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# Equilibrium Investment in High-Frequency Trading Technology: A Real Options Approach


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# Investment in High-Frequency Trading Technology: A Real Options Approach*

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**Abstract**

This paper derives an optimal timing strategy for a regular slow trader considering investing in a high-frequency trading (HFT) technology. The market is fragmented, and slow traders compete with fast traders for trade execution. Given this optimal timing rule, I then determine the equilibrium level of fast trading in the market as well as the welfare-maximising socially optimal level. I show that there is always a unique cost of investment such that the equilibrium level of fast trading and the socially optimal level coincide. Finally I discuss potential policy responses to addressing equilibrium and social optimality misalignment in HFT.

**Keywords:** Finance, High frequency trading, Fragmented markets, Real options.

**JEL Classification Numbers:** C61, G10, G20.

## 1 Introduction

Over the last decade, the state of financial markets has changed considerably. In the first instance, markets have become highly fragmented. There are now more than 50 trading venues for U.S. equities - 13 registered exchanges and 44 so called Alternative Trading Systems (see Biais et al. [4] and O'Hara and Ye [21]). Hence, traders must search across many markets for quotes and doing so can be costly as it may delay full execution of their orders.

In response to the increase in market fragmentation, so called high frequency trading (HFT) technologies have been developed to reduce the associated costs borne by traders. HFT is a type of algorithmic trading that uses sophisticated computer algorithms to implement vast amounts of trades in extremely small time intervals. For example, traders can buy colocation rights (the placement of their computers next to the exchange's servers) which gives them fast

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access to the exchange's data feed, they can invest in smart routers which can instantaneously compare quotes across all trading venues and then allocate their orders accordingly, or they can invest in high-speed connections to the exchanges via fiber optic cables or microwave signals. Proprietary trading desks, hedge funds, and so called pure-play HFT outlets are investing large sums of money into such technologies in an effort to outpace the competition. Indeed, according to Hoffman [16], recent estimates suggest that HFTs are now responsible for more than 50% of trading in U.S. equities.

In a recent paper, O'Hara [20] details the many ways in which market microstructure has changed over the past decade and calls for a new approach to research in this area which "reflects the new realities of the high frequency world". Nevertheless, there has been a growth in the finance literature on HFT in recent years (see a survey by Foucault [10]). Much of the literature is empirical and, on the whole, the consensus has been that HFT improves liquidity through lower bid-ask spreads (Hendershott et al. [15] and Hasbrouck and Saar [13]); is highly profitable (Menkveld [18] and Baron et al. [2]); and facilitates price discovery (Hendershott and Riordan [14] and Brogaard et al. [5]). The theoretical literature has also been growing in recent years. For example, Hoffman [16] presents a stylised model of HFT in a limit order market where agents differ in their trading speed; Pagnotta and Philippon [22] propose a model in which trading venues invest in speed and compete for traders who choose where and how much to trade; and Biais et al. [4] develop a model of equilibrium investment in a HFT technology in a Glosten and Milgrom [12] type framework.

While the finance literature strengthens our understanding of the nature and implication of HFT in many dimensions, to the best of my knowledge, the decision to be fast is always taken to be exogenous. However, the decision to be fast or slow is a real investment like any other. In particular, there is uncertainty associated with the payoff generated from investing in the HFT technology and the investment involves an upfront investment cost which is sunk. Moreover, the slow trader can adopt the technology at any future point in time with no terminal date. This adds a dynamic aspect to the investment decision which is not accounted for in other models. Hence, it makes sense to endogenise the investment decision and determine how the optimality of investment timing has implications for HFT in the marketplace. To this end, the problem has a place in the operations research literature.

In this paper I use a real options approach to determine analytically the optimal time for financial market traders to invest in a HFT technology such that the market is fragmented and slow traders compete with HFTs for trade execution. Given this optimal timing strategy, I then determine and compare the equilibrium level of fast trading in the market with the welfare-maximising socially optimal level. While optimal investment timing has been well-developed in the operations research literature through its application to many different types of problems (see, for example, Banerjee et al. [1]; Battauz et al. [3]; Munoz et al. [19]; Delaney and Thijssen [7]), it is the first application of the approach in a HFT environment.

There are a number of novel results generated by the model, all of which arise from the optimal timing policy derived, in particular, the inclusion of a value of waiting to invest into the slow trader's value function. The results are as follows. (i) It is optimal to wait longer to invest if the level of high frequency trading in the market increases, and early adoption is optimal if the slow trader's probability of finding a liquid venue decreases. (ii) It is also optimal to wait longer if the uncertainty of the profit process increases and, if the probability of finding a liquid venue is low, if the discount rate increases and/or the shortfall in the expected rate of return from holding the option to invest decreases. However, if the probability of finding a liquid venue is high, an increase in the discount rate and a decrease in the shortfall make early adoption optimal. These comparative static results with respect to the discount rate and the shortfall are novel from a real options perspective, which I discuss in a later section of the paper. (iii) There is always a unique equilibrium level of fast trading in the market. (iv) The equilibrium level of fast trading never increases in the probability of finding a liquid venue. (v) There can be either under-investment or over-investment in equilibrium relative to the trading industry welfare-maximising socially optimal level. Over-investment arises when the cost of investing in the technology is low, and under-investment arises when the cost is high. (vi) There is a unique cost of investment such that the socially optimal level of fast trading and the equilibrium level coincide. (vii) Increases in the discount rate and/or uncertainty over the profit process alleviate the extent of over-investment in equilibrium and exacerbate the extent of under-investment. However, both under- and over-investment are alleviated by increases in the shortfall in the expected rate of return from holding the option to invest.

Finally, I discuss the efficacy of using Pigouvian taxes or subsidies to align the equilibrium level of HFT with the socially optimal level. In the case of under-investment, subsidising slow traders' investment cost by an amount equal to the externality effect generated by fast trading appears to be an appropriate policy response to aligning these levels, but in the case of over-investment, taxing HFTs by an amount equal to the externality effect may not be the most effective response because it will not alter the prevailing level of HFT activity in the market. Instead, when there is over-investment, subsidising HFTs to exit the market by refraining from more fast trading, by an amount equal to the externality effect, may be a more effective response in that case.

The remainder of this paper is organised as follows. The set-up of the model is described in the next section. In Section 3 the solution to the optimal stopping problem is given, as well as a brief discussion on the comparative statics. Section 4 characterises and compares the market equilibrium and welfare-maximising socially optimal levels of fast trading, as well a providing some discussion on policy implications. Section 5 concludes. All proofs are placed in the appendix.

## 2 The Model

Consider a risk-neutral market trader contemplating investment in a HFT technology. Time is continuous, the horizon is infinite, and indexed by $t \in [0, \infty)$. The trader discounts the future at the risk-free rate $r > 0$. Investing in the technology incurs a sunk cost $I > 0$. Before investing, the trader is a regular (slow) trader who trades in markets where fast high frequency traders (HFTs) also trade. Hence, he is exposed to the impact such traders have on the likelihood of his orders getting executed at favourable prices. Once he invests, however, he becomes one of the HFTs.

The objective is to determine the optimal time to invest in the technology so that the trader's discounted expected payoff from investing is maximised. To solve for this optimal stopping problem, we must determine the expected present value of trading profits for the trader as if he were (i) fast and (ii) slow. These value functions depend on the trading environment which I describe in following subsection, and then I derive the value functions in accordance with this environment.

## 2.1 The Trading Environment

The market is fragmented where slow traders compete with HFTs. I capture this by assuming there is a size-one continuum of trading venues distributed on a circle and indexed clockwise from 0 to 1. When markets are fragmented, liquidity conditions vary across venues so that traders must search for quotes which can lead to delayed execution of orders. At every instant a fraction $\lambda < 1$ of the trading venues are "liquid". In the context of this model, a trading venue is liquid if the trader's order, when sent to the exchange venue, is fully executed.

At any instant, the trader can only send an order to one trading venue. His choice of venue is random and uniformly drawn from the unit circle. HFTs have extremely fast connection speeds to the market and can observe all venues instantaneously. Thus, if the trader is fast, he always finds a liquid one immediately upon order submission. He is indifferent between liquid venues. However, if the trader is slow, he must search for liquid trading venues and finding one can take time. Thus, at each instant he executes his trade with probability $\lambda$. Otherwise, with complementary probability, he must continue to search for a liquid venue.[1]

## 2.2 Valuations

The trading activities of a slow trader yields a stream of profits $X^S$ in perpetuity, and the activities of a fast trader yields a profit stream $X^F$, such that both processes depend on a stochastic process $(X_t)_{t\geq 0}$ which is a geometric Brownian motion of the form:

$$dX = \mu X dt + \sigma X dW, \tag{1}$$

for constants $0 < \mu < r$ and $\sigma > 0$, which represent the drift and volatility of the process respectively, and $W = (W_t)_{t\geq 0}$ is a standard Brownian motion.

I describe the different profit process specifications for a slow and a fast trader later in the section, but will determine a process for each type that depends positively on the stochastic component $X$. Hereafter, I refer to $X$ simply as the "profit process".

[1] This set up, in terms of a fragmented market environment, has the flavour of Biais et al. [4].

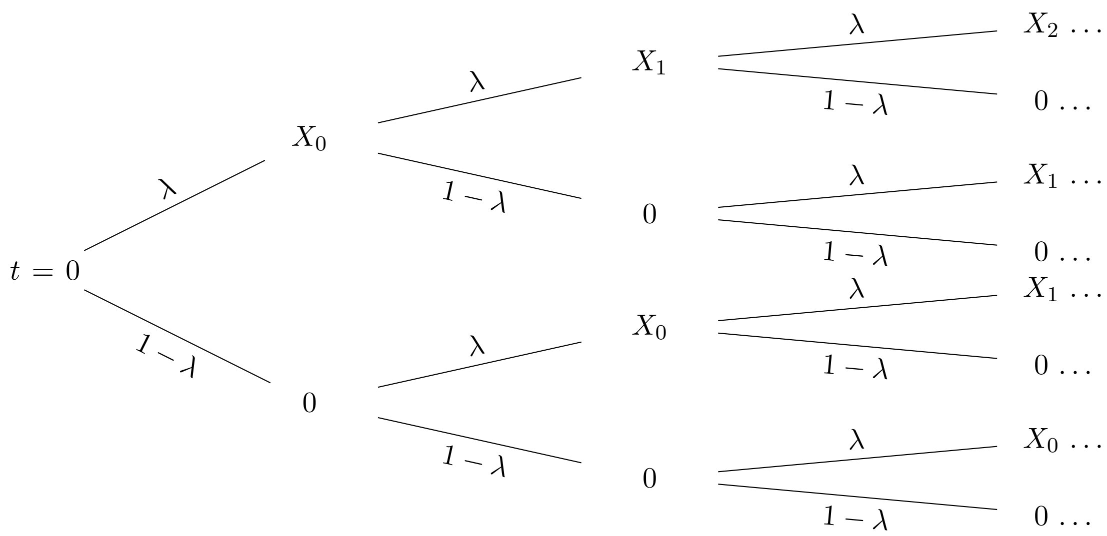


Figure 1: The Search Process

### 2.2.1 The Search Process

As discussed, if the trader is slow, he does not have access to the HFT technology and cannot observe all trading venues instantaneously. Thus, he may sometimes send orders to a trading venue which is not liquid. He can only send orders to one trading venue in any instant. If he sends an order to some venue at some instant $t$, it will be executed with probability $\lambda$. If it is not executed, I assume that the trader continues to search for a liquid venue for that order until it is executed. To describe the process, I let the dependence on time be explicit so that $X_t$ denotes the value of $X$ at time $t$. The process works as follows (a graphical depiction of which is given in Fig. 1 below):

- At $t = 0$: the trader submits an order which delivers a profit $X_0$ with probability (w.p.) $\lambda$, otherwise gets 0.
- At $t = 1$ the trader will do *one* of the following:
  1. If his order is executed at $t = 0$, he submits another order at $t = 1$ which will yield, if filled, $X_1$. It gets filled with probability $\lambda$.
  2. If, however, his $t = 0$ order is not filled, at $t = 1$ he is searching for a liquid venue for his $t = 0$ order and gets it filled w.p. $\lambda$. If filled, it yields the profit $X_0$.

  Thus,

$$\begin{aligned} E^0[X_1^S] &= \lambda^2 E^0[X_1] + (1-\lambda)\lambda X_0 \\ &= \lambda X_0 \left(\lambda e^{\mu} + (1-\lambda)\right) \end{aligned} \tag{2}$$

  since $X_1$ and $X_0$ are governed by (1).

- The process continues in this way as long as the trader is slow.

Therefore, for some arbitrary instant $t \geq 0$, the search process described implies that

$$\begin{aligned} E^0[X_t^S] =& \lambda(1-\lambda)^t X_0 \sum_{i=0}^{t} \binom{t}{i} \left(\frac{\lambda e^{\mu}}{1-\lambda}\right)^i \\ =& \lambda X_0((1-\lambda) + \lambda e^{\mu})^t \end{aligned} \tag{3}$$

by the Binomial theorem.

If, however, the trader is fast, he gets his order filled with certainty whenever he sends it to the market because he can observe which venues are liquid. Thus, he does not need to search for liquidity and at every instant $t$ that he trades, he gets a profit of $X_t$. Thus, $X_t^F \equiv X_t$ and

$$E^0[X_t^F] = X_0 e^{\mu t}, \tag{4}$$

which is, in fact, Eq. (3) for $\lambda = 1$.

Moreover, since $\lambda < 1$, $E^0[X_t^F] > E^0[X_t^S]$ for all $t$. This says if the trader is fast, he has higher trading profits in expectation than if he were slow because his execution delay costs are zero. This is intuitive and necessary because if this condition did not hold, investment in the HFT technology would never be optimal.

### 2.2.2 Prevalence of HFT in Market

Much of the literature in this area documents that HFTs generate adverse selection costs for slow traders because their accelerated access to value-relevant information for the asset implies they profit more at the expense of slow traders (see, for example, empirical studies by Baron et al. [2] and Brogaard et al. [5]). Moreover, as documented in Biais et al. [4], anecdotal evidence suggests that the profitability of HFTs has declined in recent years which may be due to the fact that the prevalence of high frequency trading in the market place has increased. I capture these stylized facts by assuming that $X_0 := (1+\alpha)^{-1}x$, for some $x > 0$, and where I define $\alpha < 1$ as the fraction of HFT activity in the market at each instant. Therefore, since the profit processes both follow a geometric Brownian motion, the current (discounted) expected profit from trading for *each type* of trader, for any future time, decreases in the level of HFT activity.

I should point out here that since a higher level of $\alpha$ is simply equivalent to a lower initial level of the profit process, this parameter is of particular interest in Section 4, where I examine the equilibrium level of fast trading in the market, as well as the trading industry value maximising level.

## 3 The Optimal Stopping Problem

If the trader decides to adopt the technology at time $\tau$, the present value of the entire investment opportunity, denoted by $V(x)$, is given by the expected discounted profit flow of a slow trader between now (i.e., $t = 0$) and time $\tau$, at which time he pays the investment cost $I$ and becomes a fast trader. From time $\tau$ onwards, the present value of the investment opportunity is the expected discounted profit flow of a fast trader[2]; i.e.;

$$V(x) = E^0\left[\int_0^\tau e^{-rt}X_t^S dt + \int_\tau^\infty e^{-rt}X_t^F dt - e^{-r\tau}I\right] \tag{5}$$

where $E^{t'}$ denotes the expectation operator applied at time $t'$.

Using the strong Markov property of diffusions, we can re-write Eq. (5) as

$$\begin{aligned} V(x) =& E^0\left[\int_0^\infty e^{-rt}X_t^S dt - \int_\tau^\infty e^{-rt}X_t^S dt + \int_\tau^\infty e^{-rt}X_t^F dt - e^{-r\tau}I\right] \\ =& E^0\left[\int_0^\infty e^{-rt}X_t^S dt\right] + E^0\left[e^{-r\tau}E^\tau\left(\int_0^\infty e^{-rt}\left(X_t^F - X_t^S\right)dt - I\right)\right]. \end{aligned} \tag{6}$$

The problem is to find a value function $V^*(x)$ and a stopping time $\tau^* > 0$ such that the following optimal stopping problem is solved:

$$V^*(x) = \sup_{\tau\in\mathcal{T}} E^0[e^{-r\tau}F(X_\tau^S, X_\tau^F)], \tag{7}$$

for $\mathcal{T}$ the set of stopping times, and

$$F(X_\tau^S, X_\tau^F) = E^\tau\left[\int_0^\infty e^{-rt}\left(X_t^F - X_t^S\right)dt\right] - I \tag{8}$$

is the trader's payoff from adopting the technology. Hereafter I denote $F(X_t^S, X_t^F)$ by $F(X_t)$ since both $X^S$ and $X^F$ are functions of $X$.

By Fubini's theorem, and using Eqs. (3) and (4), Eq. (8) can be re-written as

$$\begin{aligned} F(X_\tau) =& X_\tau\left(\frac{1}{\delta} - \lambda\int_0^\infty e^{-rt}\left((1-\lambda) + \lambda e^{r-\delta}\right)^t dt\right) - I \\ =& \Omega(r,\delta,\lambda)X_\tau - I \end{aligned} \tag{9}$$

where $\delta := r - \mu > 0$ is the convenience yield (or shortfall)[3], and the scaling term

$$\Omega(r,\delta,\lambda) = \frac{1}{\delta} - \frac{\lambda}{(r - \ln((1-\lambda) + \lambda e^{r-\delta}))} > 0 \tag{10}$$

represents the relative advantage (in terms of profits) from being fast. Note that I have replaced $\mu$ by $r - \delta$ for the purpose of analysing the solution and comparing it with that obtained in a standard real options model with the payoff flow being obtained in perpetuity. Hereafter, $\delta$ will be referred to as the "shortfall".

I present the derivation of Eq. (10) in Appendix A, where I also prove that $\Omega(r,\delta,\lambda) > 0$.

[2]This is essentially an exchange of one profit flow for another by paying a fixed cost; see for example Dixit and Pindyck [8] Chapter 9, Section 3. However, in that example, there are two investors in a particular project and the leader's profit flow is exchanged for a different (lesser) profit flow once the follower invests.

[3]In the context of real options models of this type, the convenience yield is interpreted as the shortfall in the expected rate of return from holding the option to invest rather than the HFT technology, and hence represents an opportunity cost of waiting rather than investing now (Dixit and Pindyck [8]).

The relative advantage of being fast is induced by the elimination of delayed execution arising from the need to search for quotes in fragmented markets.

**Theorem 1.** *Let $\beta_1 > 1$ be the positive root of the quadratic equation*

$$\mathcal{Q}(\beta) = \frac{1}{2}\sigma^2\beta(\beta-1) + (r-\delta)\beta - r = 0. \tag{11}$$

*Investment takes place at the first passage time $\tau^* = \inf\{t : X_t \geq X^*\}$, for some constant*

$$X^* = \frac{\beta_1}{\beta_1 - 1}\left(\Omega(r,\delta,\lambda)\right)^{-1} I. \tag{12}$$

*Moreover, the optimal stopping problem (7) is solved by*

$$V^*(x) = \begin{cases} \frac{\lambda(1+\alpha)^{-1}x}{\left(r-\ln\left((1-\lambda)+\lambda e^{r-\delta}\right)\right)} + \left(\frac{(1+\alpha)^{-1}x}{X^*}\right)^{\beta_1} F(X^*) & \text{if } x < (1+\alpha)X^* \\ \frac{(1+\alpha)^{-1}x}{\delta} - I & \text{if } x \geq (1+\alpha)X^*, \end{cases} \tag{13}$$

*where $F(X)$ is given by Eq. (9).*

**Proof.** See Appendix B. ■

## 3.1 Comparative Statics

In this subsection I discuss the economics underlying the optimal investment strategy.

First I discuss the effects of the two HFT related parameters, $\alpha$ and $\lambda$, on the optimal investment threshold $X^*$.

**Proposition 1.** *An increase in the level of fast trading activity, $\alpha$, and an increase in the probability of finding a liquid venue, $\lambda$, makes it optimal for a slow trader to wait longer before investing.*

The level of HFT activity in the market does not directly affect the profit threshold $X^*$ above which it is optimal to become fast. However, an increase in the level of $\alpha$ reduces a slow trader's current profit from trading since $X_0 := (1+\alpha)^{-1}x$. Therefore, the slow trader will have to wait longer before it is optimal to invest since an increase in $\alpha$ increases the distance between the current profit level and $X^*$.

The probability of finding a liquid venue impacts the optimal threshold via its effect on the relative advantage of being fast; in particular, through its effect on $\Omega(r,\delta,\lambda)$. The relative advantage of being fast is high when the probability of finding a liquid venue is low and thus, it is optimal to invest sooner if the probability of finding a liquid venue decreases. This is because it becomes harder for the slow traders to find quotes quickly and owing to the search process, the more they lag behind the HFTs in terms of trading profits, i.e.; $E^0[X^F] - E^0[X^S]$ decreases in the probability of finding a liquid venue.

The following proposition states the effect of the other parameters which are standard in all real option models, but not particular to a HFT environment, on the optimal investment threshold. The results, are interesting in their own right because they differ somewhat from their effects on the investment threshold in the more standard real options investment model (see for example, Dixit and Pindyck [8], Chapter 6) as I discuss below, but they are also useful in understanding the driving forces underlying the extent of over- and under-investment in the HFT technology in equilibrium, relative to the trading industry welfare maximising level of investment, which I discuss in Section 4.2.

**Proposition 2.** *The optimal investment threshold $X^*$ increases in the uncertainty in the profit process $\sigma$. Moreover, $X^*$ increases in the discount rate $r$ and decreases in the shortfall $\delta$ when the probability of finding a liquid venue is low, and vice versa when this probability is high.*

**Proof.** See Appendix C. ∎

The effect of uncertainty on $X^*$ is as expected from other standard models of investment under uncertainty (see for example, Dixit and Pindyck [8] and McDonald and Siegal [17]). It impacts the threshold via the value of waiting. In particular, the more volatile are the profits from trading, the greater is the opportunity cost of investing in the technology since the cost of doing so is high and the investment, once made, is irreversible.

The discount rate and the shortfall impact the optimal threshold $X^*$ via their effect on the value of waiting (the option effect) and their effect on the present value of the difference in perpetuity factors of fast and slow traders (i.e., its effect on $\Omega(r, \delta, \lambda)$)

The equilibrium relationship $\delta = r - \mu$ must be maintained. Therefore, if $r$ increases, so too does the rate of growth of profit from being fast $\mu$ and, hence, the expected appreciation in the value of the option to invest in the HFT technology increases making it costlier to invest immediately rather than to wait. However, if $\delta$ increases while keeping the discount rate fixed, $\mu$ will decrease implying that it is costlier to wait rather than to invest immediately because the expected appreciation in the value of the option to invest decreases.[4]

However, an increase in $r$ also increases the present value of the relative advantage of being fast when $\delta$ is fixed (cf. Eq. (C.4) in Appendix C). This is because the expected present value of a fast trader's profit flow does not change when the shortfall is constant, but an increase in $r$ reduces the expected present value of a slow trader's profit flow. Hence, the relative advantage of being fast increases implying earlier investment is optimal. On the other hand, an increase in $\delta$, while keeping $r$ fixed, reduces the present value of the relative advantage of being fast (cf. Eq. (C.6) in Appendix C) because it reduces the expected present value of a fast trader's profit flow more than it increases the expected present value of a slow trader's profit flow. This is because an increase in $\delta$ is synonymous with a decrease in the rate of profit growth from being fast. Thus, an increase in $\delta$ implies waiting longer is optimal.

[4] We can verify both of these results technically since

$$\frac{\partial}{\partial r}\left(\frac{\beta_1}{\beta_1 - 1}\right) = -\frac{1}{(\beta_1 - 1)^2}\frac{\partial \beta_1}{\partial r} > 0$$

and

$$\frac{\partial}{\partial \delta}\left(\frac{\beta_1}{\beta_1 - 1}\right) = -\frac{1}{(\beta_1 - 1)^2}\frac{\partial \beta_1}{\partial \delta} < 0$$

since $\partial\beta_1/\partial r < 0$ when $\delta$ fixed and $\partial\beta_1/\partial\delta > 0$ when $r$ fixed (cf. Appendix C).

So long as the probability of a slow trader finding a liquid venue $\lambda$ is very low, then the relative advantage of being fast will always remain high so that any change in either $r$ or $\delta$ will not significantly impact $\Omega(r, \delta, \lambda)$. Therefore, the effect of any shift in either of the parameters on the value of waiting dominates its effect on the relative advantage of being fast. This implies that an increase in $r$ makes waiting longer optimal, but an increase in $\delta$ makes earlier investment optimal when $\lambda$ is low owing to the option effect.

On the other hand, when the probability of a slow trader finding a liquid venue is high, then value of waiting will always be high and this will not be impacted significantly by any shift in $r$ or $\delta$. Hence, the effect of a change in $r$ or $\delta$ on the relative value of being fast dominates their effects on the value of waiting implying that an increase in $r$ will make earlier investment optimal and an increase in $\delta$ will make waiting longer optimal in this case, both owing to the present value effect.

The results with respect to $r$ and $\delta$ are particularly interesting from the real options perspective because they differ from more standard result on the (positive) effects of the discount rate (when $\delta$ is fixed) and the shortfall (when $r$ is fixed) on the optimal investment threshold when the post-investment profit flow is received in perpetuity rather than as a lump sum payout (see, for example, Dixit and Pindyck [8], Chapter 6).

In their model, the effect of $r$ on the threshold is always via its effect on the value of waiting when the shortfall $\delta$ is kept constant. This is because the pre- and post- investment profit flows are linear functions of the underlying geometric Brownian motion, implying that the present value of the investment is unaffected by changes in $r$ when $\delta$ is fixed. Hence, all the impact of $r$ on the threshold comes from its effect on the value of waiting. However, in my model, the post-investment profit flow is a linear function of the geometric Brownian motion describing the profit flow, but the pre-investment profit flow is not. This is because the slow trader must search for quotes in order to find a liquid venue. Hence, the present value of investing is not unaffected by a change in $r$ in this model and, therefore, the non-linearity of the pre-investment profit flows, as a result of the search procedure, implies that the threshold may increase or decrease in $r$ owing to the option or present value effects, respectively.

In the case of $\delta$, there are two opposing effects in the Dixit and Pindyck [8] model also, but the present value effect always dominates the option effect so that an increase in $\delta$ always implies waiting longer is optimal. This corresponds with my result for a high $\lambda$, but an additional effect whereby early investment is optimal will be generated in my model when the probability of finding a liquid venue is low. This is because I include in the model the search procedure for slow traders when finding liquid venues is not certain, and this produces the additional effect.

# 4 Levels of Fast Trading

In the previous section, I derived and analysed the optimal investment timing strategy for a given level of fast trading $\alpha$. In this section, I determine the equilibrium and socially optimal levels of fast trading conditional on the optimal timing strategy derived above.

To determine the equilibrium level of fast trading, I adopt the same definition of the equilibrium as Biais et al. [4]. They define it as the level of $\alpha$ such that the *ex ante* expected payoff from fast trading is exactly equal to the *ex ante* expected payoff from slow trading. Ex ante, in the context of the model, means before a trader earns any profit from trading; i.e., when he is neither fast nor slow. Therefore, prior to earning any profit, a trader knows that once he earns a profit $X_t$ at some time $t \geq 0$, if $X_t \geq X^*$, it will be optimal for him to be fast, otherwise it is optimal for him to be slow (cf. Theorem 1). (Note that $X_t$ here corresponds directly with the initial profit level $X_0$ in the optimal stopping analysis.) However, there are a range of values $X_t$ can take over which it is optimal to be slow, and another range over which it is optimal for him to be fast. To capture this variation in possible values, I denote his time $t$ profit level by $X_t^i = (1+\alpha)^{-1}x_t^i$, for $i = 1, 2, \ldots$, and such that the $x_t^i$'s are i.i.d. and continuously distributed on $[0, \bar{x}]$ with $\bar{x} > (1+\alpha)X^*$, and have cumulative distribution function $G(\cdot)$ and density $g(\cdot)$.

The trader's ex ante expected payoff from being fast and from being slow over all possible time $t$ profit levels are denoted by $V^f(\alpha)$ and $V^s(\alpha)$, respectively. Thus, conditional on the optimal strategy defined in Theorem 1:

$$\begin{aligned} V^f(\alpha) =& E_0 \left[ e^{-rt} \left( V^*(x_t^i) + I \right) \Big| x_t^i \geq (1+\alpha)X^* \right] \\ =& e^{-rt} \int_{(1+\alpha)X^*}^{\overline{x}} \left[ V^*(x_t^i) + I \Big| x_t^i \geq (1+\alpha)X^* \right] g^f(x_t^i) dx_t^i \end{aligned} \tag{14}$$

and

$$\begin{aligned} V^s(\alpha) =& E_0 \left[ e^{-rt} V^*(x_t^i) \Big| x_t^i < (1+\alpha)X^* \right] \\ =& e^{-rt} \int_0^{(1+\alpha)X^*} \left[ V^*(x_t^i) \Big| x^i < (1+\alpha)X^* \right] g^s(x_t^i) dx_t^i, \end{aligned} \tag{15}$$

where $V^*(x_t^i)$ is given in Eq. (13), and $g^f(\cdot)$ and $g^s(\cdot)$ denote the particular value of the density function which pertains to fast and slow trading, respectively. $E_0$ denotes the expectation operator applied at time 0.

I denote the *ex ante expected* relative advantage of fast trading in the market by $\Delta(\alpha)$, where

$$\Delta(\alpha) := V^f(\alpha) - V^s(\alpha).$$

### 4.1 Equilibrium Fast Trading

As mentioned above, in the manner of Biais et al. [4], I define the equilibrium level of fast trading, denoted by $\alpha^*$, as the level of $\alpha$ such that the ex ante expected relative advantage from fast trading is equal to the cost of investing. Therefore, if the level of fast trading is $\alpha^*$ at time $t$, then

$$\begin{aligned} V^f(\alpha^*) - e^{-rt} I =& V^s(\alpha^*) \\ \iff \Delta(\alpha^*) =& e^{-rt} I \end{aligned} \tag{16}$$

Thus, if the trader earns a profit at time $t$ and if the level of HFT is $\alpha^*$, if $x_t^i = (1+\alpha^*)X^*$, he will be indifferent about being fast or slow, if $x_t^i > (1+\alpha^*)X^*$, it will be optimal to become fast, otherwise it will be optimal for him to be slow. However, the level of HFT will be an equilibrium level according to the equilibrium definition.

Now from Eq. (16), it is clear that in determining $\alpha^*$, the discount factors will cancel out because the discount factor is the same irrespective of whether the trader is fast trading or slow trading. Therefore, the equilibrium results are independent of $t$, so in the analysis that follows I drop the notational dependence on $t$.

**Proposition 3.** *There is always a unique equilibrium level of fast trading. If (i)* $\Delta(1) < I < \Delta(0)$*, then* $0 < \alpha^* < 1$*; (ii)* $\Delta(0) \leq I$*,* $\alpha^* = 0$*; or (iii)* $\Delta(1) \geq I$*,* $\alpha^* = 1$.

**Proof.** See Appendix D. ■

The fact that there is always uniqueness of equilibrium is novel. The uniqueness arises from the fact that the ex ante expected relative value of being fast always decreases in $\alpha$ when $\Delta(1) < I < \Delta(0)$ which, in turn, arises from the optimal stopping rule. If $\alpha$ increases, then the initial profit level of a trader, irrespective of whether he is fast or slow, decreases. However, this increase in $\alpha$ increases the range of initial profit levels over which it is optimal to be slow ($[0,(1+\alpha)X^*)$), the overall effect being an increase in $V^s(\alpha)$. On the other hand, an increase in $\alpha$ reduces the range of profit levels over which it is optimal to be fast ($[(1+\alpha)X^*, \bar{x}]$) and thereby reduces $V^f(\alpha)$. Overall, therefore, $\Delta(\alpha)$ decreases everywhere in $\alpha$. Moreover, in this sense, we can say that HFT induces a positive externality on slow trading in the market and a negative externality on fast trading.

If $\Delta(0) \leq I$, then the ex ante expected relative value of being fast is low if all trading activity in the market is slow. Therefore, all traders are better off being slow if all other traders are also slow so that $\alpha^* = 0$.

If $\Delta(1) \geq I$, then the ex ante expected relative value of being fast is high when all trading activity in the market is fast. Hence, everyone is better off being fast so that $\alpha^* = 1$.

Multiple equilibria can arise in other financial market models because of virtuous or vicious circles. For example, in Glosten and Milgrom [12] and Dow [9], if traders anticipate the market will be liquid, they will submit many orders, and hence, the market is liquid, and vice versa. In a similar vein, in Biais et al. [4], one equilibrium can arise in which no trader invests because each expects others not to invest, and one in which all traders should become fast because all expect everyone else to be fast. This situation of multiple equilibria arising from self-fulfilling cycles could only arise in the current model if $\Delta(\alpha)$ were to increase in the level of HFT for some values of $\alpha$. In that case, it would be possible to have three equilibria at any instant because if $\Delta(\alpha)$ increases in $\alpha$, then it is possible that $\Delta(0) < I < \Delta(1)$ could arise. In that case, $\alpha^* = 0$ would be an equilibrium (see reasoning above), $\alpha^* = 1$ would be another, and a third $\alpha^* \in (0,1)$ such that $\Delta(\alpha^*) = I$. When $\Delta(\alpha)$ decreases in $\alpha$, then the three possibilities cannot arise simultaneously. Therefore, multiplicity of equilibria is ruled out in my model because the value of waiting ensures $\Delta(\alpha)$ decreases everywhere in $\alpha$.

**Proposition 4.** *The equilibrium level of fast trading never increases in the probability of finding a liquid venue.*

**Proof.** See Appendix E. ■

This result is intuitive and the reasoning is as follows. A decrease in the probability of finding a liquid venue implies earlier investment is optimal (cf. Proposition 1). Thus, the ex ante expected payoff from investing relative to not investing decreases in the probability of finding a liquid venue and, as discussed, this $\Delta(\alpha)$ decreases in $\alpha$. Hence, for equilibrium to be maintained, $\alpha^*$ must increase in response to a decrease in $\lambda$, and vice versa. However, as long as $\Delta(0) < I$, irrespective of a change in $\lambda$, $\alpha^* = 0$, and similarly, $\alpha^* = 1$ for all $\Delta(1) > I$. In the interior region where $\Delta(1) < I < \Delta(0)$, we experience the actual negative relationship between $\lambda$ on $\alpha^*$.

## 4.2 Socially Optimal versus Equilibrium Fast Trading

The socially optimal level of fast trading is the level of $\alpha$ that maximises a trader's ex ante utilitarian welfare. Say the level of HFT in the market at time $t$ is $\alpha$. Then ex ante, a trader expects the payoff from fast trading to be $V^f(\alpha) - e^{-rt}I$, which the fraction $\alpha$ of all trader's in the market earn in expectation, and $V^s(\alpha)$ is the ex ante expected payoff earned by the remaining fraction $1 - \alpha$ of traders. Hence, the ex ante utilitarian welfare is given by

$$
\begin{aligned}
W(\alpha) =& \alpha \left( V^f(\alpha) - e^{-rt}I \right) + (1-\alpha)V^s(\alpha) \\
=& \alpha \left( \Delta(\alpha) - e^{-rt}I \right) + V^s(\alpha).
\end{aligned}
\tag{17}
$$

Denoting the socially optimal level of fast trading in the market by $\alpha^{SO}$, then $\alpha^{SO}$ solves

$$
W'(\alpha^{SO}) = \Delta(\alpha^{SO}) - I - \left( -\alpha^{SO} \frac{\partial V^f(\alpha)}{\partial \alpha}|_{\alpha=\alpha^{SO}} - (1-\alpha^{SO}) \frac{\partial V^s(\alpha)}{\partial \alpha}|_{\alpha=\alpha^{SO}} \right) = 0, \tag{18}
$$

where $W'(\alpha^{SO}) = \frac{\partial W(\alpha)}{\partial \alpha}|_{\alpha=\alpha^{SO}}$, and the difference between the equilibrium level of fast trading and the socially optimal level is driven by the term in brackets. This term can be positive or negative since $V^f(\alpha)$ decreases and $V^s(\alpha)$ increases in $\alpha$ (cf. Appendix D). It will be positive (resp. negative) if the impact of $\alpha$ on $V^f(\alpha)$ (resp. $V^s(\alpha)$) dominates its impact on $V^s(\alpha)$ (resp. $V^f(\alpha)$) and the overall effect of HFT is therefore a cost (resp. benefit) to society. Hereafter, I refer to it as the externality effect and denote it by $E$.

Note that the discount factor drops out when determining $\alpha^{SO}$ in Eq. (18) , so in the following analysis of the socially optimal level, I omit the notational dependence on $t$.

The socially optimal level of HFT is not necessarily zero. If the socially optimal level were to be zero, then the following condition would have to hold: $W'(0) = \Delta(0) - I + \frac{\partial V^s(\alpha)}{\partial \alpha}\Big|_{\alpha=0} = 0$. However, since we have shown in Appendix D that $\partial V^s(\alpha)/\partial \alpha > 0$, for all $\alpha$, the condition cannot hold if $\Delta(0) - I > 0$; i.e., if the ex ante expected relative value of being fast is high when everyone else is slow. Therefore, the socially optimal level of fast trading will be non-zero when the ex ante expected relative value of being fast is high when everyone is slow. This implies that it is the relative gain from fast trading which gives HFT its social value in the model. This relative gain is owing to the reduction in delay costs which trading in fragmented markets can inflict on those without access to the technology.

On the other hand, it would be socially optimal for all traders in the market to be fast if $W'(1) = \Delta(1) - I + \left.\frac{\partial V^f(\alpha)}{\partial \alpha}\right|_{\alpha=1} = 0$ holds. However, as shown in Appendix D, $\partial V^f(\alpha)/\partial\alpha < 0$ for all $\alpha$, the condition will not be satisfied if $\Delta(1) - I < 0$; i.e., if the ex ante expected relative value of fast trading in the market is low when everyone if fast. Thus, it is not socially optimal for all traders to be fast when the expected relative value of fast trading in the market is low if everyone in the market is fast.

**Proposition 5.** *When the cost of investing in the HFT technology is low, the equilibrium level of fast trading is never lower than the socially optimal level. However, for sufficiently high levels of $I$, the equilibrium level will be lower than the socially optimal level. Moreover, there is always a cost level such that the equilibrium level and the socially optimal level coincide.*

The reasoning is as follows. If the effect of a change in $\alpha$ on $V^f(\alpha)$ dominates its effect on $V^s(\alpha)$, then the externality effect E is positive and there is an over-investment in the fast trading technology in equilibrium relative to what is socially optimal. This happens when $I$ is low because the low cost implies a low value of waiting. Thus, the ex ante expected relative value of being fast is high. From Eq. (18) and the equilibrium condition that $\Delta(\alpha^*) - I = 0$, it must be the case therefore that $\Delta(\alpha^{SO}) > \Delta(\alpha^*)$. Since $\Delta(\alpha)$ decreases in $\alpha$ (see Appendix D), then $\alpha^{SO} < \alpha^*$.

The over-investment in speed in equilibrium relative to the socially optimal level result has been found in other papers, but for different reasons to the one in this paper. See, for example, Budish et al. [6] who develop a model in which traders invest in speed so that they can quickly react to the arrival of public information but, in contrast to this model, gains from trade are not modelled. This absence in trading gains leads to a result that slow trading is always socially optimal. In Biais et al. [4], each investor also chooses his speed at which it operates in a given market, but in that paper, over-investment in speed always arises and that is because fast traders benefit from the adverse selection cost they inflict on slow traders, but they do not internalise this cost in equilibrium. This increases the level of HFT in equilibrium, but the socially optimal level does not increase in the benefits from adverse selection. Hence, over-investment arises. Moreover, Glode et al. [11] view investment in HFT as an arms race, and there is an equilibrium in which every investor chooses to invest in the technology in the fear others will also do so, which leads to over-investment.

On the other hand, if the effect of a change in $\alpha$ on $V^s(\alpha)$ dominates its effect on $V^f(\alpha)$, $E$ is negative and there is an under-investment in equilibrium relative to the socially optimal level. This occurs when $I$ is high since waiting to invest is optimal. By a similar reasoning to the over-investment case, $\alpha^{SO} > \alpha^*$; i.e., there is an under-investment in equilibrium.

There is little evidence in the literature of under-investment in the HFT technology. An extension of the model in Biais et al. [4], which includes markets where fast trading is not permitted, can lead to under-investment in equilibrium. This is because slow traders opt to trade only in slow markets which, in turn, reduces the expected profits of HFTs. In their extended model, only two types of equilibria can arise: all traders trade in slow markets or all are fast. By contrast, under-investment can occur in my model without considering the possibility of slow only markets. This is because I view investment in the HFT technology as an optimal stopping problem and, owing to the inclusion of the value of waiting, under-investment can occur in equilibrium (and not necessarily either an all fast ($\alpha^* = 1$) or all slow ($\alpha^* = 0$) equilibrium) when the cost of investing is very high. In my case, when under-investment arises, it does so because, for a given level of fast trading $\alpha^*$, it is not optimal for enough traders to avail of the reduction in delay cost which investment would provide because the benefit of this reduction is not sufficiently high to warrant paying such a high investment cost owing to the value of the option to wait.

Therefore, owing to the optimal timing policy, it is the case that for some levels of investment cost $\alpha^{SO} < \alpha^*$, and for other levels $\alpha^* > \alpha^{SO}$, implying there is a unique $I$ such that $\alpha^* = \alpha^{SO}$ and this is the level of $I$ such that the externality effect is zero; in other words, the level of $I$ such that

$$\frac{\partial V^f(\alpha)/\partial\alpha}{\partial V^s(\alpha)/\partial\alpha}\Big|_{\alpha=\alpha^{SO}} = 1 - \frac{1}{\alpha^{SO}}.$$

Another interesting question is how the regions of over and under-investment in equilibrium are related to changes in the standard real option parameters $r$, $\sigma$, and $\delta$ (via a change in $\mu$). In essence, we want to understand how shifts in these parameter values impact the extent of over- and under-investment. The following proposition summarises the impact they have.

**Proposition 6.** *The extent of over-investment decreases and the extent of under-investment increases in $r$ and $\sigma$. The extent of over-investment and the extent of under-investment decrease in $\delta$.*

**Proof.** See Appendix F. ■

This result is driven by the impact of the parameters on the externality effect $E$ (i.e., the bracketed term in Eq. (18)). As discussed, when $E > 0$, the externality cost of HFT incurred by fast traders dominates the externality benefit of HFT acquired by slow traders and there is over-investment in equilibrium. On the other hand, when $E < 0$, the externality benefit dominates the externality cost, and there is under-investment. Therefore, over-investment will be alleviated when any parameter shift leads to a reduction in $E$, and under-investment will be alleviated when there is an increase in $E$. However, if the externality effect increases or decreases, so too must $\Delta(\alpha)$ so that the socially optimal condition (18) is maintained. Thus, we can interpret the effects of the parameters on the extent of over- and under-investment by examining the impact of a parameter shift on $\Delta(\alpha)$. In essence, a parameter shift that leads to an increase in $\Delta(\alpha)$ exacerbates the extent of over-investment and alleviates the extent of under-investment in equilibrium, and vice versa, which is what intuition would suggest.

I demonstrate in Appendix F, that the effects of $r$ and $\sigma$ on $\Delta(\alpha)$ are via their impact on the optimal investment threshold $X^*$ in the over-investment scenario. In particular, we know from Proposition 2 that $X^*$ increases in $\sigma$ and in $r$ when the probability of finding a liquid venue is low. However, when $\lambda$ is low, we know from Proposition 4 that the equilibrium level of investment is high and hence, we can infer that the over-investment scenario arises in this case. Thus, when we have over-investment in equilibrium relative to the socially optimal level, $X^*$ increases in $r$ and $\sigma$. However, $\Delta(\alpha)$ decreases in $X^*$ (which is proven in Appendix F) because such an increase reduces (increases) the range of profit levels over which it is optimal to be fast (resp. slow) so that $\Delta(\alpha)$ will be lower for some fixed $\alpha$. Therefore, we can conclude that increases in $\sigma$ and/or $r$ in the over-investment scenario alleviates the extent of over-investment.

In the case of under-investment, the probability of finding a liquid venue will be high and, therefore, so too will be the optimal investment threshold $X^*$. The higher the level of the threshold, the more immaterial it is in the sense that as $\lambda \to 1$, $X^* \to \infty$, so that for very high probability levels, the investors will not contemplate investing and, hence, the impact of $X^*$ on the extent of under-investment is irrelevant. Therefore, the effect of $r$ and $\sigma$ on the extent of under-investment is via their direct impacts on $V^f(\alpha)$ and $V^s(\alpha)$. In fact, in the under-investment case, the value of being fast is very small relative to the value from being slow (since $\lambda$ is high) so that the effects of the parameters are really via their effects on $V^s(\alpha)$ and, in particular, via their effects on the value of waiting (cf. Appendix F). An increase in $r$ or $\sigma$ implies the value of waiting increases so that the ex ante expected profit from being slow increases. Therefore, $\Delta(\alpha)$ decreases (since their impact on $V^f$ is negligible) and the extent of under-investment will be exacerbated.

Regarding the shortfall $\delta$, an increase in $\delta$, owing to a decline in the rate of profit growth from being fast, leads to a decrease in the ex ante expected profit from being fast and an increase in the ex ante expected profit from being slow. When we have over-investment, $V^f(\alpha)$ is high relative to $V^s(\alpha)$, so the effect of $\delta$ on $V^f(\alpha)$ dominates its effect on $V^s(\alpha)$, and vice versa when we have under-investment. Therefore, an increase in $\delta$ leads to a reduction in $\Delta(\alpha)$ when we are in the over-investment scenario, and an increase in this value when we are in the under-investment scenario. Hence, an increase in $\delta$ alleviates the extent of both over-investment and under-investment in equilibrium.

### 4.3 Implications for Policy

The analysis so far leads to the inference that one appropriate policy response would be to tax or subsidise HFT activity in a manner akin to imposing a Pigouvian tax or subsidy. This tax (or subsidy), denoted by $T^{**}$, would align the equilibrium level of HFT with the socially optimal level when it is set equal to the externality effect given by the bracketed term in Eq. (18); i.e., $E$. Note that when $E > 0$, we have over-investment in equilibrium relative to the socially optimal level, so $T^{**}$ would take the form of a tax, but when $E < 0$ it would be a subsidy. The result from Proposition 5 indicates that the tax will be higher when the cost of investment is low because this encourages early investment in the HFT technology, and overall leads to over-investment in equilibrium relative to the socially optimal level. On the other hand, for sufficiently high levels of investment cost, the cost should actually be subsidised to encourage slow traders to invest sooner so that they avail of the reduction in delay cost that HFT provides in fragmented markets.

A caveat with this approach is that when there is over-investment, imposing a tax on HFTs when the cost is low will deter slow traders from adopting the technology as it essentially increases the cost of investing making waiting relatively more valuable, but it will not reduce the level of HFT that is already prevalent because there is no incentive with such a policy for HFTs to stop "trading fast". Such a caveat does not arise in the under-investment case because reducing the cost of investment by a subsidy will make investing relatively more attractive to slow traders.

One way of circumventing the issue in the over-investment case would be to subsidise HFTs to stop trading fast. This subsidy, $S^{**}$ would need to be large enough so that the HFTs are at least indifferent between being fast or slow; i.e., $S^{**}$ is such that

$$V^s(\alpha) + S^{**} \geq V^f(\alpha) - I.$$

If $|S^{**}| = |T^{**}|$, then the level of HFT in equilibrium will also be socially optimal.

# 5 Concluding Remarks

In recent years, the state of market microstructure has changed considerably. There are many ways in which these changes have come about, but one of the biggest changes is that markets have become highly fragmented. When markets are fragmented, traders must search across many markets for venues which will execute their orders at their specified prices. This can result in delayed or partial execution which is costly. In response to the increase in market fragmentation, there has been a demand for speed by traders, and various types of expensive technologies have been developed. Such technologies enable traders to compare all trading venues instantaneously or obtain a glimpse of the true state of the market before everyone else.

In this paper I derive a dynamic model, using techniques from real options analysis, which provides an optimal timing strategy for slow traders to invest in a high frequency trading technology. The model prescribes waiting longer to invest if the level of high frequency trading in the market increases, and it prescribes earlier adoption if the probability of finding a liquid venue decreases. It also prescribes waiting longer if the uncertainty of the profit process increases and, if the probability of finding a liquid venue is low, if the discount rate increases and/or the shortfall decreases. However, if the probability of finding a liquid venue is high, an increase in the discount rate and a decrease in the shortfall make early adoption optimal.

Based on this optimal timing strategy, I then characterise the equilibrium level of fast trading in the market as well as the welfare-maximising socially optimal level. From this analysis, the following results emerge. There is always a unique equilibrium level of fast trading in the market, and this level never increases in the probability of finding a liquid venue. There is also a socially optimal level of fast trading such that, when the cost of adopting the technology is relatively cheap, there is over-investment in equilibrium relative to what is socially optimal, and when the cost is relatively high, there is under-investment in equilibrium. This implies that there is a unique cost of investment such that the equilibrium level and the socially optimal level of fast trading coincide. Moreover, increases in the discount rate and/or uncertainty over the profit process alleviate the extent of over-investment in equilibrium and exacerbate the extent of under-investment. However, both under- and over-investment are alleviated by increases in the shortfall. All of these results are driven by the optimal timing policy.

Finally, I discuss why a Pigouvian subsidy given to slow traders to encourage investment when the cost of investment is high is a reasonable policy response to aligning the equilibrium level of HFT with the socially optimal level in the case of under-investment, but that a Pigouvian tax imposed on fast traders when the cost is low, may not be the most effective means of obtaining alignment in the case of over-investment. However, subsidising HFTs to refrain from trading fast (i.e., trading using their fast technology) may be effective in that instance.

It is clear that further research on this issue is warranted and one possibility could be to extend the model by assuming regulators impose a stochastic per period tax on HFTs which is positively correlated with the extent to which the equilibrium level and the socially optimal level are misaligned. If the fast trader had then the option to "exit the market", or in other words, revert to being slow if paying this tax became too costly relative to the benefit from having no delay costs, then this may be a more effective way of aligning market equilibrium with social optimality (than paying the HFTs to "exit" when there is over-investment) because this could then simultaneously change the *current* level of HFT as well as changing the optimal adoption threshold for slow traders. This would require a high level of surveillance to ensure that those fast traders who no longer pay the "fast tax" also no longer use co-location services, smart routers etc. The approach would also work in the under-investment case because if the subsidy is stochastic and positively correlated with the extent of under-investment, then if the under-investment problem is large, the subsidy would be large enough to encourage a high level of investment, but if the problem is small, it would be small enough to only encourage a small increase. Also, a stochastic subsidy would be unlikely to alter the level of HFT already prevalent because it would not discourage HFTs to refrain from trading fast. Certainly, investigating this type of policy response, along with the potential means of enforcement, is an interesting topic for future research.

# Appendix

## A Derivation of $\Omega(r, \mu, \lambda)$ and proof that $\Omega(r, \mu, \lambda) > 0$

### A.1 Derivation

From Eq. (9),

$$\Omega(r, \mu, \lambda) := \left( \frac{1}{\delta} - \lambda \int_0^\infty e^{-rt} \left( (1-\lambda) + \lambda e^{\mu} \right)^t dt \right)$$

To prove my result in Eq. (10), it is sufficient to simply present my derivation of $\int_0^\infty e^{-rt} \left( (1-\lambda) + \lambda e^{\mu} \right)^t dt$ as follows:

$$\int_0^{\infty} e^{-rt}\left((1-\lambda)+\lambda e^{\mu}\right)^t dt = \int_0^{+\infty} e^{-t(r-\ln((1-\lambda)+\lambda e^{\mu}))} dt$$
$$= \left[-\frac{1}{r-\ln\left((1-\lambda)+\lambda e^{\mu}\right)} e^{-t(r-\ln((1-\lambda)+\lambda e^{\mu}))}\right]_0^{+\infty}.$$

Now

$$\lim_{t\to+\infty} e^{-t(r-\ln((1-\lambda)+\lambda e^{\mu}))} = 0$$

if and only if $r-\ln\left((1-\lambda)+\lambda e^{\mu}\right)>0$. Hence, we must prove that $r-\ln\left((1-\lambda)+\lambda e^{\mu}\right)>0$.

If $\lambda=1$, then the latter expression becomes $r-\mu$ $(=\delta)$ which is positive by assumption. Therefore, if the expression decreases in $\lambda$, then it is positive everywhere. Indeed,

$$\frac{\partial}{\partial\lambda}\left(r-\ln\left((1-\lambda)+\lambda e^{\mu}\right)\right) = \frac{1-e^{\mu}}{(1-\lambda)+\lambda e^{\mu}} < 0$$

implying $r-\ln\left((1-\lambda)+\lambda e^{\mu}\right)>0$.

Therefore,

$$\int_0^{\infty} e^{-rt}\left((1-\lambda)+\lambda e^{\mu}\right)^t dt = \frac{1}{r-\ln\left((1-\lambda)+\lambda e^{\mu}\right)}.$$

### A.2 Proof that $\Omega(r,\mu,\lambda)>0$

$$\Omega(r,\mu,\lambda) = \frac{1}{\delta} - \frac{\lambda}{\left(r-\ln\left((1-\lambda)+\lambda e^{\mu}\right)\right)} > 0$$

If $\lambda=1$, then $\Omega(\cdot)=0$. Hence, if $\Omega(\cdot)$ decreases in $\lambda$, then $\Omega(\cdot)$ must be positive.

$$\frac{\partial\Omega(\cdot)}{\partial\lambda} < 0 \iff r-\ln\left((1-\lambda)+\lambda e^{\mu}\right) + \frac{\lambda\left(e^{\mu}-1\right)}{(1-\lambda)+\lambda e^{\mu}} > 0.$$

This is true since $\mu>0$ by assumption and since $r-\ln\left((1-\lambda)+\lambda e^{\mu}\right)>0$, which is proven above.

## B Proof of Theorem 1

The derivation of the optimal threshold uses well-developed standard techniques from real options theory (see, for example, Dixit and Pindyck [8]).

Once the trader invests in the technology, he obtains a flow of profits of $X_t^F \equiv X_t$ in perpetuity. Thus, the net present value from investing, denoted by $V^A(X_0)$, is given by

$$V^A(X_0) = E^0\left[\int_0^{\infty} e^{-rt}X_t dt\right] - I = \frac{X_0}{\delta} - I. \tag{B.1}$$

Prior to investing, the trader obtains a flow of profits $X^S$, as well as having the option to invest. Letting $V^B(X_0)$ denote the current value to the trader before investing, and using standard dynamic programming arguments from Dixit and Pindyck [8], $V^B(X_0)$ solves the following Bellman equation:

$$\frac{1}{2}\sigma^2 X_0^2 (V^B)''(X_0) + (r-\delta)X_0 (V^B)'(X_0) - rV^B(X_0) + X_0^S = 0, \tag{B.2}$$

where $(V^B)''(X_0) := \partial^2 V^B(\cdot)/\partial X_0^2$ and $(V^B)'(X_0) := \partial V^B(\cdot)/\partial X_0$.

The solution to this equation takes the following general form

$$V^B(X_0) = A_1 X_0^{\beta_1} + A_2 X_0^{\beta_2} + \frac{X_0^S}{\delta},$$

where $A_1$ and $A_2$ are constants to be determined, and $\beta_1$ and $\beta_2$ are the two (real) roots of the quadratic equation:

$$\frac{1}{2}\sigma^2\beta(\beta-1) + (r-\delta)\beta - r = 0.$$

Intuitively, the present value of trading for the slow trader ought to be comprised of the expected present value of his profit flow as if he were never to invest in the HFT technology, but adjusted for the fact that he has the option to invest in the technology should the expected payoff from investing become sufficiently large to warrant the cost of doing so (Dixit and Pindyck [8]). Hence, the general form of $V^B(X_0)$ should comply with this intuition. To this end the first two terms are the value of his option to invest, and the term $X_0^S/\delta$ represents the expected present value of his profit flow as if he were never to invest. Hence, it must be the case that

$$\begin{aligned}\frac{X_0^S}{\delta} \equiv & E^0\left[\int_0^\infty e^{-rt} X_t^S dt\right] \\ = & \lambda X_0 \int_0^\infty e^{-rt}\left((1-\lambda) + \lambda e^{r-\delta}\right)^t dt \\ = & \frac{\lambda X_0}{r - \ln\left((1-\lambda) + \lambda e^{r-\delta}\right)}\end{aligned} \tag{B.3}$$

(cf. Eq. (3) and Appendix A.1 for technical details.)

If the slow trader's current profit $X_0^S$ is zero, it will stay at zero forever because $X^S$ must follow a geometric Brownian motion, which has an absorbing barrier at zero, owing to its dependence on X (see Eq. (B.3)). Thus $X_0^S = 0$ iff $X_0 = 0$. However, since $X_0 \equiv X_0^F$, this implies that investing and becoming fast will never yield any profit either and, hence, that the value of the option to invest has no value. Therefore, the following condition must be satisfied at the boundary:

$$V^B(0) = 0.$$

But since $\beta_2 < 0$, this condition is satisfied iff $A_2 = 0$.

Therefore, a solution to Eq. (B.2) which satisfies to the boundary condition is given by

$$V^B(X_0) = A_1 X_0^{\beta_1} + \frac{\lambda X_0}{r - \ln\left((1-\lambda) + \lambda e^{r-\delta}\right)}. \tag{B.4}$$

Finally, there is a value of $X$ at some time $\tau^*$, which I denote by $X^*$, at which the trader is indifferent between investing and waiting. Moreover, at this threshold, the value functions before and and after investing should meet tangentially. In other words, the following conditions must be satisfied

$$V^B(X^*) = V^A(X^*)$$

and

$$(V^B)'(X^*) = (V^A)'(X^*),$$

where $(V^i)'(X^*) = \frac{\partial V^i(X_0)}{\partial X_0}\Big|_{X_0=X^*}$ (for $i = \{A, B\}$).

Together these these conditions give

$$X^* = \frac{\beta_1}{\beta_1 - 1}(\Omega(r,\delta,\lambda))^{-1}I, \tag{B.5}$$

and

$$A_1 = (\Omega(r,\delta,\lambda)X^* - I)(X^*)^{-\beta_1} \equiv (X^*)^{-\beta_1}F(X^*) \tag{B.6}$$

where

$$\Omega(r,\delta,\lambda) := \frac{1}{\delta} - \frac{\lambda}{r - \ln((1-\lambda) + \lambda e^{r-\delta})}$$

and $F(X^*)$ is given by Eq. (9). Finally, since $X_0 := (1+\alpha)^{-1}x$, the result for the value function which solves the optimal stopping problem can be verified.[5] ■

# C Proof of Proposition 2

$$X^* = \frac{\beta_1}{\beta_1 - 1}\left(\Omega(r,\delta,\lambda)\right)^{-1} I$$

where

$$\Omega(r,\delta,\lambda) = \frac{1}{\delta} - \frac{\lambda}{r - \ln((1-\lambda) + \lambda e^{r-\delta})}.$$

(Hereafter I just write $\Omega$ in the interest of preserving space.)

Let

$$\mathcal{Q}(\beta_1) := \frac{1}{2}\sigma^2\beta_1(\beta_1 - 1) + (r-\delta)\beta_1 - r = 0.$$

Then for $\zeta \in \{r, \delta, \sigma\}$:

$$\frac{\partial X^*}{\partial \zeta} = -\frac{\beta_1}{\beta_1 - 1}\frac{1}{\Omega^2}\frac{\partial \Omega}{\partial \zeta}I - \frac{1}{(\beta_1 - 1)^2}\frac{\partial \beta_1}{\partial \zeta}\frac{I}{\Omega}. \tag{C.1}$$

The first term on the right hand side measures the sensitivity of the threshold with respect to the returns on the investment and is referred to as the present value effect. The second term measures the sensitivity of the threshold with respect to $\beta_1$ and, thus, measures the option effect. This decomposition is standard in real options models.

[5] Indeed, using the result in Dixit and Pindyck [8] pp. 315-316, we know that $E^0[e^{-r\tau}] = \left(\frac{X_0}{X^*}\right)^{\beta_1}$, where $\tau$ is the time of investment. Therefore, $(X^*)^{-\beta_1} = (X_0)^{-\beta_1}E^0[e^{-r\tau}] = ((1+\alpha)^{-1}x)^{-\beta_1}E^0[e^{-r\tau}]$. Thus, we can write $A_1$ as $A_1 = E^0[e^{-r\tau}]\left((1+\alpha)^{-1}x\right)^{-\beta_1}F(X^*)$. In other words, the value of the option to invest is just the expected discounted value of the net present value at the time investment takes place, which is intuitive.

Now

$$\frac{\partial \beta_1}{\partial \zeta} = -\frac{\partial \mathcal{Q}(\beta_1)/\partial \zeta}{\partial \mathcal{Q}(\beta_1)/\partial \beta_1} = -\frac{\partial \mathcal{Q}(\beta_1)}{\partial \zeta} \frac{1}{\sigma^2 \left(\beta_1 - \frac{1}{2}\right) + r - \delta}$$

(cf. Dixit and Pindyck [8] pp. 144) so that we can re-write Eq. (C.1) as follows:

$$\frac{\partial X^*}{\partial \zeta} \frac{\chi}{X^*} = -\chi \frac{\partial \Omega}{\partial \zeta} + \frac{\Omega^2}{\beta_1} \frac{\partial \mathcal{Q}(\beta_1)}{\partial \zeta}, \tag{C.2}$$

where

$$\chi := (\beta_1 - 1) \left( \sigma^2 \left( \beta_1 - \frac{1}{2} \right) + r - \delta \right) \Omega^2.$$

From Eq. (C.2), since $\partial \Omega / \partial \sigma = 0$ and $\partial \mathcal{Q}(\beta_1)/\partial \sigma = \sigma \beta_1 (\beta_1 - 1) > 0$,

$$\frac{\partial X^*}{\partial \sigma} > 0$$

owing solely to the option effect.

Also, from Eq. (C.2), we see that

$$\frac{\partial X^*}{\partial r} > 0 \Longleftrightarrow \frac{\Omega^2}{\beta_1}(\beta_1 - 1) > \chi \frac{\partial \Omega}{\partial r} \tag{C.3}$$

where the right hand side of the second inequality in (C.3) gives the present value effect and the left hand side the option effect. Moreover,

$$\frac{\partial \Omega}{\partial r} = \frac{\lambda(1-\lambda)}{\left(r - \ln\left((1-\lambda) + \lambda e^{r-\delta}\right)\right)^2 \left(1 - \lambda + \lambda e^{r-\delta}\right)} > 0. \tag{C.4}$$

If $\Omega$ is large, then the option effect will dominate and therefore drive the effect of $r$ on $X^*$. In this case, $X^*$ will increase in $r$.

However, if $\lambda$ is high, and $\Omega$ is therefore small, then we have that $X^*$ will be impacted by $r$ owing to the PV effect because $\Omega$ increases in $r$. In this case, $X^*$ will decrease in $r$.

$$\frac{\partial X^*}{\partial \delta} > 0 \Longleftrightarrow \chi \frac{\partial \Omega}{\partial \delta} + \Omega^2 < 0 \tag{C.5}$$

where

$$\frac{\partial \Omega}{\partial \delta} = -\frac{1}{\delta^2} + \frac{\lambda^2 e^{r-\delta}}{\left(r - \ln\left((1-\lambda) + \lambda e^{r-\delta}\right)\right)^2 \left(1 - \lambda + \lambda e^{r-\delta}\right)} < 0 \tag{C.6}$$

(since $\partial \Omega / \partial \delta$ is continuous in $\lambda$ and $\partial \Omega / \partial \delta = 0$ for $\lambda = 1$ and $\partial \Omega / \partial \delta = -1/\delta^2 < 0$ for $\lambda = 0$).

As in the case of $r$, there are two effects at play. Specifically, if $\Omega$ is large, the option effect will dominate the PV effect of $\delta$ on $X^*$. In this is the case, $X^*$ will decrease in $\delta$. On the other hand, if $\Omega$ is small, the present value effect will dominate and $X^*$ will increase in $\delta$.

## D Proof of Proposition 3

First I use Leibniz rule for differentiating integrals to show that $\Delta(\alpha)$ decreases in $\alpha$.

$$
\begin{aligned}
\frac{\partial \Delta(\alpha)}{\partial \alpha} =& \frac{\partial V^f(\alpha)}{\partial \alpha} - \frac{\partial V^s(\alpha)}{\partial \alpha} \\
=& -\frac{1}{\delta(1+\alpha)^2} \int_{(1+\alpha)X^*}^{\bar{x}} x^i g^f(x^i) dx^i - \frac{(X^*)^2}{\delta} g^f((1+\alpha)X^*) \\
& - \Big[ -\frac{\lambda}{(1+\alpha)^2} \frac{1}{(r - \ln((1-\lambda) + \lambda e^{r-\delta}))} \int_0^{(1+\alpha)X^*} x^i g^s(x^i) dx^i \\
& + \frac{\lambda (X^*)^2}{(r - \ln((1-\lambda) + \lambda e^{r-\delta}))} g^s((1+\alpha)X^*) \\
& - \beta_1 (1+\alpha)^{-\beta_1 - 1} (X^*)^{-\beta_1} F(X^*) \int_0^{(1+\alpha)X^*} (x^i)^{\beta_1} g^s(x^i) dx^i \\
& + X^* F(X^*) g^s((1+\alpha)X^*) \Big],
\end{aligned} \tag{D.1}
$$

where the term in the square brackets is $\partial V^s(\alpha)/\partial \alpha$.

It is clear from the equation that $\partial V^f(\alpha)/\partial \alpha < 0$. Then it must be the case that $\partial \Delta(\alpha)/\partial \alpha < 0$ if $\partial V^s(\alpha)/\partial \alpha > 0$.

$\partial V^s(\alpha)/\partial \alpha > 0$ if

$$
(X^*)^2 g^s((1+\alpha)X^*) > \frac{1}{(1+\alpha)^2} \int_0^{(1+\alpha)X^*} x^i g^s(x^i) dx^i \tag{D.2}
$$

and if

$$
X^* g^s((1+\alpha)X^*) > \beta_1 (1+\alpha)^{-\beta_1 - 1} (X^*)^{-\beta_1} \int_0^{(1+\alpha)X^*} (x^i)^{\beta_1} g^s(x^i) dx^i. \tag{D.3}
$$

Using the integration by parts technique to evaluate the integrals in the latter two equations, it is sufficient to approximate the integrals using only the "$uv$" term in the standard formula to show the conditions hold if we assume that $\frac{\partial g^s(x^i)}{\partial x^i} \geq 0$. This is because the "$\int v du$" term is negative. Indeed, there are many specifications of $g^s(\cdot)$ such that the derivative is nonnegative. For example, assuming the $x^i$'s follow a uniform distribution, $g^s(x^i) = \frac{1}{(1+\alpha)X^*}$, the derivative of which is zero. Letting $u = g^s(x^i)$ and $dv = x^i dx^i$ or $dv = (x^i)^{\beta_1} dx^i$, then $[uv]_0^{(1+\alpha)X^*} = \frac{1}{2}(1+\alpha)^2 (X^*)^2 g^s((1+\alpha)X^*)$ for Eq. (D.2), and $[uv]_0^{(1+\alpha)X^*} = \frac{1}{\beta_1+1}(1+\alpha)^{\beta_1+1} (X^*)^{\beta_1+1} g^s((1+\alpha)X^*)$ for (D.3).

Eq. (D.2) becomes

$$
(X^*)^2 g^s((1+\alpha)X^*) > \frac{1}{2} (X^*)^2 g^s((1+\alpha)X^*), \tag{D.4}
$$

which clearly holds, and Eq. (D.3) becomes

$$
X^* g^s((1+\alpha)X^*) > \frac{\beta_1}{\beta_1 + 1} X^* g^s((1+\alpha)X^*), \tag{D.5}
$$

which also clearly holds.

Therefore, $\partial V^s(\alpha)/\partial\alpha > 0$ and $\Delta(\alpha)$ decreases in $\alpha$ everywhere.

From this, it is clear then that the equilibrium condition

$$\Delta(\alpha^*) - I = 0 \tag{D.6}$$

is satisfied if $\Delta(0) - I > 0$ and $\Delta(1) - I < 0$ so that there is a unique zero point. If both are true, then the condition must hold for some $0 < \alpha^* < 1$.

If $\Delta(0) \leq I$, then since $\Delta(\alpha)$ decreases everywhere in $\alpha$, it must also be true that $\Delta(\alpha) \leq I$. Hence, $\alpha^* = 0$. If $\Delta(1) \geq I$, then $\alpha^* = 1$ by a similar reasoning. ■

# E Proof of Proposition 4

First I determine how $\Delta(\alpha)$ is affected by $\lambda$.

$$\frac{\partial \Delta}{\partial \lambda} = \frac{\partial V^f}{\partial \lambda} - \frac{\partial V^s}{\partial \lambda}. \tag{E.1}$$

We use Liebniz rule once more and recall that $\partial X^*/\partial\lambda > 0$:

$$\begin{aligned}\frac{\partial \Delta}{\partial \lambda} =& -(1+\alpha)\frac{\partial X^*}{\partial \lambda}\frac{X^*}{\delta}g^f((1+\alpha)X^*) \\ & - \Big(\int_0^{(1+\alpha)X^*}\Big[(1+\alpha)^{-1}\frac{x^i}{(r-\ln((1-\lambda)+\lambda e^{r-\delta}))^2}\Big(r - \ln\Big((1-\lambda)+\lambda e^{r-\delta}\Big) \\ & + \frac{\lambda\left(e^{r-\delta}-1\right)}{(1-\lambda)+\lambda e^{r-\delta}}\Big) - \beta_1\left((1+\alpha)^{-1}x^i\right)^{\beta_1}(X^*)^{-\beta_1-1}\frac{\partial X^*}{\partial\lambda}F(X^*)\Big]g^s(x^i)dx^i \\ & + (1+\alpha)\frac{\partial X^*}{\partial\lambda}\Big[\frac{\lambda X^*}{(r-\ln((1-\lambda)+\lambda e^{r-\delta}))} + F(X^*)\Big]g^s((1+\alpha)X^*)\Big).\end{aligned} \tag{E.2}$$

This expression is definitely negative if

$$\begin{aligned}&(1+\alpha)\frac{\partial X^*}{\partial\lambda}\Big[\frac{\lambda X^*}{(r-\ln((1-\lambda)+\lambda e^{r-\delta}))} + F(X^*)\Big]g^s((1+\alpha)X^*) \\ >&\beta_1(1+\alpha)^{-\beta_1}(X^*)^{-\beta_1-1}\frac{\partial X^*}{\partial\lambda}F(X^*)\int_0^{(1+\alpha)X^*}(x^i)^{\beta_1}g^s((1+\alpha)x^i)dx^i.\end{aligned} \tag{E.3}$$

As in Appendix D, assuming the derivative of $g^s(x^i)$ is nonnegative, it is sufficient to approximate $\int_0^{(1+\alpha)X^*}(x^i)^{\beta_1}g^s((1+\alpha)x^i)dx^i$ by $(\beta_1+1)^{-1}(1+\alpha)^{\beta_1+1}(X^*)^{\beta_1+1}g^s((1+\alpha)X^*)$ since the remaining term is negative. Replacing for this in the latter equation gives

$$\frac{\lambda X^*}{(r-\ln((1-\lambda)+\lambda e^{r-\delta}))} > -\frac{1}{\beta_1+1}F(X^*), \tag{E.4}$$

which clearly holds. Therefore, this implies that $\Delta(\alpha)$ *decreases* in $\lambda$.

Now suppose that the market is in equilibrium for some $\lambda = \hat{\lambda}$ and $\alpha^* = \hat{\alpha}$; i.e.; $\Delta(\hat{\alpha}) = I$. If $\hat{\lambda}$ increases to say $\tilde{\lambda}$, then $\Delta(\hat{\alpha})$ will decrease and the market will no longer be in equilibrium. Since $\Delta(\alpha)$ decreases in $\alpha$ (from Proposition 3), equilibrium will only be attained at $\tilde{\lambda}$ for some

$\tilde{\alpha} < \hat{\alpha}$. Therefore, $\alpha^*$ decreases in $\lambda$. ■

# F Proof of Proposition 6

From Eq. (18), we see that if $\Delta(\alpha)$ increases owing to a shift in any of the parameter values, then so too must the externality effect $E$ (the bracketed term in this equation) so that the socially optimal condition is satisfied. When $E = 0$, then $\alpha^* = \alpha^{SO}$. If $E > 0$, over-investment in equilibrium arises and any parameter shift that forces $E$ to increase will exacerbate the extent of over-investment. When $E < 0$, under-investment arises, and any parameter shift that causes an increase in $E$ will alleviate the extent of under-investment. Therefore, if we examine the effects of $r$, $\delta$ and $\sigma$ on $\Delta(\alpha)$, we can infer how shifts in these parameters impact the extent of over- and under-investment in equilibrium.

From Eqs. (14) and (15),

$$\begin{aligned}\Delta(\alpha) = &\int_{(1+\alpha)X^*}^{\bar{x}} \frac{x^i}{(1+\alpha)\delta} g^f(x^i)dx^i \\ &- \int_0^{(1+\alpha)X^*} \left( \frac{\lambda x^i}{(1+\alpha)(r - \ln((1-\lambda)+\lambda e^{r-\delta}))} + \left(\frac{x^i}{(1+\alpha)X^*}\right)^{\beta_1} F(X^*) \right) g^s(x^i)dx^i.\end{aligned} \tag{F.1}$$

We see from this equation that the parameters impact $\Delta(\alpha)$ directly and via their impact on $X^*$. Thus, we need to determine how shifts in $X^*$ owing to shifts in $r$, $\sigma$, and $\delta$ impact on $\Delta(\alpha)$. By Leibniz rule

$$\begin{aligned}\frac{\partial\Delta(\alpha)}{\partial X^*} = &- (1+\alpha)\frac{X^*}{\delta} g^f((1+\alpha)X^*) \\ &- \Big[ \int_0^{(1+\alpha)X^*} \Big( \left(\frac{x^i}{(1+\alpha)X^*}\right)^{\beta_1} F'(X^*) - \frac{\beta_1}{(1+\alpha)X^*}\left(\frac{x^i}{(1+\alpha)X^*}\right)^{\beta_1} F(X^*) \Big) g^s(x^i)dx^i \\ &+ (1+\alpha)\left(\frac{\lambda X^*}{(r - \ln((1-\lambda)+\lambda e^{r-\delta}))} + F(X^*)\right) g^s((1+\alpha)X^*) \Big]\end{aligned}$$

But $F(X^*) = \Omega X^* - I$ and substituting for $\int_0^{(1+\alpha)X^*} (x^i)_1^{\beta} g^s(x^i)dx^i$ (see Appendix D), the equation reduces to

$$\frac{\partial\Delta(\alpha)}{\partial X^*} \approx -\left(2\frac{X^*}{\delta} - I\right)(1+\alpha)g^f((1+\alpha)X^*) - \frac{\alpha\beta_1}{\beta_1^2 - 1} I g^s((1+\alpha)X^*) < 0$$

if $2X^* > \delta I$. This always holds if it holds when $X^*$ is at a minimum. $X^*$ will have minimum value when $\lambda = 0$ and, thus, when $\Omega = \delta$. Then the latter condition becomes $2\beta_1/(\beta_1 - 1) > 1$, which is satisfied since $\beta_1 > 1$. Hence $\Delta(\alpha)$ decreases in $X^*$ everywhere.

Now we examine $\Delta(\alpha)$ more closely in the case of (i) $\lambda$ being low and (ii) $\lambda$ being high. We do this because $r$ and $\delta$ impact $X^*$ differently, depending on whether $\lambda$ is high or low (see Proposition 2). Moreover, when $\lambda$ is low, we know from Proposition 4 that $\alpha^*$ will be very high and, thus, the case of over-investment arises, and vice versa when $\lambda$ is high.

(i) For $\lambda$ low (over-investment case):

$$\lim_{\lambda \downarrow 0} \Delta(\alpha) = \int_{(1+\alpha)X^*}^{\bar{x}} \frac{x^i}{(1+\alpha)\delta} g^f(x^i)dx^i - \int_0^{(1+\alpha)X^*} \left(\frac{x^i}{(1+\alpha)X^*}\right)^{\beta_1} F(X^*)g^s(x^i)dx^i$$

and, as before, we approximate the integrals by

$$\int_{(1+\alpha)X^*}^{\bar{x}} x^i g^f(x^i)dx^i \approx \frac{1}{2}\left[\bar{x}^2 g^f(\bar{x}) - (1+\alpha)^2(X^*)^2 g^f((1+\alpha)X^*)\right]$$

and

$$\int_0^{(1+\alpha)X^*} (x^i)^{\beta_1} g^s(x^i)dx^i \approx \frac{1}{\beta_1+1}(1+\alpha)^{\beta_1+1}(X^*)^{\beta_1+1} g^s((1+\alpha)X^*)$$

so for $\lambda$ low

$$\Delta(\alpha) \approx \frac{1}{2\delta(1+\alpha)}\left(\bar{x}^2 g^f(\bar{x}) - (1+\alpha)^2(X^*)^2 g^f((1+\alpha)X^*)\right) - \frac{1+\alpha}{\beta_1+1}(X^*)F(X^*)g^s((1+\alpha)X^*)$$

Thus, we can see from this equation that impact of $r$ and $\sigma$ on the extent of over-investment will certainly be via their impact on $X^*$, but the impact of $\delta$ could be via its impact on $X^*$ or directly. However, since $X^*$ relatively low for $\lambda$ low, the effect of $\delta$ on $\frac{1}{2\delta(1+\alpha)}\bar{x}^2 g^f(\bar{x})$ will dominate its effect on the other two terms. Therefore, if $\delta$ increases, $\Delta(\alpha)$ will decrease and, as discussed above, so too will $E$. Therefore the extent of over-investment will be alleviated.

Since we know that $X^*$ increases in $\sigma$ and in $r$ when $\lambda$ is low (cf. Proposition 2), the extent of over-investment will decline in $\sigma$ and $r$. This is because if $r$ and/or $\sigma$ increase, so too will $X^*$ and, as just shown $\Delta(\alpha)$ will decrease. Hence, $E$ will decline also implying a reduction in the extent of over-investment.

(ii) For $\lambda$ high (under-investment case):

$$\lim_{\lambda \uparrow 1} \Delta(\alpha) = -\int_0^{\infty} \frac{x^i}{(1+\alpha)\delta} g^s(x^i)dx^i. \tag{F.2}$$

Taking this limit shows that when we have under-investment, or equivalently, the probability of finding a liquid venue is so high that the threshold $X^*$ is essentially immaterial, the effect of $r$, $\delta$ and $\sigma$ on $\Delta(\alpha)$ is owing to their direct effects, rather than via $X^*$. However, we need to examine the Eq. (F.1) again to understand the direct effects.

We see that, since $\frac{\lambda}{r-\ln((1-\lambda)+\lambda e^{r-\delta})} \approx \frac{1}{\delta}$ and $\left(\frac{x^i}{(1+\alpha)X^*}\right)^{\beta_1} \approx 0$ for $\lambda$ very high, the effects of $r$ and $\sigma$ on $\Delta(\alpha)$ will be via their effect on $F(X^*) = I/(\beta_1 - 1)$. So for $\zeta = \{r, \sigma\}$

$$\partial\Delta(\alpha)/\partial\zeta > 0 \Longleftrightarrow \partial F(X^*)/\partial\zeta < 0 \Longleftrightarrow \partial\beta_1/\partial\zeta > 0.$$

However, we showed in Appendix C that $\beta_1$ decreases in $r$ and $\sigma$. Therefore, an increase in $r$ or $\sigma$ will lead to a decrease in $\Delta(\alpha)$ which will exacerbate the extent of under-investment.

Finally, for $\delta$, we see directly from Eq (F.2) that $\Delta(\alpha)$ increases in $\delta$ when $\lambda$ is high, implying that the extent of under-investment will be alleviated when $\delta$ increases.

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