ON CERTAIN BLOCKS OF SCHUR ALGEBRAS

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Abstract. In this paper, we use what is known about defect 2 blocks of symmetric groups to deduce information on corresponding blocks of Schur algebras. This information includes Ext-quivers, decomposition numbers, and Loewy structures of the Weyl modules, principal indecomposable modules and tilting modules.

Schur discovered an important relationship between the complex general linear groups and the symmetric groups which he used to construct the irreducible polynomial representations of the complex general linear groups from knowledge of the irreducible characters of the symmetric groups. Green gave a modern treatment and development of Schur's ideas in [4]. In his set-up, which works over any infinite field, the general linear groups may be 'replaced' by finite-dimensional algebras called Schur algebras. Many authors have further developed and exploited Green's approach.

In [1] we obtained the characters of Young modules of defect 2 blocks of symmetric groups; by a theorem of James, this is equivalent to knowing the composition multiplicities of simple modules in Weyl modules of the corresponding blocks of Schur algebras. As we also explicitly described the Loewy series of these Young modules in [1] we are here also able to obtain Loewy series of Weyl modules, projective modules, and tilting modules in the corresponding blocks of Schur algebras. We then reverse the process, returning to the symmetric groups by applying the Schur functor; we are able to describe the Ext-quivers and the structures of the Specht modules of defect 2 blocks of symmetric group algebras.

1. Preliminaries

Let k be an algebraically closed field of prime characteristic p. The general linear group $GL_n(k)$ acts naturally on the space k^n of column vectors and therefore on the r-fold tensor product $(k^n)^{\otimes r}$ via the diagonal action; the Schur algebra $S(n,r)$ is defined to be the k-linear span of the image of $GL_n(k)$ in the endomorphism ring of $(k^n)^{\otimes r}$. General references for Schur algebras and their representation theory are [4] and [8]. Any finite-dimensional polynomial representation of $GL_n(k)$ is a direct sum of homogeneous representations, and any homogeneous polynomial representation $GL_n(k) \to GL_m(k)$, homogeneous of degree r, factors through the natural homomorphism $GL_n(k) \to S(n,r)$. So instead of studying finite-dimensional polynomial representations of $GL_n(k)$ directly, we consider finite-dimensional representations of the Schur algebras.

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We use $>$ and $\rhd$ to denote the lexicographic and dominance orders on the set of partitions (see, e.g., [6, §3]). Also, if λ is a partition, then λ' is the conjugate partition. The map $\lambda \mapsto \lambda'$ is lexicographic and dominance order-reversing, i.e. $\lambda > \mu$ (resp. $\lambda \rhd \mu$) if and only if $\mu' > \lambda'$ (resp. $\mu' \rhd \lambda'$). A partition $\lambda = (\lambda_1, \lambda_2, \ldots)$ is called p-singular if $\lambda_{i+1} = \ldots = \lambda_{i+p}$ for some i, and p-regular otherwise. It is called p-restricted if λ' is p-regular.

We name some important $S(n,r)$-mdoules. For a partition λ of r, denote by $\Delta(\lambda)$ the Weyl module associated to λ, by $L(\lambda)$ the unique simple quotient, and by $P(\lambda)$ a projective cover of $L(\lambda)$. The modules $L(\lambda)$, as λ runs over partitions of r, forms a complete set of representatives of the isomorphism classes of simple $S(n,r)$-modules. A simple module $L(\mu)$ occurs as a composition factor of the Weyl module $\Delta(\lambda)$ only if $\mu \leq \lambda$ and, $L(\lambda)$ occurs with multiplicity one. The projective module $P(\lambda)$ has a Weyl filtration, that is, it has a filtration such that the corresponding factor modules are isomorphic to Weyl modules [2, (2.2)]. In any such filtration $\Delta(\mu)$ occurs as a factor $[\Delta(\mu) : L(\lambda)]$ times. Moreover the filtration $P = P_0 \supset P_1 \supset \cdots$ may be chosen so that if $P_i/P_{i+1} \cong \Delta(\lambda_i)$, then $\lambda_0 \leq \lambda_1 \leq \cdots$.

If V is an $S(n,r)$-module then the dual space V^* can be given the structure of an $S(n,r)$-module, called the contravariant dual of V (see, e.g., [4, §2.7]). It is known that the simple $S(n,r)$-modules are self-dual (see, e.g., [4, (3.5a) and (3.3e)]).

We shall assume throughout that $n \geq r$. In this case there exists an idempotent $e \in S(n,r)$ such that $eS(n,r)e$ is isomorphic to the group algebra $k\mathfrak{S}_r$, where $\mathfrak{S}_r$ denotes the symmetric group of degree r [4, §6.1]. The Schur functor $f : S(n,r)\text{-mod} \to k\mathfrak{S}_r\text{-mod}$, takes an $S(n,r)$-module V to the $k\mathfrak{S}_r$-module eV.

We will use the standard notation for $k\mathfrak{S}_r$-modules. Given a partition $\lambda = (\lambda_1, \lambda_2, \dots)$ of r, we obtain the permutation module $M^\lambda = k{\uparrow}_{k\mathfrak{S}_\lambda}^{k\mathfrak{S}_r}$ by inducing the trivial module of the Young subgroup $\mathfrak{S}_\lambda \cong \mathfrak{S}_{\lambda_1} \times \mathfrak{S}_{\lambda_2} \times \cdots$ to $\mathfrak{S}_r$. One of the most important submodules of M^λ is the Specht module S^λ, which is a p-modular reduction of an ordinary irreducible representation of $\mathfrak{S}_r$ whose character is denoted by χ^λ. As λ runs through the partitions of r, the χ^λ give a complete list of irreducible characters of $\mathfrak{S}_r$. The Specht module S^λ has a simple, self-dual head D^λ if the partition λ is p-regular, and as λ runs through the p-regular partitions of r, the set of D^λ is a complete list of mutually non-isomorphic simple modules of $k\mathfrak{S}_r$.

It is known that $S^\lambda \otimes \mathrm{sgn}$ is isomorphic to the dual of $S^{\lambda'}$, where sgn is the signature representation [6, Theorem 8.15]. If λ is p-regular, let λ^* be the p-regular partition such that $D^\lambda \otimes \mathrm{sgn} \cong D^{\lambda^*}$. Note that the map $\lambda \mapsto \lambda'^*$ gives a 1-1 correspondence between p-restricted and p-regular partitions of r which preserves p-cores.

The Young module Y^λ is defined to be the unique indecomposable direct summand of M^λ which contains S^λ as a submodule (see, e.g., [5]). It is the p-modular reduction of a unique (up to isomorphism) ordinary representation of $\mathfrak{S}_r$; we define $[Y^\lambda : S^\mu]$ to be the multiplicity of χ^μ in this character. A Young module is self-dual, and is projective if and only if λ is p-restricted, in which case $Y^\lambda \cong P(D^{\lambda'^*})$. Note that every indecomposable projective $k\mathfrak{S}_r$-module is a Young module.

We collect together some facts:

Lemma 1.1. *Let λ and μ be partitions of r.*

(1) *We have $f\Delta(\lambda) \cong (S^\lambda)^*$, the dual of S^λ.*
(2) *We have*
$$fL(\lambda) \cong \begin{cases} D^{\lambda'^*}, & \text{if } \lambda \text{ is } p\text{-restricted;} \\ 0, & \text{otherwise.} \end{cases}$$
(3) *We have $fP(\lambda) \cong Y^\lambda$.*
(4) *The Schur functor f induces an equivalence between the category of finite-dimensional projective $S(n,r)$-modules and finite-dimensional $k\mathfrak{S}_r$-modules which are isomorphic to direct sums of Young modules.*
(5) *We have $[\Delta(\lambda) : L(\mu)] = [Y^\mu : S^\lambda]$ and if μ is p-restricted then $[\Delta(\lambda) : L(\mu)] = [S^{\lambda'} : D^{\mu'}]$.*
(6) *If λ is p-restricted, then $L(\lambda'^*)$ is the unique composition factor of $P(\lambda)$ having the largest partition (in dominance order).*
(7) *If λ is p-restricted, then $P(\lambda)$ is self-dual.*
(8) *If λ is p-regular, then $\Delta(\lambda)$ has a simple socle $L(\lambda^{*\prime})$.*

Proof. Parts (1) and (2) are proved in §6.3 and §6.4 of [4] and part (5) is proved in [5]. Parts (3), (6), (7), and (8) are (2.5), (2.8), (2.9), and (2.10) of [2]. Part (4) with 'projective' replaced by 'injective' follows from (2.4) of [2] and the discussion following it. To get part (4) from this, note that if V is an $S(n,r)$-module, then $f(V^*)$ and $(fV)^*$ are naturally isomorphic, and that Young modules are self-dual.

We remark that there is an typographical error in the statement of (2.9) in [2]. It said that $P(\lambda)$ is self-dual if λ is p-regular. However, its proof assumed that λ is *column* p-regular (i.e., p-restricted). Proposition 4.6.6 of [8] made the same mistake. $\square$

We now discuss the blocks of $k\mathfrak{S}_r$ and $S(n,r)$. By 'Nakayama's Conjecture' (see, e.g., [7, §6.1.21–6.1.42]), two Specht modules S^λ and S^μ of $k\mathfrak{S}_r$ lie in the same block if, and only if, λ and μ have the same p-core. Hence a block of $k\mathfrak{S}_r$ is determined by a p-core partition τ of $r - wp$ (where w is a nonnegative integer, known as the weight of the block). An irreducible character, Specht module, simple module, or Young module of $k\mathfrak{S}_r$ lies in this block if, and only if, its associated partition of r has p-core τ. The defect group of a block having weight w is isomorphic to $C_p \wr (\mathfrak{S}_w)_p$ where $(\mathfrak{S}_w)_p$ is a Sylow p-subgroup of $\mathfrak{S}_w$ (see, e.g., [7, 6.2.39]). The defect of such a block is thus $w + \nu_p(w!)$, where ν_p is the standard p-valuation. Hence, for example, B is a defect 2 block of $k\mathfrak{S}_r$ if, and only if, $w = 2$ and $p > 2$.

Because every indecomposable projective $k\mathfrak{S}_r$-module is a Young module, parts (3) and (4) of the preceding lemma imply that there exists a 1-1 correspondence between blocks of $k\mathfrak{S}_r$ and blocks of $S(n,r)$ such that if B is a block of $k\mathfrak{S}_r$ then the Schur functor sends $S(n,r)$-modules lying in the corresponding block S_B to $k\mathfrak{S}_r$-modules lying in B. Moreover if τ is the p-core partition associated to B then it is clear that τ is associated to S_B in the following way: a Weyl module or simple module of $S(n,r)$ lies in the block S_B if and only if its associated partition of r has p-core τ. We denote the set of such partitions by $\mathfrak{P}_B$. This labelling of the blocks of $S(n,r)$ was discovered by Donkin [2, (2.12)].

If B is a block of $k\mathfrak{S}_r$ associated to a p-core partition τ, then we denonte by B' the block of $k\mathfrak{S}_r$ associated to the conjugate partition τ'. Tensoring with the signature representation of $k\mathfrak{S}_r$ induces an equivalence between the module categories of B and B'.

If B has defect 0 then it is semisimple, and then S_B is semisimple as well; this case is not particularly interesting. If B is a block of defect 1, its structure is known completely, and from this the structure of S_B may be determined (see, e.g., [8, §5.6]).

2. Decomposition numbers

For the remainder of the paper, we let B be a block of defect 2 of $k\mathfrak{S}_r$ and let S_B be the corresponding block of $S(n,r)$. We are thus assuming $w = 2$ and $p > 2$.

In this section we point out that we can already determine the multiplicity of simple modules as composition factors of Weyl modules of S_B by combining results of Richards [9] and ours [1].

We introduce some terminology and results due to Richards [9]. Given a partition $\lambda \in \mathfrak{P}_B$, one can remove two p-hooks in succession from the diagram of λ. There may not be a unique way to do this, but the absolute value of the difference of the leglengths of the two hooks is well defined and is denoted by $\partial\lambda$. If $\partial\lambda = 0$ we say, following Richards, that λ is black if either λ has two p-hooks and the larger leg-length is even, or λ has a p-hook and a $2p$-hook and the leg-length of the $2p$-hook is congruent to 0 or 3 (mod 4), and that λ is white otherwise.

Definition 2.1. Given $\lambda \in \mathfrak{P}_B$, we shall write λ_+ (resp. λ_-) for the next larger (resp. smaller) partition in $\mathfrak{P}_B$, if it exists, having the same ∂-value as λ, and having the same color as well if this value is 0.

Remarks .

(1) [9, Lemma 4.3] λ_- is defined if, and only if, λ is p-regular.
(2) [9, Lemma 2.11 and Theorem 4.4] If λ is p-regular, then $\lambda_- = \lambda^{*\prime}$.
(3) It is clear that $(\lambda_+)_- = \lambda$ whenever λ_+ is defined and that $(\lambda_-)_+ = \lambda$ whenever λ_- is defined. Thus combining with (1) and (2) above, we see that λ_+ is defined if, and only if, λ is p-restricted, and that $\lambda_+ = \lambda'^*$.

Theorem 2.2. *Suppose $\lambda, \mu \in \mathfrak{P}_B$. Then*

$$[\Delta(\lambda) : L(\mu)] = \begin{cases} 1, & \begin{array}{l} \text{if } \lambda \in \{\mu, \mu_+\}, \text{ or} \\ \text{both } |\partial\mu - \partial\lambda| = 1 \text{ and } \mu_+ \rhd \lambda \rhd \mu; \end{array} \\ 0, & \text{otherwise.} \end{cases}$$

The dominance condition $\mu_+ \rhd \lambda$ is to be treated as vacuous if μ_+ is undefined.

Proof. If μ is p-restricted, $[\Delta(\lambda) : L(\mu)] = [S^{\lambda'} : D^{\mu'}]$ by lemma 1.1(5). It is easy to check that ∂-values and colors are preserved under conjugation of partitions. Therefore in this case the proposition is just a restatement of [9, Theorem 4.4]. blah of Richards. If μ is not p-restricted, then $[\Delta(\lambda) : L(\mu)] = [Y^\mu : S^\lambda]$, by lemma 1.1(5), and one can verify that the proposition holds, using lemma 2.1 and theorem 2.2 (including the remark following theorem 2.2) of [1]. □

3. Ext-quiver and Weyl modules

In this section we compute the Ext-quiver of S_B and describe completely the structures of Weyl modules of S_B.

Theorem 3.1.

(1) *Suppose* $\lambda, \mu \in \mathfrak{P}_B$ *and* $\lambda \geq \mu$. *Then*

$$\dim_k \operatorname{Ext}^1_{S_B}(L(\lambda), L(\mu)) = \begin{cases} 1, & \text{if } |\partial\mu - \partial\lambda| = 1 \text{ and } \mu_+ \rhd \lambda \rhd \mu; \\ 0, & \text{otherwise.} \end{cases}$$

The dominance condition $\mu_+ \rhd \lambda$ *is to be treated as vacuous if* μ_+ *is undefined.*

(2) *If* $\lambda \in \mathfrak{P}_B$ *is* p*-regular, then* $\Delta(\lambda)$ *has Loewy length* 3 *and a simple socle isomorphic to* $L(\lambda_-)$.

(3) *If* $\lambda \in \mathfrak{P}_B$ *is* p*-singular, then* $\Delta(\lambda)$ *is either simple or has Loewy length* 2.

Remarks . As the simple $S(n, r)$-modules are self-dual, we have

$$\operatorname{Ext}^1_{S_B}(L(\mu), L(\lambda)) \cong \operatorname{Ext}^1_{S_B}(L(\lambda), L(\mu)),$$

so the theorem determines the Ext-quiver of S_B.

Proof. Before beginning, we note that $\dim_k \operatorname{Ext}^1_{S_B}(L(\lambda), L(\mu))$ is equal to the multiplicity of $L(\mu)$ in the second Loewy layer of $\Delta(\lambda)$ as long as $\mu \leq \lambda$ because $P(\lambda)$ has a filtration by $\Delta(\nu)$'s with $\nu \geq \lambda$ and such that $\Delta(\lambda)$ occurs just once.

We prove all three statements by induction on λ with respect to lexicographic order. If λ is p-singular, then λ_- is not defined, so by theorem 2.2, all the ∂-values of the partitions μ for which $L(\mu)$ is a composition factor of $\operatorname{rad}(\Delta(\lambda))$ have the same parity. As all these μ's are strictly smaller than λ in the lexicographic order, we have by induction that there are no extensions between these $L(\mu)$'s, and consequently all three statements hold in this case.

Now suppose λ is p-regular. Then λ_- is defined and is p-restricted. By lemma 1.1(8) and the remarks after definition 2.1, $\Delta(\lambda)$ has a simple socle $L(\lambda_-)$. Now by theorem 2.2 the composition factors $L(\mu)$ of the heart of $\Delta(\lambda)$ satisfy $|\partial\mu - \partial\lambda| = 1$ and $\lambda \rhd \mu$. In particular, the ∂-values of the μ all have the same parity. By the induction hypothesis, these composition factors do not extend each other. Hence the theorem will be proved once we show that the heart of $\Delta(\lambda)$ is nonzero.

So assume the contrary, looking for a contradiction. In this case, $\Delta(\lambda)$ has composition length 2, with a simple head $L(\lambda)$ and a simple socle $L(\lambda_-)$. Let $\Omega = \{\mu \in \mathfrak{P}_B \mid [\Delta(\mu) : L(\lambda_-)] = 1,\ \mu \notin \{\lambda, \lambda_-\}\}$. Note that by induction, we have $\mu \in \Omega$ if, and only if, $\lambda > \mu \geq \lambda_-$ and $\operatorname{Ext}^1_{S_B}(L(\mu), L(\lambda_-)) \neq 0$. Since $P(\lambda_-)$ has a multiplicity-free Weyl filtration whose factors are

$\Delta(\lambda_-)$, $\Delta(\mu)$ ($\mu \in \Omega$), $\Delta(\lambda)$, we see that Ω must be non-empty, for $[P(\lambda_-) : L(\lambda_-)] \geq 3$ (as $[P(D^\lambda) : D^\lambda] \geq 3$ [10, Theorem I(3)]).

Let ν be the largest partition in Ω. The heart of $P(\lambda_-)$ has a filtration with factors $\mathrm{rad}(\Delta(\lambda_-))$, $\Delta(\mu)$ ($\mu \in \Omega$), $L(\lambda)$. It is clear from this filtration that $[P(\lambda_-) : L(\nu)] = 1$. And since $L(\lambda_-)$ extends $L(\nu)$, $L(\nu)$ must lie in the head of the heart of $P(\lambda_-)$. However, as $\Delta(\nu)$ is non- simple with a simple head $L(\nu)$ (as $[\Delta(\nu) : L(\lambda_-)] = 1$), $L(\nu)$ is not lying in the socle of the heart of $P(\lambda_-)$. But the heart of $P(\lambda_-)$ is self-dual since $P(\lambda_-)$ is self-dual. This gives us the required contradiction. □

Remarks . We note that the module structures of the Weyl modules of S_B are completely determined, using theorems 2.2 and 3.1.

We have two immediate corollaries to the previous theorem:

Corollary 3.2. *Group the simple modules of S_B according to the parity of the ∂- values of the associated partitions. Then this gives a partition of the the simple modules of S_B displaying the bipartite nature of the* Ext-*quiver of S_B.* □

Corollary 3.3. *Let $\lambda,\ \mu \in \mathfrak{P}_B$ with $\lambda \geq \mu$. The following statements are equivalent:*

1. $\mathrm{Ext}^1_{S_B}(L(\lambda), L(\mu))$ *is non-zero;*
2. $\mathrm{Ext}^1_{S_B}(L(\lambda), L(\mu))$ *is one-dimensional;*
3. $\mu_+ \rhd \lambda \rhd \mu$ *and* $|\partial\lambda - \partial\mu| = 1$, *the dominance condition* $\mu_+ \rhd \lambda$ *to be treated as vacuous if* μ_+ *is undefined;*
4. $[\Delta(\lambda) : L(\mu)] = 1$ *and* $\partial\lambda$ *and* $\partial\mu$ *have different parity;*
5. $[\Delta(\lambda) : L(\mu)] \neq 0$ *and* $\partial\lambda$ *and* $\partial\mu$ *have different parity;*
6. $[P(\lambda) : L(\mu)] \neq 0$ *and* $\partial\lambda$ *and* $\partial\mu$ *have different parity.*

Proof. Statements (1), (2), and (3) are equivalent by theorem 3.1, statements (3), (4), and (5) are equivalent by theorem 2.2, and (5) clearly implies (6). Finally if (6) holds, then some $\Delta(\nu)$ which occurs as a factor in a Weyl filtration of $P(\lambda)$ must have $L(\mu)$ as a composition factor; because $\Delta(\nu)$ is a factor in this filtration it also must have $L(\lambda)$ as a composition facter. Since $L(\lambda)$ and $L(\mu)$ are composition factors of $\Delta(\nu)$ with different parities, they must lie in consecutive Loewy layers of $\Delta(\nu)$, as $\Delta(\nu)$ has Loewy length at most 3 by theorem 3.1. Also, as $\Delta(\nu)$ always has a simple head and has a simple socle when $\Delta(\nu)$ has Loewy length 3, we see that $L(\lambda)$ must extend $L(\mu)$. □

4. Principal Indecomposable Modules

In this section we obtain the Loewy structures of the principal indecomposable modules of S_B. We first introduce a non-standard notation:

Notation . For a module M, we write $\ell\ell(M)$ for the Loewy length of M.

Suppose P_0 is a principal indecomposable S_B-module. Let s be the maximal integer for which there exist indecomposable projective modules $P_1, \dots, P_s$ and nonisomorphisms $\psi_i : P_i \to P_{i-1}$ such that $\psi_1 \circ \cdots \circ \psi_s \neq 0$. In the light of lemma 1.1(4), s is also the maximal integer for which there exist Young modules $Y_1, \dots, Y_s$ and nonisomorphisms $\phi_i : Y_i \to Y_{i-1}$ such that $\phi_1 \circ \cdots \circ \phi_s \neq 0$ (where $Y_0 = fP_0$). We note that $\ell\ell(P_0) = s + 1$.

Scopes [10, Theorem I(7)] showed that any projective module of B has Loewy length five and its Loewy layers and socle layers coincide, while we [1, Theorem 2.4] showed that any non-projective Young module has Loewy length 1 (in which case it is simple) or 3, and its Loewy layers and socle layers coincide as well.

Lemma 4.1. *Let $\phi : Y \to Y'$ be a homomorphism of a Young module into another which is not an isomorphism, and let M be a submodule of Y such that $\phi(M) \neq 0$. Let $l = \ell\ell(Y)$ and $l' = \ell\ell(Y')$ be the Loewy lengths of Y and Y'. We have*

$$\ell\ell(\phi(M)) \leq \ell\ell(M) - \begin{cases} 0, & \text{if } l < l'; \\ 1, & \text{if } l = l'; \\ (l - l'), & \text{if } l > l'. \end{cases}$$

Proof. We first note that since $\phi(M)$ is a quotient of M, it is clear that $\ell\ell(\phi(M)) \leq \ell\ell(M)$.

Let m be the Loewy length of M. Then $M \subseteq \operatorname{soc}^m(Y) = \operatorname{rad}^{l-m}(Y)$ and $\phi(\operatorname{rad}^{l-m} Y) \subseteq \operatorname{rad}^{l-m}(Y') = \operatorname{soc}^{l'-(l-m)}(Y')$. Therefore $\phi(M) \subseteq \operatorname{soc}^{m-(l-l')}(Y')$.

Supose $l = l'$. Then $\phi(Y) \subseteq \operatorname{rad}(Y')$: this is clear if $l = 1$ or $l = 5$, and may be checked directly if $l = 3$, using [1, Theorem 2.4]. Thus $\phi(\operatorname{rad}^{l-m} Y) \subseteq \operatorname{rad}^{l-m+1}(Y') = \operatorname{soc}^{l-(l-m+1)}(Y')$, and therefore $\phi(M) \subseteq \operatorname{soc}^{m-1}(Y')$. ☐

Now let s be a positive integer, and let $Y_0, \dots, Y_s$ be Young modules. Suppose there exist nonisomorphisms $\phi_i : Y_i \to Y_{i-1}$ such that $\Phi = \phi_1 \circ \cdots \circ \phi_s \neq 0$. For $j \in \{-4, -2, 0, 2, 4\}$, let

$$n_j = |\{i \mid l \leq i \leq s \text{ and } \ell\ell(Y_{i-1}) - \ell\ell(Y_i) = j\}|.$$

Then we have

$$\begin{aligned} s &= n_{-4} + n_{-2} + n_0 + n_2 + n_4, \quad \text{and} \\ \ell\ell(Y_0) - \ell\ell(Y_s) &= -4n_{-4} - 2n_{-2} + 2n_2 + 4n_4. \end{aligned}$$

Moreover, by the above lemma,

$$\ell\ell(\Phi(Y_s)) \leq \ell\ell(Y_s) - 4n_{-4} - 2n_{-2} - n_0.$$

Therefore

$$\begin{aligned} \ell\ell(Y_0) - \ell\ell(Y_s) &= -4n_{-4} - 2n_{-2} + 2n_2 + 4n_4 \\ &\geq -4n_{-4} - 2n_{-2} + 2(s - n_{-4} - n_{-2} - n_0) \\ &= -6n_{-4} - 4n_{-2} - 2n_0 + 2s \\ &\geq 2(\ell\ell(\Phi(Y_s)) - \ell\ell(Y_s) + s). \end{aligned}$$

Hence,

$$\begin{aligned} s &\leq \tfrac{1}{2}(\ell\ell(Y_0) + \ell\ell(Y_s)) - \ell\ell(\Phi(Y_s)) \\ &\leq \tfrac{1}{2}(\ell\ell(Y_0) + 5) - 1 \\ &= \tfrac{1}{2}(\ell\ell(Y_0) + 3). \end{aligned}$$

¿From this calculation and the comments above we deduce that:

Proposition 4.2. *The Loewy length of a projective module $P(\lambda)$ is bounded above by* 5, 4, *or* 3 *when the corresponding Young module Y^λ has Loewy length* 5, 3, *or* 1, *respectively.* □

In fact, these bounds are exact:

Theorem 4.3. *The Loewy length of a projective module $P(\lambda)$ is* 5, 4, *or* 3 *when the corresponding Young module Y^λ has Loewy length* 5, 3, *or* 1, *respectively.*

Proof. It is clear that Y^λ has Loewy length 5 if, and only if, λ is p-restricted, and since $fP(\lambda) = Y^\lambda$, $P(\lambda)$ has Loewy length at least least five.

If Y^λ has Loewy length 3, then λ is not p-restricted. Thus $fL(\lambda) = 0$, and therfore $f(\mathrm{rad}(P(\lambda))) \cong Y^\lambda$. Thus $\mathrm{rad}(P(\lambda))$ has Loewy length at least 3, and hence $P(\lambda)$ has Loewy length at least 4.

There is only one partition in $\mathfrak{P}_B$ that is both p-singular and non-p-restricted, namely that denoted by $\lambda^{(p-1)}$ in [1], and $Y^{\lambda^{(p-1)}}$ has Loewy length 3. Thus, if Y^λ has Loewy length 1, then λ is p-regular, and using theorem 3.1(2), we see that $\Delta(\lambda)$ has Loewy length 3, and so $P(\lambda)$ has Loewy length at least 3. □

Theorem 4.4. *Let $\lambda, \mu \in \mathfrak{P}_B$ with $\lambda \neq \mu$, and suppose that $\Delta(\mu)$ is a Weyl factor of $P(\lambda)$. Then i-th Loewy layer of $\Delta(\mu)$ lies completely in the j-th Loewy layer of $P(\lambda)$, where*

$$j = i + 2 - |\partial\lambda - \partial\mu|.$$

Proof. Since $\Delta(\mu)$ is a Weyl factor of $P(\lambda)$, we have $[\Delta(\mu) : L(\lambda)] \neq 0$, and using theorem 3.1(1), this implies that $|\partial\lambda - \partial\mu| = 0$ or 1. If $|\partial\lambda - \partial\mu| = 0$, then $\mu = \lambda_+$. Since λ_+ is p-regular, theorem 3.1(2) shows that $\Delta(\lambda_+)$ has Loewy length 3. Also, as λ is p-restricted, we see that $P(\lambda)$ has Loewy length 5 by the previous theorem. The bipartite nature of the Ext- quiver of S_B shows that the head $L(\lambda_+)$ of $\Delta(\lambda_+)$ must lie in the third Loewy layer. The statement now follows for this case.

If $|\partial\lambda - \partial\mu| = 1$, then by theorems 3.1(1) and 2.2, we see that $\mathrm{Ext}^1_{S_B}(L(\lambda), L(\mu)) \neq 0$, and thus the head $L(\mu)$ of $\Delta(\mu)$ must lie in the second Loewy layer of $P(\lambda)$. Since the Loewy length of $P(\lambda)$ is at most 5, together with the bipartite nature of the Ext-quiver of S_B, the statement will fail only if a composition factor, $L(\nu)$ say, of the second Loewy layer of $\Delta(\mu)$ actually occurs in the fifth Loewy layer of $P(\lambda)$. Assuming that this is indeed possible, then λ is p-restricted, and so $P(\lambda)$ is self-dual. Thus the head and socle of $P(\lambda)$ are isomorphic, in particular, they are both simple. But the fifth Loewy layer of $P(\lambda)$ is a submodule of its socle, and as λ is p-restricted, the socle of $P(\lambda)$ should come from the socle of the Weyl factor $\Delta(\lambda_+)$. □

5. Tilting modules

Tilting modules of Schur algebras are indecomposable self-dual modules having a Weyl filtration. For each partition λ of r, there is an associated tilting module $T(\lambda)$ which is characterised by $\Delta(\lambda) \subseteq T(\lambda)$, and its other Weyl factors $\Delta(\mu)$ satisfy $\mu < \lambda$. Moreover, the multiplicity of the factor $\Delta(\mu)$ in the a filtration of $T(\lambda)$ is equal to $[\Delta(\mu') : L(\lambda')]$ [3, (3.8)]. It is also clear that $T(\lambda)$ is simple if, and only if, $\Delta(\lambda)$ is simple; in which case $T(\lambda) = \Delta(\lambda) = L(\lambda)$.

If λ is a p-regular partition of r, let $\mu = \lambda^{*\prime}$. Then μ is p-restricted and by lemma 1.1(7), we see that $P(\mu)$ is a tilting module. Moreover, by lemma 1.1(6) and the known filtration of $P(\mu)$, we see that $\Delta(\mu'^{*}) = \Delta(\lambda)$ is a submodule of $P(\mu)$, so that $P(\mu) = T(\lambda)$.

In this section, we will obtain the Loewy structures of the tilting modules of S_B. From the above comments, we only need to look at the non-simple tilting modules associated to p-singular partitions. Theorem 3.1(3) showed that these partitions have associated Weyl modules having Loewy length 2.

Since the Loewy lengths of the principal indecomposable modules of S_B are bounded above by 5, any indecomposable module having Loewy length 5 must in fact be projective. In fact, it must be of the form $P(\mu)$ with μ p-restricted using theorem 4.3. Hence the Loewy lengths of the tilting modules of S_B associated to p-singular partitions are bounded above by 4.

Using the fact that the multiplicity of $\Delta(\mu)$ in a Weyl filtration of $T(\lambda)$ is $[\Delta(\mu') : L(\lambda')]$ and theorem 2.2, it is routine to enumerate the Weyl factors of $T(\lambda)$ (with λ p-singular) and hence its composition factors (with multiplicity). We find that there are at most two composition factors occurring twice and the remaining composition factors occurring once. Moreover, the ∂-values of the composition factors of the same (resp. different) multiplicity have the same (resp. different) parity. A little thought shows that $T(\lambda)$ must have Loewy length 3: the composition factors having multiplicity 2 occur in the head and in the socle, and the composition factors having multiplicity 1 lie in the semi-simple heart.

6. Analogous results for symmetric groups

In this section, we apply the Schur functor in order to get analogues of the results in section 3 for symmetric groups.

Let λ and μ be p-restricted partitions in $\mathfrak{P}_B$. We have $fL(\lambda) \cong D^{\lambda'^{*}}$ and $fL(\mu) \cong D^{\mu'^{*}}$. Suppose that $\mathrm{Ext}^1_{S_B}(L(\lambda), L(\mu)) \neq 0$. Then as $fP(\lambda) \cong P(D^{\lambda'^{*}})$, we would get $\mathrm{Ext}^1_B(D^{\lambda'^{*}}, D^{\mu'^{*}}) \neq 0$.

Conversely suppose that $\mathrm{Ext}^1_B(D^{\lambda'^{*}}, D^{\mu'^{*}}) \neq 0$. Then there exists a homomorphism $\phi : P(D^{\mu'^{*}}) \to P(D^{\lambda'^{*}})$ such that the image of ϕ is not contained in $\mathrm{rad}^2(P(D^{\lambda'^{*}}))$. By lemma 1.1(4), there exists a homomorphism $\widetilde{\phi} : P(\mu) \to P(\lambda)$ such that $f\widetilde{\phi} = \phi$. Now suppose that $\mathrm{Ext}^1_{S_B}(L(\lambda), L(\mu)) = 0$, looking for a contradiction. There the image of $\widetilde{\phi}$ is contained in $\mathrm{rad}^2(P(\lambda))$, so there exist indecomposable projective S_B-modules $P_1, \dots, P_t$, and nonisomorphisms $\widetilde{\alpha}_i : P(\mu) \to P_i$ and $\widetilde{\beta}_i : P_i \to P(\mu)$ such that $\widetilde{\phi} = \sum_{i=1}^t \widetilde{\beta}_i \circ \widetilde{\alpha}_i$. Applying the Schur functor, we get $\phi = \sum_{i=1}^t \beta_i \circ \alpha_i$, where $\alpha_i = f\widetilde{\alpha}_i : P(D^{\lambda'^{*}}) \to fP_i$ and $\beta_i = f\widetilde{\beta}_i : fP_i \to P(D^{\lambda'^{*}})$ are nonisomorphisms. If fP_i is projective then the image of $\beta_i \circ \alpha_i$ is contained in $\mathrm{rad}^2(P(D^{\lambda'^{*}}))$. If fP_i is not projective, then because it is a Young module it has Loewy length at most 3. So in this case the image of $\beta_i \circ \alpha_i$ is contained in $\mathrm{soc}^3(P(D^{\lambda'^{*}})) = \mathrm{rad}^2(P(D^{\lambda'^{*}}))$. Therefore, the image of ϕ is contained in $\mathrm{rad}^2(P(D^{\lambda'^{*}}))$, a contradiction.

So we have proved that $\mathrm{Ext}^1_{S_B}(L(\lambda), L(\mu)) \neq 0$ if and only if $\mathrm{Ext}^1_B(D^{\lambda'^*}, D^{\mu'^*}) \neq 0$. As $\mathrm{Ext}^1_{S_B}(L(\lambda), L(\mu))$ is at most one-dimensional by theorem 3.1 and the same holds for $\mathrm{Ext}^1_B(D^{\lambda'^*}, D^{\mu'^*})$ by [10, Theorem I(5)], we actually have that $\dim_k \mathrm{Ext}^1_{S_B}(L(\lambda), L(\mu)) = \dim_k \mathrm{Ext}^1_B(D^{\lambda'^*}, D^{\mu'^*})$.

Theorem 6.1. *Let ρ and σ be p-regular partitions in $\mathfrak{P}_B$ with $\rho \leq \sigma$. The following statements are equivalent:*

1. $\mathrm{Ext}^1_B(D^\sigma, D^\rho)$ *is non-zero;*
2. $\mathrm{Ext}^1_B(D^\sigma, D^\rho)$ *is one-dimensional;*
3. $\sigma \rhd \rho \rhd \sigma_-$ *and* $|\partial\sigma - \partial\rho| = 1$*;*
4. $[S^\rho : D^\sigma] = 1$*, and* $\partial\rho$ *and* $\partial\sigma$ *have different parity;*
5. $[S^\rho : D^\sigma] \neq 0$*, and* $\partial\rho$ *and* $\partial\sigma$ *have different parity;*
6. $[P(D^\rho) : D^\sigma] \neq 0$*, and* $\partial\rho$ *and* $\partial\sigma$ *have different parity.*

Proof. Let $\lambda = \rho'$ and $\mu = \sigma'$. Then λ and μ are p-restricted partitions in $\mathfrak{P}_{B'}$, and $\lambda \geq \mu$. Using corollary 3.3, we only have to show that the i-th statement of this theorem is equivalent to the i-th statement of the corollary.

For statements (1) and (2), from the comments above, we note that $\dim_k \mathrm{Ext}^1_{S_{B'}}(L(\lambda), L(\mu)) = \dim_k \mathrm{Ext}^1_{B'}(D^{\rho^*}, D^{\sigma^*}) = \dim_k \mathrm{Ext}^1_B(D^\rho, D^\sigma)$. For the remaining statements, we first note that the ∂-values are preserved under the map $\nu \mapsto \nu'$. Moreover,

$$\begin{aligned} \mu_+ \rhd \lambda \rhd \mu \;&\Leftrightarrow\; (\sigma')'^* \rhd \rho' \rhd \sigma' \\ &\Leftrightarrow\; \sigma \rhd \rho \rhd \sigma^{*\prime} = \sigma_-, \end{aligned}$$

and $[\Delta(\lambda) : L(\mu)] = [S^{\lambda'} : D^{\mu'}] = [S^\rho : D^\sigma]$ and $[P(\lambda) : L(\mu)] = [P(D^{\lambda'^*}) : D^{\mu'^*}] = [P(D^\rho) : D^\sigma]$, since λ and μ are p-restricted. □

Thus we can conclude that grouping the simple B-modules according to the parity of ∂-values of their associated partitions also gives a partition of the simple B-modules displaying the bipartite nature of the Ext- quiver of B.

Proposition 6.2. *Let $\lambda \in \mathfrak{P}_B$. The Loewy length of S^λ is bounded above by 3. Moreover, S^λ has Loewy length 3 if, and only if, λ is both p-regular and p-restricted.*

Proof. Since $f\Delta(\lambda) = (S^\lambda)^*$, and $\Delta(\lambda)$ has Loewy length at most 3 by theorem 3.1, we see that S^λ has Loewy length at most 3 as well.

If S^λ has Loewy length 3, then so has $(S^\lambda)^* = f\Delta(\lambda)$. Thus, λ is p-regular by theorem 3.1. Also, $fL(\lambda) \neq 0$, as otherwise $f(\mathrm{rad}(\Delta(\lambda))) = (S^\lambda)^*$ has Loewy length at most 2. This shows that λ is p-restricted.

If λ is both p-regular and p-restricted, then S^λ has a simple head D^λ and a simple socle $D^{\lambda'^*}$. Since the ∂-values of λ and $\lambda'^* = \lambda_+$ are the same, the head cannot extend the socle by the previous theorem. Thus, S^λ has Loewy length at least 3. □

Since S^λ has a simple head D^λ (resp. simple socle $D^{\lambda'^*}$) if λ is p-regular (resp. p-restricted), using the theorem and proposition of this section as well as [9, Theorem 4.4], we will be able to obtain the Loewy structures of the Specht modules of B.

Remarks . Instead of using the Schur functor to translate the results of Schur algebras obtained in this paper, the results of this section can also be obtained independently by considering directly the defect 2 blocks of symmetric groups.

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