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## Couplings in Affine Toda Field Theories

Andreas Fring \*

*Universidade de São Paulo,  
Caixa Postal 369, CEP 13560 São Carlos-SP, Brasil*

### Abstract

We present a systematic derivation for a general formula for the  $n$ -point coupling constant valid for affine Toda theories related to any simple Lie algebra  $\mathfrak{g}$ . All  $n$ -point couplings with  $n \geq 4$  are completely determined in terms of the masses and the three-point couplings. A general fusing rule, formulated in the root space of the Lie algebra, is derived for all  $n$ -point couplings.

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\*FRING@BR.ANSP.USP.IFQSC

# 1 Introduction

Affine Toda field theories describe  $r$ , being the rank of a particular Lie algebra  $\mathfrak{g}$ , massive scalar fields in a relativistically invariant manner. Due to their property to be integrable, they have been subject of investigation for more than a decade [1], which has recently seen some revival since it has been conjectured [2] that they might provide an explicit Lagrangian version of integrable deformations of conformal field theories. Seminal work on the latter approach has been carried out by Zamolodchikov [3], who argued that although the conformal symmetry is broken by a perturbation, for certain cases the theory would still maintain its integrability, i.e. posses an infinite number of integrals of motion as a relic from the conformal field theory. The process of conformal symmetry breaking in this context simply corresponds to an affinisation of the Lie algebra  $\mathfrak{g}$  underlying the conformally invariant Toda theories.

Although it has turned out, from an application of the thermodynamic Bethe Ansatz [4], that only those theories with purely imaginary coupling constant  $\beta$  correspond to perturbed minimal models [5], whereas those with  $\beta$  real reproduce a central charge of the Virasoro algebra equalin g the rank of the algebra, the latter theories remain a subject of interest in their own right since they might give new insight into general principles of field theory beyond perturbation theory. This hope is sustained by the fact that it has been possible to understand the on-shell physics by making reasonable Ansätze for scattering matrices [6-11,4] which satisfy all consistency requirements demanded by general principles of axiomatic quantum field theory (in particular the bootstrap equation), supported by perturbative checks [12, 6, 13].

For the simplest models of affine Toda field theories, i.e. the Sinh-Gordon theory ( $A_1$ ) and the Bullough-Dodd model ( $A_2^{(2)}$ ) [14], it has been possible to extend the knowledge off-shell and compute form factors. In this approach the proper un-

derstanding of the n-point coupling is of vital importance. Since it has turned out that many of the classical features of affine Toda field theories, in particular for the simply laced case, survive in the quantum version, it is desirable to obtain explicit values for the n-point couplings. Generalizing the work in [15] by extending it to higher order, a systematic derivation of expressions for all n-point couplings valid for any Lie algebra will be the subject of investigation in this paper.

## 2 The n-point coupling constant

Following the notation of [15, 11] we shall start with the Lagrangian density of affine Toda field theory corresponding to any simple Lie algebra  $\mathfrak{g}$

$$\mathcal{L} = tr \left( \frac{1}{2} \partial_\mu \Phi \partial^\mu \Phi - e^{\beta\Phi} E e^{-\beta\Phi} E^\dagger \right) , \quad (2.1)$$

proposed in this form originally in [16]. Here  $\Phi$  denotes a field depending on  $x, t$  and being an element of the Cartan subalgebra  $\mathfrak{h}$ . Taking the coupling constant  $\beta$  to be real and using the fact that the cyclic element  $E$  [17] commutes with its hermitian conjugate  $E^\dagger$ , the theory will possess, on the contrary to  $\beta$  being purely imaginary [18], a unique classical vacuum at  $\Phi = 0$  due to the vanishing of the linear term in  $\Phi$ .

For what follows the existence and properties of a second Cartan subalgebra  $\mathfrak{h}'$ , furnished by the set of all elements commuting with  $E$ , will be of vital importance. Taking two elements, say  $T_i$  and  $T_j$ , which commute with the cyclic element, the Jacobi identity involving this three elements yields that  $[T_i, T_j]$  commutes with  $E$  too. Furthermore from  $tr(E[T_i, T_j]) = tr(T_j[T_i, E])$  one concludes that the commutator of  $T_i$  and  $T_j$  vanishes and therefore  $\mathfrak{h}'$  is indeed a Cartan subalgebra.

Taking now  $T_3 \in \mathfrak{h}$  to be the generator of the principal  $SU(2)$ - subalgebra embedded in  $\mathfrak{g}$ , or equivalently the vector of which the scalar product with any

root gives its height, the so-called principal element

$$S := e^{\frac{2\pi iT_3}{h}} \quad (2.2)$$

can be defined, the adjoint action of which on  $\mathfrak{g}$  provides a  $\mathbf{Z}^n$  grading

$$\mathfrak{g} = \bigoplus_{t=0}^{h-1} \mathfrak{g}_t \quad S \mathfrak{g}_t S^{-1} = \omega^n \mathfrak{g}_t \quad , \quad (2.3)$$

with  $\omega = \exp(\frac{2\pi i}{h})$ . The order of  $S$  coincides with the Coxeter number of  $\mathfrak{g}$ . Denoting the generators in the Cartan-Weyl basis of  $\mathfrak{h}'$  and  $\mathfrak{h}$ , by  $h_i$  and  $H_i$ , respectively, the action of the principal element yields that  $H_i \in \mathfrak{g}_0$  and  $h_i \in \mathfrak{g}_1$ . On the other hand the adjoint action of  $S$  on the step-operators can be used to define a Coxeter transformation  $\sigma$ , a product of Weyl reflections in some complete set of simple roots  $\alpha_i$ ,  $S e_{\alpha} S^{-1} = e_{\sigma(\alpha)}$ . Since all Coxeter transformations are conjugate to each other, it is necessary to make a particular choice. As has been first pointed out in [19] it is possible to select out two special elements of the Weyl group,  $\sigma_+$  and  $\sigma_-$ , consisting out of products of Weyl reflections solely in simple “white” and “black” roots, respectively. Here the two colours are associated to the vertices of the Dynkin diagram of  $\mathfrak{g}$  in such a way that no two vertices of the same colour are connected. Selecting furthermore the colour values  $c(w) = -c(b) = 1$ , the Coxeter transformation will separate the set of all roots into  $r$  distinct orbits  $\Omega_i$ , where each orbit is represented by  $\gamma_i = c(i)\alpha_i$ .  $\sigma_+$  and  $\sigma_-$  can be used to define unambiguously a Coxeter element universally for all  $\mathfrak{g}$ , i.e.  $\sigma = \sigma_- \sigma_+$ , for more details and notation refer [15, 11]. Using this facts, we shall now define the following element in  $\mathfrak{h}$

$$A_i := \frac{1}{\sqrt{h}} \sum_{n=1}^h S^n e_{\gamma_i} S^{-n} = \frac{1}{\sqrt{h}} \sum_{n=1}^h e_{\sigma^n(\gamma_i)} \quad (2.4)$$

in which we choose to expand the field  $\Phi$

$$\Phi = \sum_{i=1}^r A_i \phi_i \quad . \quad (2.5)$$

Here  $\phi_i$  are the  $r$ , denoting the rank of the Lie algebra  $\mathfrak{g}$ , scalar fields of the theory having the property  $\phi_i^* = \phi_{\bar{i}}$ .

Expanding now the potential term in (2.1), the  $n^{\text{th}}$  order coupling  $C_{l_1 \dots l_n}$  is defined as usual via the relation

$$\frac{1}{n!} C_{l_1 \dots l_n} \phi_{l_1} \dots \phi_{l_n} . \quad (2.6)$$

For the Lagrangian (2.1) it turns out to be

$$C_{l_1 \dots l_n} = (-1)^{n+1} \beta^n \text{tr} \left( E^\dagger K_{l_1 \dots l_n} \right) \quad (2.7)$$

where we introduced the tower of commutators

$$K_{l_1 \dots l_n} := \left[ \left[ \dots \left[ E, A_{l_1} \right] \dots \right], A_{l_{n-1}} \right], A_{l_n} \right] . \quad (2.8)$$

Or alternatively to (2.7), avoiding to compute  $K$  to order  $n$ , by exploiting the properties of the trace we obtain

$$C_{l_1 \dots l_n} = (-1)^{n+1} \beta^n \begin{cases} \text{tr} \left( K_{l_1 \dots l_{\frac{n}{2}}} K_{l_{\frac{n}{2}+1} \dots l_n}^\dagger \right) & \text{for } n \text{ even} \\ \text{tr} \left( [K_{l_1 \dots l_{\frac{n-1}{2}}}, K_{l_{\frac{n+1}{2}} \dots l_{n-1}}^\dagger] A_{l_n} \right) & \text{for } n \text{ odd} . \end{cases} \quad (2.9)$$

We shall now attempt to compute the  $K_{l_1 \dots l_n}$  to all orders. Since  $E \in \mathfrak{h}'$  we can expand it in terms of the basis of  $\mathfrak{h}'$ , i.e.  $E = q(1) \cdot h$  with  $q(n)$  denoting the eigenvector of the Coxeter element with eigenvalue  $\omega^{s(n)}$

$$q(n) = \sum_{k \in B} x_k(n) \alpha_k + e^{-\frac{i\pi}{h} s(n)} \sum_{k \in W} x_k(n) \alpha_k , \quad (2.10)$$

$s(n)$  labeling the exponents of  $\mathfrak{g}$  and  $x_k(n)$  being the left eigenvector of the Cartan matrix. We then compute with the usual commutation relations in the Cartan-Weyl basis

$$K_i = [E, A_i] = \frac{q(1) \cdot \gamma_i}{\sqrt{h}} \sum_{n=1}^h \omega^{-n} e_{\sigma^n(\gamma_i)} . \quad (2.11)$$

Evidently  $K_i \in \mathfrak{g}_1$ , having furthermore the properties

$$\text{tr}(K_i K_j^\dagger) = (q(1) \cdot \gamma_i)(q^*(1) \cdot \gamma_j) \delta_{ij} \quad \text{tr}(K_i E) = \text{tr}(K_i E^\dagger) = 0 \quad (2.12)$$

Next we compute recursively

$$K_{ij} = \frac{q(1) \cdot \gamma_i}{h} \sum_{p,q=1}^h \omega^{-p} S^p \left[ e_{\gamma_i}, e_{\sigma^q(\gamma_j)} \right] S^{-p} . \quad (2.13)$$

There can only be two possibilities for this to be non-zero, that is either the roots of the two step-operators add up to a third or to zero. Starting with the former case, we know from the solution of the fusing rule [11] that there exist only two possibilities for this to happen, that is

$$\left[ e_{\gamma_i}, e_{\sigma^{\zeta(j)-\zeta(i)}(\gamma_j)} \right] = \varepsilon(i, j, \bar{k}) e_{\sigma^{\zeta(\bar{k})-\zeta(i)}(\gamma_{\bar{k}})} \quad (2.14)$$

and a similar commutator after replacing  $\varepsilon(i, j, \bar{k}) \rightarrow \varepsilon'(i, j, \bar{k})$  and  $\zeta(t) \rightarrow \zeta'(t) = -\zeta(t) + \frac{c(t)-1}{2}$  for  $t = i, j, k$ . Here the two triplets of integers  $(\zeta(i), \zeta(j), \zeta(k))$  and  $(\zeta'(i), \zeta'(j), \zeta'(k))$  denote the two inequivalence classes of solutions of the fusing rule for a three particle process. From the fact that  $A_i$  and  $A_j$  commute we obtain that  $\varepsilon$  and  $\varepsilon'$  add up to zero and we can carry out the sum over  $q$  in (2.13), which after some re-arrangement yields

$$\begin{aligned} & \frac{2i}{\sqrt{h}} \sum_{\bar{k}} \varepsilon(i, j, \bar{k}) \frac{m_i}{m_j} \sin \left( 2\zeta(\bar{k}) - 2\zeta(i) + \frac{c(i) - c(\bar{k})}{2} \right) K_{\bar{k}} \\ &= \frac{1}{\beta} \sum_{\bar{k}} \frac{c_{ij\bar{k}}}{m_{\bar{k}}^2} K_{\bar{k}} . \end{aligned} \quad (2.15)$$

The other possibility for the commutator in (2.13) to be non-zero is when  $j \in \Omega_{\bar{i}}$  in which case the only contribution in the sum over  $q$  comes from  $q = \frac{h}{2} + \frac{c(\bar{i})-c(i)}{4}$ . By defining the quantity

$$L_i := \frac{q(1) \cdot \gamma_i}{h} \sum_{p=1}^h \omega^{-p} \sigma^p(\gamma_i) \cdot h \quad (2.16)$$

the total contribution in (2.13) will be

$$K_{ij} = \frac{1}{\beta} \sum_{\bar{k}} \frac{c_{ij\bar{k}}}{m_{\bar{k}}^2} K_{\bar{k}} + L_i \delta_{ij} . \quad (2.17)$$

It turns out to be un-practical to carry on with this expression to higher order or even to take the trace at this stage will not give an obvious expression involving quantities which possess an obvious physical interpretation. As so often in this context the solution lies in a change of base.

From the adjoint action of  $S$  on  $L_i$  we obtain that  $L_i \in \mathbf{h}'$ , and further with the fact that they are linearly independent and commute we obtain that they furnish a basis of  $\mathbf{h}'$ . However taking the trace we observe that it is not orthogonal. Nonetheless, we can construct such a base by means of a particular element in  $\mathfrak{g}_{s(n)}$ , defined as

$$E_n := q(n) \cdot h \quad . \quad (2.18)$$

From the complex conjugation of the eigenvalue equation for  $q(n)$  and the equation for the inner product of  $q(n)$  with  $\gamma_i$ , [11] equation (3.7), we derive

$$q^*(n) = q(r+1-n) \quad . \quad (2.19)$$

and further from the unitarity of the Coxeter element we obtain

$$q^\dagger(n)q(m) = \delta_{nm} \quad . \quad (2.20)$$

Therefore

$$E_n^\dagger = E_{r+1-n} \quad \text{and} \quad \text{tr} \left( E_n^\dagger E_m \right) = \delta_{nm} \quad , \quad (2.21)$$

Hence we have obtained a base of  $\mathbf{h}'$  with the property  $\text{tr}(h_i h_j) = \delta_{ij}$

$$h_i = \sum_{n=1}^r q_i^*(n) E_n \quad . \quad (2.22)$$

Thus on substituting this into (2.16) we obtain

$$L_i = iq(1) \cdot \gamma_i \sum_{n=1}^r \alpha_i^2 x_i(r+1-n) \sin \left( \frac{2\pi}{h} \right) e^{-is(r+1-n)\frac{1-c(i)}{2}} \sum_{p=1}^h e^{-\frac{2\pi i}{h}(s(r+1-n)+1)p} E_n \quad .$$

The sum over  $p$  will always vanish except when  $n = 1$  and therefore this becomes

$$iq(1) \cdot \gamma_i \alpha_i^2 x_i(r) \sin \frac{2\pi}{h} e^{-i\frac{2\pi is(r)}{h} \frac{1-c(i)}{2}} E = \frac{m_i^2}{\beta^2} E \quad (2.23)$$



an so we can re-write (2.13) in a more suitable way as

$$K_{ij} = \frac{1}{\beta} \sum_{\bar{k}} \frac{C_{ij\bar{k}}}{m_{\bar{k}}^2} K_{\bar{k}} + \frac{m_i^2}{\beta^2} E \delta_{ij} . \quad (2.24)$$

Thus further K's can now be obtained recursively

$$K_{ijk} = \frac{1}{\beta^2} \sum_{\bar{l}, \bar{m}} \frac{C_{ij\bar{l}} C_{l\bar{k}\bar{m}}}{m_{\bar{l}}^2 m_{\bar{m}}^2} K_{\bar{m}} + \frac{m_i^2}{\beta^2} \delta_{ij} K_k + \frac{C_{ijk}}{\beta^3} E \quad (2.25)$$

$$K_{ijkl} = \frac{1}{\beta^3} \sum_{\bar{i}, \bar{n}, \bar{s}} \frac{C_{ij\bar{i}} C_{i\bar{k}\bar{n}} C_{n\bar{l}\bar{s}}}{m_{\bar{i}}^2 m_{\bar{n}}^2 m_{\bar{s}}^2} K_{\bar{s}} + \frac{1}{\beta^3} m_i^2 \delta_{ij} \sum_{\bar{t}} \frac{C_{kl\bar{t}}}{m_{\bar{t}}^2} K_{\bar{t}} + \frac{C_{ijk}}{\beta^3} K_l \quad (2.26)$$

$$\frac{1}{\beta^4} \left( \sum_{\bar{t}} \frac{C_{ij\bar{t}} C_{t\bar{k}l}}{m_{\bar{t}}^2} + m_i^2 \delta_{ij} m_k^2 \delta_{\bar{k}l} \right) E .$$

From the commutation relations it is now evident how this generalizes and we find the following Feynman like rules for the construction of a general  $K_{l_1 \dots l_n}$ : Start with the contraction of some indices always from the left hand side. A fusing of two indices to a third, say  $ij$  to  $k$ , contributes a factor

$$\frac{1}{\beta} \sum_{\bar{k}} \frac{C_{ij\bar{k}}}{m_{\bar{k}}^2} ,$$

whereas the annihilation of two indices,  $i$  and  $j$  say, contributes the mass matrix

$$\frac{m_i^2 \delta_{ij}}{\beta^2} .$$

If at the end of the contraction process there is still an index left it will contribute a  $K_i$ , otherwise if all indices are consumed we close with an  $E$ . Performing all possible contractions then gives the final answer for  $K_{l_1 \dots l_n}$ .

The proof of this rule follows by induction. Clearly, it is true for  $K_{l_1 l_2}$ , since we can either contract the two indices to a third or annihilate them, and thus reproduce equation (2.24). Assuming the validity of the rule for  $K_{l_1 \dots l_{n-1}}$  we can employ the fact that  $K_{l_1 \dots l_n} = [K_{l_1 \dots l_{n-1}}, A_{l_n}]$  and deduce the expression for  $K_{l_1 \dots l_n}$ . According to the rule the terms in  $K_{l_1 \dots l_{n-1}}$  can only finish with  $K_i$  or  $E$ , corresponding to the situation in which the last index is left or consumed, respectively. In the former

case the additional index  $l_n$  can either be contracted to a third or annihilated, which corresponds to an extension of this terms by  $[K_i, A_{l_n}] = K_{il_n}$ . Whereas in the latter case there is no index left and we have to extend the terms with  $K_{l_n} = [E, A_{l_n}]$

According to (2.7) or (2.9) we can now compute the  $n$ -point coupling constant. We recover correctly the mass matrix  $C_{ij} = -m_i^2 \delta_{ij}$  the three point coupling and further

$$C_{ijkl} = -\sum_{\bar{i}} \frac{C_{ij\bar{i}} C_{\bar{i}kl}}{m_{\bar{i}}^2} - m_i^2 \delta_{ij} m_k^2 \delta_{\bar{k}l} \quad (2.27)$$

$$C_{ijkln} = \sum_{\bar{i}, \bar{u}} \frac{C_{ij\bar{i}} C_{\bar{i}k\bar{u}} C_{\bar{u}ln}}{m_{\bar{i}}^2 m_{\bar{u}}^2} + m_i^2 \delta_{ij} C_{kln} + C_{ijk} m_l^2 \delta_{\bar{l}n} \quad (2.28)$$

Equation (2.27) confirms the result of [9, 13] derived from the assumption that certain projection operators exist.

So we can cast the formula for the  $n$ -point coupling into the following compact form

$$C_{l_1 \dots l_n} = (-1)^{n+1} \sum_{t=1}^{n-2} \sum_{\overleftrightarrow{x}} \frac{x_1 \dots x_t}{\mathcal{N}_t} m_{l_{n-1}}^2 \delta_{\bar{l}_{n-1} l_n} \quad (2.29)$$

where the  $x_i$  for each particular  $t$  are given by

$$x_1, \dots, x_{2(t+1)-n} = \sum_{\bar{k}} \frac{C_{\mu\nu\bar{k}}}{m_{\bar{k}}^2} \quad x_{2t+3-n}, \dots, x_t = m_{l_\nu}^2 \delta_{\bar{l}_\nu l_\mu} \quad (2.30)$$

Here  $\sum_{\overleftrightarrow{x}}$  denotes the sum over all possible permutations of the  $x_i$ . The factor  $\mathcal{N}_t$  takes care of the overcounting when symmetric terms are permuted. Its explicit value is given by

$$\mathcal{N}_t = (2t + 2 - n)!(n - t - 2)! \quad (2.31)$$

Having carried out the sums in (2.29), the greek indices  $\mu, \nu, \dots$  have to be substituted in the same order as on the left hand side of this equation. A three-point coupling will always be connected to its neighbouring term on the right via a sum over a dummy variable and a ‘‘zero momentum propagator’’.

This formula follows simply from the fact that  $\frac{1}{\beta^n} C_{l_1 \dots l_n}$  corresponds to the term in front of  $E$  in  $K_{l_1 \dots l_n}$ . Since the last two indices are always annihilated in this

term by construction and all the other terms are symmetric in the  $C$ 's and  $\delta$ 's, up to a permutation of the  $C$ 's and  $\delta$ 's among each other we obtain (2.29).

### 3 Fusing Rules

It appears now natural to pose the question of how this rules can be formulated in the rootspace of the Lie algebra, that is how the fusing rule known for the three-point coupling [10, 15, 11] generalizes to higher order. In the same way as in [15] we check what is implied for the root space, from a non-vanishing term in the coupling constant resulting from some non-zero commutator. Starting at the lowest order, the second term in (2.24) is only non-zero, and therefore  $C_{ij}$ , if there exist two roots in the orbits  $\Omega_i$  and  $\Omega_j$  which add up to zero

$$\sigma^{\zeta(i)}\gamma_i + \sigma^{\zeta(j)}\gamma_j = 0 \quad . \quad (3.32)$$

This can only be true if particle  $j$  is the anti-particle of  $i$ , i.e.  $j = \bar{i}$ , in which case the mass matrix will be diagonal. The three-point coupling constant  $C_{ijk}$  is known to be non-vanishing if there are three orbits  $\Omega_i$ ,  $\Omega_j$  and  $\Omega_k$  such that

$$\sigma^{\zeta(i)}\gamma_i + \sigma^{\zeta(j)}\gamma_j + \sigma^{\zeta(k)}\gamma_k = 0 \quad . \quad (3.33)$$

From the previous section we obtain that  $C_{l_1\dots l_n}$  for  $n \geq 4$  is non-zero if and only if there exist  $n$  roots in  $\Omega_i$  for  $i = 1, \dots, n$  which add up to zero

$$\sum_{i=1}^n \sigma^{\zeta(i)}\gamma_i = 0 \quad , \quad (3.34)$$

together with the additional constraint that the following  $n - 2$  triangles exist

$$\begin{aligned} \sigma^{\zeta(1)}\gamma_1 + \sigma^{\zeta(2)}\gamma_2 &= \sigma^{\zeta(\bar{t}_1)}\gamma_{\bar{t}_1} \\ \sigma^{\zeta(\bar{t}_1)}\gamma_{\bar{t}_1} + \sigma^{\zeta(3)}\gamma_3 &= \sigma^{\zeta(\bar{t}_2)}\gamma_{\bar{t}_2} \\ &\vdots \\ \sigma^{\zeta(\bar{t}_{n-4})}\gamma_{\bar{t}_{n-4}} + \sigma^{\zeta(n-2)}\gamma_{n-2} &= \sigma^{\zeta(\bar{t}_{n-3})}\gamma_{\bar{t}_{n-3}} \\ \sigma^{\zeta(\bar{t}_{n-3})}\gamma_{\bar{t}_{n-3}} + \sigma^{\zeta(n-1)}\gamma_{n-1} &= \sigma^{\zeta(n)}\gamma_n \quad . \end{aligned} \quad (3.35)$$

Here the intermediate particles  $\gamma_{\bar{i}}$  are elements of  $\Omega_i$  for  $i = 0, 1, \dots, r$ . By  $\Omega_0$  we denote the “orbit” just consisting out of the zero vector, which means it is allowed that some of the triangles collapse. In the same fashion as in [11] we can show that each of the equations (3.35) possesses two inequivalent solutions of equivalence classes, such that for  $C_{l_1 \dots l_n}$  there will be  $2n - 4$  solutions. We illustrate the four possible solutions for a particular  $\bar{t}$  of the fusing rule giving rise to the first term in (2.24) in figure 1. Setting  $\bar{t} \rightarrow 0$  will give rise to the additional term involving the masses.

We can make use as well of the relation between roots and weights, i.e.  $\gamma_i = (\sigma_- - \sigma_+) \lambda_i$  and reformulate the whole set of equations in the weight space, which might be used to find alternative formulations in terms of representations analogously to [20].

Taking the inner product of the whole set of vectors with  $q(1)$  and projecting equations (3.34) - (3.35) into the velocity plane will lead to n-gons, bounded by the masses  $m_1, \dots, m_n$  of the fusing particles. The further condition on this is that it has to be possible to triangulate the n-gon by using as secants only the masses present in the theory. We illustrate this for the nine-gon in figure 2 for  $C_{ijklmnopqr}$ . Notice the possibility that some of the vertices might superimpose in case some of the inner secants collapse.

## 4 Conclusions

Starting from the affine Toda field theory Lagrangian density we presented a systematic derivation for the n-point coupling constant valid for any simple Lie algebra  $\mathfrak{g}$ . The masses and the three-point couplings turn out to be the fundamental quantities in which all the higher couplings can be completely expressed. Fusing rules which provide a selection rule for non-vanishing n-point couplings have been formulated in the root space, where again the fundamental identities are the fusing

rules for the non-vanishing three-point coupling, i.e. that is the "triangle rule". Using the result that the classical fusing rule for a non-vanishing three-point coupling becomes quantum mechanically equivalent to the bootstrap equation [11], we might expect this to generalize. However, provided the analogy holds a "bootstrap equation" involving more than three particles will only contain redundant information what the on-shell physics concerns and "everything" can be extracted from the three-particle bootstrap. However, off-shell the information about the possible  $n$ -particle interactions will be of importance and the classical information may be used to formulate a general theory of form factors for affine Toda theories.

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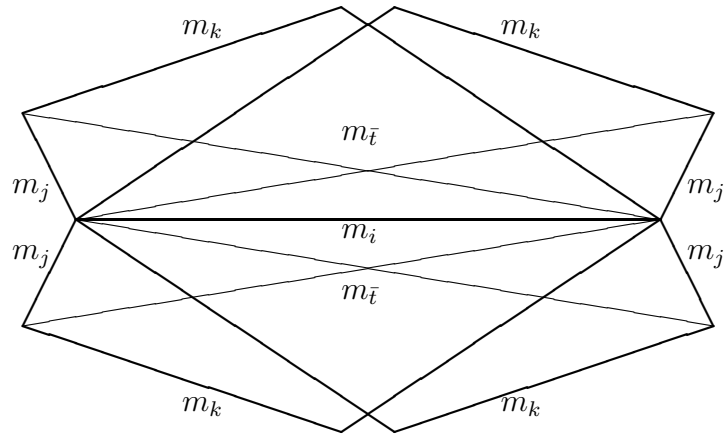


Figure 1: Four-gons in the velocity plane for  $C_{ijkl}$

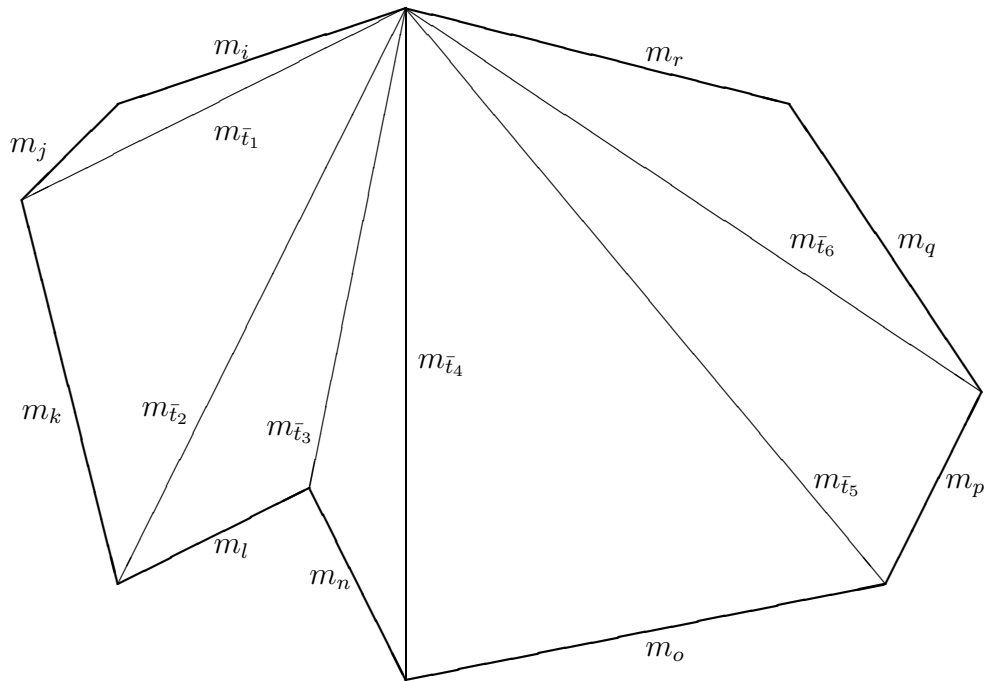


Figure 2: Triangulated nine-gon in the velocity plane for  $C_{ijklmnopqr}$