparisons between rebalanced and unrebalanced portfolios, since even when the portfolios are initially identical the composition of the unrebalanced portfolio tends to shift over time. We derive like-for-like comparisons between rebalanced and unrebalanced portfolios and demonstrate analytically that in the absence of mean reversion in relative asset prices the greater expected growth of rebalanced strategies is entirely explained by their lower portfolio volatilities rather than – as is claimed – being due to the rebalancing trades themselves being profitable. We also demonstrate that the apparent advantages of such rebalanced strategies over infinite horizons are very misleading indicators of performance over the finite horizons that are likely to be of interest to investors.

This has important implications for investors. The misleading claims that are made for rebalancing encourage investors to increase the scale of their rebalancing trades by holding volatile assets and rebalancing frequently. More efficient portfolios can be constructed simply by diversifying effectively. Investors might rationally prefer rebalancing strategies where they anticipate mean reversion in relative asset prices, but this is very different from the claims in the theoretical literature that rebalancing directly boosts returns even in the absence of any such time structure in returns.

The rest of the paper is organised as follows. Section 2 reviews the theoretical and empirical literature on rebalancing. Section 3 shows, using analytic and simulation approaches, that under standard assumptions the expected growth rate of a portfolio of risky assets (either rebalanced or B&H) is entirely explained by diversification, with no additional "rebalancing return". Section 4 examines the impact of rebalancing in the widely-cited case of a portfolio consisting of one risky and one risk-free asset. Conclusions are drawn in Section 5.

2. Literature Review

Rebalancing is important in a wide range of situations. It is inherent in any portfolio weighting strategy other than capitalisation-weighting and plays a part in the debate over fundamental and alternative forms of equity indexing (Arnott, Hsu and Moore 2005, Kaplan 2008, Hsu et al. 2011) and universal portfolios (Cover 1991). Rebalancing also has a significant impact on the growth rates of portfolios of commodity futures (Gorton and Rouwenhorst 2006a and 2006b, Erb and Harvey 2006).

In the theoretical literature Cheng and Deets (1971) compare the performance of B&H and rebalancing strategies assuming that risky asset prices follow random walks and are IID except for their different mean returns μ_i. The B&H strategy and the rebalancing strategy give the same expected terminal wealth when the mean returns of all risky assets are equal. However, if at least one pair of assets has different mean returns $\mu_i \neq \mu_j$ then an initially equally weighted B&H strategy always gives a higher expected terminal wealth than the corresponding rebalancing strategy (intuitively, a B&H portfolio is likely over time to give increasing weight to the assets with higher μ_i). This relative superiority of B&H in terms of expected wealth is found to be larger the greater the dispersion of the μ_i, the longer the investment horizon and the more frequently the rebalanced portfolio is returned to equal weights.

A specific situation which is widely analysed (e.g. Fernholz and Shay 1982, Perold and Sharpe 1988, Luenberger 1997, and Gabay and Herlemont 2007) is a portfolio which is rebalanced each period to keep a fixed proportion $1-\pi$ in a risk-free asset which pays zero interest. The remainder is invested in a risky asset which follows the lognormal diffusion process $S_t = \exp(gt + \sigma W(t))$ where $g = \mu - \sigma^2 / 2$ and $W(t)$ is a Wiener process. Hence, if $g = 0$ the long term growth rate of this risky asset approaches zero as t→∞. However, the terminal wealth of the rebalanced portfolio is:

$$V_t^{reb} = (S_t)^\pi e^{g^* t} \quad \text{where} \quad g^* = \pi(1-\pi)\sigma^2 / 2 \tag{1}$$

The term g^* is referred to as the "excess growth rate" of the portfolio (in excess of the weighted average of the growth rates of the component assets, which in this case is zero). The expected portfolio excess growth rate is positive for $0 < \pi < 1$, is maximized for $\pi = 1/2$ and increases with volatility σ. The corresponding B&H portfolio has the same initial weights, but does not rebalance. This has terminal wealth:

$$V_t^{B\&H} = (1-\pi) + \pi S_t \tag{2}$$

Using (1) and (2), for the risk-free plus single risky asset portfolio we have:

$$\ln \frac{V_t^{reb}}{V_t^{B\&H}} = \frac{1}{2}\sigma^2 \pi(1-\pi)t + \ln \frac{S_t^\pi}{1-\pi+\pi S_t} \tag{3}$$

Thus the log of relative wealth for the two strategies is stochastic, but depends positively on the time horizon and the volatility of the risky asset (hence the term "volatility pumping" given to the rebalancing strategy). Dempster, Evstigneev and Schenk-Hoppe (2007) use a more general setting of stationary stochastic processes for returns to show that an equally-weighted rebalanced portfolio of N risky assets also generates excess growth. They too note that if individual assets have zero long-run growth, then this rebalancing strategy seems to provide "something for nothing".

One distinctive feature of this literature is the use of the expected geometric portfolio growth rate (or the corresponding discrete time geometric mean return, GM) as the performance metric instead of more conventional metrics such as the arithmetic mean return (AM), Sharpe ratio, Fama-French 4-factor alpha, expected utility or certainty equivalent return.[1] Our objective in this paper is to clear up the confusion which is apparent in the literature over the effects of rebalancing on some of these metrics.

Rebalancing clearly has important implications for portfolio risk[2], but our key interest in this paper is in the claims that are made for the growth rates and terminal wealth achieved by rebalanced portfolios (either in expectation or asymptotically as $t \to \infty$). In particular, we examine the claims originally made in Fernholz and Shay (1982) and endorsed by an increasing number of

[1] Using the geometric growth rate or GM as a performance metric raises a number of questions (see, for example, Elton and Gruber 1974, Brennan and Schwartz 1985). For example, maximising the GM will maximise the welfare of an investor whose utility is a logarithmic function of terminal wealth. It is less clear that it is an appropriate target for investors with other utility functions. This topic has previously been the subject of a long and rancorous debate, which we do not wish to revisit here. For our purposes it is sufficient to note that investors are encouraged to choose rebalanced strategies on the basis of their expected growth rates or GM returns. This paper seeks to clarify how these apparently attractive returns come about.

[2] Perold and Sharpe (1988) note that a constant-weight rebalanced strategy which buys more risky assets after they have underperformed does the opposite of a portfolio insurance strategy, and could thus be seen as selling such portfolio insurance to other investors. By contrast, rebalancing a portfolio composed entirely of equally risky assets can help ensure that diversification is maintained and portfolio variance kept low.

subsequent papers (e.g. Luenberger 1997, Mulvey et al. 2007, Stein et al. 2009, Qian 2012, Willenbrock 2012, Bouchey et al. 2012, Hallerbach 2014), that even when returns have no predictable time structure (for example when asset prices evolve following geometric Brownian motion), rebalanced portfolios must be expected to outperform because they generate "rebalancing returns" which are entirely absent in unrebalanced portfolios.

In parallel to this theoretical literature, a large body of empirical research has investigated the effects of rebalancing. This is testament to the practical importance of this issue to investors. One sizeable strand of the empirical literature seeks to identify the optimal rebalancing frequency or no-trade interval around desired portfolio weights (e.g. Buetow et al. 2002, Masters 2003, Smith and Desormeau 2006, McLellan et al 2009), but it arrives at no clear consensus. Many studies attempt to assess the effects of rebalancing on a simple two asset equity/bond portfolio. The results have varied, depending on the different performance metrics used, different markets and time horizons considered, the frequency of rebalancing and the extent to which transactions costs are taken into account. Some studies find that a rebalancing strategy outperforms a B&H strategy (Arnott and Lovell 1993, Tsai 2001, Donohue and Yip 2003, Harjoto and Jones 2006, Tokat and Wicas 2007, Bolognesi et al. 2013), whilst Jaconetti, Kinniry and Zilbering (2010) find the reverse. Plaxco and Arnott (2002) conclude that a B&H strategy "may appear to outperform in a strongly trending market", but that rebalancing outperforms on a risk-adjusted basis.

Dichtl, Drobetz and Wambach (2012) note that most empirical studies consider at most a small number of different sets of realised equity and bond returns, and hence cannot attribute any statistical significance to their findings. Instead, they use a block bootstrap approach, finding that rebalancing boosts returns “only marginally” and not consistently.

By contrast, Plyakha et al. (2012) construct portfolios by randomly sampling 100 stocks from within the S&P500 index and find that rebalanced equally-weighted portfolios outperform in terms of mean return and four factor alpha as well as risk-adjusted measures. They attribute this to the fact that rebalancing is “a contrarian strategy that exploits reversal in stock prices”, consistent with earlier evidence of such reversals (eg. Jegadeesh 1990, and Jegadeesh and Titman, 1993 and 2002). However, the finding that rebalancing strategies may outperform because of such empirical regularities is very different from the claim made in the theoretical literature that rebalancing strategies automatically outperform even when asset returns follow a geometric Brownian motion, and hence have no identifiable time structure. In this paper we address the theoretical claims made in favour of rebalancing, and demonstrate that they are very misleading.

3. Rebalancing Multiple Asset Portfolios

In this section we first define the terms “volatility drag”, “diversification returns” and “rebalancing returns”, as used in this literature. Second we demonstrate analytically that the concept of “rebalancing returns” is misleading since unrebalanced portfolios also generate such returns. Finally, we show using simulations that the expected growth of a rebalanced or unrebalanced portfolio is entirely explained by portfolio volatility and there is no additional effect from rebalancing.

As discussed above, a number of theoretical papers which use expected growth rates as their performance metric claim that rebalanced strategies outperform corresponding unrebalanced strategies. Use of the expected growth rate means that we must take account of "volatility drag". This effect will be familiar to many investors, and can be understood directly from the standard relationship between the arithmetic and geometric mean returns:[3]

$$E[GM] \approx E[AM] - \frac{\sigma^2}{2} \tag{4}$$

This tells us that, all else equal, an asset or portfolio will generate a lower expected GM if it has a higher volatility. This is because the expected GM is a concave function of terminal wealth. If we compare two strategies with identical expected terminal wealth, the one with the more dispersed distribution of wealth outturns will generate a lower E[GM] because this concave relationship effectively penalizes both exceptionally high and exceptionally low terminal wealth outturns (by contrast, terminal wealth is a linear function of the AM). Figure 1 illustrates this volatility drag. The same volatility drag effect is seen for the continuous time geometric growth

[3] Fama and Booth (1992) derive this relationship using a Taylor expansion for the expected continuously compounded return E[log(1+r)] around E[r]. This derivation makes no assumption about the nature of the asset or portfolio which generates these returns (except that the distribution of returns is differentiable with finite derivatives), yet it is an accurate approximation when compounding over small time periods for which E[r] is also small. The precise expression derived by Fama and Booth is $\mathrm{E[GM]} = \mathrm{E[AM]} - \frac{\sigma^2}{2(1+\mathrm{E[r]})^2}$[GM], but we use it here in the form in which it is most normally cited, which is of course still a good approximation for small E[r].

rate, which is a similarly concave function.[4] We demonstrate this when we examine continuous time applications in the next section.

[Figure 1]

Volatility drag is inherent in the relationship between periodic returns and terminal wealth and, as Equation 4 shows, is merely a function of the volatility of returns. Volatility drag thus affects both rebalanced and unrebalanced portfolios, although rebalancing may indirectly affect expected portfolio growth to the extent that it keeps the portfolio better diversified, with lower variance. Another entirely different effect is that portfolio rebalancing could boost expected terminal wealth if the rebalancing trades themselves tend to be profitable. A fixed weight rebalancing strategy sells some of each asset that outperformed in the most recent period and buys assets that underperformed, so it will profit if these relative price movements tend to reverse in future as a result of negative autocorrelation in relative asset returns.

These two effects are entirely distinct: One is an increase in the *expected growth rate* as a result of the reduction in portfolio volatility (but, as Chambers and Zdanowicz 2014 show, this does not affect expected terminal wealth), the other is an increase in *expected terminal wealth* which occurs only if there is (a) rebalancing, and (b) negative autocorrelation in relative asset returns. We argue below that proponents of rebalancing strategies tend to confuse these two effects

[4] $GM=(W_T/W_0)^{1/T}-1$ in discrete time (where W_T= Terminal wealth, W_0= Initial wealth). The equivalent continuous time growth rate is $^1/_T\log(W_T/W_0)$. Both are concave functions of terminal wealth.

by claiming that rebalancing generates excess growth because of "rebalancing returns" which are entirely absent from unrebalanced portfolios, but which are not dependent on any time structure in asset returns.

Equation (4) holds for any asset or portfolio. We can apply it to a diversified portfolio *p* and also to a portfolio containing a single asset *i*. Subtracting one of the resulting equations from the other gives us:[5]

$$E[GM_p] - E[GM_i] \approx (E[AM_p] - \tfrac{1}{2}\sigma_p^2) - (E[AM_i] - \tfrac{1}{2}\sigma_i^2) \tag{5}$$

For simplicity we assume all asset returns are IID and not autocorrelated[6]. Without this assumption, assets with larger expected returns are likely over time to comprise a larger proportion of an unrebalanced portfolio, raising E[AM] of the portfolio as a whole, as shown by Cheng and Deets (1971). By removing this effect, our assumption of IID asset returns not only simplifies the analysis, it is also generous to rebalanced portfolios. We show below that even with this assumption, unrebalanced portfolios give expected geometric returns equal to those of rebalanced portfolios with equal levels of volatility. Without it, unrebalanced portfolios would tend to outperform.

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[5] Applying equation (2) to a portfolio *p* gives us $E[GM_p] \approx E[AM_p] - \sigma_p^2/2$. A similar equation relates the expected return and variance (σ_i^2) of a single asset *i*: $E[GM_i] \approx E[AM_i] - \sigma_i^2/2$

[6] Other studies such as Fernholz and Shay (1982) and Luenberger (2013) similarly assume that all assets have identical expected growth rates, with no autocorrelation of returns.

By definition, the expected portfolio AM is the weighted average of the expected AM of its component assets. With E[AM] assumed equal for every asset $E[AM_p]=E[AM_i]$ regardless of the composition of the portfolio (we assume zero leverage, so portfolio weights always sum to unity). This gives us:

$$E[GM_p] - E[GM_i] \approx \tfrac{1}{2}(\sigma_i^2 - \sigma_p^2) \tag{6}$$

Booth and Fama (1992) define the “diversification return” as the degree to which the expected GM of a portfolio is greater than the weighted average of the expected GMs of its component assets. Our assumption of IID asset returns means that every asset has an identical E[GM] so the weighted average of these component returns is the same for any unleveraged portfolio of these assets. Equation 6 thus represents the diversification return of shifting from a single asset to a portfolio of similar assets. It also tells us that this diversification return is entirely due to the associated reduction in portfolio volatility (volatility drag).

The "rebalancing return" is portrayed as resulting from a very different process. Fernholz and Shay (1982) claim that even when asset returns follow a geometric Brownian motion an equally-weighted portfolio of IID assets generates “excess growth” (an expected growth rate which is greater than the weighted average growth of its component assets) and so will tend to outperform the corresponding unrebalanced portfolio, which they claim has the same expected growth rate as its component assets (i.e. zero excess growth). They explain this higher growth as resulting from the rebalancing process “buying on downticks and selling on upticks”. This claim has been endorsed by subsequent papers (e.g. Qian, 2012, Luenberger, 1997, Bouchey et al., 2012, Willenbrock, 2012, Stein et al. 2012) and in the practitioner literature.

We can assess this claim directly and with complete generality by taking an unrebalanced portfolio P_u and modelling it as two sub-portfolios which are initially of equal value. These sub-portfolios have growth paths described by the random variables x and y:

$$P_u = 0.5e^{x_t} + 0.5e^{y_t} \tag{7}$$

Using the Taylor expansion:

$$P_u \approx \frac{1}{2} + \frac{x_t}{2} + \frac{x_t^2}{4} + \frac{1}{2} + \frac{y_t}{2} + \frac{y_t^2}{4} \tag{8}$$

$$\Rightarrow P_u \approx 1 + \frac{x_t+y_t}{2} + \frac{1}{2}\left(\frac{x_t+y_t}{2}\right)^2 + \frac{1}{8}(x_t - y_t)^2 \approx e^{\frac{x_t+y_t}{2}} + \frac{1}{8}(x_t - y_t)^2 \tag{9}$$

If x and y are always identical (so that the portfolio in effect consists of only one asset) then $log(P_u) \approx (x_t + y_t)/2$, so the portfolio grows at a rate which is close to the average growth rate of its component parts, with no excess growth.[7] If instead P_u can be divided into two non-identical

[7] For clarity the analysis above expanded only to the quadratic terms, but the excess growth clearly remains positive if we add the third and fourth power terms of the expansion. Adding the extra terms to equation (5) gives us the following (for simplicity we omit the *t* subscripts):

$$P_u \approx \frac{1}{2} + \frac{x}{2} + \frac{x^2}{4} + \frac{x^3}{12} + \frac{x^4}{48} + \frac{1}{2} + \frac{y}{2} + \frac{y^2}{4} + \frac{y^3}{12} + \frac{y^4}{48}$$

$$= 1 + \frac{x+y}{2} + \frac{1}{2}\left(\frac{x+y}{2}\right)^2 + \frac{1}{6}\left(\frac{x+y}{2}\right)^3 + \frac{1}{24}\left(\frac{x+y}{2}\right)^4 + \frac{1}{8}(x-y)^2 + \frac{3}{48}(x+y)(x-y)^2 + \frac{6}{384}(x^2-y^2)^2$$

$$+ \frac{1}{384}(x-y)^4$$

sub-portfolios then the $\frac{1}{8}(x-y)^2$ term is unambiguously positive. Thus any diversified portfolio has "excess growth" even if it is not rebalanced.[8] This directly contradicts the explicit claims made by proponents of rebalancing strategies. Labelling these as "rebalancing returns" is extremely misleading – they are better regarded as the result of reduced portfolio volatility, since they clearly arise for any diversified portfolio.

This gives us all the terms up to the fourth power in the Taylor expansion of $\exp\left(\frac{x+y}{2}\right)$ plus additional terms which represent "excess growth" by which growth in the portfolio exceeds the average (equally weighted in this case) of the growth rates of its components. If our portfolio consists of a single asset then x and y will be identical and, by inspection, the excess growth terms will all be equal to zero. If instead the portfolio is diversified then x and y are non-identical and these excess growth terms are:

$$Excess\ growth \approx \frac{1}{8}(x-y)^2 + \frac{3}{48}(x+y)(x-y)^2 + \frac{6}{384}(x^2-y^2)^2 + \frac{1}{384}(x-y)^4$$

$$= \frac{1}{64}(x-y)^2\left(4 + (x+y+2)^2 + \frac{1}{6}(x-y)^2\right)$$

This excess growth is clearly always positive for x≠y. We cannot rule out the possibility that over long horizons higher order terms will become significant, but at least for modest investment horizons (where x and y are well below 1) these can be safely ignored. Thus, contrary to what proponents of rebalancing explicitly claim, diversification unambiguously raises the expected growth rate of an unrebalanced portfolio.

[8] The growth rate g_u of a B&H portfolio over a horizon T can be expressed in terms of the growth rates g_i of its component assets as the following, since the terminal portfolio value is simply the sum of the terminal values of the component assets: $(1+g_u)^T = \sum_{i=1}^{n} w_i(1+g_i)^T$ where w_i are the initial portfolio weights of the assets. However, this complex non-linear relationship does not in general imply the simpler relationship $g_u = \sum_{i=1}^{n} w_i g_i$ that is sometimes alleged and which would if true imply that by definition a B&H portfolio generates no "excess growth".

Simulation

We can also demonstrate the relationship between expected portfolio growth and portfolio volatility using simulation. Such numerical techniques have been little used in this literature since without rebalancing, portfolio asset weights tend to vary over time making it difficult to derive like-for-like comparisons between rebalanced and unrebalanced strategies.

We show below that expected portfolio growth for both rebalanced and unrebalanced portfolios is entirely explained by portfolio volatility and there is no *additional contribution* to expected growth from rebalancing. Even when asset returns are IID and have no time structure there are two effects which might be thought to cause E[GM] of rebalanced and unrebalanced portfolios to differ. First, unrebalanced portfolios which are initially equally weighted are likely to become less diversified over time, leading to higher portfolio volatility and hence a lower E[GM]. Second, it is widely claimed that there is an additional effect on E[GM] which is directly due to the act of rebalancing – a claim that we reject.

We consider returns that are IID and not autocorrelated so E[AM] is the same for both strategies. To simulate rebalanced and unrebalanced strategies with a wide range of different portfolio variances we use two approaches. We first consider portfolios containing N risky assets which we vary from N=2 to 100. For each value of N we simulate (i) an equally-weighted $1/N$ portfolio with rebalancing each month and (ii) an unrebalanced portfolio with initial weights $1/N$, but with the weights then evolving in line with relative asset returns (i.e. B&H). Monthly asset returns are assumed to be IID and not autocorrelated, with annualized (arithmetic) mean 10% and standard deviation 20% per annum for each asset. For each portfolio we conduct 10,000 simulations, each over a horizon of 100 years.

The results are shown in Figure 2. Panel A shows that the variances of rebalanced and unrebalanced portfolios fall as *N* rises. However for each level of *N* the unrebalanced portfolios have higher average variances. Asset returns are all assumed identically distributed, so an equally weighted rebalanced portfolio gives the minimum portfolio variance. Without rebalancing, the weights on each asset tend to diverge over time, leaving the portfolio less effectively diversified. The expected AM return remains constant for every portfolio (since the portfolio is unleveraged and all assets have identical expected AMs), but the geometric mean returns of these portfolios increase as their variances decrease, consistent with $E[GM] \approx E[AM] - \sigma^2/2$.

Panel B presents the same results, but with the average variance for each set of simulations plotted against the corresponding expected GM. The results for the rebalanced and unrebalanced portfolios now coincide. This shows that rebalancing affects the expected GM only to the extent that it affects the portfolio variance, and hence generates different levels of volatility drag. By contrast, if rebalancing generated returns by “buying low and selling high” as proponents suggest, we should expect different GMs for these portfolios even after correcting for their different variances.

[Figure 2 around here]

Next we examine the relationship between the expected GM and portfolio variance for portfolios of two risky assets with a range of different initial asset weights (Figure 3). Panel A shows that a fixed 50:50 weighting is the minimum variance portfolio, with unrebalanced portfolios seeing higher variances as the portfolio weights subsequently drift over time. However in the unrebalanced portfolios, if the initial portfolio weights are highly unequal (with one asset accounting for 86% or more of the portfolio), then the drift in these portfolio weights reduces average portfolio variance because it can result in the weights becoming substantially more equal over time.

Panel B plots the same results in terms of mean realised variance versus mean GM return for each set of simulations. The results coincide for rebalanced and unrebalanced portfolios just as they did for our earlier simulations. Furthermore, these results and those in Figure 2 all describe the same linear relationship ($E[GM] \approx E[AM] - \sigma^2/2$), confirming that rebalancing only affects the average GM to the extent that it affects the average portfolio variance.[9] Thus the different expected growth rates of the rebalanced and unrebalanced portfolios can be entirely explained by the resulting portfolio volatilities and there are no additional "rebalancing returns". For robustness we

[9] Annex 1 shows that even though the portfolio variance shifts over time for an unrebalanced portfolio, when compounding over multiple short periods equation 1 is still a good approximation for the whole-horizon expected GM as a function of the average expected AM and average variance over this horizon. Using monthly (rather than continuous) compounding is inherently an approximation, but these results show that in these simulations it is a very good approximation. The average GMs of our simulated portfolios differ by a maximum of only 0.8 basis points from those implied by equation (4).

repeated the simulations using asset returns drawn from uniform and t distributions. These show the same result: That portfolios with the same variance generated the same expected GM regardless of whether they are rebalanced.

[Figure 3 around here]

Fernholz and Shay (1982) state that a fixed weights portfolio "buys on a downtick and sells on an uptick", and Luenberger (1997) agrees that it will automatically "buy low and sell high". These authors claim that these effects boost profits even if asset returns follow a geometric Brownian motion, but Dempster et al. (2009) rightly note that any such profits are conditional on presumed negative autocorrelation of returns. Rebalancing will by construction sell some of an asset after a period in which it outperformed the rest of the portfolio, but this sale is only profitable if it takes place before a period (of whatever duration) of relative underperformance. It is worth noting that if there were such negative autocorrelation then rebalancing would increase the expected AM as well as the expected GM, but none of the proponents of rebalancing that we cite above claim such an increase in E[AM].

Similarly, Willenbrock (2011) argues that "the underlying source of the diversification return is the rebalancing", and Qian (2012) states that a "diversified portfolio, if left alone and not rebalanced, does not provide diversification return". These statements are incorrect. Rebalancing can be used to keep the portfolio at its minimum-variance weights and hence maximize the diversification return (minimizing volatility drag), but this does not imply that the diversification return will be zero in unrebalanced portfolios. Equation (9) and our simulations both clearly show that unrebalanced portfolios achieve higher expected GMs than their component assets as long as they retain some element of diversification. This is confirmed in Annex 2 where we derive expressions for the excess growth of unrebalanced portfolios.

Rebalancing will be profitable in markets which tend to mean-revert, and loss-making in markets which tend to show momentum (as assets which have underperformed in the past ─ and so are bought by the rebalancing strategy ─ tend to continue underperforming in the future, and vice versa). A large empirical literature has sought to establish the extent to which asset returns in various markets tend to show such predictable autoregressive properties. Adding significantly to this huge empirical literature is beyond the scope of the present paper. Instead our objective is to address the misleading claims that are made in the theoretical literature where there is assumed to be no such autoregression. This is important because these misleading claims are likely to encourage investors to pursue strategies which are inappropriate for the markets concerned.

4. Portfolios of One Risky Asset and One Risk-Free Asset

The previous section showed that even unrebalanced portfolios generate "rebalancing returns" (defined in this literature as expected growth rates that are greater than the weighted average of the expected growth rates of the component assets). We demonstrated this algebraically in complete generality, and by simulation for portfolios of assets with returns that are assumed IID (the assumption which is least favourable to unrebalanced portfolios) and not autocorrelated. In this section we return to a portfolio consisting of a risk-free deposit and a single risky asset with variance σ_a^2. This example is important, since it is widely used in the academic and practitioner literature and because the misleading conclusions drawn from it encourage investors to hold volatile assets so as to maximise the scale of the rebalancing trades.

In this section we show analytically that rebalanced and unrebalanced portfolios initially have equal expected growth rates. Second, using simulations, we show that the relative outperformance claimed for rebalancing over infinite investment horizons does not apply over finite horizons. Indeed, we show that expected portfolio final wealth for an unrebalanced portfolio is greater than for a rebalanced portfolio, so on this performance criterion an unrebalanced portfolio is superior.

The risk free and the risky asset are generally assumed to have identical expected growth rates (e.g. Fernholz and Shay, 1982, and Mulvey et al., 2007) and for simplicity we follow Dempster et al. (2007) and Qian (2012) in also normalising these rates to zero.[10] Under these assumptions the expected growth rate of a rebalanced portfolio which is 50% risky asset and 50% risk-free is shown to be $\sigma_a^2/8$. Thus two assets which each have zero expected growth combine to

[10] Without this normalisation, the AM and GM figures in Table 1 would all be increased by the risk-free rate. However the key result would remain unchanged: that the risky asset must by implication have E[AM] which is greater than the risk-free rate, and that this should be seen as the underlying source of the positive expected GM on the 50/50 portfolio, which has half the E[AM] of the risky asset but only one quarter of the volatility drag.

give positive expected growth in a rebalanced portfolio. This excess growth is interpreted as resulting directly from the rebalancing trades. As Fernholz and Shay (1982) put it:

> "...a balanced cash-stock portfolio will buy on a downtick and sell on an uptick. The act of rebalancing the portfolio is like an infinitesimal version of buying at the lows and selling at the highs. The continuous sequence of fluctuations in the price of the stock produces a constant accrual of revenues to the portfolio."

However, the authors explicitly assume a geometric Brownian motion in which the risky asset has an expected growth rate of zero, so after the price has diverged from its original value, the expected geometric return on any shares bought or sold at this new price is zero. This applies in every period, so any rebalancing trade shifts wealth from one asset into another which is as likely to outperform as underperform the asset it replaces. This shift will raise the expected portfolio growth rate if it improves diversification and so reduces volatility drag, but the language used by proponents of rebalancing strategies suggests that a very different effect is at work.

An alternative interpretation is suggested by deriving the size of this increase in expected growth without using any dynamic expressions, but instead merely using the standard arithmetic/geometric mean relationship ($E[GM] \approx E[AM] - \sigma^2/2$). As discussed above, this relationship captures the effects of portfolio diversification (volatility drag) on portfolio growth. This equation is of general applicability, and makes no presumption about rebalancing − it applies to the returns of both rebalanced and unrebalanced portfolios, and indeed to all positive numbers. In this case it tells us that the risk-free asset must have zero expected AM (since it is assumed to have zero GM and zero variance). The risky asset is assumed to have variance σ_a^2, but zero expected growth rate, so this equation tells us that it must have a positive expected AM of $\sigma_a^2/2$ (see Table 1). This positive arithmetic mean is generally not made explicit in discussions of this strategy, thus helping to maintain the impression that the expected geometric mean return on the 50:50 portfolio is caused by the rebalancing trades buying low and selling high.

Table 1: Expected AM and GM returns derived using E[GM]≈E[AM] - $\sigma^2/2$

	E[AM]	Variance	E[GM]
Risky asset	$\sigma_a^2/2$	σ_a^2	0
Risk-free asset	0	0	0
50:50 fixed weight portfolio	$\sigma_a^2/4$	$\sigma_a^2/4$	$\sigma_a^2/8$

The 50:50 portfolio has an expected AM equal to half that of the risky asset, but only one quarter of the variance. Thus it must have an expected growth rate of $\sigma_a^2/8$. Fernholz and Shay (1982) derive this result using stochastic calculus to model the dynamics of the portfolio, but it follows directly from the standard AM:GM relationship which applies for all assets and portfolios, regardless of whether they are rebalanced. This shows that a positive arithmetic mean return on the risky asset is vital if we are to see positive expected portfolio growth. For the 100% risky asset portfolio this positive expected AM is perfectly offset by volatility drag, leaving E[GM]=0. By contrast, the 50/50 portfolio has an expected AM which is half as large, but it suffers only one quarter of the volatility drag, leaving it a positive expected GM. The positive E[GM] of the

rebalanced portfolio can thus be explained entirely by the reduced volatility drag, rather than being evidence of the buy-low/sell-high effects which are claimed.

The fact that risky asset returns are assumed to follow a geometric Brownian motion suggests that volatility drag is the more appealing of these two explanations. Nevertheless, observational equivalence is still a potential problem here since the results in Table 1 require the asset weights to be constant at 50%, which requires rebalancing. To prove that there are no such buy low/sell high effects we need to consider explicitly the extent to which unrebalanced portfolios generate similar gains.

In practice if there is no rebalancing then the proportion π of the portfolio which is held in this single risky asset is likely to vary from one period to the next. E[AM] in any period varies in direct proportion to π, whilst the variance of the portfolio varies with π^2. Thus the expected GM in any period has a quadratic relationship[11] with π, with the maximum expected GM at $\pi = 0.5$ as shown in Figure 4.

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[11] $E[GM] = E[AM] - \sigma_p^2/2 = \pi\sigma_a^2/2 - \pi^2\sigma_a^2/2 = \pi(1-\pi)\,\sigma_a^2/2$. This result is derived by Qian (2012) and, in continuous time form, by Fernholz and Shay (1982). If there is only a single risky asset, with an expected GM equal to the risk-free rate, then the 50:50 fixed weight portfolio will indeed be the most attractive option for an investor who wishes to maximise the expected portfolio GM. But in practice there are likely to be alternative risky assets which are less than perfectly correlated. This allows clearly superior strategies to be constructed. If, for example, two or more assets have the same expected AM and variance then an unleveraged portfolio of them (however weighted) will have the same expected AM, but a lower variance, and thus a higher expected GM. Combining this multi-asset portfolio with a fixed π% of cash will (for any $\pi>0$) generate an expected GM which is greater than that shown in Figure 4.

[Figure 4 around here]

Rebalancing is required to maximise E[GM] by keeping the portfolio composition at 50/50. If π falls to zero then the portfolio is composed entirely of risk-free asset, with zero GM. If π rises to 1 then the portfolio consists entirely of the risky asset and volatility drag will completely offset the positive E[AM], leaving E[GM] zero. However, for short time horizons the proportion of the risky asset in the portfolio should not be expected to diverge substantially from its initial 50%, so the expected GM for an unrebalanced portfolio will be only slightly below that for the rebalanced portfolio. Annex 2 demonstrates that this results also holds in continuous time, with an unrebalanced portfolio which is initially 50% cash generating expected growth of approximately $\frac{\sigma^2}{8} - \frac{\sigma^4 t}{192}$, compared to $\frac{\sigma^2}{8}$ for the equivalent rebalanced portfolio. Thus expected growth for the unrebalanced portfolio is not the zero that is claimed. For short time horizons the $\frac{\sigma^4 t}{192}$ term will be negligible, showing that the rebalanced and unrebalanced portfolios initially have near-identical expected growth. This gradual divergence of expected growth rates for these two portfolios is inconsistent with the claim that one always generates rebalancing returns and the other never does. By contrast, it is entirely consistent with the different growth rates being due to the gradual increase in volatility (and hence volatility drag) as the unrebalanced portfolio's composition gradually drifts away from its initial equal weights.

If instead of following geometric Brownian motion the risky asset tends to revert to previous levels, then the rebalancing trades will generate additional profits that are not available to a B&H strategy. This can be illustrated using the simplest possible example of two consecutive periods of duration t over which the risky asset price evolves according to a geometric binomial distribution, either rising or falling by a factor $\sigma\sqrt{t}$. The portfolio is initially equally-weighted, with 0.5 in the risky asset and 0.5 in the (zero return) risk-free asset. If the risky asset rises in the first period, then the portfolio has $0.5(1+\sigma\sqrt{t})$ in the risky and 0.5 in the risk-free asset. The rebalancing trade then sells half of the first period gains on the risky asset (an amount $0.25\sigma\sqrt{t}$) in order to return to equal weights. If the risky asset price falls in the second period (by $\sigma\sqrt{t}$) then this trade generates a profit of $\sigma^2 t/4$ over the two periods, and this is the amount by which the rebalanced portfolio outperforms the unrebalanced portfolio (which simply returns to its starting value). The rebalancing strategy makes an identical profit if the risky asset falls in the first period and then rebounds in the second. Conversely, the rebalanced portfolio would underperform if the risky asset either rises in both periods (in which case the rebalancing trade sold some of this asset at the end of the first period before it continued to outperform in the second) or falls in both periods (so the rebalancing trade buys more risky asset at the end of period 1 before its price falls in the second). Thus over this two period horizon a rebalancing strategy generates greater terminal wealth than the B&H strategy at a rate of $\sigma^2/8$ per period, but this is strictly contingent on the assumption that the risky asset price reverts to exactly its starting value at the end of the horizon.

The same result can be derived if we initially assume that the price of the risky asset follows a standard geometric Brownian motion: $S_t = S_0 e^{\left(\mu - \frac{\sigma^2}{2}\right)t + \sigma W(t)}$. Without loss of generality we normalise $S_0=1$, and we follow Fernholz and Shay (1982) and other authors in assuming that the risky asset has zero expected growth rate, implying that $\mu=\sigma^2/2$ and $S_t = e^{\sigma W(t)}$. A portfolio P_r which is constantly rebalanced to keep 50% in the risky asset and 50% in the risk-free asset will always have half the upward drift rate and half the standard deviation of the risky asset itself ($\mu=\sigma^2/4$ and standard deviation $\sigma/2$), giving us $P_r = e^{\frac{\sigma^2}{8}t+\frac{\sigma}{2}W(t)}$.

Fernholz and Shay (1982) note that whenever the unrebalanced portfolio returns to its initial weights (i.e. $W(t)=0$, implying $S_t=S_0$), the rebalanced portfolio will have outperformed by a factor $e^{\frac{\sigma^2}{8}t}$ (since $P_r = e^{\frac{\sigma^2}{8}t}$ whilst the unrebalanced portfolio $P_u = 0.5 + 0.5S_t$ will merely have returned to its initial value of 1). Thus the rebalanced portfolio has 'excess growth' of $\sigma^2/8$ per period compared to the zero growth rate of each of its component assets, but again this is contingent on the assumption that the risky asset returns at the end of the horizon to exactly its starting value.

One reason for the apparent confusion in the literature may be that this equally weighted risky/risk-free asset portfolio generates two very different forms of return which happen to be of equal magnitude. First, we have the extra terminal wealth generated by the rebalancing trades if the risky asset reverts to its starting value. Second, as we saw above, we have the diversification return which means that the rebalanced portfolio generates an expected growth rate of $\sigma^2/8$ whereas each component asset has a growth rate of zero.[12] These two effects have the same magnitude, but they use different metrics. The first is an increase in *terminal wealth* (a move along the horizontal

[12] The assumption that $\mu=\sigma^2/2$ means that the risky asset has zero growth as the volatility drag perfectly offsets the positive drift μ, but the equally-weighted portfolio has half the drift ($\mu/2=\sigma^2/4$) and only one quarter of the volatility, implying volatility drag of only $\sigma^2/8$ and a portfolio growth rate of $\sigma^2/8$.

axis in Figure 1) whilst the second is an increase in the *growth rate* (the vertical axis). Moreover the first is contingent on the risky asset price reverting to its starting value, whilst the second is not.

Fernholz/Shay (1982) is still very widely cited in support of rebalancing strategies. It supports its claim that such strategies benefit from “buying on downticks and selling on upticks” by noting, as above, that every time that the B&H portfolio returns to equal proportions the corresponding rebalanced portfolio will have outperformed, and also noting that as $t \to \infty$ “this will occur infinitely often with probability one”. This statement is formally correct, and appears to imply that every rebalancing trade must eventually end up generating a profit, but this is far from true over the finite horizons that are likely to be of interest to investors. Figure 5 shows that for typical parameter values it takes several millennia for the probability that the rebalanced portfolio outperforms the unrebalanced portfolio to get anywhere close to unity. Over horizons of up to 100 years P_r outperforms P_u in less than 70% of our 100,000 simulations.[13]

[Figure 5 around here]

Furthermore, even though the proportion of paths which at some point return to $S_t = S_0$ tends to 1, each period the subset of paths which have not yet returned to S_0 will on average have diverged further from S_0 than in the previous period, so that a rebalancing trade made when it first diverged

[13] Gabay and Herlemont (2007) derive a closed form solution for the single risky asset case which shows similarly slow convergence to unity of the probability that the rebalanced portfolio outperforms.

from S_0 will on average be recording ever-larger losses.[14] It is only by taking both these subsets into account that we can make meaningful statements about the expected return on the rebalanced portfolio. Figure 6 shows that the unrebalanced portfolio P_u outperforms at both tails of the distribution. Rebalancing trades are profitable on average on paths where S_t tends to mean revert, leaving only small positive or negative cumulative returns. Conversely, when S_t makes large cumulative moves in either direction (i.e. tends instead to trend) then P_r underperforms P_u.

[Figure 6 around here]

The right tail of the distribution thus shows P_u generating more extreme outturns than P_r. Over time this tail represents a smaller and smaller probability space, but the average size of (P_u - P_r) in these cases keeps increasing. It is straightforward to demonstrate that the expected terminal wealth is in fact greater for the unrebalanced portfolio than the rebalanced.[15] This ever-more-

[14] Similarly, we have the standard result that the expected length of time required for a geometric Brownian motion with zero expected growth rate ($\mu=\sigma^2/2$) to converge to any arbitrarily distant level is infinite. The time to convergence for many paths may be very short, and the proportion of paths which reach this level inevitably rises over time, but the remaining paths that have not yet converged will on average have moved in the opposite direction.

[15] As above, the underlying asset $S_t = e^{\sigma W(t)}$ where *W(t)~N(0,1).* The upward skew of this lognormal distribution gives $\mathrm{E}\left[e^{\sigma W(t)}\right] = e^{\frac{\sigma^2}{2}t}$. Thus $E[P_u] = E[0.5 + 0.5S_t] = 0.5 + 0.5e^{\frac{\sigma^2 t}{2}}$. As above, the rebalanced portfolio $P_r = e^{\frac{\sigma^2}{8}t+\frac{\sigma}{2}W(t)}$, where $E\left[e^{\frac{\sigma}{2}W(t)}\right] = e^{\frac{\sigma^2}{8}t}$ and hence $E[P_r] = e^{\frac{\sigma^2 t}{4}}$. By inspection, the ratio $E[P_r]/E[P_u]$ clearly tends to zero as the time horizon increases, and our simulations confirm that the average terminal value is higher for unrebalanced portfolios. Cheng and Deets (1971) demonstrate a similar result in discrete time and we noted its antecedents in continuous time in the introduction.

extended, but ever-less-likely right tail explains why $E[P_u]>E[P_r]$ even though the probability that $P_r>P_u$ tends to 1 as time passes. It also tells us that over finite investment horizons it is very misleading to ignore this right tail by assuming (as Fernholz and Shay (1982) explicitly does) that P_r always outperforms P_u. This is a property that is only true for infinite investment horizons.

5. Conclusion

A sizeable theoretical literature has now developed concerning the effects of rebalancing. Within this literature it is widely claimed that rebalancing strategies automatically generate rebalancing returns by "buying low and selling high" even when asset returns show no predictable time structure. This paper demonstrates instead that the difference between the expected growth rates of rebalanced and unrebalanced portfolios of IID assets is entirely explained by portfolio diversification (volatility drag), with no evidence of any additional rebalancing returns. This paper also shows that the arguments used by key proponents of rebalancing strategies based on infinite horizons are not applicable over finite investor lifetimes.

We also demonstrate that all diversified portfolios generate expected growth rates greater than the average growth of their component assets (the definition of "excess growth" used in this literature). This contradicts the claims made by proponents of rebalancing strategies that such excess growth is the direct result of the process of rebalancing, and so is entirely absent from unrebalanced portfolios. By contrast, we show that unrebalanced portfolios initially generate the same expected growth as the corresponding rebalanced portfolios and these growth rates only diverge as the composition of the unrebalanced portfolio evolves.

The misleading arguments that are widespread in the literature have important implications, since they lead to a misinterpretation of the benefits of rebalancing. Specifically, they encourage investors to hold portfolios which are concentrated in volatile assets so as to increase the scale of the resulting rebalancing trades (e.g. “The pumping effect is obviously most dramatic when the original variance is high. After being convinced of this, you will likely begin to enjoy volatility, seeking it out for your investment rather than shunning it” Luenberger, 1997). Frequent rebalancing is likely to be costly due to transaction and market impact costs. Furthermore, the desire to maximise these transactions may push investors into sub-optimal asset allocations. Investors would be better advised to seek to minimize volatility drag by diversifying effectively and to rebalance no more than is necessary to keep their portfolio compositions adequately close to their target allocations.

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Annex 1

Fama and Booth (1992) show that the continuously compounded holding period return is well approximated by the expected return expressed in continuously compounded terms minus a fraction of the variance of the simple returns.

$$E\left[\log(1+r)\right] \approx \log(1+E[r]) - \frac{\sigma^2}{2(1+E[r])^2} \tag{A1}$$

This equation holds in each period, so we can sum each side over periods 1 to *T*:

$$\sum_{t=1}^{T} E\left[\log(1+r_t)\right] \approx \sum_{t=1}^{T} \log(1+E[r_t]) \;-\; \sum_{t=1}^{T} \frac{\sigma_t^{\;2}}{2(1+E[r_t])^2} \tag{A2}$$

Rearranging and dividing through by *T*:

$$\frac{1}{T} E[\log\left(\prod_{t=1}^{T}(1+r_t)\right)] \approx \frac{1}{T}\sum_{t=1}^{T} \log(1+E[r_t]) \;-\; \frac{1}{T}\sum_{t=1}^{T} \frac{\sigma_t^{\;2}}{2(1+E[r_t])^2} \tag{A3}$$

$$\approx E[r_t] - \frac{1}{2T}\sum_{t=1}^{T} \sigma_t^{\;2} \tag{A4}$$

If each period is short then $E[r_t]$ will be small and the continuously-compounded GM and AM above will be close to their more commonly-used discretely compounded equivalents. Thus we end up with a form of the standard relationship $E[GM] \approx E[AM] - \sigma^2/2$ which applies even if the distribution of r_t varies over time: the expected GM over the whole multi-period horizon is approximately equal to the average expected return over these periods minus half of the average variance. The linear relationships shown in Figures 2 and 3 confirm that this relationship holds for the unrebalanced portfolio, whose E[AM] and variance shift over time.

Annex 2: Expected Growth Rates Of Rebalanced and Unrebalanced Portfolios

(a) Single Risky Asset

The risky asset S follows $\frac{dS}{S} = \mu dt + \sigma dW \quad \Rightarrow S = e^{\left(\mu - \frac{\sigma^2}{2}\right)t + \sigma W(t)}$ where $\mu=\sigma^2/2$, $S_0 \equiv 1$. To keep the notation simple we suppress the t subscripts in the following.

The rebalanced portfolio P_r follows $\frac{dPr}{Pr} = \frac{\mu}{2}dt + \frac{\sigma}{2}dW \Rightarrow P_r = e^{\frac{\sigma^2}{8}t + \frac{\sigma}{2}W(t)}$

Thus generating expected growth rate: $\frac{1}{t}E[\log(P_r)] = \frac{\sigma^2}{8}$ (A5)

For the unrebalanced portfolio $P_u = 0.5 + 0.5S_t = 0.5 + 0.5e^{\sigma W(t)}$

$$\Rightarrow \log(P_u) = \log(0.5 + 0.5e^{\sigma W(t)}) \tag{A6}$$

A Taylor expansion of the exponent up to terms in $\sigma^4 W(t)^4$ gives:

$$\log(P_u) \approx \log\left(\frac{1}{2} + \frac{1}{2}\left(1 + \sigma W(t) + \frac{\sigma^2 W(t)^2}{2} + \frac{\sigma^3 W(t)^3}{6} + \frac{\sigma^4 W(t)^4}{24}\right)\right) \tag{A7}$$

A Taylor expansion of the log up to terms in $\sigma^4 W(t)^4$ gives:

$$\log(P_u) \approx \frac{\sigma W(t)}{2} + \frac{\sigma^2 W(t)^2}{4} + \frac{\sigma^3 W(t)^3}{12} + \frac{\sigma^4 W(t)^4}{48} - \frac{1}{2}\left(\frac{\sigma^2 W(t)^2}{4} + \frac{\sigma^3 W(t)^3}{4} + \frac{\sigma^4 W(t)^4}{16} + \frac{\sigma^4 W(t)^4}{12}\right) + \frac{1}{3}\left(\frac{\sigma^3 W(t)^3}{8} + \frac{3\sigma^4 W(t)^4}{16}\right) - \frac{1}{4}\left(\frac{\sigma^4 W(t)^4}{16}\right) \tag{A8}$$

We cannot rule out the possibility that over long horizons higher order terms will become significant, but for modest investment horizons (where w(t) is small) these can be safely ignored. Substituting in standard assumptions for the cumulative Wiener term in geometric Brownian motion $E[W(t)] \equiv E[W(t)^3] \equiv 0$, $E[W(t)^2]=t$, $E[W(t)^4] = 3t^2$:

$$E[\log(P_u)] \approx E\left[\frac{\sigma^2 W(t)^2}{8} + \sigma^4 W(t)^4\left(\frac{1}{48} - \frac{1}{32} - \frac{1}{24} + \frac{1}{16} - \frac{1}{64}\right)\right] = \frac{\sigma^2 t}{8} - \frac{\sigma^4 t^2}{192} \tag{A9}$$

Hence the expected growth rate is $\frac{1}{t}\mathrm{E}[\log(P_u)] \approx \frac{\sigma^2}{8} - \frac{\sigma^4 t}{192}$. For small horizons the $\frac{\sigma^4 t}{192}$ term will be negligible, so the expected growth of P_u is identical to that of P_r. This is inconsistent with the key claim made by proponents of rebalancing that P_r generates expected excess growth at a constant rate of $\sigma^2/8$ per period, whilst P_u has no rebalancing returns and hence zero excess growth (implying zero total growth in this case since both the risky and risk-free asset are assumed to have expected growth rates of zero). By contrast, the gradual decline of the expected growth rate of P_u is entirely consistent with our contention that this decline is due to increased volatility drag as the composition of the portfolio drifts away from the optimal 50:50 mix.

(b) <u>Portfolio Of N Risky Assets</u>

Each asset is assumed to follow a geometric Brownian motion $\frac{dS_i}{S} = \mu_i dt + \sigma_i dW$, implying $S_i = e^{\left(\mu_i - \frac{\sigma_i^2}{2}\right)t + \sigma W(t)}$ where the initial value of S_0 is normalized to 1. A <u>rebalanced portfolio</u> P_r gives equal weight to each of n such assets, so the growth rate of P_r is the weighted average of the growth rates of the component assets (we again suppress the t subscripts in the following; summations are over the n component assets):

$$\frac{dPr}{Pr} = \frac{1}{n}\sum \frac{dS_i}{S_i} = \sum \frac{\mu_i}{n} dt + \sum \frac{\sigma_i}{n} dW_i \qquad \text{(A10)}$$

These assets are assumed IID. The dW_i are standard N(0,1) normal variates and are independent, so $\Sigma dW_i \sim N(0,n)$ and the portfolio growth rate is distributed $N(\mu_i, \sigma_i^2/n)$. We can express this in terms of another standard N(0,1) variate dW:

$$\Rightarrow \frac{dPr}{Pr} = \mu_i dt + \frac{\sigma_i}{n}\sum dW_i = \mu_i dt + \frac{\sigma_i}{\sqrt{n}} dW \qquad \text{(A11)}$$

Thus application of Ito's lemma gives a result for P_r which is similar to S, and simply reflects the lower standard deviation $\sigma_i/\sqrt{n}$ of the rebalanced portfolio (and hence the reduced volatility drag).

$$\Rightarrow P_r = e^{(\mu_i - \frac{\sigma_i^2}{2n})t + \frac{\sigma_i}{\sqrt{n}}\sum W(t)} \qquad \text{(A12)}$$

$$\Rightarrow \frac{1}{t} E[\log(P_r)] = \mu_i - \frac{\sigma_i^2}{2n} \tag{A13}$$

The <u>unrebalanced portfolio</u> $P_u = \frac{1}{n}\sum S_i = \frac{1}{n}\sum e^{(\mu_i - \frac{\sigma_i^2}{2})t + \sigma_i W_i(t)}$

However, these assets are assumed *IID*:

$$\Rightarrow P_u = e^{(\mu_i - \frac{\sigma_i^2}{2})t} \frac{1}{n}\sum e^{\sigma_i W_i(t)} \tag{A14}$$

$$\log(P_u) = \left(\mu_i - \frac{\sigma_i^2}{2}\right) t + \log\left(\frac{1}{n}\sum e^{\sigma_i W_i(t)}\right) \tag{A15}$$

Taylor expansion of the exponent up to the fourth power gives:

$$\log(P_u) \approx \left(\mu_i - \frac{\sigma_i^2}{2}\right) t + \log\left(\frac{1}{n}\sum\left(1 + \sigma_i W_i(t) + \frac{\sigma_i^2 W_i^2(t)}{2} + \frac{\sigma_i^3 W_i^3(t)}{6} + \frac{\sigma_i^4 W_i^4(t)}{24}\right)\right) \tag{A16}$$

Taylor expansion of the log term to the fourth power gives:

$$\log(P_u) \approx \left(\mu_i - \frac{\sigma_i^2}{2}\right) t + \frac{\Sigma\sigma_i W_i(t)}{n} + \frac{\Sigma\sigma_i^2 W_i^2(t)}{2n} + \frac{\Sigma\sigma_i^3 W_i^3(t)}{6n} + \frac{\Sigma\sigma_i^4 W_i^4(t)}{24n} - \frac{1}{2}\left(\frac{\left(\Sigma\sigma_i W_i(t)\right)^2}{n^2} + \frac{\left(\Sigma\sigma_i^2 W_i^2(t)\right)^2}{4n^2} + \frac{\left(\Sigma\sigma_i W_i(t)\right)\left(\Sigma\sigma_i^2 W_i^2(t)\right)}{n^2} + \frac{\left(\Sigma\sigma_i W_i(t)\right)\left(\Sigma\sigma_i^3 W_i^3(t)\right)}{3n^2}\right) + \frac{1}{3}\left(\frac{\left(\Sigma\sigma_i W_i(t)\right)^3}{n^3} + \frac{3\left(\Sigma\sigma_i W_i(t)\right)^2\left(\Sigma\sigma_i^2 W_i^2(t)\right)}{2n^3}\right) - \frac{1}{4}\left(\frac{\left(\Sigma\sigma_i W_i(t)\right)^4}{n^4}\right) \tag{A17}$$

Note that for this standard Weiner process E[W_i(t)]≡E[W_i^3(t)]≡0. Hence E[off-diagonal terms]=0 so, for example, E[(ΣW_i(t))^2]= E[ΣW_i^2(t)]=nt:

$$\mathrm{E}[\log(P_u)] \approx \left(\mu_i - \frac{\sigma_i^2}{2n}\right) t + \mathrm{E}\left[\frac{\Sigma\sigma_i^4 W_i^4(t)}{24n}\right] - \frac{1}{2}\,\mathrm{E}\left[\frac{\left(\Sigma\sigma_i^2 W_i^2(t)\right)^2}{4n^2} + \frac{\left(\Sigma\sigma_i W_i(t)\right)\left(\Sigma\sigma_i^3 W_i^3(t)\right)}{3n^2}\right] + \mathrm{E}\left[\frac{\left(\Sigma\sigma_i W_i(t)\right)^2\left(\Sigma\sigma_i^2 W_i^2(t)\right)}{2n^3}\right] - \frac{1}{4}\mathrm{E}\left[\frac{\left(\Sigma\sigma_i W_i(t)\right)^4}{n^4}\right] \tag{A18}$$

Noting that $E[W_i^4(t)]=3t^2$ for this standard Wiener process and simplifying:

$$\mathrm{E}[\log(P_u)] \approx \left(\mu_i - \frac{\sigma_i^2}{2n}\right)t + \frac{\sigma_i^4 t^2}{8} \; - \frac{\sigma_i^4}{2}\left(\frac{3t^2}{4n} + \frac{(n-1)t^2}{4n} + \frac{t^2}{n}\right) + \; \sigma_i^4\left(\frac{3t^2+(n-1)t^2}{2n^2}\right) - \frac{\sigma_i^4}{4}\left(\frac{3nt^2+3n(n-1)t^2}{n^4}\right) \qquad \text{(A19)}$$

$$\Rightarrow \quad \frac{1}{t}\mathrm{E}[\log(P_u)] \approx \left(\mu_i - \frac{\sigma_i^2}{2n}\right) + \frac{\sigma_i^4 t}{4n}\left(\frac{1}{n} - 1\right) \qquad \text{(A20)}$$

For small t the $\sigma_i^4 t$ term is negligible, so the unrebalanced and rebalanced portfolios have the same expected growth rate: $\frac{1}{\mathrm{t}}\mathrm{E}[\log(P_u)] = \frac{1}{\mathrm{t}}\mathrm{E}[\log(P_r)] = \left(\mu - \frac{\sigma_i^2}{2n}\right)$. Thereafter the $\sigma_i^4 t$ term becomes significant so $\mathrm{E}[\log(P_u)] < \mathrm{E}[\log(P_r)]$. An intuitive interpretation is that P_u and P_r initially have equal growth rates since they start with identical composition. Thereafter the arithmetic mean of the two portfolios remains identical (since the underlying assets are assumed IID), but P_u becomes less well diversified over time, so it suffers from increasing volatility drag. But the claim that P_u never shows any excess growth is not true over finite horizons.

FIGURE 1

Illustration of Volatility Drag

This chart shows how volatility in terminal wealth results in a lower expected growth rate, in accordance with equation (4). A zero-volatility terminal wealth outturn of A generates a growth rate B, but although two equally likely outturns equidistant (in terms of terminal wealth) above and below A would generate the same expected terminal wealth, they would generate lower expected growth C because the concavity of the log function effectively penalises both exceptionally high and exceptionally low outturns.

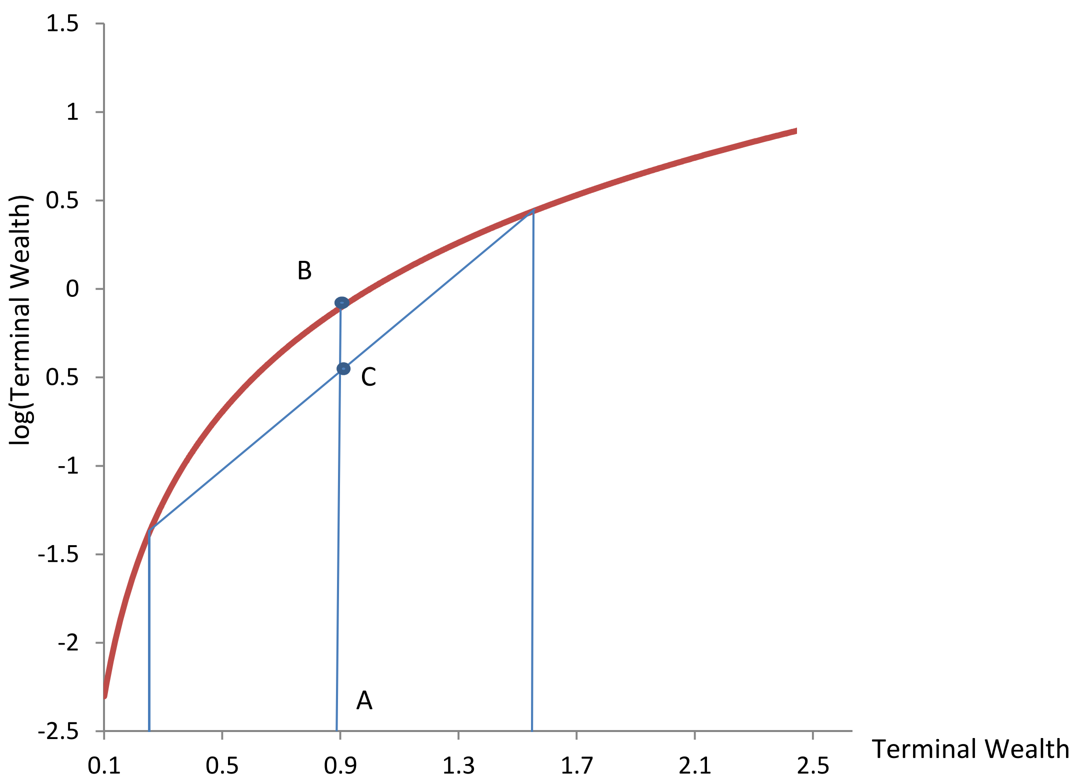
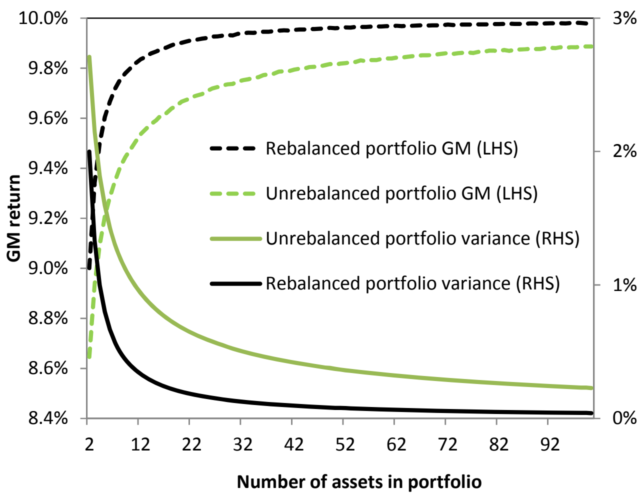


FIGURE 2

Rebalanced and unrebalanced portfolios – varying number of assets

These charts show the average annualized GM and variance of rebalanced and unrebalanced portfolios comprising different numbers (from 2 to 100) of component assets. The unrebalanced portfolios start with equal weights in the chosen number of assets, but these weights are then allowed to evolve in line with relative asset returns. For the rebalanced portfolio the weights are returned to equality at the end of each period. Asset returns are assumed normal, IID and not autocorrelated, with annualized arithmetic mean 10% and variance 4%. Each path is calculated over 100 years for 10,000 simulated paths.

Panel A



Panel B

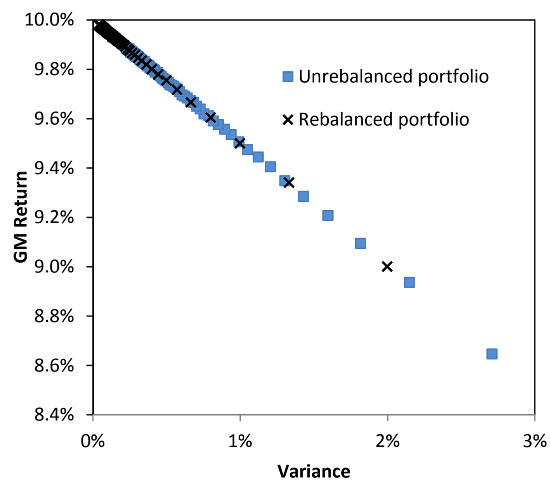
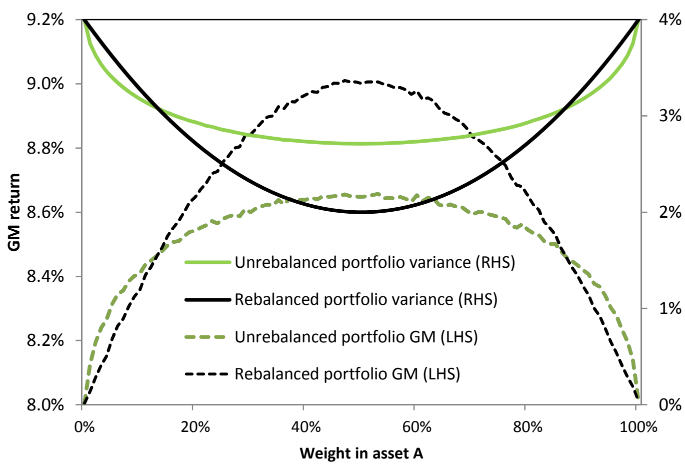


FIGURE 3

Rebalanced and Unrebalanced Portfolios – Varying Initial Portfolio Weights

These charts show the average annualized GM and variance of portfolios comprising two assets. The unrebalanced portfolios initially start with the weight shown for asset A, but the weights are then allowed to evolve in line with relative asset returns. For the rebalanced portfolio the weight of asset A is returned at the end of each period to its initial value. Initial portfolio weights are given 101 different values (from 0% to 100% asset A) for both rebalanced and unrebalanced portfolios – a total of 202 variants. Asset returns are assumed normal, IID and not autocorrelated, with annualized arithmetic mean 10% and variance 4%. Each path is calculated over 100 years for 10,000 simulated paths.

Panel A



Panel B

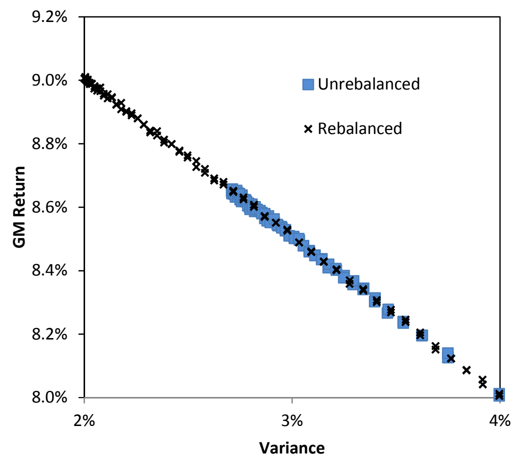


FIGURE 4

Volatility Pumping – Single Risky Asset

This chart shows the arithmetic and geometric means of a portfolio comprising a risk-free asset (zero AM return and zero variance) and a risky asset (expected AM return 2%, and variance 4%) which is rebalanced to keep the proportion invested in the risky asset fixed at the indicated percentage.

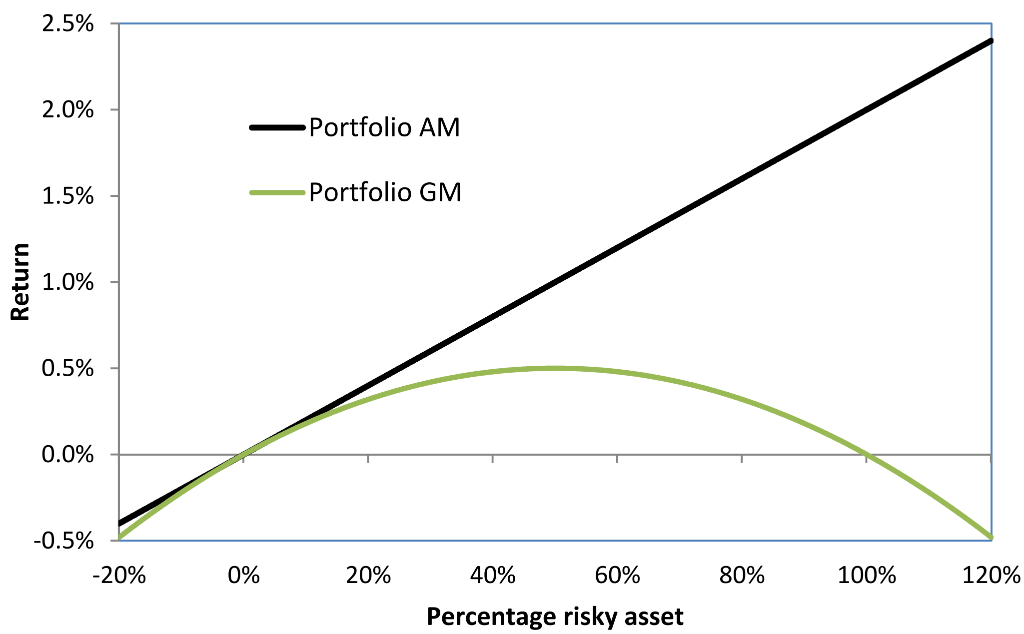


FIGURE 5

Proportion of Outcomes where $P_r > P_u$

This chart shows the proportion of the simulated paths for which the terminal wealth of the rebalanced portfolio P_r is greater than the unrebalanced portfolio P_u. This proportion is shown for simulations with a wide range of different time horizons. We follow Fernholz and Shay (1982) in assuming asset returns follow a geometric Brownian motion with zero expected geometric return. We consider two portfolios with starting value $1. Both invest $0.50 in a risk-free asset which has an interest rate of zero and $0.50 in the risky asset. The rebalanced portfolio rebalances back to 50/50 asset mix every month. We assume σ is 10% per annum for the risky asset (other simulations, not reported here, show our results are robust to alternative assumptions).

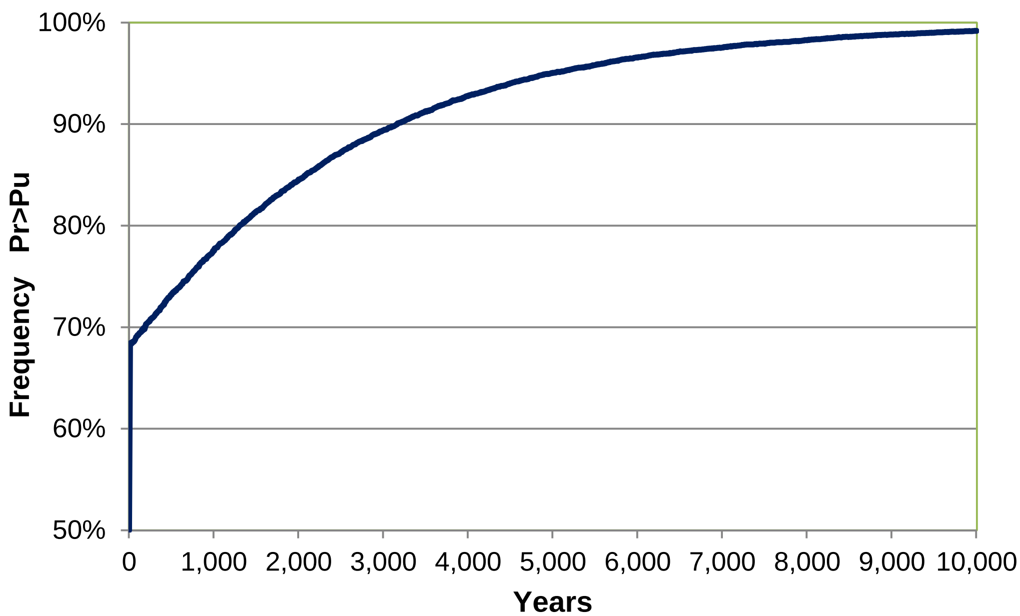


FIGURE 6

Probability Density of the Terminal Wealth of Rebalanced and Unrebalanced Portfolios after 100 Years

The chart shows the distribution of terminal wealth over 100,000 simulated paths of a 100 year time period for rebalanced and unrebalanced portfolios which are initially identical. The parameters of the simulated values are the same as for Figure 5.