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Abstract

We derive the optimal contract in a two period costly state verification model with repeated loans, where in each period, the borrower invests in an identical project. We allow the borrower to switch lenders at the end of the first period. We show that while the second period optimal contract continues to be a standard debt contract, the optimal contract for the first project need not be. Regardless of the form of the first period contract, there is less monitoring in the first period and total monitoring costs are strictly lower, relative to a sequence of short term contracts. We illustrate our results assuming a uniform distribution for firm revenue and fixed monitoring costs. In particular, we show that either there is no monitoring in the first period or maximum possible amount consistent with the outside option is collected in the second period.

1 Introduction

This paper examines the nature of the optimal contract in a two period model with costly state verification and possibility to reinvest the residual from the first period in a second, identical project. The static environment is essentially that of Gale and Hellwig (1985). That is, an entrepreneur borrows from a lender to invest in a project that yields random returns at the end of one period. These returns are privately observed by the entrepreneur but can be observed by the lender at a cost, known as monitoring or verification cost. The cost is possibly a function of revenue. The entrepreneur reports revenue and the lender decides whether to monitor and the borrower repays the prespecified amount corresponding to the reported revenue. Gale and Hellwig (GH from now on) show that the optimal contract is a standard debt contract (SDC) in which the lender sets the repayment to be a fixed amount and monitors if and only if the borrower reports a revenue less than this fixed amount. In states that are monitored, the lender seizes the entire revenue but incurs monitoring costs. We assume that the entrepreneur is faced with an identical project in the second period. A second loan may or may not be needed depending on the realization of revenue in the first period. If there is a second loan, the sequence of events of the first period is repeated. Thus, the borrower reports revenue at the end of the first period, repays according to the initial contract and then invests again in another, identical project. We assume that the revenues of the two projects are independently and identically distributed. If a second loan is not needed, the entrepreneur self-finances the project and there is no second contract. We also assume that at date 1, the borrower has the option to borrow from a different lender. The loan market is competitive so that the lender maximizes the borrower's expected payoff subject to various constraints.

The aim of the paper is to derive the optimal contract between the lender and the borrower in a dynamic setting with asymmetric information and uncertainty. In particular, we want to examine whether the GH result that the static optimal contract is an SDC holds in a dynamic framework and how the corresponding total monitoring costs change. We show that the first period optimal contract need not be an SDC while the second period optimal contract must be. Further, we show that the first period repayment function is somewhat irrelevant over the range of unmonitored states where a second loan is needed, except at the limits. This is because the lender breaks-even over two periods and thus, a higher (lower) repayment simply offsets a higher loan for the next period. We find that adding the second period, even though the two periods' projects are independent, weakens the incentive constraint of the borrower in the first period and thus, changes the properties of the first period contract. Intuitively, this occurs because of the need to borrow again. If the borrower reports a lower revenue than the true level, it must borrow a larger amount than needed and thus, agree to a larger expected repayment in the second period than needed.

One noteworthy feature of our two period setting is that the definition of feasibility of a report changes. In the static setting, a report is feasible as long as the corresponding repayment is less than the actual revenue. Thus, a borrower can report a higher revenue as long as the required repayment is less than the actual revenue. In our setting, this condition is not sufficient for feasibility. Indeed, reporting a higher revenue is possible only if the borrower intends to take the money and run, and not borrow again. The reason is simple - if the borrower reports a higher revenue than actual, it obtains a lower loan than needed and thus cannot undertake the second project. This has an important implication for determining optimal contract. Unlike the static setting, here it is non-trivial to show that the incentive constraint binds.

We show that the optimal dynamic contract entails less monitoring in the first period, relative to the sequence of two short term GH contracts. Since this sequence is also feasible, we conclude that the dynamic contract strictly dominates the sequence of GH contracts, and lowers the total expected monitoring costs. Further, the total net expected repayment at any state that is not monitored at date one is constant. This constant is defined relative to the amount the bank decides to collect in the second period if the borrower is monitored in the first period. In particular, if this amount is the cost of the project, the constant payment in the first period is simply the threshold revenue as in GH. However, we show that this is not optimal. The first period optimal contract entails monitoring up to a threshold, with full recovery and a fixed amount above the threshold when the second loan becomes unnecessary. However, between these two thresholds, the repayment function need satisfy only the condition that it is sufficiently high so that a second loan is needed. However, despite this indeterminacy, because of the second loan, it is as if the borrower repays what it reports and receives the full amount of the loan. This is also true in the monitored states. We illustrate our results by assuming that firm revenue is uniformly distributed and the monitoring cost is fixed. Further, we show in this example that there is less monitoring in the first period than in the static contract. Thus, a dynamic relationship between the lender and the borrower allows leniency in the first period. Indeed, for some parameter values, there is no monitoring in the first period.

In most of our analysis, we assume that the borrower's decision to stay with the current bank is observed by the bank. Further, if the borrower chooses to leave, we assume that the bank monitors and seizes all revenue. When this assumption is relaxed, so that the bank is either unable to observe and thus unable to punish, or simply unable to punish, results change in an intuitive way. The bank is no longer able to be lenient in the first period. Indeed, the optimal dynamic contract in this case is exactly the sequence of two GH contracts. Thus, dynamics lead to no advantage if the borrower and lender cannot enter into a long term relationship. In the intermediate case, where the bank can observe whether the entrepreneur borrows again from it but not whether there is another simultaneous borrowing from another bank, we show that our results from the observable case continue to hold.

We also show that if the second period loan is different and small, it is possible for the lender to break even if a dynamic contract is allowed but not if only a sequence of contracts is allowed. However, this is possible only if the borrower is punished for leaving at date one.

Costly state verification (CSV) models have received some attention in the literature, since the seminal work of GH and Townsend (1979). Renou (2008) studies multi-lender coalitions in a static CSV framework. In a recent paper, Monnet and Quintin (2005) examine dynamic optimal contracts in a CSV framework[1]. A key difference between this work and their work is the presence of savings. Thus, in their model, all output is consumed in each period. This in our view is a restrictive assumption especially in the context of banking, where savings are a fundamental source of bank deposits and loans. Indeed, once savings are allowed, their main result, Theorem 1, does not provide much insight into properties of optimal contracts as there can be many contracts satisfying that result, including the ones that are not optimal. Further, in their model, the bilateral relationship between a principal and agent is given, with no possibility of the agent walking away whereas in our framework, the extent of the relationship in the second period depends on the first period contract.

Our paper is closely related to Chang (1990). He considers one project that yields cash flows over two periods and there is no second loan or project.

[1] Also see Smith and Wang (1998) in the context of dynamic risk sharing.

Further, he assumes in the main analysis that the residual from first period is available for the lender to seize in the second period. That is, the residual cannot be consumed nor used for other loans and thus, there is no limited liability at date 2. He then argues that the optimal contract essentially remains the same if the borrower were to consume all of the residual in the first period and the borrower prefers to enter into a covenant with the lender wherein such consumption is prohibited. This is not the case in our model since the second loan is associated with a second project and it is strictly better to invest in this project rather than consume or put it aside for avoiding bankruptcy. We argue that the scenario analyzed by Chang is rather restrictive since the borrower must commit to putting aside the residual from the first period. In addition, despite some similarity in our first period optimal contracts, our model captures the dynamic nature of loan contracts. Thus, by adding a second loan as well as allowing the borrower to switch lenders at date 1, our work enriches and complements Chang's analysis while at the same time allows us to compare the sequence of GH contracts with the dynamic one. Webb (1992) considers an environment similar to ours. There is one project at each date. We extend this work in several ways. First, we allow the borrower to switch lenders at date 1. Given that the loan market is competitive, and there is no commitment, the borrower ought to be able to choose who to borrow from at each date. This has important effects on the optimal contract. Second, we allow the lender to break even over two time periods. This again is natural to assume given the two period relationship between the lender and borrower. We also characterize the solution.

Our paper also contributes to the wider literature on dynamic loan contracts. Bolton and Scharfstein (1990) study a two period environment in which outcome of the first report determines the probability with which the borrower gets another loan. They show that the threat to cut-off funding in the second period induces the borrower to report truthfully in the first. Thus, dynamics weaken the incentive problem of the borrower. However, their main concern is the issue of predation.

The remainder of the paper is as follows: the model is presented in section 2; date 0 problem is analyzed in section 3, a special case is analyzed in section 4. The example is presented in Section 5. The Appendix contains some proofs.

2 Model

There are two time periods. An entrepreneur with no wealth needs to borrow L for the first project and may need to borrow for the second if the residual from the first project is not sufficient to self-finance. We denote the second loan by $l \leq L$, taken at date 1. Note that L is the investment needed at each date. The loan market is competitive. Thus, the lender is willing to lend provided the expected repayment over two periods exceeds the total loan amount over two periods. Let $z_t, t = 1, 2$, denote the random revenue realized at date t, assumed independently and identically over the two periods with a continuous density function $f(.)$, whose support is $[0, H]$. It is assumed that Ez_t is greater than L, so that the project is profitable. We also assume that these returns are private information of the borrower/entrepreneur. The lender can observe it only at a cost, given by $c(z_t)$. We assume that this function is continuous and differentiable.

At date 1, z_1 is realized. The entrepreneur observes z_1 and repays according to $R_1(z_1)$. Monitoring occurs according to $M_1(z_1)$, which is either 0 (no monitoring) or 1 (monitoring). The entrepreneur borrows again if $z_1 - R_1(z_1) < L$, receives the expected return of $E(z_2)$ and repays according to $R_2(z_1, z_2, l)$ and gets monitored according to $M_2(z_1, z_2, l)$. If $z_1 - R_1(z_1) \geq L$, the entrepreneur self-finances. Let $\bar{z}$ denote the lowest revenue at which the entrepreneur self-finances. We make the following assumption on the repayment function to ensure that monitoring occurs in any one period. In the static setting, this is trivially met since the borrower always reports the lowest realization in the absence of monitoring.

Assumption 1: If there is no monitoring, $ER_1(z_1) < L$.

The entrepreneur has the option of borrowing from another lender at date 1, since the loan market is competitive. In the benchmark model, we assume that the bank monitors if the borrower does not borrow again even though the reported revenue is such that a second loan is needed. Thus, we assume that the bank punishes the borrower for not borrowing again from it.[2]

3 Date 0 Problem

We first state the static GH problem: the bank maximizes $E(z - R(z))$ subject to $R(z) \leq z, z - R(z) \geq z - R(z'), \forall z'$ such that $M(z') = 0$, and $R(z') \leq z$, and the bank breaks-even, that is, $E(R(z) - c(z)) = L$. GH show that the optimal contract is an SDC in which $R(z) = \widehat{z}$ and $M(z) = 0$ for all $z \geq \widehat{z}$ and $R(z) = z$ and $M(z) = 1$ for all $z < \widehat{z}$. That is, low states are monitored at a cost; there is full recovery in such states and repayment is a constant in the unmonitored states, the constant being equal to the threshold of monitoring.

The sequence of two short-term GH contracts amounts to the bank lending L to the firm at date 0 and $L - (z - R(z))$ at date 1. The bank breaks even per period and does not take into account the possibility that the same firm may borrow from it again. That is, the bank treats the relationship as a one period relationship. In this case, the maximization problem in each period is identical to the static problem, except that in the second period, the loan amount is different (lower in the solvent states). Thus, the first period monitoring threshold remains $\widehat{z}$ while the second period threshold decreases with the first period z because the second loan equals $L - (z - \widehat{z})$.

[2]The borrower has the option to underreport and borrow again from the bank to escape this punishment. This is already covered by the incentive constraint IC1 below. The borrower may also consider using the residual to borrow from another lender but it can be shown that given only one project, that option is strictly inferior to staying with the bank.

We now study the dynamic contracting problem in which the bank enters into a long term relationship with the firm, taking into account the possibility that the firm has an incentive to switch lenders at date 1. We first consider the scenario in which the bank observes what the firm does at date 1 and then examine the unobservable case.

It is straightforward to show that the last period's problem is similar to the one period problem of GH. That is, given z_1 and the loan l, the bank decides on the threshold $z_{2d}(z_1, l)$ such that it monitors if $z_2 < z_{2d}(z_1, l)$ and $R_2(z_2, z_1, l) = z_2$ and does not monitor if $z_2 \geq z_{2d}(z_1, l)$ and $R_2(z_2, z_1, l) = z_{2d}(z_1, l)$. Then, the expected revenue is

$$E\left(R_2|(z_1, l)\right) = \int_0^{z_{2d}(z_1,l)} z_2 dF + z_{2d}(z_1, l)\left[1 - F(z_{2d}(z_1, l))\right]$$

and the expected monitoring cost is a function of the expected revenue $E\left(R_2|(z_1, l)\right)$,

$$C_{z2}(z_{2d}(z_1, l)) = \int_0^{z_{2d}(z_1,l)} c\left(z_2\right) dF = C_{R2}(E\left(R_2|(z_1, l)\right))$$

A difference from the static model is that the second period problem does not require the bank to break even in the single period. Thus, $E\left(R_2|(z_1, l)\right) - C_{R2}(E\left(R_2|(z_1, l)\right))$ need not equal l. Instead, the bank fixes $E\left(R_2|(z_1, l)\right)$, determines $z_{2d}(z_1, l)$, taking into account the fact that the contract must be a standard debt contract, and then computes the expected monitoring cost. *Note that C_{R2} is an increasing function of ER_2.*

In the first period, the entrepreneur receives z_1, makes a report to the lender and asks for a second loan of $l \geq 0$. Then, assuming truthful reporting for the time being, the first period payoff to the entrepreneur, conditional on z_1 and l, is,

$$z_1 - R_1(z_1) - [L - l].$$

In the 2nd period, conditional on z_1 and l, the entrepreneur pays to the bank, in expected value, $E\left(R_2|(z_1, l)\right)$. Therefore, the expected profit in the 2nd period is $Ez_2 - E\left(R_2|(z_1, l)\right)$. Therefore, the total expected profit of the entrepreneur in two periods, conditional on z_1 and l, is

$$V(z_1, l) = z_1 - R_1(z_1) - L + l + Ez_2 - E\left(R_2|(z_1, l)\right).$$

The break-even constraint: The bank's two period expected profit conditional on z_1 and l, is,

$$\pi(z_1, l) = R_1(z_1) - l + E\left(R_2|(z_1, l)\right) - c(z_1 M_1(z_1)) - C_{R2}(E\left(R_2|(z_1, l)\right)) - L$$

Substituting the bank's profit equation into the entrepreneur's utility function, we obtain,

$$V(z_1, l) = z_1 + Ez_2 - \pi(z_1, l) - c(z_1 M_1(z_1)) - C_{R2}(ER_2|(z_1, l)) - 2L$$

Because the bank breaks even, $E(\pi(z_1, l)) = 0$, and thus, by taking expectation over z_1, we obtain

$$EV = Ez_1 + Ez_2 - Ec_1 - EC_{R2} - 2L \tag{1}$$

where Ec_1 denotes expected monitoring costs in the first period and EC_{R2} denotes expected monitoring costs in the second period. *Because C_{R2} is increasing in $E(R_2|(z_1, l))$*, loan l should be chosen so as to minimize $E(R_2|(z_1, l))$. The break-even constraint implies that the higher the loan l, the higher has to be the revenue. Therefore, the bank induces the entrepreneur, who reports z_1, to ask for the lowest loan possible, $l^*(z_1) = L + R_1(z_1) - z_1$.[3] Thus, we now drop the argument l, for convenience.

The bank designs the two-period contract in order to maximize (1), or equivalently, minimize the sum of expected monitoring costs, subject to,

Feasibility constraints:

$$z_1 - R_1(z_1) \geq 0,$$

Dynamic Incentive compatibility constraints:

IC1: Suppose the entrepreneur stays with the same bank at date 1. Then,

$$Ez_2 - E(R_2|z_1) \geq Ez_2 - E(R_2|z') + z_1 - z',$$

for all z_1, $z_1 > z'$ such that $z_1 - R_1(z_1) < L$ and z' is not observed. Note that, $z_1 - R_1(z') \geq 0$, by feasibility. The condition $z_1 > z'$ is required because otherwise, the lie is not feasible.[4] To see this, note that if $z' > z_1$, where z_1 is the true state, overreporting implies that the firm must borrow $L + R_1(z') - z' < L + R_1(z') - z_1$, the amount of loan needed. Thus, the firm is unable to carry out the second project if it overreports, and therefore, unlike the static GH or Chang (1990), in our model, we do not need the corresponding incentive constraint. The LHS of IC1 incorporates the fact that the borrower reinvests the residual from the first period. We can show that it is better for the borrower to do so than consume. The RHS assumes that the borrower consumes the residual $z_1 - z'$ because there is no other investment opportunity available.

Simplifying IC1 leads to,

$$E(R_2|z_1) \leq E(R_2|z') + z' - z_1, \tag{2}$$

[3] Suppose in the first period, the entrepreneur reports z and asks for the loan l. The minimum loan in this case is $l^*(z) = L + R_1(z) - z$, and thus, by setting up $E(R_2|(z, l))$ to be

$$E(R_2|(z, l)) = E(R_2|z) + f(z, \Delta l), \Delta l = l - l^*(z)$$

where

$$f_{\Delta l}(x, \Delta l) \geq 1,$$

then, the entrepreneur, given report z, will ask for a loan of $l^*(z) = L + R_1(z) - z$.

[4] Notice that this is slightly different from the intertemporal incentive compatibility constraint of Chang (1990), even when he allows for consuming all the residual.

for all $z_1 > z'$, such that $M_1(z') = 0$.

IC2: This constraint prevents the borrower from going to another lender. Let

$$Ec(l) \equiv \int_0^{\hat{z}(l)} c(z_1) dF(z)$$

where $\hat{z}(l)$ solves,

$$\int_0^{\hat{z}} (z - c(z)) dF(z) + \int_{\hat{z}}^{H} \hat{z} dF(z) = l$$

Let

$$ER(l) = \int_0^{\hat{z}(l)} (z) dF(z) + \int_{\hat{z}(l)}^{H} \hat{z} dF(z)$$

be the expected repayment corresponding to a loan of l in the static model. Note that $ER(l) = l + Ec(l)$. We shall also use the notation introduced earlier for date 2, namely, $C_{R1}(ER)$. Given $ER, \hat{z}$ is a function of ER, and thus, expected monitoring costs is a function of ER. Indeed, this function is the same as derived for date 2 and thus, where convenient, we shall drop the reference to the time period.

Now, $Ec(L)$ is the expected cost of monitoring in a one-period GH contract with loan L. Then IC2 is,

$$Ez_2 - E(R_2|z_1) \geq -L + Ez_2 - Ec(L)$$

for all z_1 that are not monitored (provided the borrower stays with the current lender) and for which a second loan is needed. The RHS presumes that if the borrower leaves, the lender monitors, seizing everything so that the borrower must borrow the full amount from another lender. Note that the inequality also holds for states that are monitored, in order for the bank to keep the borrower for the second period. This constraint simplifies to,

$$E(R_2|z_1) \leq L + Ec(L)$$

Suppose that z_d is the lowest state that is not monitored.[5] IC1 implies that $E(R_2|z)$ is decreasing in z. Then IC1 and IC2 imply that

$$\begin{aligned} E(R_2|z_1) &\leq L + Ec(L), z_1 \leq z_d && (3) \\ E(R_2|z_1) &\leq E(R_2|z') + z' - z_1, z_1 \geq z' \geq z_d, z_1 - R_1(z_1) < L. && (4) \end{aligned}$$

IC3

So far, we assumed that the unmonitored states were such that a second loan was needed. If z_1 is high enough so that $z_1 - R_1(z_1) \geq L$, a second loan is not needed. The incentive constraints are different for this case. Here our

[5] We show later that this must be the case in an optimal contract given the assumptions on the monitoring cost function.

assumption is that if the reported state is such that a second loan is not needed, the bank does not monitor. We show later that there exists a value $\bar{z}$, such that for all $z_1 \geq \bar{z}$, a second loan is not needed so that the IC constraint for this type, for $z_d \leq z' < \bar{z}$ is:

$$z_1 - R_1(z_1) - L + Ez_2 \geq z_1 - R_1(z') - (z' - R_1(z')) + Ez_2 - E\left(R_2|z'\right)$$

$$R_1(z_1) \leq z' + ER_2|z' - L \tag{5}$$

If z' is also above $\bar{z}$, then for all $z_1, z' \geq \bar{z}$, we need,

$$z_1 - R_1(z_1) - L + Ez_2 \geq z_1 - R_1(z') - L + Ez_2$$

Thus,

$$R_1(z_1) = R_1(z')$$

for all z_1, z' above $\bar{z}$.

To prevent $z < \bar{z}$, such that $z - R_1(\bar{z}) \geq 0$, from pretending to be $\bar{z}$, we must have,

$$Ez_2 - E(R_2|z) \geq \max[z - R_1(\bar{z}), Ez_2 - ER_2(L - z + R_1(\bar{z}))] = Ez_2 - ER_2(L - z + R_1(\bar{z}))$$

The first term in the max operator is the amount obtained by overreporting and not operating the second project and the second term is the amount from overreporting and borrowing from another lender. It can be shown that the project is profitable enough to ensure the last equality. By the one-period break-even constraint of the other lender, we obtain,

$$L - z + R_1(\bar{z}) + Ec(L - z + R_1(\bar{z})) \geq E(R_2|z)$$

$$R_1(\bar{z}) \geq E(R_2|z) + z - L - Ec(L - z + R_1(\bar{z}))$$

for all $z < \bar{z}$ such that $z - R_1(\bar{z}) \geq 0$. Combined with the earlier result (5), we obtain that for $z_d \leq z < \bar{z}$,

$$E(R_2|z) + z - L - Ec(L - z + R_1(\bar{z})) \leq R_1(\bar{z}) \leq E\left(R_2|z\right) + z - L \tag{6}$$

The two period break-even constraint of the lender can now be simplified as follows:

$$ER_1 + E(E(R_2)) \geq L + E(l) + Ec_1 + E(Ec_2))$$

Here,

$$\begin{aligned} E(Ec_2) &= \int_0^{\bar{z}} \int_0^{z_{2d}(z,l)} c(z_2) dF(z_2) dF(z_1) \\ E(E(R_2)) &= \int_0^{\bar{z}} E\left(R_2|z\right) dF(z) \end{aligned}$$

Ec_1 is the expected monitoring cost incurred in the first period and $E(Ec_2)$ in the second period. Now, $l = \max[L+R_1(z)-z, 0]$, so we have, letting $R_1(\bar{z}) = \bar{R}$,

$$\int_0^{\bar{z}} R_1(z)dF + \int_{\bar{z}}^H \bar{R}dF + E(E(R_2)) = L + \int_0^{\bar{z}} (L + R_1(z) - z)dF + Ec_1 + E(Ec_2))$$

$$\int_{\bar{z}}^H \bar{R}dF + E(E(R_2)) = L + \int_0^{\bar{z}} (L - z)dF + Ec_1 + E(Ec_2) \tag{7}$$

Notice that $R_1(z)$ does not appear in the constraint, except for those realizations where the second loan is not needed. However, we require that $z - R_1(z) < L$, for all $z < \bar{z}$, a condition trivially met for the monitored states. Further, although $R_1(z)$ does not appear explicitly, the additional loan amount in state z of $L - z$ suggests that it is as if the firm surrenders all it has in all states, monitored or not, and then borrows the full amount L again. That is, it is as if $R_1(z) = z$ and $l(z) = L$ for all $z < \bar{z}$. Or, one can also think that the firm pays nothing to the bank at date 1 and instead uses its first period revenue towards reducing the loan for the second period. That is, it is as if $R_1(z) = 0$ and $l(z) = L - z$ for all $z \leq \bar{z} = L$. The two interpretations are equivalent from the bank's point of view. This constraint also implies that there is no difference between monitored and non-monitored states in terms of what is repaid and what is borrowed for the second time.

Lemma 1 *Let H be arbitrarily high. There exists $\overline{z}$, such that for all $z \geq \overline{z}$, a second loan is not needed.*

Proof. Because of IC1 and IC2 constraints (that is, (3) and (4)), for any $z \geq z_d$,

$$ER_2|z \leq ER_2|z_d - (z - z_d).$$

For sufficiently large z, RHS is negative, which is not permitted, and thus in this case, $ER_2|\overline{z} = 0$ for $\overline{z} = ER_2|z_d + z_d$ and for all $z \geq \overline{z}$. For these states, truth-telling fails if $R(z)$ is such that a loan is needed. This is because for all such states, the second period expected repayment is zero and the current period net payoff from truth telling is also zero. However, reporting $\overline{z}$ when the true state is higher, yields a net gain of $z - \overline{z}$. Thus, for all $z \geq \overline{z} = ER_2|z_d + z_d$, we must have $L = 0$ and $R(z) \leq ER_2|z_d + z_d - L$. Note that this satisfies IC3. ■

Proposition 1 *The sequence of GH contracts satisfies all the constraints in the two period maximization problem.*

Proof. The feasibility constraints are obviously satisfied. The second period ICC are also satisfied because the second period contract is an SDC. We now check the first period ICC.

IC1 requires $E(R_2|z') \geq E(R_2|z) + z - z'$. In GH, $E(R_2|z) = l(z) + Ec_{l(z)}$, because the break-even constraint holds per period. Substituting, we get, $l(z') + Ec_{l(z')}) \geq l(z) + Ec_{l(z)} + z - z'$, for all $z' < z$. Since $l(z') - l(z) = z - z'$, IC1 reduces to $Ec_{l(z')} \geq Ec_{l(z)}$, which holds because $l(z') > l(z)$.

IC2 holds by definition since GH set up is period by period.

IC3 also holds because in an SDC, states are monitored upto a threshold and then there is a constant required repayment in solvent states. Substituting for $ER_2|z$ and the first period threshold $\widehat{z}$, we find that the lower bound on $R(z)$ is $\widehat{z}$ while the upper bound is $\widehat{z} + Ec_{l(z)}$. Thus, the constraint is met.

Finally, the GH contract satisfies the break even constraint per period and thus satisfies the two period break even constraint. ■

This Proposition shows that the optimal contract is at least as good as the sequence of two short-term standard debt contracts. We later show that the optimal dynamic contract is strictly better.

Lemma 2 *Suppose that $\frac{c(z)f(z)}{1-F(z)}$ is an increasing function of z. Then, the constraint*

$$E[R_2|z'] \leq E[R_2|z] + z - z' \ \forall z' > z \geq z_d$$

is binding at the optimum.

Proof. See the Appendix. ■

The Lemma derives sufficient conditions for IC1 to bind. Recall that unlike the static GH, in our model, it is not possible to overreport. Thus, the inequality does not hold in both directions and therefore, does not trivially bind. The condition in the Lemma implies that the expected monitoring cost function is a convex function of the amount to be raised. That is, $C_{R2}(ER_2)$ is convex.

Corollary 1 *The optimal dynamic contract strictly dominates the sequence of GH contracts.*

Proof. From Proposition 1 and Lemma 2, and the fact that the incentive constraint IC1 is slack when the sequence of GH contracts is used, the result follows. ■

This Corollary implies that when the lender can observe the borrower's decision to stay or leave, it is possible to design a dynamic contract that lowers the total monitoring costs relative to the sequence of two GH contracts. In what follows, we derive some properties of the optimal contract.

Lemma 3 *There exists $z_d \in [0, H]$ such that $M_1(z) = 1 \forall z < z_d$ and $M_1(z) = 0 \forall z \geq z_d$.*

Proof. Suppose not. Then WLOG, there exist values $z_1 > 0$ and $H > z_2 > z_1$, such that $M_1(z) = 1 \ \forall z \in (z_1, z_2)$ and $M_1(z) = 0$ for all $z \notin (z_1, z_2)$. Let $E[R_2|0] = R_2^0$. Then, by the binding IC1, $E[R_2|z'] = R_2^0 - z' \ \forall z' \in [0, R_2^0]$. Notice that this must apply to $z \in (z_1, z_2)$ because it is possible for the firm to understate and escape monitoring. However, this expected second period revenue is the same as one where there is no monitoring. Further, the lender's break-even constraint remains unchanged in all other respects. Thus, $z_1 = 0$ and $z_2 \geq 0$. ■

Note that when only low states are monitored, the expected second period revenue function is no longer necessarily $E[R_2|z'] = R_2^0 - z'$ because of the inability to mimic up and the inability to report the state to be a monitored one when it is actually not monitored.

Let $E(R_2|z) \equiv R_2$, for all $z \leq z_d$. Thus, we assume that the same amount is raised in the second period for all states upto and including the monitoring threshold.

Lemma 4 *Under the assumptions of Lemma 2,* $R_1(\bar{z}) \equiv \overline{R} = R_2 + z_d - L$.

Proof. Recall (6): for all $z \in [z_d, \bar{z})$ such that $z - \overline{R} \geq 0$

$$E(R_2|z) + z - L - Ec\left(L - z + \overline{R}\right) \leq \overline{R} \leq E(R_2|z) + z - L$$

By definition, we have,

$$\overline{R} + L = \bar{z}.$$

Consider, for arbitrarily small positive ϵ, $\hat{z}$ such that

$$\hat{z} = \overline{R} + L - \epsilon.$$

Then, $\hat{z} < \bar{z}$ and by the first inequality and Lemma 2, we have,

$$E(R_2|\hat{z}) = R_2 + z_d - \hat{z} \leq \overline{R} + L - \hat{z} + Ec\left(\overline{R} + L - \hat{z}\right) = \epsilon + Ec(\epsilon)$$

$$\Rightarrow \hat{z} = \overline{R} + L - \epsilon \geq R_2 + z_d - \epsilon - Ec(\epsilon)$$

$$\Rightarrow \overline{R} + L \geq R_2 + z_d - Ec(\epsilon)$$

Now, from the second inequality, we get to see that

$$\overline{R} + L \leq E(R_2|z) + z = R_2 + z_d$$

Since $c(.)$ is continuous, with $c(0) = 0$, and these two inequalities hold for any small ϵ, we have shown that

$$\overline{R} = R_2 + z_d - L. \tag{8}$$

■

This Lemma shows that it is optimal to ask for the maximum amount possible in all states in which the firm does not need a second loan. At $z = R_2 + z_d$, we saw that to ensure truth-telling, the bank must leave sufficient residual for the firm to self-finance. This Lemma implies that this residual must be minimum possible. Of course, the residual is higher for all revenue levels that are higher than $R_2 + z_d$, because the repayment must be constant for such states to induce truth-telling, as in the static model.

Corollary 2 *For all* $z \geq z_d$, $R_1(z) + E(R_2|z) - (L + R_1(z) - z) = R_2 + z_d - L$.

Proof. Since IC1 binds, we have

$$\begin{aligned} E[R_2|z] &= E[R_2|z'] + z' - z \ \forall z > z' \geq z_d, \\ &\Rightarrow E[R_2|z] = E[R_2|z_d] + z_d - z \end{aligned}$$

Thus, $R_1(z) + E(R_2|z) - (L + R_1(z) - z) = E[R_2|z_d] + z_d - z - L + z = R_2 + z_d - L$. ■

The Corollary shows that the net total payment to the lender is constant for all realizations not monitored. This result is similar to Chang (1990) and is intuitive. In order to provide incentives to the firm to report truthfully, the lender ends up specifying the same net total repayment over two time periods for all unmonitored states. This result preserves the spirit of the static GH where repayment is constant in all unobserved states.

We next show that under the assumptions of Lemma 2, the optimal first period monitoring threshold is strictly less than the static threshold, denoted as z_{GH}.

Proposition 2 *Suppose that* $\frac{c(z)f(z)}{1-F(z)}$ *is an increasing function of* z. *Then,* $z_d < z_{GH}$.

Proof. See the Appendix. ■

As we can see, in our dynamic model, in the first period, the monitoring threshold is $z_d < z_{GH}$, i.e. the bank is more "lenient" in the first period. This is in contrast to the common results in the literature on dynamic contracts such as Bolton and Scharfstein, Monnet and Quintin, etc. This is because the net benefit in the first period obtained by the entrepreneur is not used as savings to reduce the second period loan, which reduces the overall financing cost of the bank as well. In standard dynamic contracts, the per-period net profit of the firm is only used as consumption, only benefitting the firm. We can highlight the contrast by considering the case where the bank cannot observe whether the firm will stay with the bank or not when he offers the 2nd period contract. In the above model, the bank could observe separation by the firms, and thus could punish it with monitoring. But in this case the bank does not know whether the firm uses the first period net revenue as savings at the same bank or take it to the other banks. Then, it is no longer optimal for the bank to be lenient in the first period. The following Proposition formalizes the argument.

Lemma 5 *In the unobservable case,* $z_d \geq z_{GH}$.

Proof. See the Appendix. ■

We now show that the optimal dynamic contract in the unobservable case must be the sequence of GH contracts. That is, if the bank cannot observe if the borrower stays with it or not, it is as if there is no long term contract. Let $\widehat{z}_2(z)$ denote the second period monitoring threshold under GH, as a function of the first period realization.

Proposition 3 *In the unobservable case,* $z_d = z_{GH}$ *and for all* $z, z_{2d}(z) = \widehat{z}_2(z)$.

Proof. The dynamic incentive constraint in this case is modified as:

$$z + E\left(R_2|z\right) \leq min\{z' + ER_2|z', L + z_d + Ec(L + z_d - z)\}, L + z_d \geq z > z' \geq z_d$$

The first element in the min function is the same as in the observable case. The second part ensures that the borrower stays for each z. Now, suppose that $z_d > z_{GH}$ and that

$$ER_1 + E\left(E(R_2|z)\right) = L + Ec(L) + E\left(L + z_{GH} - z + Ec(L + z_{GH} - z)\right).$$

That is, total expected revenue is the same in the dynamic case and the GH case. Then, the total expected cost difference between the two is

$$\begin{aligned} & EC_{R1}(L + Ec(L)) + EC_{R2}\left(L + z_{GH} - z + EC_{R1}(L + z_{GH} - z)\right) - EC_{R1}\left(ER_1\right) - EC_{R2}\left(ER_2|z\right) \\ = \; & E\left[\frac{\partial C_R(R^*)}{\partial R}\left(L + z_{GH} - z + EC_R(L + z_{GH} - z) - ER_2|z\right)\right] + \frac{\partial C_R(R^{**})}{\partial R}\left(L + Ec(L) - ER_1\right) \\ = \; & E\left[\frac{\partial C_R(R^{**})}{\partial R} - \frac{\partial C_R(R^*)}{\partial R}\right]\left(L + Ec(L) - ER_1\right) < 0 \end{aligned}$$

The last inequality follows because $L + Ec(L) - ER_1 < 0$ since $z_d > z_{GH}$, and given that,

$$R^{**} \in \left(L + Ec(L), ER_1\right), R^* \in \left(ER_2|z, L + Ec(L)\right)$$

by convexity of $C_R(ER)$,

$$\frac{\partial C_R(R^{**})}{\partial R} > \frac{\partial C_R(R^*)}{\partial R}.$$

Thus, $z_d = z_{GH}$. Then, the bank breaks even in the first period. Thus, the bank must also break even in the second period. That is,

$$E\left(ER_2|z\right) = E\left(L + z_{GH} - z + Ec(L + z_{GH} - z)\right)$$

By the incentive constraint presented above,

$$ER_2|z \leq L + z_d - z + Ec(L + z_d - z) \forall L + z_d \geq z \geq z_d$$

Since $z_d = z_{GH}$, we have,

$$ER_2|z \leq L + z_{GH} - z + Ec(L + z_{GH} - z) \forall L + z_{GH} \geq z \geq z_{GH}$$

Thus, for the bank to break-even in the second period, we must have this constraint bind. This means that the second period contract is identical to the GH contract. ■

The unobservable case leads to an interesting question: what if the first period bank is only a one-period bank while the entrepreneur lives for two periods and the bank knows it? In this case, the first period bank maximizes the entrepreneur's expected utility $E(z_1 - R_1(z_1))$ subject to,

$$z_1 - R_1(z_1) \geq 0,$$

$$E(R_2|z_1) = E(R_2|z') + z_1 - z',$$

for all z_1, $z_1 > z'$ such that $z_1 - R_1(z_1) < L$ and z' is not observed. We are assuming here that the entrepreneur cannot use $z_1 - z'$ towards the second loan from a second bank. That is, $z_1 - z'$ must be consumed. Thus IC1 remains the same even though the first bank lives for one period. The idea is that truth-telling for bank 1 needs to take into account the fact that the entrepreneur borrows again from another bank.

The break-even constraint of the first bank is

$$ER_1 = L + Ec_1$$

In order to solve its maximization problem, the first bank must solve for $E(R_2|z)$, which requires solving the second bank's maximization problem. We know that the second period optimal contract is a standard debt contract satisfying the break-even condition

$$E(R_2|z) = l(z) + Ec_{l(z)}$$

where $l(z) = L + R_1(z) - z$.

We argue that the first bank can implement the dynamic contract by setting $E(R_2|z) = R_2 + z_d - z$, solving for $l(z)$ and then setting $R_1(z)$ accordingly. Since the two-period break-even constraint is satified, and the second period break-even constraint is satisfied by construction, the first period break-even constraint must be satisfied as well. Thus, we have the following result:

Proposition 4 *If the entrepreneur must consume the residual from misrepresentation of revenue, even if the first period bank is a one-period bank, as long as the entrepreneur borrows again, the dynamic contract can be implemented. This implies that the first period contract is different from the GH contract.*

So far, we have considered two cases. In one, the bank perfectly observes the amount of loans from the other banks and thus can punish the borrower, and in the second, the unobservable case, the bank cannot punish any deviation from the dynamic relationship. We now assume that the bank can observe if the firm decides not to make *any* loan contract in the 2nd period and punish the deviation with monitoring. But the bank does not observe whether the entrepreneur has opened another loan contract or not, and therefore cannot punish that behavior. Then, the firm in the 1st period receiving z can deviate by reporting z' to the bank and pay to the bank

$$ER_2|z'$$

and at the same time, go to the other bank for the loan contract with the loan of $l = L + R_1(z') - z$ and pay to the bank

$$L + R_1(z') - z + Ec(L + R_1(z') - z).$$

Because the entrepreneur only has one project, the outcome of the project is still z_2 only. Therefore, the new truth telling constraint is

$$ER_2|z \leq ER_2|z' + L + R_1(z') - z + Ec(L + R_1(z') - z), \forall z \geq z_d$$

Now, let the bank set the first period revenue to be $R_1(z) = z$, the maximum possible level. Then, if the entrepreneur stays with the bank, its objective function, and thus the outcome will not change. However, the above constraint changes to

$$ER_2|z \leq ER_2|z' + L + z' - z + Ec(L + z' - z), \forall z \geq z_d$$

Rewriting it, we obtain

$$ER_2|z + z \leq ER_2|z' + z' + L + Ec(L + z' - z), \forall z \geq z_d$$

we can immediately see that this constraint is weaker than IC1 for $z' \leq z$, and furthermore, misreporting $z' > z$ is not feasible. Therefore, given IC1, this constraint is slack.
Thus, in the intermediate case where the borrower may borrow from multiple lenders to escape monitoring, the optimal dynamic contract does not change.

4 A special case

So far, we have seen that in the observable case, the bank can reduce monitoring cost by offering a dynamic loan contract which offers leniency in the first period and toughness in the second period. Even if the bank cannot observe the continuation of the contract by the entrepreneur, the bank can offer a dynamic contract that is different from static GH contract and reduce the total monitoring cost by offering tough initial period monitoring threshold and leniency in the 2nd period. Tough monitoring threshold prevents the entrepreneur from underreporting their first period status and obtain assets to carry over to other banks, and leniency in the second period increases the attractiveness of staying with the same bank. In both cases, we can see that the dynamic contract improves on the loan market incompleteness by reducing the overall monitoring cost. In this section, we show that in the observable case, under dynamic contract, entrepreneurs with projects that are profitable but not profitable enough to obtain a GH loan contract are able to get funding.

Assume that the first period loan (investment) is $L + \delta > L$, where L is the second period investment needed. The amount δ is such that

$$\int_0^{z_*} (z - c(z))dF(z) + \int_{z_*}^{H} z_* dF(z) = L + \delta$$

Here z_* is the maximizer of $\int_0^{\hat{z}} (z - c(z))dF(z) + \int_{\hat{z}}^H \hat{z}dF(z)$ with respect to $\hat{z}$. Thus, we are assuming that in a one-period GH, the lender monitors the maximum possible and just breaks-even. If δ is slightly higher, the one-period GH becomes unviable. Thus, the lender is indifferent between lending and not lending. Assume that it does not lend, just for one period.

The second period loan being smaller allows the lender to break-even without maximum monitoring if the borrower were to go to another lender, rather than stay with the current one. Thus, the outside option continues to be defined in the same way as above. And thus, the incentive compatibility constraints remain unchanged, leading to the same ER_2 as derived earlier and the same $E(R(z))$ and $l(z)$ expressions.

We now show that if the lender were allowed to offer a two-period dynamic contract, the borrower is able to borrow in the first period as well. That is, while the GH one-period contract fails, the two period contract survives.

The two-period break-even constraint (setting $R_2 = L + Ec(L)$) changes to:

$$\int_0^{\hat{z}_d} (z - c(z))dF(z) + (1 - F(\hat{z}_d)\hat{z}_d + Ec(L) = L + \delta + E(Ec_2)$$

By definition of z_*, setting $\hat{z} = z_*$, we get,

$$0 = E(Ec_2) - Ec(L)$$

Since $Ec(L) \geq E(Ec_2)$, because $l(z) \leq L$, the equality does not hold or rather, the break-even constraint is slack at z_*. Thus, there exists $\hat{z} < z_*$ which makes the lender just break even without maximum monitoring in the first period. This leads to the following result.

Proposition 5 *Suppose that the first period loan is $L+\delta$ while the second period loan is L. There exist values of δ, such that a one-period SDC does not allow the lender to break-even whereas a two-period contract in which the lender lends in both periods, does.*

5 Uniform Distribution and fixed monitoring cost

Let $c(z) = k > 0$ for all z and let $f(z) = \frac{1}{H}$. Thus, the monitoring cost is fixed and the distribution of revenues is uniform. Let $L = 1$.

The static monitoring threshold $\hat{z}$ is given by,

$$\int_0^{\hat{z}} (z - k)dz + \hat{z}(H - \hat{z}) = H. \tag{9}$$

The solution is,

$$\hat{z} = H - k - \sqrt{(H - k)^2 - 2H}$$

Similarly, the second period monitoring threshold, which corresponds to a static threshold except for a loan amount $l = 1 + \hat{z} - z$, is given by,

$$\hat{z}_2 = H - k - \sqrt{(H - k)^2 - 2H(1 + \hat{z} - z)}, \hat{z} \leq z \leq 1 + \hat{z}$$

Note that $Ec(L) = \frac{k\hat{z}}{H}$. Also, the following parameter restrictions are implied:

$$\sqrt{(H-k)^2 - 2H} \geq 0 \iff k \leq H - \sqrt{2H}$$

Note also that $H > 2$, for the upper bound on k to be positive.

We also need $1 + \hat{z} \leq H$, for the loan to become zero at some revenue level. This is not essential but assumed in our analysis. This condition boils down to

$$\begin{aligned} 1 + H - k - \sqrt{(H-k)^2 - 2H} &\leq H \\ 1 - k &\leq \sqrt{(H-k)^2 - 2H} \end{aligned}$$

Recall that $E(R_2|z_d) = R_2$.Then, the second-period threshold in the dynamic case is

$$\begin{aligned} z_{2d} &= H - \sqrt{H^2 - 2H(R_2 + z_d - z)}, R_2 + z_d \geq z \geq z_d, \\ \widehat{z}^* &= H - \sqrt{(H)^2 - 2HR_2}, z \leq z_d. \end{aligned}$$

The first period threshold can be derived from the break-even constraint (7). Letting $c = \frac{k\widehat{z}^*}{H}$,

$$EEc_2 = z_d \frac{c}{H} + \int_{z_d}^{R_2+z_d} k \frac{z_{2d}}{H^2} dz$$

Thus, the break even constraint becomes:

$$-z_d^2 + 2(H - k - c)z_d - (4LH - 2HR_2 + \frac{2k}{H} \int_{\hat{z}_d}^{\bar{z}_d} \hat{z}_{2d} dz) = 0$$

$$z_d = H - k - c - \sqrt{(H-k-c)^2 - 2(2H - HR_2 + k\frac{\beta}{H})}$$

where $\beta = \int_{\hat{z}_d}^{\hat{z}_d+R_2} \hat{z}_{2d} dz = HR_2 - \frac{1}{3H}\left(H^3 - (H - \widehat{z}^*)^3\right)$.

Note that parameters must be constrained to ensure that $z_d \geq 0$, for all values of R_2. In particular, if $R_2 = 0$, $z_d = H - k - \sqrt{(H-k)^2 - 4H}$, so that $k \leq H - 2\sqrt{H}$. We also need $2(2H - HR_2 + k\frac{\beta}{H}) \geq 0 \Rightarrow 2H - HR_2 + \frac{k}{H}\left(HR_2 - \frac{1}{3H}H^3 - (\sqrt{(H)^2 - 2HR_2})^3\right) \geq 0$. This implies that R_2 cannot be too high. Finally, we also need $(H-k-c)^2 - 2(2H - HR_2 + k\frac{\beta}{H}) \geq 0$.

Total monitoring costs under two-period GH (no dynamics) are:

$$M_{gh} = \frac{\hat{z}}{H}k + \frac{\hat{z}}{H}(k\frac{\hat{z}}{H}) + \frac{k}{H^2}\int_{\hat{z}}^{\hat{z}+1} \hat{z}_2 dz$$

and total monitoring costs under two period with dynamics are,

$$M_d = \frac{\hat{z}_d}{H}k + \frac{\hat{z}_d}{H}(k\frac{\hat{z}^*}{H}) + \frac{k}{H^2}\int_{\hat{z}_d}^{\hat{z}_d+R_2} \hat{z}_{2d} dz$$

Despite a multitude of parameter restrictions, we can derive the following result.

Proposition 6 *Dynamic monitoring costs are strictly decreasing in R_2.*

Proof. We show that M_d is a strictly decreasing function of R_2 over $[0, 1 + Ec(L)]$. Differentiating M_d, we obtain

$$\frac{dM_d}{dR_2} = \frac{k}{H}[(1 + \frac{\hat{z}*}{H})z_d' + \frac{z_d}{H}\hat{z}^{*\prime}] + \frac{k}{H^2}\beta'$$

Now, $\hat{z}^{*\prime} = \frac{H}{H - z^*} > 0$, $\beta' = H - \frac{(H-\hat{z}^*)^2}{H}\hat{z}^{*\prime} = \hat{z}^* > 0$. Substituting,

$$\frac{dM_d}{dR_2} < 0 \iff z_d' + \frac{z_d}{H - \hat{z}^*} + \frac{\hat{z}^*}{H}(1 + z_d') < 0$$

Now,

$$z_d' = -\frac{k}{H - \hat{z}^*} + \frac{k\frac{H-k-c}{H-\hat{z}^*} + \frac{k}{H}\hat{z}^* - H}{\sqrt{(H - k - c)^2 - 2(2H - HR_2 + k\frac{\beta}{H})}}$$

Next, we show that $1 + z_d' < 0$.

$$1 + z_d' = 1 - \frac{k}{H - \hat{z}^*} + \frac{k\frac{H-k-c}{H-\hat{z}^*} + c - H}{H - k - c - z_d}.$$

or

$$1 + z_d' < 0 \iff (H - \hat{z}^*)(-k - z_d) + kz_d < 0$$

This holds if $H - \hat{z}^* > k$. Substituting for $\hat{z}^* = \sqrt{H^2 - 2HR_2}$, we get $H - \hat{z} - k > 0$ if and only if $\frac{H^2-k^2}{2H} > R_2$. Now it suffices to show that $\frac{H^2-k^2}{2H} > L + k\frac{\hat{z}}{H} = \max R_2$. This follows because the inequality reduces to $(H - k)^2 > 2(H - k\sqrt{(H - k)^2 - 2H}$ and $(H - k)^2 > 2H$.

Next, we show that $z_d' + \frac{z_d}{H-\hat{z}^*} < 0$.

$$z_d' + \frac{z_d}{H - \hat{z}^*} < 0 \iff -k(H-k-c-z_d) + (H-k-c)k + (c-H)(H-\hat{z}^*) + z_d(H-k-c-z_d) < 0$$

or

$$(c - H)(H - \hat{z}^*) + z_d(H - c - z_d) < 0$$

or

$$(c - H)(H - \hat{z}^* - z_d) - z_d^2 < 0$$

Thus, it suffices to show that $H - \hat{z}^* - z_d > 0$. Substituting for $H - \hat{z}^* = \sqrt{H^2 - 2HR_2}$ and z_d, we need,

$$H - k - c - \sqrt{(H - k - c)^2 - 2(2H - HR_2 + k\frac{\beta}{H})} < \sqrt{H^2 - 2HR_2}$$

Simplifying, we obtain,

$$H^2 - 4H - \frac{2k\beta}{H} + 2\sqrt{(H - k - c)^2 - 2(2H - HR_2 + k\frac{\beta}{H})}\sqrt{H^2 - 2HR_2} > 0$$

We now show that this holds because $H^2 - 4H - \frac{2k\beta}{H} > 0$. Recall that,

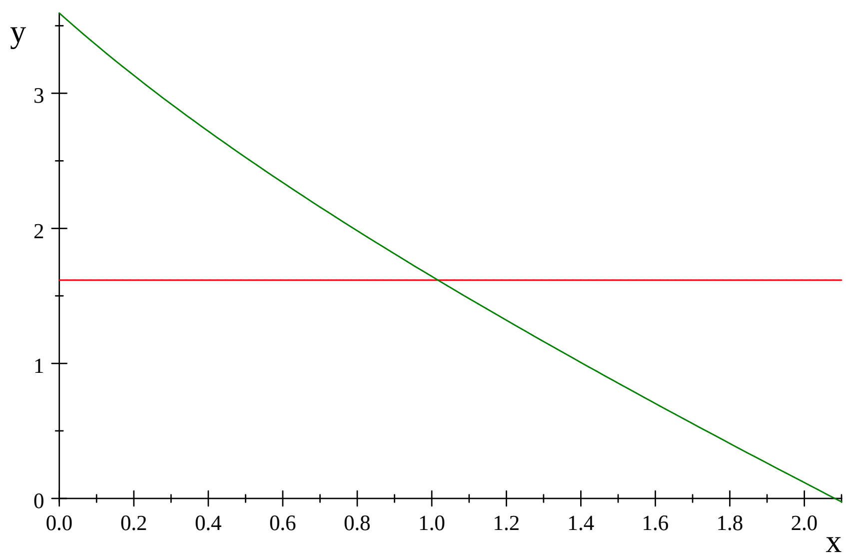
$$\beta = \int_{z_d}^{z_d + R_2} z_{2d}(z)dz < \hat{z}^* R_2 < \hat{z}^* \frac{H}{2} < \frac{H^2}{2}.$$

Thus,

$$4H + \frac{2k\beta}{H} < 4H + kH < H^2 \iff H > 4 + k$$

This holds because by assumption $k < H - 2\sqrt{H} < H - 4$, because for $k > 0, H$ must be greater than 4. ■

The graph below shows total monitoring costs in GH v/s in the dynamic case for $H = 25, L = 1$ and $k = 14$. It is clear that the dynamic case entails lower monitoring costs. Note that, for these parameter values, $1 + Ec(L) = 2.44$. However, monitoring costs become negative at this level because z_d becomes sufficiently negative. This is not allowed and thus, optimal value of R_2 is the lower of the two values, $1 + Ec(L)$ and the value at which z_d becomes zero.



M_d (green) and M_{gh} (red) as a function of R_2

Corollary 3 $z'_d < 0$.

Proof. It suffices to show that $k\frac{H-k-c}{H-\hat{z}^*} + c - H < 0$, or $-(H-c)(h-\hat{z}-k) - k^2 < 0$. This holds if $H - c > 0$ and $H - \hat{z} - k > 0$. Now, $H > c$ and the second inequality has been shown to hold in the previous proof. ■

Thus, as the bank collects more in the second period, and thus monitors more in the second period, the first period monitoring threshold declines. This is intuitive given the trade-off between monitoring in the first period and monitoring in the second.

Corollary 4 *The objective function is strictly increasing in R_2.*

Proof. is straightforward. ■

Thus, as R_2 increases, total monitoring costs fall and this implies that the borrower's total expected profits are maximized. However, since z_d falls as well, either the optimal value of R_2 is $1 + Ec(L)$ or it is less but z_d is zero. That is, either there is no monitoring in the first period or the bank collects the maximum amount possible given the outside option, in the second period.

The following result has been proved in general, assuming that expected monitoring costs are an increasing and convex function of the amount collected. It can be shown that this assumption is satisfied here. However, we can also prove it directly in a straightforward way.

Corollary 5 $z_d < \hat{z}$.

Proof. There are two possibilities for the optimal value of R_2: $R_2^* = 1 + Ec(L)$ or R_2^* is such that $z_d = 0$. Since $\hat{z} > 0$, we only need to show that $z_d \leq \hat{z}$ if $R_2^* = 1 + Ec(L)$. Now, using the results from the main part of the paper, namely that the net total repayment is constant in solvent states at $R_2 + z_d - L$, and substituting for R_2, the break-even constraint simplifies to $z_d + Ec(L) = L + Ec_1 + EEc_2$. Or $z_d - Ec_1 = L + EEc_2 - Ec(L)$. If $z_d = \hat{z}$, we have $\hat{z} - Ec(L) = L + EEc_2 - Ec(L)$, or $\hat{z} = L + EEc_2 \leq L + Ec(L)$. Since $\hat{z} > \int_0^{\hat{z}} (z)dF + \hat{z} \int_{\hat{z}}^{H} dF$, and RHS is smaller than in the static model, the break-even constraint is slack and thus, z_d must be strictly less than $\hat{z}$.Hence the result. ■

In the example considered above, the optimal value of R_2 is 2.048 4 because $z_d < 0$ for all $R_2 > 2.048\,4$, and $1 + Ec(L) = 2.441\,4$. Also, $\widehat{z} = 2.5739$. This means that the second loan becomes zero only at 3.5739 in the GH case whereas in the dynamic model, the second loan is zero starting at 2.0484. Thus, at the optimum, we have $z_d = 0$, so that there is no monitoring at date 1. The dynamic contract achieves reduces total monitoring costs to almost zero while the sequence of two GH contracts entails a much higher monitoring cost. This is an extreme but interesting scenario. All monitoring occurs in the second period to minimize the overall expected repayment by the borrower. This need not always be the case. For example, if $H = 6, k = 0.3, L = 1$, $R_2^* = 1 + Ec(L) = 1.058$ and $z_d = 1.121\,8 < \hat{z} = 1.1734$. Here, the second loan becomes zero approximately at the same z in both cases: 2.1738.

Thus, we have shown that in the example of uniform distribution and fixed monitoring costs, the optimal contract entails less monitoring in the first period than in the static Gale-Hellwig contract. The result is lower total monitoring costs in the dynamic case than in the static case. In the simulations, we see that the second period monitoring threshold is also lower, for all date 1 realizations, in the dynamic case and thus, the lender is able to be lenient in both periods when a long term contract is allowed.

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Appendix

Proof of Lemma 2. Let

$$E\left[R_2(z)|z > z_d\right] = \hat{R}$$

Then, the contract is minimizing the following expected cost

$$E\left[c(z)|z > z_d\right]$$

given the following two constraints

$$E\left[R_2(z)|z > z_d\right] \geq \hat{R}$$

and

$$E[R_2|z'] \leq E[R_2|z] + z - z' \;\forall z' > z \geq z_d$$

Since

$$C_{z2}(\hat{z}_2) = \int_0^{\hat{z}_2} c(u)dF(u),\;\; \frac{\partial C_{z2}(\hat{z}_2)}{\partial \hat{z}_2} = c(\hat{z}_2)f(\hat{z}_2)$$

and

$$E[R_2(\hat{z}_2)] = \int_0^{\hat{z}_2} udF(u) + \hat{z}_2\left[1 - F(\hat{z}_2)\right],\;\; \frac{\partial E[R_2(\hat{z}_2)]}{\partial \hat{z}_2} = 1 - F(\hat{z}_2)$$

$$\frac{\partial C_{R2}\left(ER_2\right)}{\partial ER_2} = \frac{c(\hat{z}_2(ER_2))f(\hat{z}_2(ER_2))}{1 - F(\hat{z}_2(ER_2))}$$

and since by assumption this is increasing in ER_2, $C_{R2}\left(ER_2\right)$ is a convex function of R_2.

Now, notice that once the first period z is realized, then if $z > z_d$, the bank sets the next period threshold $\hat{z}_2(z)$ and the next period expected revenue as follows:

$$E[R_2|z] = \int_0^{\hat{z}_2(z)} c(u)dF(u) + \hat{z}_2(z)\left[1 - F(\hat{z}_2(z))\right]$$

The set of $E[R_2|z]$ satisfying the two constraints

$$E\left[R_2|z > z_d\right] \geq \hat{R}$$

$$E[R_2|z'] \leq E[R_2|z] + z - z' \;\forall z' > z \geq z_d, E[R_2|z'] > 0, E[R_2|z] > 0$$

is a convex set. Therefore, a solution exists.

Now, assume that the incentive constraint IC1 does not bind, i.e. there exist z_1 and z_2 such that $z_1 < z_2$ and

$$E\left(R_2|z_2\right) < E\left(R_2|z_1\right) + z_1 - z_2$$

First, suppose that ER_2 is continuous in $[z_1, z_2]$. Then, from the intermediate value theorem, there exists a point $\tilde{z} \in (z_1, z_2)$ such that

$$\int_{z_1}^{\tilde{z}} [(u-\tilde{z}) - (ER_2|\tilde{z} - ER_2|u)]\, f(u)du = \int_{\tilde{z}}^{z_2} [(\tilde{z}-u) - (ER_2|u - ER_2|\tilde{z})]\, f(u)du. \tag{10}$$

To see why, let $\tilde{z} = z_1$. Because IC1 is satisfied, for any $u \in (z_1, z_2)$,

$$E(R_2|u) - E(R_2|z_1) \leq z_1 - u$$

and from the continuity of $E(R_2|z)$, there exists $z < z_2$ close enough to z_2 such that for any $u \in [z, z_2]$

$$ER_2|u - ER_2|z_1 < z_1 - u$$

Therefore,

$$\begin{aligned} RHS &= \int_{z_1}^{z_2} [(z_1 - u) - (ER_2|u - ER_2|z_1)]\, f(u)du \\ &\geq \int_{z}^{z_2} [(z_1 - u) - (ER_2|u - ER_2|z_1)]\, f(u)du > 0 = LHS \end{aligned}$$

Similarly, from the continuity of $E(R_2|z)$, there exists $z > z_1$ close enough to z_1 such that for any $u \in [z_1, z]$

$$E(R_2|z_2) - E(R_2|u) < u - z_2$$

Hence, for $\tilde{z} = z_2$

$$\begin{aligned} LHS &= \int_{z_1}^{z_2} [(u - z_2) - (E(R_2|z_2) - E(R_2|u))]\, f(u)du \\ &\geq \int_{z_1}^{z} [(u - z_2) - (E(R_2|z_2) - E(R_2|u))]\, f(u)du > 0 = RHS. \end{aligned}$$

Therefore, the claim holds. Let

$$\widehat{ER_2}|u = ER_2|u$$

for $u \notin [z_1, z_2]$ and

$$\widehat{ER_2}|u = E(R_2|\tilde{z}) + \tilde{z} - u.$$

for $u \in [z_1, z_2]$. Then, *from equation (9),* we can see that

$$\begin{aligned} & \int_{z_1}^{z_2} \left[\widehat{ER_2}|u - E(R_2|u)\right] f(u)du \\ = & \int_{z_1}^{z_2} [E(R_2|\tilde{z}) - E(R_2|u) + \tilde{z} - u]\, f(u)du \\ = & \int_{z_1}^{\tilde{z}} [ER_2|\tilde{z} - ER_2|u + \tilde{z} - u]\, f(u)du + \int_{\tilde{z}}^{z_2} [ER_2|\tilde{z} - ER_2|u + \tilde{z} - u]\, f(u)du = 0 \end{aligned}$$

Therefore, one can show that $\widehat{ER_2}|u$ satisfies the incentive constraints and

$$E\left[\widehat{ER_2}|z > z_d\right] = \hat{R}.$$

Now, consider the expected monitoring cost. There exists $ER^* \in (ER|z_1, ER|\tilde{z}) > ER^{**} \in (ER|\tilde{z}, ER|z_2)$ and thus, once again, using the intermediate value theorem, we obtain,

$$\begin{aligned}
& \int_{z_1}^{z_2} C_{R2}(ER|u)f(u)du \\
= & \int_{z_1}^{\tilde{z}} \left[C_{R2}(\widehat{ER}|u) + \frac{c(z(ER^*))f(z(ER^*))}{1-F(z(ER^*))}\left(ER|u - \widehat{ER}|u\right)\right] f(u)du \\
& + \int_{\tilde{z}}^{z_2} \left[C_{R2}(\widehat{ER}|u) + \frac{c(z(ER^{**}))f(z(ER^{**}))}{1-F(z(ER^{**}))}\left(ER|u - \widehat{ER}|u\right)\right] f(u)du \\
> & \int_{\tilde{z}}^{z_2} C_{R2}(\widehat{ER}|u)f(u)du.
\end{aligned}$$

The last inequality is from the assumption that

$$\frac{c(\hat{z}(ER^*))f(\hat{z}(ER^*))}{1-F(\hat{z}(ER^*))} > \frac{c(\hat{z}(ER^{**}))f(\hat{z}(ER^{**}))}{1-F(\hat{z}(ER^{**}))} \ ,\hat{z}(ER^*) > \hat{z}(ER^{**})$$

Next, assume that $ER|u$ is not continuous in $[z_1, z_2]$. Then, for some $\tilde{z}$, there exists $\delta > 0$ such that

$$inf_{z<\tilde{z}}E[R_2|z] - sup_{z'>\tilde{z}}E[R_2|z'] > \delta$$

Then, define $\tilde{ER}$ such that

$$\tilde{ER_2}|u = (ER_2|\tilde{z}) + \tilde{z} - u.$$

where

$$\tilde{ER_2}|\tilde{z} = sup_{z'>\tilde{z}}E[R_2|z'] + \frac{\delta}{2}$$

Then, choose η_1, η_2 such that $\eta_1 < \tilde{z} < \eta_2$, $\eta_1, \eta_2 \in [z_1, z_2]$ and

$$\int_{\eta_1}^{\tilde{z}} \left[ER_2|u - \tilde{ER_2}|u\right] f(u)du = \int_{\tilde{z}}^{\eta_2} \left[\tilde{ER_2}|u - ER_2|u\right] f(u)du$$

Because of the continuity of the integral, and the monotonicity of the LHS and RHS functions with respect to η_1 and η_2, such η_1, η_2 can be found. Then, define $\widehat{ER_2}$ such that

$$\widehat{ER_2}|u = \tilde{ER_2}|u$$

for $u \in [\eta_1, \eta_2]$ and

$$\widehat{ER_2}|u = ER_2|u$$

otherwise. Then, again, one can show that

$$\begin{aligned} E\left[\widehat{ER}_2|z>z_d\right] &= E\left[ER_2|z>z_d\right]+\int_{\eta_1}^{\eta_2}\left[\widehat{ER}_2|u-ER_2|u\right]f(u)du \\ &= \int_{\eta_1}^{\tilde{z}}\left[\widehat{ER}_2|u-ER_2|u\right]f(u)du+\int_{\tilde{z}}^{\eta_2}\left[\widehat{ER}_2|u-ER_2|u\right]f(u)du=\widehat{R}. \end{aligned}$$

$$\begin{aligned} & \int_{\eta_1}^{\eta_2} C_{R2}(ER_2|u)f(u)du \\ = & \int_{\eta_1}^{\tilde{z}}\left[C_{R2}(\widehat{ER}_2|u)f(u)du+\frac{c(z(ER_2^*))f(z(ER_2^*))}{1-F(z(ER_2^*))}\left(ER_2|u-\widehat{ER}_2\right)\right]f(u)du \\ & +\int_{\tilde{z}}^{\eta_2}\left[C_{R2}(\widehat{ER}_2|u)f(u)du+\frac{c(z(ER_2^{**}))f(z(ER_2^{**}))}{1-F(z(ER^{**}))}\left(ER_2|u-\widehat{ER}_2|u\right)\right]f(u)du \\ > & \int_{\eta_1}^{\eta_2} C_{R2}(\widehat{ER}|u)f(u)du. \end{aligned}$$

The last inequality is due to the convexity of $c()$ resulting in

$$\frac{c(z(ER^*))f(z(ER^*))}{1-F(z(ER^*))}>\frac{c(z(ER^{**}))f(z(ER^{**}))}{1-F(z(ER^{**}))}\ ,z(ER^*)>z(ER^{**})$$

Now, let

$$\gamma=\int_{z_1}^{z_2} C_{R2}(ER|u)f(u)du-\int_{z_1}^{z_2} C_{R2}(\widehat{ER}|u)f(u)du$$

Now, define for some small positive ϵ,

$$\bar{ER}_2|z=Max\left\{\widehat{ER}_2|z-\epsilon,0\right\}$$

for $z\in[z_1,z_2]$. Then, by construction

$$E\left[\bar{ER}_2|z>z_d\right]<\hat{R}$$

and because of continuity,

$$\int_{z_1}^{z_2} C_{R2}\left(\bar{ER}_2|u\right)f(u)du\le\int_{z_1}^{z_2} C_{R2}\left(\widehat{ER}_2|u\right)f(u)du-D\epsilon$$

for some constant $D>0$. Thus once can choose ϵ small enough such that

$$(1-D)\epsilon<\gamma$$

Therefore, one can slightly decrease expected revenue such that

$$\begin{aligned} & \left[\bar{ER}_2|z>z_d\right]-E\left[C_{R2}\left(\bar{ER}_2|z\right)|z>z_d\right] \\ = & \left[ER_2|z>z_d\right]-E\left[C_{R2}\left(ER_2|z\right)|z>z_d\right]+\gamma-[1-D]\epsilon \\ > & \left[ER_2|z>z_d\right]-E\left[C_{R2}\left(ER_2|z\right)|z>z_d\right], \end{aligned}$$

i.e. higher than the original expected net revenue, which contradicts the assumption that the original revenue function minimized equation 1. ■

Proof of Proposition 6. *From previous results, w*e can simplify the break-even constraint as follows:

$$
\begin{aligned}
& \int_{\bar{z}}^{H} \bar{R} dF + EER_2 - \int_{0}^{\bar{z}} (L-z) dF - L - Ec_1 - EEc_2 \\
= \quad & R_2 - F(z_d) C_{R2}(R_2) - \int_{z_d}^{R_2+z_d} C_{R2}(R_2 + z_d - z) dF - L \\
& + \int_0^H Min\left\{z_d, z\right\} dF - L - \int_0^{z_d} c(z) dF = 0
\end{aligned}
$$

Now, denote

$$R_1(z) = \int_0^H Min\left\{z, u\right\} dF(u)$$

Then, total revenue minus total cost for dynamic model can be expressed as follows

$$R_2 + R_1(z_d) - F(z_d) C_{R2}(R_2) - \int_{z_d}^{R_2+z_d} C_{R2}(R_2 + z_d - z) dF - 2L - C_{R1}(R_1(z_d))$$

Notice that the Gale Hellwig one period threshold z_{GH} and its corresponding $R_{1GH} = R_1(z_{GH})$ satisfies

$$\int_0^H Min\left\{z_{GH}, z\right\} dF - L - \int_0^{z_{GH}} c(z) dF = 0$$

or

$$R_{1GH} - L - C_{R1}(R_{1GH}) = 0$$

Then, the corresponding $R_2 = R_{2GH}$ would be such that

$$R_{2GH} - F(z_{GH}) C_{R2}(R_{2GH}) - \int_{z_{GH}}^{R_{2GH}+z_{GH}} C_{R2}(R_{2GH} + z_{GH} - z) dF - L = 0$$

and

$$R_{2GH} < L + C_L(L)$$

because if we let

$$\bar{R}_2 = L + C_L(L)$$

$$\bar{R}_2 - F(z_d) C_{R2}(\bar{R}_2) - \int_{z_d}^{\bar{R}_2+z_d} C_{R2}(\bar{R}_2 + z_d - z) dF - L > \bar{R}_2 - C_{R2}(\bar{R}_2) - L = 0$$

Now, consider two values, R_L and R_H such that $R_L < R_H \leq L + C_L(L)$. First, let

$$R_1 = R_H, R_2 = R_L.$$

Then, the first period threshold corresponding to $R_1 = R_H$ is z_{1H}, such that

$$R_H = \int_0^H Min\{z_{1H}, z\}\, dF$$

Then, total revenue minus total cost is

$$R_H + R_L - C_{R1}(R_H) - F(z_{1H})C_{R2}(R_L) - \int_{z_{1H}}^{R_L+z_{1H}} C_{R2}(R_L + z_{1H} - z)dF - 2L$$

On the other hand, if we let $R_1 = R_L$ and $R_2 = R_H$, Then, the total revenue minus total cost is

$$R_L + R_H - C_{R1}(R_L) - F(z_{1L})C_{R2}(R_H) - \int_{z_{1L}}^{R_H+z_{1L}} C_{R2}(R_H + z_{1L} - z)dF - 2L$$

Furthermore, because $R_L < R_H$, the first period threshold is $z_{1L} < z_{1H}$. then the total monitoring cost is

$$\begin{aligned}
& C_{R1}(R_L) + F(z_{1L})C_{R2}(R_H) + \int_{z_{1L}}^{R_H+z_{1L}} C_{R2}(R_H + z_{1L} - z)dF && (11)\\
\leq\ & C_{R1}(R_H) + F(z_{1L})C_{R2}(R_L) + \int_{z_{1L}}^{R_H+z_{1L}} C_{R2}(R_L + z_{1L} - z)dF && (12)\\
<\ & C_{R1}(R_H) + F(z_{1H})C_{R2}(R_L) + \int_{z_{1H}}^{R_L+z_{1H}} C_{R2}(R_L + z_{1H} - z)dF && (13)
\end{aligned}$$

Below, we show the inequality. First, wlog, define

$$C_{R2}(R) = 0 \ if R < 0$$

Then, because of convexity, for $z_{1L} \leq z$

$$\begin{aligned}
EC(R_H) - EC(R_L) &= \int_{R_L}^{R_H} \frac{\partial C_{R2}(R)}{\partial R} dR \\
&\geq \int_{R_L+z_{1L}-z}^{R_H+z_{1L}-z} \frac{\partial C_{R2}(R)}{\partial R} dR \\
&= C_{R2}(R_H + z_{1L} - z) - C_{R2}(R_L + z_{1L} - z)
\end{aligned}$$

Then,

$$\begin{aligned}
(2) - (1) &= (1 - F(z_{1L}))\,[C_{R2}(R_H) - C_{R2}(R_L)] \\
&\quad - \int_{z_{1L}}^{R_H+z_{1L}} [C_{R2}(R_H + z_{1L} - z) - C_{R2}(R_L + z_{1L} - z)]\, dF \\
&= (1 - F(R_H + z_{1L}))\,[C_{R2}(R_H) - C_{R2}(R_L)] \\
&\quad + \int_{z_{1L}}^{R_H+z_{1L}} [(C_{R2}(R_H) - C_{R2}(R_L)) - (C_{R2}(R_H + z_{1L} - z) - C_{R2}(R_L + z_{1L} - z))]\, dF \\
&\geq 0
\end{aligned}$$

Next, we show (3) > (2).

$$
\begin{aligned}
(3)-(2) &= \int_{z_{1L}}^{z_{1H}} \left[C_{R2}(R_L) - C_{R2}(R_L + z_{1L} - z)\right] dF \\
&\quad + \int_{z_{1H}}^{R_L+z_{1L}} \left[C_{R2}(R_L + z_{1H} - z) - C_{R2}(R_L + z_{1L} - z)\right] dF \\
&\quad + \int_{R_L+z_{1L}}^{R_L+z_{1H}} C_{R2}(R_L + z_{1H} - z) dF \\
&\quad - \int_{R_L+z_{1H}}^{R_H+z_{1H}} C_{R2}(R_L + z_{1L} - z) dF \\
&\geq 0
\end{aligned}
$$

The first and second terms are positive because of $C_{R2}(R)$ being an increasing function, and the fourth term is zero because for $z \geq R_L + z_{1H}$, $C_{R2}(R_L + z_{1L} - z) = 0$. So far, we have shown that for the same total $R_2 + R_1(z_d)$, it is better to set R_2 higher than R_1 rather than the other way around.The same argument holds if we substitute $R^H = L + C_L(L)$ and $R_L = R_{2GH}$. Thus, it is better to set $R_2 = L + C_L(L)$ and $R_1 = R_{2GH} < L + C_L(L)$. This implies that $z_d < z_{GH}$ since $R_1 = L + C_L(L)$ under static GH and it gives lower monitoring cost with the same revenue. This implies that the GH sequence of contracts cannot be optimal and further that, $z_d < z_{GH}$, since if not, then $R_2 < R_1$ and that cannot be optimal.

Next, consider the case where $z_{1H} > z_{GH}$, i.e. $R_1 = R_H > L + C_L(L)$, and $R_L < R_H$. Then, we cannot use the flipping argument because if we flip R_L and R_H, then $R_2 = R_H > L + EC(L)$, violating the IC4. I this case, consider the alternative where $\tilde{R}_1 = R_H - \Delta > \tilde{R}_2 = R_L + \Delta$, thus

$$
\tilde{R}_1 + \tilde{R}_2 = R_H + R_L
$$

where $\Delta = R_H - L - Ec(L)$. Then, again, we can focus on the total sum of the expected monitoring cost. We first show that

$$
\begin{aligned}
& C_{R2}(R_H) + F(z_{1H}) C_{R2}(R_L) + \int_{z_{1H}}^{R_L+z_{1H}} C_{R2}(R_L + z_{1H} - z) dF \\
\geq\ & C_{R2}(\tilde{R}_1) + F(z_{1H}) C_{R2}(\tilde{R}_2) + \int_{z_{1H}}^{R_L+z_{1H}} C_{R2}(\tilde{R}_2 + z_{1H} - z) dF
\end{aligned}
$$

This is because

$$
C_{R2}(R_H) - C_{R2}(\tilde{R}_1) = \Delta \frac{\partial C_{R2}(\hat{R})}{\partial \hat{R}}|_{\hat{R} \geq \tilde{R}_1} \tag{14}
$$

$$
\begin{aligned}
& F(z_{1H}) \left[C_{R2}(\tilde{R}_2) - C_{R2}(R_L)\right] + \int_{z_{1H}}^{R_L+z_{1H}} \left[C_{R2}(\tilde{R}_2 + z_{1H} - z) - C_{R2}(R_L + z_{1H} - z)\right] dF \\
=\ & F(z_{1H}) \Delta \frac{\partial C_{R2}(\hat{R})}{\partial \hat{R}}|_{\hat{R} \leq \tilde{R}_1} + \Delta \int_{z_{1H}}^{R_2+z_{1H}} \left[\frac{\partial C_{R2}(\hat{R})}{\partial \hat{R}}|_{\hat{R} \leq \tilde{R}_1}\right] dF \qquad (15)
\end{aligned}
$$

Because of convexity,

$$\frac{\partial C_{R2}(\hat{R})}{\partial \hat{R}}|_{\hat{R}\geq\tilde{R}_1} \geq \frac{\partial C_{R2}(\hat{R})}{\partial \hat{R}}|_{\hat{R}\leq\tilde{R}_1}$$

therefore,

$$(4) \geq (5)$$

and we have shown the inequality. Furthermore, because $C_{R2}()$ is increasing, and $z_{1H} > \tilde{z}_1$

$$\begin{aligned} & C_{R2}(\tilde{R}_1) + F(z_{1H})C_{R2}(\tilde{R}_2) + \int_{z_{1H}}^{R_2+z_{1H}} C_{R2}(\tilde{R}_2 + z_{1H} - z)dF \\ > \quad & C_{R2}(\tilde{R}_1) + F(\tilde{z}_1)C_{R2}(\tilde{R}_2) + \int_{\tilde{z}_1}^{\tilde{R}_2+\tilde{z}_1} C_{R2}(\tilde{R}_2 + \tilde{z}_1 - z)dF \end{aligned}$$

Therefore, $z_1 > z_{GH}$ does not result in lower expected 2 period monitoring cost.
■

Proof of Proposition 7. To make the borrower stay at date 1, the bank must ensure that

$$Ez_2 - ER_2|z \geq Ez_2 - E(R|z, z'), \forall z, z' \geq z_d, \tag{1}$$

where

$$E(R|z, z') = L + R(z') - z + Ec(L + R(z') - z), z \leq L + R(z').$$

Simplifying, and letting $\underline{R} = \min_{z'}(R(z'))$,

$$ER_2|z \leq L + \underline{R} - z + Ec(L + \underline{R} - z)\forall L + \underline{R} \geq z \geq z_d \tag{1'}$$

$$\begin{aligned} ER_2|z - Ec(L + R(z) - z) \quad &\leq \quad L + \underline{R} - z + Ec(L + \underline{R} - z) - Ec(L + R(z) - z) \\ &\leq \quad L + \underline{R} - z \leq L + z_d - z \end{aligned}$$

Therefore, for $z \geq z_d$,

$$ER_2|z - EC(ER_2|z) - (L - z) \leq z_d$$

Similarly, for $z < z_d$,

$$\begin{aligned} ER_2|z - EC(ER_2|z) \quad &= \quad ER_2|z - Ec(L) \leq L = L - z + z \\ ER_2|z - EC(ER_2|z) - (L - z) \quad &\leq \quad z. \end{aligned}$$

Integrating over $z \in [0, \overline{z}]$, we obtain

$$EER_2 - EEC_2 - \int_0^{\overline{z}} (L - z)dF \leq \int_0^{z_d} zdF + z_d\left(F(\overline{z}) - F(z_d)\right)$$

Now, the break-even constraint requires

$$\int_{\bar{z}}^{H} \bar{R}dF + E(E(R_2)) - L - \int_{0}^{\bar{z}} (L - z)dF - Ec_1 - E(Ec_2) = 0.$$

Note that $\bar{R} \leq ER_2|z + z - L \leq \underline{R} + Ec(L + \underline{R} - z)$ for all $z \in [z_d, L + \underline{R}]$. Thus, $\bar{R} \leq \underline{R} \leq z_d$. Thus,

$$\begin{aligned}
& \int_{\bar{z}}^{H} \bar{R}dF + EER_2 - EEC_2 - \int_{0}^{\overline{z}} (L - z)dF \\
\leq \quad & \int_{0}^{z_d} zdF + z_d \left(F(\overline{z}) - F(z_d)\right) + z_d \left(1 - F(\overline{z})\right) \\
= \quad & \int_{0}^{z_d} zdF + z_d \left(1 - F(z_d)\right)
\end{aligned}$$

Therefore, for $z_d < z_{GH}$

$$\begin{aligned}
& \int_{\bar{z}}^{H} \bar{R}dF + EER_2 - EEC_2 - \int_{0}^{\overline{z}} (L - z)dF - Ec_1 - L \\
\leq \quad & \int_{0}^{z_d} zdF + z_d \left(1 - F(z_d)\right) - Ec_1 - L \\
< \quad & \int_{0}^{z_{GH}} zdF + z_{GH} \left(1 - F(z_{GH})\right) - Ec_{1,GH} - L = 0
\end{aligned}$$

resulting in negative profit. Therefore, $z_d \geq z_{GH}$ holds, and thus, the dynamic bank has to be tougher in the 1st period and more lenient in the 2nd. ■