FILTRATIONS IN ROUQUIER BLOCKS OF SYMMETRIC GROUPS AND SCHUR ALGEBRAS

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Abstract. We study Rouquier blocks of symmetric groups and Schur algebras in detail, and obtain explicit description for the radical layers of the principal indecomposable, Weyl, Young and Specht modules of these blocks. At the same time, the Jantzen filtrations of the Weyl modules are shown to coincide with their radical filtrations. We also address the conjectures of Martin, Lascoux-Leclerc-Thibon-Rouquier and James for these blocks.

1. Introduction

In the course of finding character-theoretic evidence for Broue's abelian defect group conjecture, Rouquier in 1991 singled out some blocks of symmetric groups which he believed should have special properties. In the abelian defect case he moreover conjectured Morita equivalences with certain wreath products (see [23]). Rouquier's conjecture, which may be regarded as an important first step in proving Broue's conjecture for the symmetric groups, was proved in [2].

In this paper we use the Morita equivalences of [2] to give a detailed account, in terms of explicit descriptions of radical filtrations of distinguished modules, of Rouquier's blocks of symmetric groups and the corresponding blocks of Schur algebras. We give graded composition multiplicities for Weyl modules, principal indecomposable modules, Specht modules, and Young modules in these blocks. Moreover we show that in these blocks the Jantzen filtrations of Weyl modules coincide with radical filtrations. The description of radical layers of the principal indecomposable modules for Rouquier blocks of symmetric groups in particular shows that Martin's conjecture on their common radical length holds. We are also able to show that a conjecture due to Lascoux, Leclerc, Thibon, and Rouquier, that the v-decomposition numbers arising from v-deformed Fock spaces describe Jantzen filtrations of Weyl modules, holds for Rouquier blocks. This implies as well for these blocks an important conjecture of James on decomposition numbers.

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Here is an indication of the organisation of this paper:

Section 2 is devoted to a review of the representation theory of Schur algebras and symmetric groups.

In section 3, we introduce some notation for representations of wreath products $\Gamma(w) = \Gamma \wr \mathfrak{S}_w$ of an arbitrary algebra Γ, and go on to consider in detail the cases $\Gamma = A$ and $\Gamma = \mathbf{S}_A$, where A is the principal p-block of $\mathfrak{S}_p$ and $\mathbf{S}_A$ is the corresponding block of the Schur algebra $S(p,p)$. The results stated here are direct applications of the general theory developed in [4].

In section 4 we introduce Rouquier's p-core partitions which label well-behaved blocks B_w of symmetric groups. We reexamine the Morita equivalence between B_w and $A(w)$ (for $w < p$) constructed in [2], and determine the $A(w)$-modules corresponding to simple modules and Young modules in B_w.

In section 5 we prove a Schur algebra analogue of the Morita equivalence of [2]: $\mathbf{S}_{B_w}$ and $\mathbf{S}_A(w)$ are Morita equivalent, where $\mathbf{S}_{B_w}$ and $\mathbf{S}_A$ are the Schur algebra blocks corresponding to B_w and A. Moreover we determine the $\mathbf{S}_A(w)$-modules corresponding to the Weyl modules in $\mathbf{S}_{B_w}$, and use this to show that the Jantzen filtrations and radical filtrations on these Weyl modules coincide.

Finally in section 6 we present formulas for composition factors of radical layers of various important modules in the blocks B_w and $\mathbf{S}_{B_w}$. These are derived from general formulas for filtrations of modules for wreath products presented in [4]. We end by giving positive answers to special cases of some conjectures.

Remark. An analogue of the Morita equivalence of B_w and $A(w)$ for finite general linear groups has been proved independently by Turner [25] and by Miyachi [22]. Hida and Miyachi [8] have then used this to prove analogues of Proposition 4.4, the last statement in Theorem 5.2, and the second statement of Theorem 6.2.

2. Preliminaries

Fix an odd prime p and let $\mathcal{O}$ be a complete discrete valuation ring with quotient field K of characteristic 0 and residue field $k = \mathcal{O}/\pi$ of characteristic p. We usually write $\otimes$ in place of $\otimes_{\mathcal{O}}$, and if X is an $\mathcal{O}$-module we write $\overline{X} = k \otimes X$ and $\widehat{X} = K \otimes X$.

We will need to consider both left and right modules, but let us agree on left modules as the default.

Let Γ be an $\mathcal{O}$-algebra, finitely generated as an $\mathcal{O}$-module. We denote the category of finitely generated left Γ-modules as Γ-mod and the category of finitely generated right Γ-modules as mod-Γ.

If M is a left Γ-module let $M^\vee = \mathrm{Hom}_{\mathcal{O}}(M, \mathcal{O})$ be the dual space equipped with a right Γ-module structure via $(\phi\gamma)(m) = \phi(\gamma m)$ ($\phi \in M^\vee,\ \gamma \in \Gamma,\ m \in M$). We can define analogously a left Γ-module $N^\vee$

given any right Γ-module N. These dualities give inverse antiequivalences between full subcategories of modules which are free and of finite rank over $\mathcal{O}$.

We will have occasion to use the following well-known fact: If M and N are finitely generated Γ-modules, and N is $\mathcal{O}$-free, then $\mathrm{Hom}_{\overline{\Gamma}}(\overline{M}, \overline{N}) = 0$ implies $\mathrm{Hom}_{\Gamma}(M, N) = 0$.

If Γ_1 and Γ_2 are $\mathcal{O}$-algebras, then by a (Γ_1, Γ_2)-bimodule we mean a $\Gamma_1 \otimes_{\mathcal{O}} \Gamma_2^{\mathrm{op}}$-module.

2.1. **Partitions.** Let Λ be the set of partitions. We use $>$ and $\rhd$ to denote the lexicographic and dominance orders on Λ (see, e.g., [12, §3]). Given a partition $\lambda = (\lambda_1, \lambda_2, \ldots)$, we denote by λ' the conjugate partition and write $|\lambda| = \sum_j \lambda_j$. We call λ p-singular if $\lambda_{i+1} = \ldots = \lambda_{i+p} \neq 0$ for some i and p-regular otherwise. It is called p-restricted if λ' is p-regular.

Given a partially order set $(I, \geq)$, let Λ^I denote the set of I-tuples of partitions, and given in addition a nonnegative integer w, let $\Lambda^I_w = \{(\lambda^i)_{i \in I} \in \Lambda^I \mid \sum_{i \in I} |\lambda^i| = w\}$. We define a partial order $\succeq$ on Λ^I_w by $\boldsymbol{\lambda} \succeq \boldsymbol{\mu}$ if and only if $\boldsymbol{\lambda} = \boldsymbol{\mu}$ or

$$\sum_{\substack{j \in I \\ j \geq i}} |\lambda^j| \geq \sum_{\substack{j \in I \\ j \geq i}} |\mu^j|$$

for all $i \in I$, and the inequality is strict for at least one i.

For the case where $I = \{0, 1, \ldots, n-1\}$ ($n \in \mathbb{N}$) with the natural ordering, we write Λ^n for Λ^I. If $n' < n$, we identify $\Lambda^{n'}_w$ with the subset $\{(\lambda^0, \ldots, \lambda^{n-1}) \in \Lambda^n_w \mid \lambda^i = \emptyset \ \forall i \geq n'\}$.

To each $\lambda \in \Lambda$ we associate a p-core $\mathrm{core}(\lambda) \in \Lambda$ and p-quotient $\mathrm{quot}(\lambda) = (\lambda^0, \ldots, \lambda^{p-1}) \in \Lambda^p$; these are easily determined when λ is displayed on a James's p-abacus (see, e.g., [12, §2.7]). When the numbers of beads in the abacus display is a multiple of p, and the runners are labelled 0 to $p-1$ from left to right, we can read off λ^i from runner i. If $\mathrm{quot}(\lambda) \in \Lambda^p_w$ we say λ has p-weight w. We denote $\Lambda(\kappa, w)$ the set of partitions with p-core κ and p-weight w.

If $\lambda, \mu \in \Lambda$ we write $\mu \nearrow \lambda$ if λ is obtained by adding one node to μ. If $\boldsymbol{\lambda}, \boldsymbol{\mu} \in \Lambda^n$ and $i \in \{0, \ldots, n-1\}$, we write $\boldsymbol{\mu} \nearrow_i \boldsymbol{\lambda}$ to mean $\mu^i \nearrow \lambda^i$ and $\mu^j = \lambda^j$ for $j \neq i$.

2.2. **Schur algebras.** We briefly introduce some notation relating to Schur algebras, referring the reader to [7] for details. Given any positive integer r, let $\mathbf{S}_r = S_{\mathcal{O}}(r, r)$ be the Schur algebra over $\mathcal{O}$ associated to homogeneous polynomial representations of GL_r of degree r [7, §2.3].

Denote by $\Delta(\lambda)$ the Weyl module associated to a partition λ of r; it is free and of finite rank over $\mathcal{O}$ [7, §5.1, (5.4e)]. The modules $\widehat{\Delta(\lambda)}$, as λ ranges over the partitions of r, form a complete set of nonisomorphic (absolutely) simple modules for the semisimple algebra $\widehat{\mathbf{S}}_r$.

Let $L(\lambda)$ be the unique simple quotient of the $\overline{\mathbf{S}}_r$-module $\overline{\Delta(\lambda)}$ and let $P(\lambda)$ be a projective cover of $L(\lambda)$ as an $\mathbf{S}_r$-module (so that $\overline{P(\lambda)}$ is a projective cover as $\overline{\mathbf{S}}_r$-module). The modules $L(\lambda)$, as λ runs over partitions of r, form a complete set of nonisomorphic (absolutely) simple $\overline{\mathbf{S}}_r$-modules. All composition factors of $\mathrm{rad}(\overline{\Delta(\lambda)})$ are of the form $L(\mu)$ with $\mu < \lambda$.

Let J be the involutory anti-automorphism of $\mathbf{S}_r$ defined in [7, §2.7]. If M is an $\mathbf{S}_r$-module then the contravariant dual $M^\circ = \mathrm{Hom}_{\mathcal{O}}(M, \mathcal{O})$ is an $\mathbf{S}_r$-module by the rule $(\xi\phi)(x) = \phi(J(\xi)x)$ ($\phi \in M^\circ,\ \xi \in \mathbf{S}_r,\ x \in M$). Contravariant duality gives a self-antiequivalence of the category of $\mathbf{S}_r$-modules free and of finite rank over $\mathcal{O}$.

We've so far mentioned only left modules but right modules $\Delta'(\lambda)$, $L'(\lambda)$, etc., can be constructed analogously. The following gives a useful characterisation of the Weyl modules as well as a relationship between left and right module versions.

Proposition 2.1.

(1) *(Donkin) The Weyl module $\Delta(\lambda)$ is isomorphic to the quotient of $P(\lambda)$ by the sum of the images of all homomorphisms $P(\mu) \to P(\lambda)$ for all $\mu > \lambda$. An analogous statement holds for $\Delta'(\lambda)$.*

(2) *We have $\Delta(\lambda)^\circ \cong \Delta'(\lambda)^\vee$.*

Proof.

(1) Donkin shows in [5, (2.2)] that there exists a filtration
$$P(\lambda) = P^0 \supseteq P^1 \supseteq \cdots \supseteq P^s = 0$$
such that $P^0/P^1 \cong \Delta(\lambda)$ and for each $i = \{1, 2, \ldots, s-1\}$ there exists $\mu_i > \lambda$ such that $P^i/P^{i+1} \cong \Delta(\mu_i)$. The statement is therefore proved so long as $\mathrm{Hom}(P(\mu), \Delta(\lambda)) = 0$ for all $\mu > \lambda$, and this holds because $\Delta(\lambda)$ is $\mathcal{O}$-free and $\mathrm{Hom}(\overline{P(\mu)}, \overline{\Delta(\lambda)}) = 0$.

(2) The simple modules $L(\lambda)$ and $L'(\lambda)$ are characterised by the leading terms of their characters [7, §3.5]. On the other hand it is easy to check that the dualities $\vee$ and $\circ$ preserve characters, so we have $L(\lambda)^{\circ\vee} \cong L'(\lambda)$. It follows that $P'(\lambda)$ is isomorphic to $P(\lambda)^{\circ\vee}$ and therefore has a filtration by $\Delta(\mu)^{\circ\vee}$'s with $\mu \geq \lambda$ in which $\Delta(\lambda)^{\circ\vee}$ appears only once, at the top. Therefore both $\Delta(\lambda)^{\circ\vee}$ and $\Delta'(\lambda)$ are isomorphic to the quotient of $P'(\lambda)$ by the sum of images of all maps $P'(\mu) \to P'(\lambda)$ with $\mu > \lambda$. Applying $\vee$ now yields the desired result.

□

2.3. **Schur functors and symmetric groups.** There exists an idempotent $e \in \mathbf{S}_r$ such that $e\mathbf{S}_r e$ can be identified with the group algebra

$\mathcal{O}\mathfrak{S}_r$, where $\mathfrak{S}_r$ denotes the symmetric group of degree r [7, §6.1]. The exact functor

$$f : \mathbf{S}_r\text{-mod} \to \mathcal{O}\mathfrak{S}_r\text{-mod}$$

taking an $\mathbf{S}_r$-module V to the $\mathcal{O}\mathfrak{S}_r$-module eV is called the Schur functor.

The anti-automorphism J sends $e\mathbf{S}_r e$ into itself, inducing on $\mathcal{O}\mathfrak{S}_r$ the anti-automorphism sending σ to σ^{-1} for each $\sigma \in \mathfrak{S}_r$. Thus for any $\mathbf{S}_r$-module V we have $f(V^\circ) \cong f(V)^*$, where $*$ is the usual duality for modules over group algebras.

For any partition λ of r let S^λ be the Specht module for $\mathcal{O}\mathfrak{S}_r$ associated to λ [9, 8.4]. For us it will be more convenient to work with the dual module $(S^\lambda)^*$, which we denote as $S(\lambda)$. The modules $\widehat{S(\lambda)}$, as λ ranges over partitions of r, form a complete set of nonisomorphic (absolutely) simple modules of the semisimple algebra $K\mathfrak{S}_r$. If λ is a p-restricted partition then $\overline{S(\lambda)}$ has unique simple quotient $D(\lambda)$ and as λ ranges over p-restricted partitions, the $D(\lambda)$'s form a complete set of nonisomorphic (absolutely) simple $k\mathfrak{S}_r$-modules.

Note. $S^\lambda \cong S(\lambda') \otimes \mathrm{sgn}$ and therefore for λ p-regular, $D^\lambda \cong D(\lambda') \otimes \mathrm{sgn}$, where D^λ is James's simple module, defined as the unique simple quotient of S^λ, and sgn is the sign representation [7, §6.4].

Let $Y(\lambda)$ be the Young module of $\mathcal{O}\mathfrak{S}_r$ associated to λ; the Young modules are precisely the indecomposable summands of permutation modules on Young subgroups of $\mathfrak{S}_r$, i.e. subgroups of the form $\mathfrak{S}_{r_1} \times \cdots \times \mathfrak{S}_{r_s}$ [10]. The Young module $Y(\lambda)$ is characterised as the unique indecomposable summand of the permutation module on $\mathfrak{S}_{\lambda_1} \times \mathfrak{S}_{\lambda_2} \times \cdots$ containing $S(\lambda)$.

We collect together some useful facts:

Proposition 2.2. *Let λ and μ be partitions of r.*

(1) *We have $f(\Delta(\lambda)) \cong S(\lambda)$,*
(2) *We have $f(L(\lambda)) \cong D(\lambda)$ if λ is p-restricted and $f(L(\lambda)) = 0$ if λ is not p-restricted.*
(3) *We have $f(P(\lambda)) \cong Y(\lambda)$, and if λ is p-restricted, then $Y(\lambda)$ is an $\mathcal{O}\mathfrak{S}_r$-projective cover of $D(\lambda)$.*
(4) *The functor f induces an isomorphism*

$$\mathrm{Hom}(P(\lambda), P(\mu)) \to \mathrm{Hom}(Y(\lambda), Y(\mu)).$$

(5) *The multiplicity of $\widehat{S(\mu)}$ in $\widehat{Y(\lambda)}$ is 1 if $\mu = \lambda$ and is 0 if $\mu \not\trianglerighteq \lambda$.*

Proof. For (1) and (2) see §6.3 and §6.4 in [7], for (3) and (4) see (2.5) and (2.4) in [5], and for (5) see [10]. □

2.4. **Jantzen filtrations.** Let M and N be free $\mathcal{O}$-modules of finite rank. To any homomorphism $\eta : M \to N$ we can associate a filtration

$$M = M^{\eta,0} \subseteq M^{\eta,1} \subseteq \cdots$$

where $M^{\eta,r} = \eta^{-1}(\pi^r N)$. This induces a filtration

$$\overline{M} = \overline{M}^{\eta,0} \subseteq \overline{M}^{\eta,1} \subseteq \cdots$$

where $\overline{M}^{\eta,s} = (M^{\eta,s} + \pi M)/\pi M$ is the image of $M^{\eta,s}$ in $\overline{M}$.

If A is an $\mathcal{O}$-algebra and η is a homomorphism of A-modules, then these are filtrations of the A-module M and the $\overline{A}$-module $\overline{M}$.

The Jantzen filtration

$$\overline{\Delta(\lambda)} = \overline{\Delta(\lambda)}^0 \subseteq \overline{\Delta(\lambda)}^1 \subseteq \cdots$$

of the Weyl module $\overline{\Delta(\lambda)}$ is classically defined (e.g., see [7, §5.5]) using a certain nondegenerate bilinear form $\langle , \rangle : \Delta(\lambda) \times \Delta(\lambda) \to \mathcal{O}$ satisfying $\langle \xi x, y \rangle = \langle x, J(\xi) y \rangle$ for all $x, y \in \Delta(\lambda)$ and $\xi \in \mathbf{S}_r$, with

$$\Delta(\lambda)^s = \{x \in \Delta(\lambda) \mid \langle x, y \rangle \in \pi^s \ \forall y \in \Delta(\lambda)\} \quad \text{and}$$
$$\overline{\Delta(\lambda)}^s = \frac{\Delta(\lambda)^s + \pi\Delta(\lambda)}{\pi\Delta(\lambda)}.$$

We can define an $\mathbf{S}_r$-homomorphism $\vartheta : \Delta(\lambda) \to \Delta(\lambda)^\circ$ by $\vartheta(x)(y) = \langle x, y \rangle$. Then $\Delta(\lambda)^s = \Delta(\lambda)^{\vartheta,s}$. Thus, the Jantzen filtration of $\overline{\Delta(\lambda)}$ is induced by a homomorphism $\vartheta : \Delta(\lambda) \to \Delta(\lambda)^\circ$ satisfying $\vartheta(\Delta(\lambda)) \nsubseteq \pi\Delta(\lambda)^\circ$ (since $\overline{\Delta(\lambda)}^1 = \operatorname{rad} \overline{\Delta(\lambda)}$). This is in fact a characterisation of the Jantzen filtration:

Lemma 2.3. *Suppose λ is a partition of r. Let $\eta : \Delta(\lambda) \to \Delta(\lambda)^\circ$ be an $\mathbf{S}_r$-homomorphism such that $\eta(\Delta(\lambda)) \nsubseteq \pi\Delta(\lambda)^\circ$ (equivalently, $\overline{\Delta(\lambda)}^{\eta,1} \subsetneq \overline{\Delta(\lambda)}$). Then for any nonnegative integer s, we have*

$$\overline{\Delta(\lambda)}^{\eta,s} = \overline{\Delta(\lambda)}^s.$$

Proof. Let $\vartheta : \Delta(\lambda) \to \Delta(\lambda)^\circ$ be defined by $\vartheta(x)(y) = \langle x, y \rangle$, where $\langle , \rangle$ is the nondegenerate bilinear form used classically to define the Jantzen filtration. Now, $\widehat{\operatorname{Hom}}(\Delta(\lambda), \Delta(\lambda)^\circ) \cong \operatorname{Hom}(\widehat{\Delta(\lambda)}, \widehat{\Delta(\lambda)^\circ})$ is one-dimensional because $\widehat{\Delta(\lambda)} \cong \widehat{\Delta(\lambda)^\circ}$ is irreducible. Therefore $\eta = \zeta\vartheta$ for some $\zeta \in K$. Since $\mathcal{O}$ is a valuation ring, we have $\zeta \in \mathcal{O}$ or $\zeta^{-1} \in \mathcal{O}$. But as the images of ϑ and η are both not contained in $\pi\Delta(\lambda)^\circ$, we see that $\zeta, \zeta^{-1} \notin \pi$. Thus ζ is a unit in $\mathcal{O}$, and hence for any negative integer s,

$$\overline{\Delta(\lambda)}^{\eta,s} = \overline{\Delta(\lambda)}^{\vartheta,s} = \overline{\Delta(\lambda)}^s.$$

$\square$

2.5. **Blocks.** By 'Nakayama's Conjecture' (see, e.g., [12, §6.1.21–6.1.42]), the modules $S(\lambda)$ and $S(\mu)$ lie in the same block if, and only if, λ and μ have the same p-core. Hence a block of $\mathcal{O}\mathfrak{S}_r$ is determined by a p-core partition κ of $r - wp$ (where w is a nonnegative integer, known as the weight of the block). A Specht module, simple module, or Young module of $\mathcal{O}\mathfrak{S}_r$ lies in this block if, and only if, its associated partition of r has p-core κ.

Because every indecomposable projective $\mathcal{O}\mathfrak{S}_r$-module is a Young module, Proposition 2.2(3,4) implies that there exists a 1-1 correspondence between blocks of $\mathcal{O}\mathfrak{S}_r$ and blocks of $\mathbf{S}_r$ such that if B is a block of $\mathcal{O}\mathfrak{S}_r$ then the Schur functor f sends $\mathbf{S}_r$-modules lying in the corresponding block $\mathbf{S}_B$ of $\mathbf{S}_r$ to $\mathcal{O}\mathfrak{S}_r$-modules lying in B. Let e_B be the component in $\mathbf{S}_B$ of the idempotent e. Then $B = e_B\mathbf{S}_B e_B$, and we let

$$f_B : S_B\text{-mod} \to B\text{-mod}$$

be the functor taking V to $eV = e_BV$. We will abuse notation, using f_B to denote as well the associated functors over K and k.

If κ is the p-core partition associated to B then κ is associated to $\mathbf{S}_B$ in the following way: a Weyl module or simple module of $\mathbf{S}_r$ lies in the block $\mathbf{S}_B$ if and only if its associated partition of r has p-core κ. This labelling of the blocks of $\mathbf{S}_r$ was discovered by Donkin [5, (2.12)].

3. Wreath products

3.1. **General theory.** We introduce standard constructions of modules for wreath products of algebras (see, e.g., [4]). Suppose $R \in \{K, \mathcal{O}, k\}$ and let Γ be an R-algebra, free and of finite rank as an R-module. In this section, we write $\otimes$ for $\otimes_R$. We define for any $w < p$, an R-algebra

$$\Gamma(w) := \Gamma^{\otimes w} \otimes R\mathfrak{S}_w$$

with multiplication

$$(\gamma_1 \otimes \cdots \otimes \gamma_w \otimes \sigma)(\delta_1 \otimes \cdots \otimes \delta_w \otimes \tau) = \gamma_1\delta_{\sigma^{-1}(1)} \otimes \cdots \gamma_w\delta_{\sigma^{-1}(w)} \otimes \sigma\tau \qquad (\gamma_i, \delta_i \in \Gamma; \sigma, \tau \in \mathfrak{S}_w).$$

For example, if Γ is the group algebra of a group G, then $\Gamma(w)$ is isomorphic to the group algebra of the wreath product $G \wr \mathfrak{S}_w$.

If $w_0 + \cdots + w_{n-1} = w$, then $\Gamma^{\otimes w} \otimes R(\mathfrak{S}_{w_0} \times \cdots \times \mathfrak{S}_{w_{n-1}})$ is a subalgebra of $\Gamma(w)$ isomorphic to $\Gamma(w_0) \otimes \cdots \otimes \Gamma(w_{n-1})$. If V is an $\Gamma(w)$-module then by restriction of scalars we get a $\Gamma(w_0) \otimes \cdots \otimes \Gamma(w_{n-1})$-module which we denote by $\mathrm{Res}^w_{w_0,\ldots,w_{n-1}}(V)$. Similarly for any $\Gamma(w_0) \otimes \cdots \otimes \Gamma(w_{n-1})$-module we write

$$\mathrm{Ind}^w_{w_0,\ldots,w_{n-1}}(W) = \Gamma(w) \otimes_{\Gamma(w_0)\otimes\cdots\otimes\Gamma(w_{n-1})} W.$$

If M is a Γ-module, then $M^{\otimes w}$ is a $\Gamma^{\otimes w}$-module, and this action extends to $\Gamma(w)$ by letting $\mathfrak{S}_w$ act by place permutation. We denote this $\Gamma(w)$-module by $T^{(w)}(M)$. If $\phi : M \to N$ is a Γ-homomorphism, then $\phi^{\otimes w} : T^{(w)}(M) \to T^{(w)}(N)$ is a $\Gamma(w)$-homomorphism.

If V is an $\Gamma(w)$-module and X is an $R\mathfrak{S}_w$-module then $V \otimes X$ becomes an $\Gamma(w)$-module in the following way:

$$(\alpha \otimes \sigma)(v \otimes x) = (\alpha \otimes \sigma)v \otimes \sigma x \qquad (\alpha \in \Gamma^{\otimes w}, \sigma \in \mathfrak{S}_w, v \in V, x \in X).$$

We denote this $\Gamma(w)$-module by $V \oslash X$. If Γ is the group algebra of a group G, then $\Gamma(w)$ is isomorphic to the group algebra of the wreath product $G \wr \mathfrak{S}_w$ and X may be viewed as an $\Gamma(w)$-module via the natural epimorphism $R(G \wr \mathfrak{S}_w) \to R\mathfrak{S}_w$. In this situation $V \oslash X$ is just the usual inner tensor product of two modules over a group algebra.

If $\{M(i) \mid i \in I\}$ is a finite set of Γ-modules, we define for each $\boldsymbol{\lambda} \in \Lambda^I_w$, a $\Gamma(w)$-module

$$M(\boldsymbol{\lambda}) = \mathrm{Ind}^w_{(|\lambda^i| : i \in I)} \left(\bigotimes_{i \in I} T^{(|\lambda_i|)}(M(i)) \oslash S(\lambda^i) \right).$$

This is a functorial construction: if $\{N(i) \mid i \in I\}$ is another set of Γ-modules, and for each $i \in I$, we have a homomorphism $\phi_i : M(i) \to N(i)$, then we define for each $\boldsymbol{\lambda} \in \Lambda^I_w$ a homomorphism $\phi(\boldsymbol{\lambda}) : M(\boldsymbol{\lambda}) \to N(\boldsymbol{\lambda})$ by

$$\phi(\boldsymbol{\lambda}) = \mathrm{Ind}^w_{(|\lambda^i| : i \in I)} \left(\bigotimes_{i \in I} \phi(i)^{\otimes |\lambda^i|} \oslash \mathrm{id}_{S(\lambda^i)} \right).$$

All of these constructions are well-behaved with respect to base change: If $R = \mathcal{O}$, then $\overline{\Gamma(w)} \cong \overline{\Gamma}(w)$ and $\widehat{\Gamma(w)} \cong \widehat{\Gamma}(w)$, and for all $\boldsymbol{\lambda} \in \Lambda^I_w$, we have $\overline{M(\boldsymbol{\lambda})} \cong \overline{M}(\boldsymbol{\lambda})$ and $\widehat{M(\boldsymbol{\lambda})} \cong \widehat{M}(\boldsymbol{\lambda})$.

We gather some results of this construction of $\Gamma(w)$-modules.

Lemma 3.1. Let $R = \mathcal{O}$ and let $\{M(i) \mid i \in I\}$ be a finite set of finitely generated Γ-modules, and let $\boldsymbol{\lambda} \in \Lambda^I_w$. Assume that k is a splitting field for $\overline{\Gamma}$.

(1) If $\{M(i) \mid i \in I\}$ is a complete set of pairwise non-isomorphic simple Γ-modules, then $\{M(\boldsymbol{\lambda}) \mid \boldsymbol{\lambda} \in \Lambda^I_w\}$ is a complete set of pairwise non-isomorphic simple $\Gamma(w)$-modules.

(2) If every $M(i)$ has a simple head $L(i)$, and $L(i) \not\cong L(j)$ whenever $i \neq j$, then $M(\boldsymbol{\lambda})$ has a simple head isomorphic to $L(\boldsymbol{\lambda})$.

(3) If a Γ-module M can be expressed as a direct sum in which every summand is isomorphic to some $M(i)$, and each $M(i)$ arises this way, then $M(\boldsymbol{\lambda})$ is a direct summand of $\mathrm{Ind}^w_{(1^w)}(M^{\otimes w})$.

In particular, if every $M(i)$ is projective, then $M(\boldsymbol{\lambda})$ is projective.

(4) Let $\geq$ be a partial order on I, and let $\{N(i) \mid i \in I\}$ be a set of Γ-modules which are $\mathcal{O}$-free of finite rank. Suppose that each $M(i)$ has a filtration in which each subquotient is isomorphic to $N(j)$ for some $j < i$ with $N(i)$ occurring exactly once. Then $M(\boldsymbol{\lambda})$ has a filtration in which each subquotient is isomorphic to $N(\boldsymbol{\mu})$ for some $\boldsymbol{\mu} \preceq \boldsymbol{\lambda}$ with $\boldsymbol{\lambda}$ occurring exactly once.

(5) If $\overline{\Gamma}$ is a quasihereditary algebra with standard modules $\{\overline{M(i)} \mid i \in I\}$, with respect to a partial order $\geq$ on I, then $\overline{\Gamma(w)}$ is quasihereditary, with standard modules $\{\overline{M(\boldsymbol{\lambda})} \mid \boldsymbol{\lambda} \in \Lambda^I_w\}$, with respect to the partial order $\succeq$ on Λ^I_w.

(6) If s is a nonnegative integer, we have

$$\operatorname{rad}^s(\overline{M(\boldsymbol{\lambda})}) = \operatorname{Ind}^w_{(|\lambda^i|:i\in I)} \bigotimes_{i\in I} \left(\sum_{l_1+\cdots+l_{|\lambda^i|}=s} \left(\bigotimes_{j=1}^{|\lambda^i|} \operatorname{rad}^{l_j} \overline{M(i)} \right) \oslash \overline{S(\lambda^i)} \right)$$

Proof. We note first that the simple Γ-modules and the simple $\overline{\Gamma}$-modules coincide. this implies moreover that the head of an A-module M coincides with that of the $\overline{A}$-module $\overline{M}$.

We then deduce part (1) from [4, Proposition 3.6] or [16, p. 204], and part (2) from [4, Lemma 4.5] . Parts (3)–(5) are from [4, Lemma 3.7, Proposition 4.7, §6] respectively. Part (6) is an application of [4, Lemma 3.5]. □

3.2. **Weight** 1 **case.** Let A be the principal block of $\mathcal{O}\mathfrak{S}_p$ and let $\mathbf{S}_A$ be the corresponding block of $\mathbf{S}_p$. These are the blocks of weight 1 associated to the empty p-core, and the associated partitions are precisely the hook partitions. Define $\mathbf{S}_A$-modules

$$\begin{aligned}
\Omega(i) &:= \Delta(i+1, 1^{p-i-1}) && (i = 0, \ldots, p-1),\\
\mho(i) &:= \Delta(i+1, 1^{p-i-1})^\circ && (i = 0, \ldots, p-1),\\
\mathcal{L}(i) &:= L(i+1, 1^{p-i-1}) && (i = 0, \ldots, p-1),\\
\mathcal{P}(i) &:= P(i+1, 1^{p-i-1}) && (i = 0, \ldots, p-1),
\end{aligned}$$

and A-modules

$$\begin{aligned}
\mathcal{S}(i) &:= S(i+1, 1^{p-i-1}) && (i = 0, \ldots, p-1),\\
\mathcal{D}(i) &:= D(i+1, 1^{p-i-1}) && (i = 0, \ldots, p-2),\\
\mathcal{Y}(i) &:= Y(i+1, 1^{p-i-1}) && (i = 0, \ldots, p-1).
\end{aligned}$$

We also define right module versions $\Omega'(i)$, $\mho'(i)$, etc.

The structure of these blocks is well known (see, e.g., [17, proof of Theorem 5.6.3]:

Lemma 3.2.

(1) For $i = 0, \ldots, p-2$, $\mathcal{P}(i)$ is an extension of $\Omega(i)$ by $\Omega(i+1)$, and $\mathcal{P}(p-1) \cong \Omega(p-1)$.
(2) For $i = 0, \ldots, p-2$, $\mathcal{Y}(i)$ is the A-projective cover of $\mathcal{D}(i)$ and is an extension of $\mathcal{S}(i)$ by $\mathcal{S}(i+1)$; and $\mathcal{Y}(p-1) \cong \mathcal{S}(p-1)$.
(3) We have $\overline{\Omega(0)} \cong \mathcal{L}(0)$, and for $i = 1, \ldots, p-1$, $\overline{\Omega(i)}$ is a nonsplit extension of $\mathcal{L}(i)$ by $\mathcal{L}(i-1)$.
(4) We have $\overline{\mathcal{S}(0)} \cong \mathcal{D}(0)$, $\overline{\mathcal{S}(p-1)} \cong \mathcal{D}(p-2)$, and for $i = 1, \ldots, p-2$, $\overline{\mathcal{S}(i)}$ is a nonsplit extenison of $\mathcal{D}(i)$ by $\mathcal{D}(i-1)$.

From Lemma 3.2(1,3), we see that $\overline{\mathbf{S}}_A$ is quasihereditary, with standard modules $\overline{\Omega(i)}$'s, with respect to the natural order on $\{0, 1, \ldots, p-1\}$.

We also have

Lemma 3.3. The Jantzen filtrations of $\Omega(i)$'s are just the radical filtrations.

Proof. This is an easy exercise using Jantzen's sum formula (see, for example, [21, §5.32]). $\square$

Now we apply the results of Lemma 3.1, noting that Schur algebras and symmetric group algebras split over any field.

Proposition 3.4.

(1) *The modules $\{\mathcal{L}(\boldsymbol{\lambda}) \mid \boldsymbol{\lambda} \in \Lambda^p_w\}$ form a complete set of nonisomorphic simple $\mathbf{S}_A(w)$-modules, and for all $\boldsymbol{\lambda} \in \Lambda^p_w$, $\mathcal{P}(\boldsymbol{\lambda})$ is the $\mathbf{S}_A(w)$-projective cover of $\mathcal{L}(\boldsymbol{\lambda})$.*

(2) *The modules $\{\mathcal{D}(\boldsymbol{\lambda}) \mid \boldsymbol{\lambda} \in \Lambda^{p-1}_w\}$ form a complete set of nonisomorphic simple $A(w)$-modules.*

(3) *If $\boldsymbol{\lambda} \in \Lambda^{p-1}_w$, then $\mathcal{Y}(\boldsymbol{\lambda})$ is an $A(w)$-projective cover of $\mathcal{D}(\boldsymbol{\lambda})$.*

(4) *For all $\boldsymbol{\lambda}, \boldsymbol{\mu} \in \Lambda^p_w$, $[\widehat{\mathcal{Y}(\boldsymbol{\lambda})} : \widehat{\mathcal{S}(\boldsymbol{\mu})}] \neq 0$ implies that $\boldsymbol{\lambda} \preceq \boldsymbol{\mu}$. Moreover $[\widehat{\mathcal{Y}(\boldsymbol{\lambda})} : \widehat{\mathcal{S}(\boldsymbol{\lambda})}] = 1$.*

(5) *The set $\{\overline{\Omega(\boldsymbol{\lambda})} \mid \boldsymbol{\lambda} \in \Lambda^p_w\}$ forms the standard modules of the quasihereditary algebra $\overline{\mathbf{S}_A(w)}$ with respect to the partial order $\succeq$ on Λ^p_w.*

(6) *For all $\boldsymbol{\lambda}, \boldsymbol{\mu} \in \Lambda^p_w$, $\operatorname{Hom}(\mathcal{P}(\boldsymbol{\mu}), \Omega(\boldsymbol{\lambda})) \neq 0$ implies that $\boldsymbol{\mu} \preceq \boldsymbol{\lambda}$.*

(7) *For all $\boldsymbol{\lambda} \in \Lambda^p_w$, $\Omega(\boldsymbol{\lambda})$ is isomorphic to the quotient of $\mathcal{P}(\boldsymbol{\lambda})$ by the sum of the images of all homomorphisms $\mathcal{P}(\boldsymbol{\mu}) \to \mathcal{P}(\boldsymbol{\lambda})$ for all $\boldsymbol{\mu} \succ \boldsymbol{\lambda}$.*

Proof. Parts (1)–(3) follow from Lemma 3.1(1–3), part (4) follows from Lemma 3.2(2) and 3.1(4), and part (5) from Lemma 3.1(5).

Part (5) then shows that $\operatorname{Hom}(\overline{\mathcal{P}(\boldsymbol{\mu})}, \overline{\Omega(\boldsymbol{\lambda})}) \neq 0$ implies that $\boldsymbol{\mu} \preceq \boldsymbol{\lambda}$. Part (6) thus follows.

By Lemma 3.1(4), $\mathcal{P}(\boldsymbol{\lambda})$ has a filtration in which $\Omega(\boldsymbol{\lambda})$ appears exactly once as a factor, and the other factors are of the form $\Omega(\boldsymbol{\mu})$ with $\boldsymbol{\mu} \succ \boldsymbol{\lambda}$. Since $\mathcal{P}(\boldsymbol{\lambda})$ is the projective cover of $\mathcal{L}(\boldsymbol{\lambda})$ by part (1), and $\Omega(\boldsymbol{\mu})$ has simple head $\mathcal{L}(\boldsymbol{\mu})$ for all $\boldsymbol{\mu} \in \Lambda^p_w$ by Lemma 3.1(2), we see that the factor $\Omega(\boldsymbol{\lambda})$ occurs at the top. This together with part (6) thus gives us part (7). $\square$

Let e_A be the idempotent of $\mathbf{S}_A$ such that $e_A \mathbf{S}_A e_A = A$ and let

$$f_A : \mathbf{S}_A\text{-mod} \to A\text{-mod}$$

be the associated Schur functor. By [4, §5] there is an idempotent $e'_A \in \mathbf{S}_A(w)$ such that $e'_A \mathbf{S}_A(w) e'_A = A(w)$. Let

$$g = f'_A : \mathbf{S}_A(w)\text{-mod} \to A(w)\text{-mod}$$

be the associated functor.

Lemma 3.5.

(1) Let $\{M(i) \mid i \in I\}$ be a set of $\mathbf{S}_A$-modules, and write $f_A(M(i)) = N(i)$. Then $g(M(\boldsymbol{\lambda})) = N(\boldsymbol{\lambda})$ for all $\boldsymbol{\lambda} \in \Lambda_w^I$.
(2) We have a ring isomorphism $\mathbf{S}_A(w) \cong \mathrm{End}_{A(w)}(g(\mathbf{S}_A(w)))$ given by right multiplication of $\mathbf{S}_A(w)$ on $g(\mathbf{S}_A(w)) = e'_A\mathbf{S}_A(w)$.

Proof. Part (1) follows from [4, Proposition 5.1(4)], while part (2) follows from [4, Proposition 5.2] and 2.2(4). □

Proposition 3.6. *Let* $\boldsymbol{\lambda}, \boldsymbol{\mu} \in \Lambda_w^p$.

(1) *We have*

$$g(\mathcal{P}(\boldsymbol{\lambda})) \cong \mathcal{Y}(\boldsymbol{\lambda}).$$
$$g(\Omega(\boldsymbol{\lambda})) \cong \mathcal{S}(\boldsymbol{\lambda}).$$
$$g(\mathcal{L}(\boldsymbol{\lambda})) \cong \begin{cases} \mathcal{D}(\boldsymbol{\lambda}), & \text{if } \boldsymbol{\lambda} \in \Lambda_w^{p-1}; \\ 0, & \text{otherwise.} \end{cases}$$

(2) *The functor* g *induces an isomorphism*

$$\mathrm{Hom}_{\mathbf{S}_A(w)}(\mathcal{P}(\boldsymbol{\lambda}), \mathcal{P}(\boldsymbol{\mu})) \to \mathrm{Hom}_{A(w)}(\mathcal{Y}(\boldsymbol{\lambda}), \mathcal{Y}(\boldsymbol{\mu})).$$

Proof. Part (1) follows from Lemmas 2.2(1–3) and 3.5(1). By Lemma 3.5(2), the homomorphism

$$\mathrm{Hom}_{\mathbf{S}_A(w)}(V, W) \to \mathrm{Hom}_{A(w)}(gV, gW)$$

induced by g is an isomorphism when V and W are both the free $\mathbf{S}_A(w)$-module of rank 1. The same is therefore true when V and W are summands of this module. Hence part (2) holds by Lemma 3.4(1). □

4. Rouquier blocks of symmetric groups

4.1. **Rouquier cores.** Throughout the rest of the paper fix a positive integer z, and let $\rho = \rho(z)$ be the partition which contains

each of the parts $1, 2, \ldots, z-1$ $\quad p-1$ times,
each of the parts $z+1, z+3, \ldots, 3z-3$ $\quad p-2$ times,

$\vdots$

each of the parts $\frac{i(i-1)}{2}(z-1)+i, \frac{i(i-1)}{2}(z-1)+2i, \ldots, \frac{i(i+1)}{2}(z-1)$ $\quad p-i$ times,

$\vdots$

each of the parts $\frac{(p-1)(p-2)}{2}(z-1)+(p-1), \ldots, \frac{(p-1)p}{2}(z-1)$ $\quad$ once.

The partition ρ is a self-conjugate p-core partition and is represented on a James's p-abacus with no beads on the first (leftmost) runner,

$z-1$ beads on the second runner, $2(z-1)$ beads on the 3rd runner,..., $(p-1)(z-1)$ beads on the p-th (rightmost) runner.

If w is a nonnegative integer, then any partition of $\Lambda(\rho, w)$ can be displayed on a James's p-abacus with $l+wp$ beads, where l is the length of ρ. For such a partition λ, with p-quotient $(\lambda^0, \dots, \lambda^{p-1})$, λ^i can be read off from runner i in this display.

We list some properties of partitions with p-core ρ:

Lemma 4.1. Let w be a nonnegative integer $\leq z$, and let $\lambda \in \Lambda(\rho, w)$ with p-quotient $(\lambda^0, \dots, \lambda^{p-1})$.

(1) The partition λ is p-restricted if and only if $\lambda^{p-1} = \emptyset$ and is p-regular if and only if $\lambda^0 = \emptyset$.
(2) If $\mu \in \Lambda(\rho, w)$, then $\mathrm{quot}(\lambda) \prec \mathrm{quot}(\mu)$ implies that $\lambda < \mu$.
(3) The conjugate partition λ' has p-quotient $((\lambda^{p-1})', \dots, (\lambda^0)')$.
(4) If $w \geq 1$, and $\mu \in \Lambda(\rho, w-1)$, then $\mu \subset \lambda$ implies that there exists $i \in \{0, \dots, p-1\}$ such that $\mathrm{quot}(\mu) \nearrow_i \mathrm{quot}(\lambda)$ and $\lambda/\mu = (i+1, 1^{p-i-1})$.

Proof. Parts (1)–(3) are easily checked using James's p-abacus. Part (4) is [2, Lemma 3(2)]. □

The reason for the following notation will become apparent in Proposition 4.4 below.

Notation 4.2. Given $\boldsymbol{\lambda} \in \Lambda^p_w$, we define λ to be the partition with p-core ρ and p-quotient $\boldsymbol{\lambda}$. Thus $\boldsymbol{\lambda} \mapsto \lambda$ defines a bijection between Λ^p_w and $\Lambda(\rho, w)$. Pre-composing this with the involution $^\dagger : \Lambda^p_w \to \Lambda^p_w$ defined by

$$(\lambda^0, \lambda^1, \lambda^2, \lambda^3, \dots, \lambda^{p-2}, \lambda^{p-1})^\dagger = (\lambda^0, \lambda^{1'}, \lambda^2, \lambda^{3'}, \dots, \lambda^{p-2'}, \lambda^{p-1}),$$

we obtain another bijection $\Lambda^p_w \to \Lambda(\rho, w)$; $\boldsymbol{\lambda} \mapsto \lambda^\dagger$; i.e. $\lambda^\dagger$ denotes the partition in $\Lambda(\rho, w)$ with p-quotient $\boldsymbol{\lambda}^\dagger$.

If $w \leq z$, this bijection $\boldsymbol{\lambda} \mapsto \lambda^\dagger$ restricts to a bijection of Λ^{p-1}_w onto the set of p-restricted partitions in $\Lambda(\rho, w)$ by Lemma 4.1(1); furthermore, $\boldsymbol{\lambda} \prec \boldsymbol{\mu}$ implies $\lambda^\dagger < \mu^\dagger$ by Lemma 4.1(2).

4.2. **Rouquier blocks.** For each nonnegative integer $w \leq z$, let B_w be the block of $\mathcal{O}\mathfrak{S}_{|\rho|+wp}$ associated to the p-core $\rho = \rho(z)$.

The block B_w can be interpreted as the 'largest' block of weight w, in the following way. Scopes defined an equivalence relation on the set of p-blocks of symmetric groups of a fixed weight $w \geq 0$ in terms of what she called $[w : k]$-pairs. She showed that the number of equivalence classes is finite and that blocks in the same class are Morita equivalent. Let $\Theta_1, \dots, \Theta_s$ be the Scopes classes of p-blocks of weight w, and for each $i = 1, \dots, s$, define $n(i)$ to be the least integer n such that Θ_i contains a block of $\mathcal{O}\mathfrak{S}_n$. We may assume that $n(1) \geq n(2) \geq \dots \geq n(s)$. It turns out that $n(1) > n(2)$ and that Θ_1 contains the blocks associated to the p-cores $\rho(w), \rho(w+1), \dots$; in particular, while B_w is defined in terms of the core $\rho(z)$, it is up to Morita equivalence independent of z ($\geq w$).

Rouquier (see [23]) introduced the core ρ in 1991 and suggested that the blocks B_w should have good properties; in particular he conjectured, in the abelian defect group case ($w < p$), a Morita equivalence with the wreath product $A(w)$. This conjecture was proved in [2]. In this subsection we show the compatibility of the Morita equivalences for different weights.

For the remainder of this paper, we fix a nonnegative integer $w \leq \min(p-1, z)$.

We denote the symmetric group on a set U by $\mathfrak{S}(U)$. Let V be a set of cardinality $pw+|\kappa|$, let $U_1, \ldots, U_w$ be disjoint subsets of V of cardinality p, and let U be the union of these subsets. In what follows, all groups we consider will be viewed as subgroups of $\mathfrak{S}(V)$ in an obvious way. For $i = 1, \ldots, w$, let D_i be a Sylow p-subgroup of $\mathfrak{S}(U_i)$, and let a_i be the principal block idempotent of $\mathcal{O}\mathfrak{S}(U_i)$. For $i = 0, \ldots, w$, let e_{w-i} be the block idempotent of $\mathcal{O}\mathfrak{S}(U_{i+1} \cup \ldots \cup U_w \cup (V - U))$ corresponding to the p-core ρ, let

$$G_i = \mathfrak{S}(U_1) \times \cdots \times \mathfrak{S}(U_i) \times \mathfrak{S}(U_{i+1} \cup \ldots \cup U_w \cup (V-U))$$

and let

$$b_i = a_1 \otimes \cdots \otimes a_i \otimes e_{w-i},$$

a block idempotent of $\mathcal{O}G_i$. We have

$$G_i \cong \underbrace{\mathfrak{S}_p \times \cdots \times \mathfrak{S}_p}_{i} \times \mathfrak{S}_{(w-i)p+|\rho|}.$$

We set $G = G_0$, $b = b_0$, $L = G_w$, and $f = b_w$. Let $D = D_1 \times \cdots \times D_w$. Let M be the subgroup of $\mathfrak{S}(U)$ consisting of permutations sending each U_i into some U_j; we note that M is isomorphic to the wreath product $\mathfrak{S}_p \wr \mathfrak{S}_{w-1}$. Set $N = M \times S(V-U)$, a subgroup of G containing $N_G(D)$ and L and normalizing L.

Theorem 4.3 ([2, §4])**.** *There is up to isomorphism a unique summand X of $\mathcal{O}Gb$ as $(G \times N)$-module with vertex $\delta D = \{(x, x) \in G \times N \mid x \in D\}$; all other summands have strictly smaller vertices. The bimodule X induces a Morita equivalence between $\mathcal{O}Gb$ and $\mathcal{O}Nf$. Furthermore, the restriction of X to $G \times L$ is indecomposable with vertex δD and is isomorphic to $\mathcal{O}Gb_0 \ldots b_w$.*

Now $\mathcal{O}Gb = B_w$, and as e_w is a block of defect 0, $\mathcal{O}Nf$ is canonically Morita equivalent to $\mathcal{O}M(a_1 \otimes \cdots \otimes a_w) = A(w)$. Let

$$\mathcal{F} : A(w)\text{-mod} \to B_w\text{-mod}$$

be the equivalence gotten by composing the equivalence of $A(w)$-mod and $\mathcal{O}Nf$-mod with the Morita equivalence $(X \otimes_{\mathcal{O}Nf} -)$.

When $w \geq 1$, we want to be able to compare the equivalence $\mathcal{F}$ with the corresponding one in weight $w-1$. So, for $i = 1, \ldots, w-1$, let

$$\widetilde{G_i} = \mathfrak{S}(U_2) \times \cdots \times \mathfrak{S}(U_i) \times \mathfrak{S}(U_{i+1} \cup \ldots \cup U_w \cup (V - U))$$

and let

$$\widetilde{b_i} = a_2 \otimes \cdots \otimes a_i \otimes e_{w-i},$$

a block idempotent of $\mathcal{O}\widetilde{G_i}$. Note that $G_i = \mathfrak{S}(U_1) \times \widetilde{G_i}$ and $b_i = a_1 \otimes \widetilde{b_i}$. We have

$$\widetilde{G_i} \cong \underbrace{\mathfrak{S}_p \times \cdots \times \mathfrak{S}_p}_{i-1} \times \mathfrak{S}_{(w-i)p+|\rho|}.$$

We set $\widetilde{G} = \widetilde{G_1}$, $\widetilde{b} = \widetilde{b_1}$, $\widetilde{L} = \widetilde{G_w}$, and $\widetilde{f} = \widetilde{b_w}$. Let $\widetilde{D} = D_2 \times \cdots \times D_w$. Let $\widetilde{M}$ be the subgroup of $\mathfrak{S}(U)$ consisting of permutations sending each U_i into some U_j; we note that $\widetilde{M}$ is isomorphic to the wreath product $\mathfrak{S}_p \wr \mathfrak{S}_{w-1}$. Set $\widetilde{N} = \widetilde{M} \times \mathfrak{S}(V - U)$, a subgroup of $\widetilde{G}$ containing $N_{\widetilde{G}}(\widetilde{D})$ and $\widetilde{L}$ and normalizing $\widetilde{L}$.

By the proposition above, applied in the weight $w-1$, there is a unique summand $\widetilde{X}$ of $\mathcal{O}\widetilde{G}\widetilde{b}$ as a $(\widetilde{G} \times \widetilde{N})$-module with vertex $\delta\widetilde{D}$, and $\widetilde{X}$ induces a Morita equivalence between $\mathcal{O}\widetilde{G}\widetilde{b}$ and $\mathcal{O}\widetilde{N}\widetilde{f}$. Furthermore, the restriction of $\widetilde{X}$ to $\widetilde{G} \times \widetilde{L}$ is indecomposable with vertex $\Delta\widetilde{D}$ and is isomorphic to $\mathcal{O}\widetilde{G}\widetilde{b}_1 \ldots \widetilde{b}_w$.

Denote by Y the restriction of X to $G \times (\mathfrak{S}(U_1) \times \widetilde{N})$. This is indecomposable with vertex δD, because the same holds upon further restriction to $G \times L$. In addition, Y is a direct summand of $\mathcal{O}Gb$ and all other summands have strictly smaller vertices. Now consider the $G \times (\mathfrak{S}(U_1) \times \widetilde{N})$-module $Z = \mathcal{O}Gb \otimes_{\mathcal{O}G_1} (\mathcal{O}\mathfrak{S}(U_1)a_1 \otimes_{\mathcal{O}} \widetilde{X})$. This is a summand of $\mathcal{O}Gb$ and is indecomposable with vertex containing δD because its restriction to $G \times L$ is $\mathcal{O}Gb \otimes_{\mathcal{O}G_1} (\mathcal{O}\mathfrak{S}(U_1)a_1 \otimes_{\mathcal{O}} \mathcal{O}\widetilde{G}\widetilde{b}_1 \ldots \widetilde{b}_w) \cong \mathcal{O}Gb_0 \ldots b_w$. We conclude that $Z \cong Y$.

Let

$$\widetilde{\mathcal{F}} : A(w-1)\text{-mod} \to B_{w-1}\text{-mod}$$

be the equivalence gotten by composing the equivalence of $A(w-1)$-mod and $\mathcal{O}\widetilde{N}\widetilde{f}$-mod with the Morita equivalence $(\widetilde{X} \otimes_{\mathcal{O}\widetilde{N}\widetilde{f}} -)$. The argument above shows that we have a diagram of functors, commutative up to natural equivalence:

$$\begin{array}{ccc} A(w)\text{-mod} & \xrightarrow{\mathcal{F}} & B_w\text{-mod} \\ {\scriptstyle \mathrm{Res}}\big\downarrow & & {\scriptstyle \mathrm{Res}}\big\downarrow \\ A(1, w-1)\text{-mod} & \xrightarrow{\mathrm{id}_A \otimes_{\mathcal{O}} \widetilde{\mathcal{F}}} & A \otimes_{\mathcal{O}} B_{w-1}\text{-mod} \end{array}$$

The lefthand vertical functor is just restriction from $A(w)$ to the subalgebra $A(1, w-1)$, and the righthand vertical functor is given by restriction from the group $\mathfrak{S}_{|\rho|+wp}$ to the Young subgroup $\mathfrak{S}_p \times \mathfrak{S}_{|\rho|+(w-1)p}$.

4.3. **Identification of simple modules and Young modules.**

Proposition 4.4. *Suppose* $\boldsymbol{\lambda} \in \Lambda^p_w$ *and* $\boldsymbol{\sigma} \in \Lambda^{p-1}_w$. *Then*

(1) $\widehat{\mathcal{F}(\mathcal{S}(\boldsymbol{\lambda}))} \cong \widehat{S(\lambda^\dagger)}$.
(2) $\mathcal{F}(\mathcal{Y}(\boldsymbol{\lambda})) \cong Y(\lambda^\dagger)$.
(3) $\mathcal{F}(\mathcal{D}(\boldsymbol{\sigma})) \cong D(\sigma^\dagger)$.

Note: we are following Notation 4.2.

Proof.

(1) We induct on w. The cases $w = 0$ and $w = 1$ are trivial. The case $w = 2$ is [1, Proposition 6.6]. Assume $w \geq 3$. By [4, Lemma 3.3(1)], we have, for $\boldsymbol{\mu} \in \Lambda^p_{w-1}$ and $i \in \{0, \dots, p-1\}$,

$$\operatorname{Ind}^w_{1,w-1}\left(\widehat{\mathcal{S}(i)} \otimes_K \widehat{\mathcal{S}(\boldsymbol{\mu})}\right) \cong \bigoplus_{\substack{\boldsymbol{\lambda} \in \Lambda^p_w \\ \boldsymbol{\mu} \nearrow_i \boldsymbol{\lambda}}} \widehat{\mathcal{S}(\boldsymbol{\lambda})}.$$

Thus by Frobenius reciprocity, we have for any $\boldsymbol{\lambda} \in \Lambda^p_w$,

$$\operatorname{Res}^w_{1,w-1}(\widehat{\mathcal{S}(\boldsymbol{\lambda})}) \cong \bigoplus_{i=0}^{p-1}\left(\widehat{\mathcal{S}(i)} \otimes_K \bigoplus_{\substack{\boldsymbol{\mu} \in \Lambda^p_{w-1} \\ \boldsymbol{\mu} \nearrow_i \boldsymbol{\lambda}}} \widehat{\mathcal{S}(\boldsymbol{\mu})}\right),$$

and by the Littlewood-Richardson rule and Lemma 4.1(3) we have the analogous formula

$$\operatorname{Res}^{\widehat{B_w}}_{\widehat{A} \otimes_K \widehat{B_{w-1}}}(\widehat{S(\lambda^\dagger)}) \cong \bigoplus_{i=0}^{p-1}\left(\widehat{\mathcal{S}(i)} \otimes_K \bigoplus_{\substack{\boldsymbol{\mu} \in \Lambda^p_{w-1} \\ \boldsymbol{\mu} \nearrow_i \boldsymbol{\lambda}}} \widehat{S(\mu^\dagger)}\right).$$

The desired statement will follow by induction from these two formulas along with the diagram of functors in §4.2, as long as we can show that $\boldsymbol{\lambda} = (\lambda^0, \dots, \lambda^{p-1}) \in \Lambda^p_w$ is determined by the set of $\boldsymbol{\mu} \in \Lambda^p_{w-1}$ such that $\boldsymbol{\mu} \nearrow_i \boldsymbol{\lambda}$ for some i. This is clear if there exist $0 \leq i < j \leq p-1$ such that $\lambda^i \neq \emptyset$ and $\lambda^j \neq \emptyset$. On the other hand if there is a unique nonempty λ^i then $|\lambda^i| \geq 3$, and we can use the easy fact that a partition α of n for $n \geq 3$ is determined by the set of partitions β such that $\beta \nearrow \alpha$.

(2) The A-module $\mathcal{Y} = \oplus_{i=0}^{p-1}\mathcal{Y}(i)$ is a direct sum of summands of permutations modules on Young subgroups of $\mathfrak{S}_p$. It follows that the $A(w)$-module $\operatorname{Ind}^w_{(1^w)}(\mathcal{Y}^{\otimes w})$ is a direct sum of summands of permutations modules on Young subgroups of $(\mathfrak{S}_p)^w$. Now $\mathcal{Y}(\boldsymbol{\lambda})$ is a summand of $\operatorname{Ind}^w_{(1^w)}(\mathcal{Y}^{\otimes w})$ by Lemma 3.1(6) and is indecomposable since $\operatorname{End}_{A(w)}(\mathcal{Y}(\boldsymbol{\lambda}))$ is local by Proposition

3.6(2). As the equivalence $\mathcal{F}$ is a summand of an induction functor, $\mathcal{F}(\mathcal{Y}(\boldsymbol{\lambda}))$ is an indecomposable summand of a permutation module on a Young subgroup, i.e., $\mathcal{F}(\mathcal{Y}(\boldsymbol{\lambda})) = Y(\mu^\dagger)$ for some $\mu \in \Lambda(\rho, w)$. Using part (1), Lemma 2.2(5), and Proposition 3.4(5), we have

$$[\widehat{Y(\mu^\dagger)} : \widehat{S(\lambda^\dagger)}] = [\widehat{\mathcal{Y}(\boldsymbol{\lambda})} : \widehat{\mathcal{S}(\boldsymbol{\lambda})}] = 1$$

which implies $\mu^\dagger \leq \lambda^\dagger$, and

$$[\widehat{\mathcal{Y}(\boldsymbol{\lambda})} : \widehat{\mathcal{S}(\boldsymbol{\mu})}] = [\widehat{Y(\mu^\dagger)} : \widehat{S(\mu^\dagger)}] = 1$$

which implies $\boldsymbol{\lambda} \preceq \boldsymbol{\mu}$, and hence $\lambda^\dagger \leq \mu^\dagger$.

(3) This follows from part (2), Lemma 2.2(3) and Proposition 3.4(3).

□

5. Rouquier blocks of Schur algebras

Let $f = f_{B_w} : \mathbf{S}_{B_w}\text{-mod} \longrightarrow B_w\text{-mod}$ be the Schur functor (see subsection 2.3), $g : \mathbf{S}_A(w)\text{-mod} \to A(w)\text{-mod}$ be the functor defined in subsection 3.2, and $\mathcal{F} : A(w)\text{-mod} \to B_w\text{-mod}$ be the equivalence of subsection 4.2.

By Lemma 2.2(3,4), if $\lambda, \mu \in \Lambda(\rho, w)$, we have

$$f(P(\lambda)) \cong Y(\lambda)$$

and isomorphisms

$$\mathrm{Hom}_{\mathbf{S}_{B_w}}(P(\lambda), P(\mu)) \cong \mathrm{Hom}_{B_w}(Y(\lambda), Y(\mu))$$

induced by f. At the same time, by Corollary 3.6 and Proposition 4.4(2), if $\boldsymbol{\lambda}, \boldsymbol{\mu} \in \Lambda^p_w$, we have

$$\mathcal{F}g(\mathcal{P}(\boldsymbol{\lambda})) \cong Y(\lambda^\dagger)$$

and isomorphisms

$$\mathrm{Hom}_{\mathbf{S}_{A(w)}}(\mathcal{P}(\boldsymbol{\lambda}), \mathcal{P}(\boldsymbol{\mu})) \cong \mathrm{Hom}_{B_w}(Y(\lambda^\dagger), Y(\mu^\dagger))$$

induced by $\mathcal{F}g$. Consequently we deduce a Schur algebra analogue of [2]:

Theorem 5.1. *$\mathbf{S}_{B_w}$ and $\mathbf{S}_A(w)$ are Morita equivalent.*

5.1. **Identification of Weyl modules and Specht modules.** In fact, our argument shows that there exists a commutative diagram of functors

$$\begin{array}{ccc} \mathbf{S}_{B_w}\text{-mod} & \xrightarrow{f} & B_w\text{-mod} \\ {\scriptstyle\mathcal{G}}\big\uparrow & & {\scriptstyle\mathcal{F}}\big\uparrow \\ \mathbf{S}_A(w)\text{-mod} & \xrightarrow{g} & A(w)\text{-mod} \end{array}$$

where $\mathcal{G}$ is an equivalence such that $\mathcal{G}(\mathcal{P}(\boldsymbol{\lambda})) \cong P(\lambda^\dagger)$ for all $\boldsymbol{\lambda} \in \Lambda^p_w$. Also, $\mathcal{G}$ induces isomorphisms $\mathrm{Hom}_{\mathbf{S}_{A(w)}}(\mathcal{P}(\boldsymbol{\lambda}), \mathcal{P}(\boldsymbol{\mu})) \cong \mathrm{Hom}_{\mathbf{S}_{B_w}}(P(\lambda^\dagger), P(\mu^\dagger))$ for all $\boldsymbol{\lambda}, \boldsymbol{\mu} \in \Lambda^p_w$.

Theorem 5.2. *Let $\boldsymbol{\lambda} \in \Lambda^p_w$. We have*

$$\begin{aligned}\mathcal{G}(\Omega(\boldsymbol{\lambda})) &\cong \Delta(\lambda^\dagger),\\ \mathcal{G}(\mho(\boldsymbol{\lambda})) &\cong \Delta(\lambda^\dagger)^\circ,\\ \mathcal{F}(\mathcal{S}(\boldsymbol{\lambda})) &\cong S(\lambda^\dagger).\end{aligned}$$

Note: We are following Notation 4.2.

Proof. By Proposition 3.4(6,7), we see that $\Omega(\boldsymbol{\lambda})$ can be defined to be the quotient of $P(\boldsymbol{\lambda})$ by the sum of images of homomorphisms $P(\boldsymbol{\mu}) \to P(\boldsymbol{\lambda})$ for all $\boldsymbol{\mu} \succ \boldsymbol{\lambda}$ or for all $\boldsymbol{\mu} \not\preceq \boldsymbol{\lambda}$. Applying the functor $\mathcal{G}$, we see that $\mathcal{G}(\Omega(\boldsymbol{\lambda}))$ is the quotient of $P(\lambda^\dagger)$ by the sum of images of homomorphisms $P(\mu^\dagger) \to P(\lambda^\dagger)$ for all $\mu^\dagger > \lambda^\dagger$. Thus $\mathcal{G}(\Omega(\boldsymbol{\lambda})) \cong \Delta(\lambda^\dagger)$ by Lemma 2.1(1). We then obtain $\mathcal{F}(\mathcal{S}(\boldsymbol{\lambda})) \cong S(\lambda^\dagger)$ as well, by the commutative diagram of functors above along with Proposition 2.2(1) and Proposition 3.6(1).

The functor $\mathcal{G}$ is naturally equivalent to $(X \otimes_{\mathbf{S}_A(w)} -)$ for some $(\mathbf{S}_{B_w}, \mathbf{S}_A(w))$-bimodule X. Then

$$\mathcal{G}' = \mathrm{Hom}_{\mathbf{S}_A(w)}(X, -) : \text{mod-}\mathbf{S}_A(w) \to \text{mod-}\mathbf{S}_{B_w}$$

is an equivalence of categories of right modules, and for any right $\mathbf{S}_A(w)$-module M, free and of finite rank over $\mathcal{O}$, we have $\mathcal{G}(M^\vee) \cong (\mathcal{G}'(M))^\vee$.

Now $\mathcal{G}'(\mathcal{P}'(\boldsymbol{\lambda})) \cong P'(\lambda^\dagger)$, so repeating the argument above (using right-module versions of Proposition 3.4(6,7) and Lemma 2.1(1)), we have $\mathcal{G}'(\Omega'(\boldsymbol{\lambda})) \cong \Delta'(\lambda^\dagger)$. By Lemma 2.1(2) we have $\Omega'(i)^\vee \cong \mho(i)$ for $i \in \{0, \dots, p-1\}$, and thus $\Omega'(\boldsymbol{\lambda})^\vee \cong \mho(\boldsymbol{\lambda})$ (see remark in [4, end of §3]). Hence,

$$\begin{aligned}\mathcal{G}(\mho(\boldsymbol{\lambda})) &\cong \mathcal{G}(\Omega'(\boldsymbol{\lambda})^\vee)\\ &\cong \mathcal{G}'(\Omega'(\boldsymbol{\lambda}))^\vee\\ &\cong \Delta'(\lambda)^\vee\\ &\cong \Delta(\lambda)^\circ.\end{aligned}$$

□

5.2. **Jantzen filtrations in Rouquier blocks.** By Lemma 2.3 and Proposition 5.2, we may try to calculate Jantzen filtrations of Weyl modules in $\mathbf{S}_{B_w}$ by constructing homomorphisms $\Omega(\boldsymbol{\lambda}) \to \mho(\boldsymbol{\lambda})$ for $\boldsymbol{\lambda} \in \Lambda^p_w$.

In the following lemmas, we use the notations $M^{\eta,r}$ and $\overline{M}^{\eta,r}$ introduced in section 2.4.

Lemma 5.3. Let the maximal ideal π of $\mathcal{O}$ be generated by x.

(1) Let $\eta : M \to N$ and $\eta' : M' \to N'$ be homomorphisms of $\mathcal{O}$-modules, which are all $\mathcal{O}$-free of finite rank. Then for all

nonnegative integer s, we have

$$(M \otimes M')^{\eta\otimes\eta',s} = \sum_{t+u=s} M^{\eta,t} \otimes M'^{\eta',u},$$
$$\phi\left(\left(\overline{M \otimes M'}\right)^{\eta\otimes\eta',s}\right) = \sum_{t+u=s} \overline{M}^{\eta,t} \otimes_k \overline{M'}^{\eta',u},$$

where ϕ is the isomorphism $\overline{M \otimes M'} \to \overline{M} \otimes_k \overline{M'}$; $\overline{m \otimes m'} \mapsto \overline{m} \otimes_k \overline{m'}$.

(2) Let A and B be $\mathcal{O}$-algebras, and if M is an $\mathcal{O}$-module, let $\varphi_M : M \to M$, $m \to xm$, and $p_{M,s} : M \to M/\pi^s M$ be the natural projection. Suppose $\mathsf{F} : A\text{-mod} \to B\text{-mod}$ is an exact left functor such that $\mathsf{F}(\varphi_M) = \varphi_{\mathsf{F}(M)}$ for all A-modules M. Let $\eta : M \to N$ be an A-homomorphism. We have

$$\mathsf{F}(M^{\eta,s}) = (\mathsf{F}(M))^{\mathsf{F}(\eta),s},$$
$$\psi(\mathsf{F}(\overline{M}^{\eta,s})) = (\overline{\mathsf{F}(M)})^{\mathsf{F}(\eta),s},$$

where $\psi : \mathsf{F}(\overline{M}) \to \overline{\mathsf{F}(M)}$ is an B-isomorphism satisfying $\psi \circ \mathsf{F}(p_{M,1}) = p_{\mathsf{F}(M),1}$.

Proof.

(1) The images of η and η' are submodules of free modules, and hence free. Therefore they are split surjections onto their images, and using the structure theorem for finitely generated modules over PIDs, we can find bases $\{\alpha_1, \dots, \alpha_m\}$, $\{\alpha'_1, \dots, \alpha'_{m'}\}$, $\{\beta_1, \dots, \beta_n\}$, $\{\beta'_1, \dots, \beta'_{n'}\}$ for M, M', N and N' respectively, such that

$$\eta'(\alpha_j) = \begin{cases} x^{i_j}\beta_j, & \text{if } j \le l; \\ 0, & \text{otherwise} \end{cases} \qquad \text{and} \qquad \eta'(\alpha'_{j'}) = \begin{cases} x^{i'_{j'}}\beta'_{j'}, & \text{if } j' \le l'; \\ 0, & \text{otherwise.} \end{cases}$$

Note that $M^{\eta,t}$ has basis $\{y_1\alpha_1, \dots, y_m\alpha_m\}$ where $y_j \in \mathcal{O}$ equals the least power of x such that $y_j\alpha_j \in M^{\eta,t}$, and we have a similar expression for a basis for $M'^{\eta',u}$. Thus $\sum_{t+u=s} M^{\eta,t} \otimes M^{\eta',u}$ has a basis $\{z_{j,j'}(\alpha_j \otimes \alpha'_{j'})\}$, where $z_{j,j'} \in \mathcal{O}$ equals the least power of x such that $z_{j,j'}(\alpha_j \otimes \alpha'_{j'}) \in (M \otimes M')^{\eta\otimes\eta',s}$.

Now, $\{\alpha_j \otimes \alpha_{j'} \mid 1 \le j \le m, 1 \le j' \le m'\}$ is a basis for $M \otimes M'$, and $\{\beta_j \otimes \beta_{j'} \mid 1 \le j \le n, 1 \le j' \le n'\}$ is a basis for $N \otimes N'$, and

$$(\eta \otimes \eta')(\alpha_j \otimes \alpha'_{j'}) = \begin{cases} x^{i_j + i'_{j'}}\beta_j \otimes \beta'_{j'}, & \text{if } 1 \le j \le l, 1 \le j' \le l' \\ 0, & \text{otherwise.} \end{cases}$$

Thus, $\{z_{j,j'}(\alpha_j \otimes \alpha'_{j'})\}$ is a basis for $(M \otimes M')^{\eta\otimes\eta',s}$ too. This proves the first statement.

For the second statement, we list statements which are easily seen to be equivalent:

- The image of $z_{j,j'}(\alpha_j \otimes \alpha'_{j'})$ in $\overline{(M \otimes M')}^{\eta\otimes\eta',s}$ is nonzero.
- $(\alpha_j \otimes \alpha'_{j'}) \in (M \otimes M')^{\eta\otimes\eta',s}$.
- $\alpha_j \in M^{\eta,t}$ and $\alpha'_{j'} \in M'^{\eta',u}$ for some $t+u=s$.
- The images of $y_j\alpha_j$ and $y'_{j'}\alpha'_{j'}$ in $\overline{M}^{\eta,t}$ and $\overline{M'}^{\eta',u}$ respectively are nonzero for some $t+u=s$.

(2) Since $M^{\eta,s} = \ker(p_{M,s} \circ \eta)$ and F is exact, we have $\mathsf{F}(M^{\eta,s}) = \ker(\mathsf{F}(p_{M,s}) \circ \mathsf{F}(\eta))$. But $\ker(p_{M,s}) = \varphi_M^s(M)$, so that $\ker(\mathsf{F}(p_{M,s})) = \varphi^s_{\mathsf{F}(M)}(\mathsf{F}(M)) = \ker(p_{\mathsf{F}(M),s})$. The first statement thus follows.

For the second statement, we have an isomorphism $\psi : \mathsf{F}(\overline{M}) \to \overline{\mathsf{F}(M)}$ making the following diagram commute:

$$\begin{tikzcd}
0 \arrow[r] & \mathsf{F}(M) \arrow[r, "\mathsf{F}(\varphi_M)"] \arrow[d, "\mathrm{id}_{\mathsf{F}(M)}"'] & \mathsf{F}(M) \arrow[r, "\mathsf{F}(p_{M,1})"] \arrow[d, "\mathrm{id}_{\mathsf{F}(M)}"'] & \mathsf{F}(\overline{M}) \arrow[r] \arrow[d, "\psi"'] & 0 \\
0 \arrow[r] & \mathsf{F}(M) \arrow[r, "\varphi_{\mathsf{F}(M)}"] & \mathsf{F}(M) \arrow[r, "p_{\mathsf{F}(M),1}"] & \overline{\mathsf{F}(M)} \arrow[r] & 0
\end{tikzcd}$$

Now $\overline{M}^{\eta,s} = p_{M,1}(M^{\eta,s})$, so that $\mathsf{F}(\overline{M}^{\eta,s}) = \mathsf{F}(p_{M,1})(\mathsf{F}(M^{\eta,s})) = \mathsf{F}(p_{M,1})\left((\mathsf{F}(M))^{\mathsf{F}(\eta),s}\right)$. The second statement thus follows.

□

Lemma 5.4. Let Γ be an $\mathcal{O}$-algebra, free and of finite rank as an $\mathcal{O}$-module, and $\overline{\Gamma}$ splits over k. Let

$$\eta(i) : M(i) \to N(i) \qquad (i \in \{0, \dots, n-1\})$$

be homomorphisms of Γ-modules which are all free and of finite rank over $\mathcal{O}$. Suppose that for each $i \in \{0, 1, \dots, n-1\}$ and nonnegative integer s, we have

$$\overline{M(i)}^{\eta(i),s} = \mathrm{rad}^s_{\overline{\Gamma}}\, \overline{M(i)}.$$

Then for each $\boldsymbol{\lambda} \in \Lambda^n_w$ and nonnegative integer s, we have

$$\overline{M(\boldsymbol{\lambda})}^{\eta(\boldsymbol{\lambda}),s} = \mathrm{rad}^s_{\overline{\Gamma(w)}}\, \overline{M(\boldsymbol{\lambda})}.$$

Proof. We have

$$\begin{aligned}
\operatorname{rad}^s(\overline{M(\boldsymbol{\lambda})}) &= \operatorname{Ind}^w_{(|\lambda^i|)} \bigotimes_{i=0}^{n-1} \left(\sum_{l_1+\cdots+l_{|\lambda^i|}=s} \left(\bigotimes_{j=1}^{|\lambda^i|} \operatorname{rad}^{l_j} \overline{M(i)} \right) \oslash \overline{S(\lambda^i)} \right) \\
&= \operatorname{Ind}^w_{(|\lambda^i|)} \bigotimes_{i=0}^{n-1} \left(\sum_{l_1+\cdots+l_{|\lambda^i|}=s} \left(\bigotimes_{j=1}^{|\lambda^i|} \overline{M(i)}^{\eta(i),l_j} \right) \oslash \overline{S(\lambda^i)} \right) \\
&\cong \operatorname{Ind}^w_{(|\lambda^i|)} \bigotimes_{i=0}^{n-1} \left(\overline{M(i)^{\otimes|\lambda^i|}}^{\eta(i)^{\otimes|\lambda^i|},s} \oslash \overline{S(\lambda^i)}^{\mathrm{id}_{S(\lambda^i)},0} \right) \\
&\cong \operatorname{Ind}^w_{(|\lambda^i|)} \bigotimes_{i=0}^{n-1} \overline{M(i)^{\otimes|\lambda^i|} \oslash S(\lambda^i)}^{\eta(i)^{\otimes|\lambda^i|}\oslash \mathrm{id}_{S(\lambda^i)},s} \\
&\cong \overline{M(\boldsymbol{\lambda})}^{\eta(\boldsymbol{\lambda}),s},
\end{aligned}$$

where the first equality is given by Lemma 3.1(6), the third and fourth isomorphisms by Lemma 5.3(1), and the last isomorphism by Lemma 5.3(2). Moreover, Lemma 5.3 shows that the net isomorphism $\operatorname{rad}^s \overline{M(\boldsymbol{\lambda})} \cong \overline{M(\boldsymbol{\lambda})}^{\eta(\boldsymbol{\lambda}),s}$ is induced from an automorphism of $\overline{M(\boldsymbol{\lambda})}$. Since every automorphism of $\overline{M(\boldsymbol{\lambda})}$ preserves $\operatorname{rad}^s \overline{M(\boldsymbol{\lambda})}$, the Lemma follows. □

Theorem 5.5. *The Jantzen filtrations of Weyl modules in $\mathbf{S}_{B_w}$ coincide with radical filtrations, i.e., for all $\lambda \in \Lambda(\rho, w)$ and nonnegative integer s,*

$$\overline{\Delta(\lambda)}^s = \operatorname{rad}^s \overline{\Delta(\lambda)}.$$

Proof. For each $i = \{0, \dots, p-1\}$, let $\eta(i) : \Omega(i) \to \mho(i)$ be the homomorphism associated to the contravariant form on $\Omega(i)$ which defines the Jantzen filtration. Then by Lemma 3.3, if s is a nonnegative integer, we have $\overline{\Omega(i)}^{\eta(i),s} = \operatorname{rad}^s \overline{\Omega(i)}$. Therefore by the preceding Lemma, we have

$$\overline{\Omega(\boldsymbol{\lambda})}^{\eta(\boldsymbol{\lambda}),s} = \operatorname{rad}^s \overline{\Omega(\boldsymbol{\lambda})}$$

for all $\boldsymbol{\lambda} \in \Lambda^p_w$. Now, applying the functor $\mathcal{G}$ we get by Theorem 5.2 homomorphisms $\mathcal{G}(\eta(\boldsymbol{\lambda})) : \Delta(\lambda^\dagger) \to \Delta(\lambda^\dagger)^\circ$. By Lemma 5.3(2), we have $\mathcal{G}(\overline{\Omega(\boldsymbol{\lambda})}^{\eta(\boldsymbol{\lambda}),s}) = \overline{\Delta(\lambda^\dagger)}^{\mathcal{G}(\eta(\boldsymbol{\lambda})),s}$, where we identify $\mathcal{G}(\overline{\Omega(\boldsymbol{\lambda})})$ with $\overline{\Delta(\lambda^\dagger)}$. On the other hand, $\mathcal{G}$ induces an equivalence between the module categories, and thus sends radicals to radicals. Hence,

$$\overline{\Delta(\lambda^\dagger)}^{\mathcal{G}(\eta(\boldsymbol{\lambda})),s} = \operatorname{rad}^s \overline{\Delta(\lambda^\dagger)}.$$

The desired result is then a consequence of Lemma 2.3. □

6. Formulas and verification of conjectures

6.1. **Radical series.** Throughout this subsection, if λ is a partition in $\Lambda(\rho, w)$, we denote its p-quotient as $(\lambda^0, \dots, \lambda^{p-1})$.

Define, for $\lambda, \mu, \sigma \in \Lambda(\rho, w)$ with σ p-restricted, polynomials

$$\begin{aligned} \mathrm{rad}_{\Delta,\lambda,\mu}(v) &= \sum_{s \geq 0} [\mathrm{rad}^s(\overline{\Delta(\lambda)}) / \mathrm{rad}^{s+1}(\overline{\Delta(\lambda)}) : L(\mu)] v^s, \\ \mathrm{rad}_{P,\lambda,\mu}(v) &= \sum_{s \geq 0} [\mathrm{rad}^s(\overline{P(\lambda)}) / \mathrm{rad}^{s+1}(\overline{P(\lambda)}) : L(\mu)] v^s, \\ \mathrm{rad}_{S,\lambda,\sigma}(v) &= \sum_{s \geq 0} [\mathrm{rad}^s(\overline{S(\lambda)}) / \mathrm{rad}^{s+1}(\overline{S(\lambda)}) : D(\sigma)] v^s, \\ \mathrm{rad}_{Y,\lambda,\sigma}(v) &= \sum_{s \geq 0} [\mathrm{rad}^s(\overline{Y(\lambda)}) / \mathrm{rad}^{s+1}(\overline{Y(\lambda)}) : D(\sigma)] v^s. \end{aligned}$$

We have the following formulas for analogous polynomials describing the radical series of $\mathcal{P}(\boldsymbol{\lambda})$, $\Omega(\boldsymbol{\lambda})$, $\mathcal{Y}(\boldsymbol{\lambda})$ and $\mathcal{S}(\boldsymbol{\lambda})$, using results obtained in [4].

Theorem 6.1. *We have*

$$\mathrm{rad}_{\Omega,\boldsymbol{\lambda},\boldsymbol{\mu}}(v) = v^{\delta(\boldsymbol{\lambda},\boldsymbol{\mu})} \sum_{\substack{\alpha^0,\dots,\alpha^p \\ \beta^0,\dots,\beta^{p-1}}} \prod_{j=0}^{p-1} c(\lambda^j; \alpha^j, \beta^j) c(\mu^j; \beta^j, \alpha^{j+1}),$$

where

$$\delta(\boldsymbol{\lambda}, \boldsymbol{\mu}) = \sum_{j=1}^{p-1} j(|\lambda^j| - |\mu^j|).$$

Moreover

$$\mathrm{rad}_{\mathcal{P},\boldsymbol{\lambda},\boldsymbol{\mu}}(v) = \sum_{\boldsymbol{\nu} \in \Lambda^p_w} \mathrm{rad}_{\Omega,\boldsymbol{\nu},\boldsymbol{\lambda}}(v) \, \mathrm{rad}_{\Omega,\boldsymbol{\nu},\boldsymbol{\mu}}(v).$$

Furthermore,

$$\begin{aligned} \mathrm{rad}_{\mathcal{S},\boldsymbol{\lambda},\boldsymbol{\sigma}}(v) &= v^{-|\lambda^{p-1}|} \, \mathrm{rad}_{\Omega,\boldsymbol{\lambda},\boldsymbol{\sigma}}(v); \\ \mathrm{rad}_{\mathcal{Y},\boldsymbol{\lambda},\boldsymbol{\sigma}}(v) &= v^{-|\lambda^{p-1}|} \, \mathrm{rad}_{\mathcal{P},\boldsymbol{\lambda},\boldsymbol{\sigma}}(v). \end{aligned}$$

The presentation of the formula for $\mathrm{rad}_{\Omega,\boldsymbol{\lambda},\boldsymbol{\mu}}$ is due to Leclerc and Miyachi [14].

Proof. The formulas for $\mathrm{rad}_{\Omega,\boldsymbol{\lambda},\boldsymbol{\mu}}(v)$ and $\mathrm{rad}_{\mathcal{P},\boldsymbol{\lambda},\boldsymbol{\mu}}(v)$ are obtained in [4, Proposition 7.1]. The formulas for $\mathrm{rad}_{\mathcal{S},\boldsymbol{\lambda},\boldsymbol{\sigma}}(v)$ and $\mathrm{rad}_{\mathcal{Y},\boldsymbol{\lambda},\boldsymbol{\sigma}}(v)$ can be obtained in an entirely similar manner, and by comparing with the formulas obtained for $\mathrm{rad}_{\Omega,\boldsymbol{\lambda},\boldsymbol{\sigma}}(v)$ and $\mathrm{rad}_{\mathcal{P},\boldsymbol{\lambda},\boldsymbol{\sigma}}(v)$, we get the desired presentations. □

We now translates this theorem to $\mathbf{S}_{B_w}$ and B_w; the second statement indicates that, in Rouquier blocks, the Schur functor preserves radical filtrations (up to a shift) of Weyl modules and projective modules.

Theorem 6.2.

(1) *We have*

$$\mathrm{rad}_{\Delta,\lambda,\mu}(v) = v^{\delta(\lambda,\mu)} \sum_{\substack{\alpha^0,\dots,\alpha^p \\ \beta^0,\dots,\beta^{p-1}}} \prod_{j=0}^{p-1} c(\lambda^j;\alpha^j,\beta^j)c(\mu^j;\beta^j,(\alpha^{j+1})'),$$

where

$$\delta(\lambda,\mu) = \sum_{j=1}^{p-1} j(|\lambda^j| - |\mu^j|).$$

Moreover

$$\mathrm{rad}_{P,\lambda,\mu}(v) = \sum_{\nu\in\Lambda(\rho,w)} \mathrm{rad}_{\Delta,\nu,\lambda}(v)\,\mathrm{rad}_{\Delta,\nu,\mu}(v).$$

(2) *We have*

$$\mathrm{rad}_{S,\lambda,\sigma}(v) = v^{-|\lambda^{p-1}|}\,\mathrm{rad}_{\Delta,\lambda,\sigma}(v);$$
$$\mathrm{rad}_{Y,\lambda,\sigma}(v) = v^{-|\lambda^{p-1}|}\,\mathrm{rad}_{P,\lambda,\sigma}(v).$$

Proof. We apply the functor $\mathcal{F}$ and $\mathcal{G}$ to the formulas obtained in the last Theorem, and use the results of §5.1. □

Theorem 6.3 (Ext-quiver)**.**

(1) *Let $\lambda,\mu \in \Lambda(\rho,w)$ with $\lambda \geq \mu$. Then* $\mathrm{Ext}^1(L(\lambda),L(\mu)) = 0$ *unless there exists $j \in \{1,\dots,p-1\}$, such that*
- $\lambda^i = \mu^i$ *whenever* $i \neq j, j-1$,
- $\lambda^{j-1} \nearrow \mu^{j-1}$,
- $\mu^j \nearrow \lambda^j$,

in which case $\dim \mathrm{Ext}^1(L(\lambda),L(\mu)) = 1$.

(2) *Let $\sigma,\tau \in \Lambda(\rho,w)$ be p-restricted partitions. Then*

$$\dim \mathrm{Ext}^1(D(\sigma),D(\tau)) = \dim \mathrm{Ext}^1(L(\sigma),L(\tau)).$$

(3) *The* Ext*-quivers of $\mathbf{S}_{B_w}$ and B_w are bipartite.*

Proof. Here we are just using Theorem 6.2 to obtain the coefficients of v in $\mathrm{rad}_{P,\lambda,\mu}(v)$ and $\mathrm{rad}_{Y,\sigma,\tau}(v)$. Since $\mathbf{S}_{B_w}$ is quasihereditary, the coefficients of v in $\mathrm{rad}_{P,\lambda,\mu}(v)$ and $\mathrm{rad}_{\Delta,\lambda,\mu}(v)$ are equal; and the latter gives the formula in part (1). Part (2) follows immediately from the Theorem 6.2(2).

For part (3), define the parity of $L(\lambda)$ to be the parity of $\sum_{j \text{ odd}} |\lambda^j|$. By part (1), if $L(\lambda)$ extends $L(\mu)$, then they have different parities. Thus grouping the simple modules of $\mathbf{S}_{B_w}$ according to their parities displays the bipartite nature of its Ext-quiver. The bipartite nature of the Ext-quiver of B_w then follows by part (2). □

6.2. **A conjecture of Martin.** S. Martin [18] conjectured that the principal indecomposable modules in a weight w p-block of symmetric group algebra with $w < p$ have a common radical length $2w+1$; this is clear for $w = 0$ and $w = 1$, and J. Scopes [24] proves the case of $w = 2$, while Martin and the second author [19, 20] provide a partial proof for $w = 3$.

This conjecture holds for Rouquier blocks:

Theorem 6.4. *The principal indecomposable modules of B_w have a common radical length $2w + 1$.*

Proof. Let $\lambda \in \Lambda(\rho, w)$ be p-restricted. The radical length of $Y(\lambda)$ is one more than the degree of the polynomial

$$\mathrm{rad}_{Y,\lambda,\lambda}(v) = \mathrm{rad}_{P,\lambda,\lambda} = \sum_{\nu\in\Lambda(\rho,w)} (\mathrm{rad}_{\Delta,\nu,\lambda}(v))^2.$$

By Theorem 6.2(1), the degree of $\mathrm{rad}_{\Delta,\nu,\lambda}(v)$ is bounded above by w, with equality when $\nu^{i+1} = \lambda^i$ for all $i \in \{0, 1, \ldots, p-2\}$. Thus, the theorem follows. □

6.3. **A conjecture of Lascoux, Leclerc, Thibon, and Rouquier.** Let $d_{\lambda\mu}(v)$ (λ, $\mu \in \Lambda$) be the 'v-decomposition numbers' arising from the canonical basis in the Fock space representation of the quantum affine algebra $U_q(\widehat{\mathfrak{sl}}_p)$ [15]. There are various algorithms for calculating these polynomials, but a general closed formula is not known. However for $\lambda, \mu \in \Lambda(\rho, w)$ Leclerc and Miyachi [14] have shown (via a formula like the one in Theorem 6.2, derived in the context of finite general linear groups by Miyachi [22]) that

$$(*) \qquad d_{\lambda',\mu'}(v) = \mathrm{rad}_{\Delta,\lambda,\mu}(v).$$

An independent proof in the case that μ is p-restricted is given in [3].

Because we have shown that the radical filtrations and Jantzen filtrations coincide (Proposition 5.5) we deduce that for $\lambda, \mu \in \Lambda(\rho, w)$,

$$d_{\lambda',\mu'}(v) = \sum_{s\geq 0} \left[\frac{\overline{\Delta(\lambda)}^s}{\overline{\Delta(\lambda)}^{s+1}} : L(\mu)\right] v^s.$$

Thus the conjecture of Lascoux, Leclerc, Thibon, and Rouquier [15, Conjecture 5.3] (see also [13, §9]) holds in the blocks $\mathbf{S}_{B_w}$. But note that their conjecture is for Weyl modules of q-Schur algebras at complex roots of unity, while our result is for Weyl modules of ordinary Schur algebras over a field of characteristic p.

6.4. **A conjecture of James.** By putting $v = 1$ in equation $(*)$ we see that $d_{\lambda',\mu'}(1)$ describes the decomposition numbers $[\Delta(\lambda) : L(\mu)]$ in the block $\mathbf{S}_{B_w}$. On the other hand, by a result of Varagnolo-Vasserot [26], $d_{\lambda',\mu'}(1)$ describes the analogous decomposition numbers in the corresponding block of a q-Schur algebra at a complex p-th root of unity. This coincidence of decomposition numbers has been conjectured by James to hold for q-Schur algebras of degree less than p^2, and more generally, in blocks of q-Schur algebras of weight less than p. For background and more precise statements of James's conjecture, see [11] and [6].

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