Strong stability of discrete-time systems

G. Halikias[1], L. Dritsas[2], A. Pantelous[3] and V. Tsoulkas[4]

Abstract: The paper introduces a new notion of stability for internal autonomous system descriptions in discrete-time, referred to as "strong stability", which extends a parallel notion introduced in the continuous-time case. This is a stronger notion of stability compared to alternative definitions (asymptotic, Lyapunov), which prohibits systems described by natural coordinates to have overshooting responses for arbitrary initial conditions in state-space. Three finer notions of strong stability are introduced and necessary and sufficient conditions are established for each one of them. The invariance of strong stability under orthogonal transformations is also shown, and this enables the characterization of the property in terms of the invariants of the Schur form of the system's state matrix. The class of discrete-time systems for which strong and asymptotic stability coincide is characterized and links between the skewness of the eigen-frame and the violation of strong stability property are obtained. Connections between the notions of strong stability in the continuous and discrete-domains are derived. Finally, as application, the strong stability property is studied in the context of balanced realizations, general similarity transformations and state/output-feedback stabilization problems.

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1. Introduction

Stability is a crucial system property that has been extensively studied from many aspects [2], [15], [16], [24], [13], [11]. Here we examine a new form of stability of internal (state-space) autonomous system descriptions, defined as "strong stability", which depends on the selection of a state coordinate frame in which states represent physical variables, referred to as a physical-system representations. The definitions given here extend similar notions established for continuous-time systems to the discrete-time case. Essentially, strong stability prohibits "overshoots" in the autonomous trajectory of the system, defined in state-space, for arbitrary initial conditions. Non-overshooting response is a desirable property in many applications and can be considered as a special case of constrained control. Thus, the notion of strong stability introduced here is relevant to many real-time applications where a human operator may interpret an overshooting response as an early indication of instability, and taking

[1]Control Engineering Research Centre, School of Engineering and Mathematical Sciences, City University, Northampton Square, London EC1V 0HB, United Kingdom, e-mail: G.Halikias@city.ac.uk

[2]Hellenic Air Force Academy, Division of Automatic Control, Athens, GR and Electrical and Computer Engineering Department, University of Patras, Rio, Achaia, Greece, e-mail: ldri@otenet.gr

[3]Department of Mathematical Sciences, University of Liverpool, Liverpool L69 7ZL, United Kingdom, e-mail: A.Pantelous@liverpool.ac.uk

[4]General Secretariat for Research and Technology and Department of Mathematics, University of Athens, Athens, Greece, e-mail: btsu@gsrt.gr

corrective actions which may destabilize the system. Note that non-overshooting responses separate clearly a stable from an unstable behaviour, if the diagnosis is based on a finite, early observation horizon of the system's time response.

The notion of "strong stability" introduced in the paper is a stronger version of classical notions of stability, such as asymptotic or Lyapunov stability. In this work we restrict ourselves to the autonomous linear time-invariant (LTI) discrete-time case and derive necessary and sufficient conditions for three refined notions of strong stability in terms of the *spectral norm* of the state matrix, the *spectral radius* of the state-matrix and an *observability property* of a matrix pair constructed directly from the state-matrix of the system. The dependence of the strong-stability property on general coordinate transformations is noted, along with the existence of special coordinate systems for which the system can not be strongly stable. It is also shown that this property is invariant under orthogonal transformations, which leads to the use of the Schur canonical form, established under orthogonal transformations, as the basis for investigating further the parametrisation of strongly stable state matrices. The role of the skewness of the eigen-frame of the state-matrix on the violation of the strong stability property, resulting in state-space overshoots, is established. Relations between strong stability properties in the discrete and continuous domains are derived. Finally, the preservation or violation of strong stability is studied for systems subjected to arbitrary coordinate transformations, balancing transformations and state/output feedback stabilizing transformations.

The definition of "strong stability" introduced here is related to the transient response of a system, e.g. its overshooting behaviour, initial exponential growth or its transient energy [13], [29] and could prove useful for analysing stability properties of systems under switching regimes [25]. Other refined stability notions proposed in the literature related to strong stability include qualitative (sign) stability, D-stability, total stability and R-stability (see [2], [20] for a survey of these stability notions).

The paper is organized as follows: The remaining part of section 1 defines the notation used in the paper and section 2 reviews the main definitions and properies of strong stability in the continuous-time case. Section 3 defines the notion of strong stability in discrete-time, develops numerous necessary and sufficient conditions for three refined strong stability notions and establishes connections between strong stability in the continuous and discrete domains via the bilinear transformation. Section 4 deals with numerous properties of strongly stable systems, establishes the invariance of strong stability under orthogonal transformations and characterizes the class of discrete systems for which strong and asymptotic stability are identical or approximately equivalent notions. Connections between strong stability and skewness of eigen-frame of the state-matrix are also developed in this section, and the Schur form is used for defining parameter-dependent conditions for strong stability. Section 4 also examines examines strong stability for systems subjected to arbitrary coordinate and balancing transformations. Section 5 poses and solves three variants of the strong stabilization problem under state feedback, output injection and output feedback, using easily verifiable necessary and sufficient conditions and gives a complete parametrization of the family of all optimal solutions in each case.

The notation used in the paper is standard and is summarized here for convenience. $\mathcal{N}$, $\mathcal{R}$ and $\mathcal{C}$ denote the sets of natural, real and complex numbers, respectively. The set of complex numbers with negative real part is denoted by $\mathcal{C}_-$ and is referred to as the *open-left-half-plane*. The set of complex numbers with non-positive real part is denoted by $\bar{\mathcal{C}}_-$ and is referred to as the *closed-left-half-plane*. $\mathcal{R}^{m\times n}$ ($\mathcal{C}^{m\times n}$) denotes the space of all $m \times n$ real (complex) matrices. For a real or complex matrix A,

A^t denotes the transpose of A and A^* the complex conjugate transpose of A. For a square invertible matrix A, A^{-1} is the inverse of A and $A^{-t} = (A^{-1})^t = (A^t)^{-1}$. If A is a square matrix, then $\lambda(A)$ denotes the spectrum of A, i.e. the set of its eigenvalues and $\rho(A)$ is the spectral radius of A. If $x \in \mathcal{R}^n$ or $x \in \mathcal{C}^n$, then $\|x\|$ denotes the Euclidian norm of x. For a real or complex matrix A, $\|A\|$ is the induced 2-norm (spectral norm or largest singular value). For a Hermitian or symmetric matrix A, $\lambda_{max}(A)$ denotes the largest eigenvalue of A and $\lambda_{min}(A)$ the smallest eigenvalue of A. A positive definite matrix A (positive semi-definite, negative definite, negative semi-definite) is denoted as $A < 0$ ($A \geq 0$, $A < 0$, $A \leq 0$, respectively). Finally, the left and right null-spaces of a matrix A are denoted as $\mathcal{N}_l(A)$ and $\mathcal{N}_r(A)$, respectively, while the range (column-span) of A is denoted as $\mathcal{R}(A)$. A left annihilator of A, denoted by $A_l^{\perp}$, is a matrix with the maximum possible number of linear independent rows such that $A_l^{\perp} A = 0$. Similarly, a right annihilator of A, denoted by $A_r^{\perp}$, is a matrix with the maximum possible number of linear independent columns such that $AA_r^{\perp} = 0$.

2. Review of Strong Stability for Continuous-time Systems

In this section we review the three notions of strong stability which have been introduced for the continuous-time case. Consider the autonomous LTI continuous-time system:

$$S_c(A) : \dot{x}(t) = Ax(t), \;\; x(0) = x_0$$

in which $A \in \mathcal{R}^{n \times n}$ is the state-matrix. For this system, the basic notions of asymptotic and Lyapunov stability are well established and the eigenvalues of A provide a simple characterisation of such properties, whereas the properties of the eigenframe have no influence. We start by quoting the classical notions of stability (e.g. see [16]).

Definition 2.1***:*** *For the linear system $\mathcal{S}_c(A)$ we define:*

1. *$\mathcal{S}_c(A)$ is* ***Lyapunov stable*** *if and only if for each $\epsilon > 0$ there exists $\delta(\epsilon) > 0$ such that $\|x(t_0)\| < \delta(\epsilon)$ implies that $\|x(t)\| < \epsilon$ for all $t \geq t_0$.*
2. *$\mathcal{S}_c(A)$ is* ***asymptotically stable*** *if and only if it is Lyapunov stable and $\delta(\epsilon)$ in part (1) of the definition can be selected so that $\|x(t)\| \to 0$ as $t \to \infty$.* □

For the autonomous LTI continuous-time system $\mathcal{S}_c(A)$, a necessary and sufficient condition for asymptotic stability is that the spectrum of A is contained in the open left-half plane (all eigenvalues have negative real parts); a necessary and sufficient condition for Lyapunov stability is that the spectrum of A lies in the closed left-half plane (Re$(s) \leq 0$) and, in addition, any eigenvalue on the imaginary axis has simple structure (i.e. equal algebraic and geometric multiplicity) [16]. Note that asymptotic stability is here taken to mean that the origin is the unique equilibrium point and that it is asymptotically stable (in the sense of Definition 2.1 part 2).

We refine these two stability notions (asymptotic and Lyapunov stability) by introducing the following definition of "strong stability":

Definition 2.2***:*** *For the system $S_c(A)$ we say that:*

1. *The system $\mathcal{S}_c(A)$ is* ***strongly Lyapunov stable*** *if and only if* $\|x(t)\| \leq \|x(t_0)\|, \ \forall t > t_0$ *and* $\forall\, x(t_0) \in \mathcal{R}^n$.

2. *The system $\mathcal{S}_c(A)$ is* ***strongly asymptotically stable w.s.*** *(in the wide sense), if and only if* $\|x(t)\| < \|x(t_0)\|, \ \forall t > t_0$ *and* $\forall\, x(t_0) \neq 0$.

3. *The system $\mathcal{S}_c(A)$ is* ***strongly asymptotically stable s.s.*** *(in the strict sense, or simply strongly asymptotically stable) if and only if* $\frac{d\|x(t)\|}{dt} < 0, \ \forall t \geq t_0$ *and* $\forall\, x(t_0) \neq 0$. □

The three notions of "strong stability" defined above are related to autonomous trajectories of the LTI system $S_c(A)$ in $\mathcal{R}^n$, whose distance from the origin (measured via the Euclidian norm) is a non-increasing (decreasing) function of time, for arbitrary initial conditions.

More precisely, strong Lyapunov stability does not allow state trajectories to exit (at any time) the (closed) hyper-sphere with centre the origin and radius the norm of the initial state vector $r_0 = \|x(t_0)\|$ (although motion on the boundary of the sphere $\|x(t)\| = r_0$ is allowed, e.g. an oscillator's trajectory).

For strong asymptotic stability (strict sense) the system's trajectory is allowed to enter each hyper-sphere $\|x(t)\| = r \leq r_0$ from a non-tangential direction, whereas for systems which are strongly asymptotically stable (wide-sense), tangential entry is allowed.

It is clear that strong Lyapunov stability implies Lyapunov stability and strong asymptotic stability (in either sense) implies asymptotic stability. Moreover, strong asymptotic stability s.s. implies strong asymptotic stability w.s. which in turn implies strong Lyapunov stability. For further discussion and concrete examples of each type of strong stability see [17] and [18].

Each notion of strong stability is equivalent to certain properties of the "state" matrix A, stated in the following Theorem.

Theorem 2.1 [18]: For the system $\mathcal{S}_c(A)$, the following properties hold true:

(i) $\mathcal{S}_c(A)$ is strongly asymptotically stable s.s. if and only if $A + A^t < 0$.

(ii) $\mathcal{S}_c(A)$ is strongly asymptotically stable w.s. if and only if one of the following two equivalant conditions hold:

  (a) $A + A^t \leq 0$ and A is asymptotically stable.

  (b) $A + A^t \leq 0$ and the pair $(A, A + A^t)$ is observable.

(iii) $\mathcal{S}_c(A)$ is strongly Lyapunov stable, if and only if $A + A^t \leq 0$.

3. Strong Stability for Discrete-time Systems

Consider the autonomous LTI discrete-time system:

$$\Sigma_d(A) : x_{k+1} = Ax_k, k \in \mathcal{N}_o(\triangleq \mathcal{N} \cup \{0\}), x_0 \in \mathcal{R}^n$$

Then we have the following standard definitions:

Definition 3.1 *For the system $\Sigma_d(A)$ the equilibrium $x = 0$ is said to be:*

(i) Lyapunov-stable if for every $\epsilon > 0$ there exists $\delta = \delta(\epsilon) > 0$ such that $\|x_k\| < \epsilon$ for all $k \in \mathcal{N}_o$ whenever $\|x_0\| < \delta$.

(ii) Asymptotically stable, if it is Lyapunov-stable and there exists $\eta > 0$ such that, if $\|x_0\| < \eta$ then $\lim_{k\to\infty} \|x_k\| = 0$.

(iii) Asymptotically stable in the large (or globally asymptotically stable), if $x = 0$ is asymptotically stable and its domain of attraction is the whole of $\mathcal{R}^n$. (An equilibrium point $x = 0$ which satisfies (ii) is called "attractive", and its "domain of attraction" is the set of all $x_0 \in \mathcal{R}^n$ for which $x = 0$ is attractive).

(iv) Exponentially stable if there exists $\alpha > 0$ and for every $\epsilon > 0$ there exists $\delta(\epsilon) > 0$ such that $\|x_k\| < \epsilon \exp(-\alpha k)$ for all $k \in \mathcal{N}_o$, whenever $\|x_0\| < \delta(\epsilon)$.

(v) Exponentially stable in the large (or globally exponentially stable) if there exists $\alpha > 0$ and for any $\beta > 0$ there exists a $k(\beta) > 0$ such that $\|x_k\| \le k(\beta)\|x_0\| \exp(-\alpha k)$ for all $k \in \mathcal{N}$, whenever $\|x_0\| < \beta$.

In this work we consider only autonomous LTI time-invariant discrete-time systems, for which we have the following results:

***Theorem** 3.1[1]: For the case of linear, time-invariant discrete-time systems $\Sigma_d(A)$ the following results hold:*

(i) The equilibrium $x = 0$ of $\Sigma_d(A)$ is Lyapunov stable if and only if the state-trajectory $\{x_k = A^k x_0, k = 0, 1, 2, \ldots\}$ is bounded.

(ii) The following three statements are equivalent:

(a) The equilibrium $x = 0$ is asymptotically stable.

(b) The equilibrium $x = 0$ is asymptotically stable in the large.

(c) The equilibrium $x = 0$ is exponentially stable.

(d) $\lim_{k\to\infty} \|A^k\| = 0$

Thus, in the LTI discrete-time case, the fundamental stability distinction is between Lyapunov and asymptotic stability. The following Theorem gives necessary and sufficient conditions for each of these two types of stability in terms of certain properties of the state matrix.

***Theorem** 3.2 [1]: (i) The equilibrium $x = 0$ of $\Sigma_d(A)$ is asymptotically stable if and only if all eigenvalues of A are within the unit circle of the complex plane (i.e. $\rho(A) < 1$). In this case we say that A is Schur-stable or asymptotically stable. (ii) The equilibrium $x = 0$ of $\Sigma_d(A)$ is Lyapunov-stable if and only if $\rho(A) \le 1$ and for each eigenvalue λ_j of A with $|\lambda_j| = 1$ having multiplicity $n_j > 1$, it is true that*

$$\lim_{z\to\lambda_j} \left\{ \frac{d^{n_j-1-l}}{dz^{n_j-1-l}} [(z - \lambda_j)^{n_j} (zI - A)^{-1}] \right\} = 0, \quad l = 1, 2, \ldots, n_j$$

Alternatively, $\Sigma_d(A)$ is Lyapunov-stable if and only if all eigenvalues of A are within or on the unit circle of the complex plane, and every eigenvalue that is on the unit circle has an associated Jordan block of order 1. *In this case A is said to be Lyapunov-stable or simply stable.*

Next, we introduce the following definitions of discrete-time strong stability. Each of these notions corresponds to a notion of strong stability introduced in section 2 for continuous-time systems.

Definition 3.2***:*** *$\Sigma_d(A)$ is strongly asymptotically stable s.s. (in the strict sense or simply strongly asymptotically stable) if and only if $\|x_{k+1}\| < \|x_k\|$ for all $k \in \mathcal{N}_o : x_k \neq 0\}$.*

Remark 3.1***:*** Note that convergence to zero in finite number of steps (dead-beat response) is allowed by the definition, provided, for each initial condition $x_0 \neq 0$, the norm of the state decreases monotonically from its initial value $\|x_0\|$ at time zero until the first time, say $N(x_0) \geq 0$, at which $x_{N(x_0)} = 0$, and stays at zero thereafter, i.e. $x_m = 0$ for all $m \geq N(x_0)$.

Proposition 3.1***:*** *$\Sigma_d(A)$ is strongly asymptotically stable (s.s.) if and only if $\|A\| < 1$, where $\|\cdot\|$ denotes the spectral norm (largest singular value).*

Proof: Consider the sequence of equivalences:

$$\begin{aligned}
\Sigma_d(A) \text{ strongly asymptotically stable s.s.} &\Leftrightarrow x_k^t A^t A x_k < x_k^t x_k \quad \text{for all } k \in \mathcal{N}_o, x_k \neq 0 \\
&\Leftrightarrow x_k^t (I_n - A^t A) x_k > 0 \text{ for all } k \in \mathcal{N}_o, x_k \neq 0 \\
&\Leftrightarrow I_n - A^t A > 0 \\
&\Leftrightarrow \|A\| < 1
\end{aligned}$$

which prove the result. □

Corollary 3.1***:*** *Strong asymptotic stability (s.s.) of $\Sigma_d(A)$ implies asymptotic stability of $\Sigma_d(A)$.*

Proof: Follows since if $\Sigma_d(A)$ is strongly asymptotically stable then $\rho(A) \leq \|A\| < 1$, while asymptotic stability for autonomous LTI discrete-time systems is equivalent to condition $\rho(A) < 1$. An alternative proof by Lyapunov-function arguments is also possible. □

Corollary 3.2***:*** *$\Sigma_d(A)$ is strongly asymptotically stable s.s. if and only if $\|A^n\| < 1$ for all $n \geq 1$.*

Proof: If $\Sigma_d(A)$ is strongly asymptotically stable s.s., then $\|A\| < 1$ and hence $\|A^n\| \leq \|A\|^n < 1$ for all $n \geq 1$. Conversely, suppose that $\|A\|^n < 1$ for all $n \geq 1$. Setting $n = 1$ gives $\|A\| < 1$ which from Proposition 3.1 implies strong asymptotic stability (s.s.) of $\Sigma_d(A)$. □

Strong Lyapunov stability in autonomous LTI discrete-time $\Sigma_d(A)$ is defined next:

Definition 3.3***:*** *$\Sigma_d(A)$ is strongly Lyapunov stable if and only if $\|x_{k+1}\| \leq \|x_k\|$ for all $k \in \mathcal{N}_o$.*

Proposition 3.2***:*** *$\Sigma_d(A)$ is strongly Lyapunov stable if and only if $\|A\| \leq 1$, where $\|\cdot\|$ denotes the spectral norm (largest singular value).*

Proof: Similar to the proof for strong asymptotic stability s.s. or via a direct Lyapunov type argument. Note that an oscillator falls in this category, so strong Lyapunov stability does not imply asymptotic stability. □

Example 3.1***:*** Every square orthogonal matrix is strongly Lyapunov stable.

Although the next result is immediate from Proposition 3.2, we give an independent proof, which is also used in the Proof of Proposition 3.5 below.

Proposition 3.3***:*** *The condition $\|A\| \leq 1$ implies that A is a Lyapunov matrix, i.e. a matrix with all eigenvalues having magnitude less than or equal to one, with those eigenvalues of magnitude equal to one having identical algebraic and geometric multiplicity.*

Proof: Note first that $\|A\| \leq 1$ implies that $\rho(A) \leq 1$. If $\rho(A) < 1$ then A is a Schur matrix and hence also a Lyapunov matrix, as required. Hence, assume that $\rho(A) = 1$. Introduce a Schur transformation,

$$UAU^* = \begin{pmatrix} \Lambda & \beta \\ 0 & B \end{pmatrix}$$

where U is unitary, Λ is an upper triangular matrix with diagonal entries $(\lambda_1, \ldots, \lambda_s)$, where $|\lambda_1| = \ldots |\lambda_s| = \rho = 1$, and B is an upper triangular matrix with diagonal entries $(\lambda_{s+1}, \ldots, \lambda_n)$ where $|\lambda_{s+1}| \leq \ldots \leq |\lambda_n| < 1$. Next note that since $\|A\| \leq 1$ and the spectral norm is unitarily invariant, we have that $\Lambda = \text{diag}(\Lambda)$, $\beta = 0$ and $\|B\| \leq 1$. Thus the eigenvalues of A which have modulus equal to one have simple Jordan blocks, and thus A is a Lyapunov matrix. $\square$

Next, we define strong asymptotic stability in the wide sense (w.s.) for $\Sigma_d(A)$.

Definition 3.4***:*** *$\Sigma_d(A)$ is strongly asymptotically stable in the wide sense (w.s.) iff it is asymptotically stable and $\|x_{k+1}\| \leq \|x_k\|$ for all $k \in \mathcal{N}_o$.*

Proposition 3.4***:*** *$\Sigma_d(A)$ is strongly asymptotically stable (w.s.) if and only if $\rho(A) < 1$ and $\|A\| \leq 1$.*

Proof: Follows immediately from Definition 3.4, the fact that $\Sigma_d(A)$ is asymptotically stable if and only if $\rho(A) < 1$ and the fact that $\|x_{k+1}\| \leq \|x_k\|$ if and only if $\|A\| \leq 1$. $\square$

From the above definitions and results it follows that strong asymptotic stability s.s. implies strong asymptotic stability w.s., which in turn implies strong Lyapunov stability. Also Strong asymptotic stability w.s. implies asymptotic stability (directly from definition) and strong Lyapunov stability implies Lyapunov stability. A strongly asymptotically stable w.s. system which is not strongly asymptotically stable s.s. is demonstrated in the example below:

Example 3.2***:*** Consider the discrete-time system $\Sigma_d(A) : x_{k+1} = Ax_k$:

$$\begin{pmatrix} x_{k+1}^{(1)} \\ x_{k+1}^{(2)} \\ x_{k+1}^{(3)} \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_k^{(1)} \\ x_k^{(2)} \\ x_k^{(3)} \end{pmatrix}$$

with $\rho(A) = 0$ and $\|A\|$=1. Note that for every $x_k \in \mathcal{R}^3$, $x_{k+3} = A^3 x_k = 0$ and hence the system is asymptotically stable (with a "dead-beat" response). Further,

$$\|x_{k+1}\|^2 = \left(x_k^{(2)}\right)^2 + \left(x_k^{(3)}\right)^2 \leq \left(x_k^{(1)}\right)^2 + \left(x_k^{(2)}\right)^2 + \left(x_k^{(3)}\right)^2 = \|x_k\|^2$$

is an equality when $x_k^{(1)} = 0$. Thus the system $\Sigma_d(A)$ is strongly asymptotically stable w.s. but not s.s.

Next we give an alternative characterization of *strongly asymptotically stable w.s.* systems. We first need a preliminary result, presented in Proposition 3.5 below. The proof of part of the Proposition is adapted from [5], [26], although the main arguments contained in the proof presented here have a distinct "system-theoretic" flavour.

Proposition 3.5: *Let $A \in \mathcal{R}^{n\times n}$ with $\|A\| = 1$ and assume that A has $r \geq 0$ eigenvalues of modulus one. Let $\kappa(A,k)$ denote the number of singular values of A^k which are less than one and define $\kappa(A,\infty) = \lim_{k\to\infty} \kappa(A,k)$. Let W be a square-root of $I_n - A^tA$, so that $I_n - A^tA = W^tW$ and define*

$$\Gamma_o(A,k) = \begin{pmatrix} W \\ WA \\ \vdots \\ WA^{k-1} \end{pmatrix}.$$

Then,

(i) Any eigenvalue of A with modulus one is unobservable through W.

(ii) The integer sequence $\kappa(A,k)$ is non-decreasing with upper bound:

$$Rank[I_n - (A^t)^n A^n] = \kappa(A,n) = \kappa(A,n+1) = \kappa(A,n+2) = \ldots = \kappa(A,\infty) = n - r$$

(iii) For each $k \geq n$, $\kappa(A,k) = Rank[\Gamma_o(A,k)] = Rank[\Gamma_o(A,n)] = \kappa(A,n)$. In particular $r = 0$ if and only if the pair (A,W) is observable.

Proof: Since $\|A\| = 1$ the matrix $I_n - A^tA$ is positive semi-definite and hence we can write $I_n - A^tA = W^tW$. Let $\exp(j\phi)$ be an eigenvalue of A and $u \neq 0$ a corresponding (right) eigenvector so that

$$Au = \exp(j\phi)u \tag{1}$$

Then

$$u^*W^tWu = u^*(I_n - A^tA)u = u^*u - \exp(-j\phi)u^*u\exp(j\phi) = 0$$

and hence

$$Wu = 0 \Rightarrow W^tWu = 0 \Rightarrow (I - A^tA)u = 0 \tag{2}$$

Thus, using equations (1) and (2),

$$\begin{pmatrix} \exp(j\phi)I_n - A \\ I_n - A^tA \end{pmatrix} u = 0 \Rightarrow \begin{pmatrix} \exp(j\phi)I_n - A \\ W^tW \end{pmatrix} u = 0 \Rightarrow \begin{pmatrix} \exp(j\phi)I_n - A \\ W \end{pmatrix} u = 0$$

and hence $\exp(j\phi)$ is unobservable through W proving (i). Equations (1) and (2) further imply that

$$u^*(I_n - A^tA) = 0 \Rightarrow u^* - \exp(-j\phi)u^*A = 0 \Rightarrow u^*A = \exp(j\phi)u^*$$

and hence u is both the left and right eigenvector of A corresponding to the eigenvalue $\exp(j\phi)$. Thus A is a Lyapunov matrix (see also Proposition 3.3) and hence there exists a unitary matrix U such that

$$U^*AU = \text{diag}(\exp(j\phi_1), \exp(j\phi_2), \ldots, \exp(j\phi_r)) \oplus \hat{A}$$

where $\oplus$ is the direct sum with $\rho(\hat{A}) < 1$. For any integer k,

$$\begin{aligned} I_n - (A^t)^k A^k &= I - A^t A + [A^t A - (A^t)^2 A^2] + \ldots + [(A^t)^{k-1} A^{k-1} - (A^t)^k A^k] \\ &= W^t W + A^t W^t W A + \ldots + (A^t)^{k-1} W^t W A^{k-1} \\ &= \Gamma_o^t(A,k)\Gamma_o(A,k) \end{aligned}$$

and hence

$$\kappa(A,k) = \text{Rank}[I_n - (A^t)^k A^k] = \text{Rank}[\Gamma_o(A,k)].$$

Now, using the Cayley-Hamilton theorem we conclude that for every $k \geq n$

$$\text{Rank}[I_n - (A^t)^n A^n] = \kappa(A,n) = \kappa(A,n+1) = \kappa(A,n+2) = \ldots = \kappa(A,\infty)$$

On noting that since $\rho(\hat{A}) < 1$ we have $\lim_{k\to\infty} \hat{A}^k = 0$ and hence $\kappa(A,\infty) = n - r$, which proves part (ii). The first equality in part (iii) also follows from the Cayley-Hamilton theorem, since for $k \geq n$ $\text{Rank}[\Gamma_o(A,k)] = \text{Rank}[\Gamma_o(A,n)]$. Recognizing $\Gamma_o(A,n)$ as the observability matrix of the pair (A,W), it follows that $\kappa(A,n) = n - r$ is equal to the number of observable modes of (A,W). In particular, (A,W) is completely observable if and only if $r = 0$, i.e. if and only if $\rho(A) < 1$. $\square$

The Corollary given below gives an alternative characterization of the family of $\Sigma_d(A)$ which are strongly asymptotically stable w.s. and are *not* strongly asymptotically stable s.s..

Corollary 3.3***:*** *$\Sigma_d(A)$ is strongly asymptotically stable w.s. but not strongly asymptotically stable s.s. if and only if $\|A\| = 1$ and the pair $(A, I_n - A^t A)$ is observable.*

Proof: From Proposition 3.4 and Proposition 3.1 it follows that $\Sigma_d(A)$ is strongly asymptotically stable (w.s.) and not strongly asymptotically stable (s.s.) if and only if $\|A\| = 1$ and $\rho(A) < 1$. From Proposition 3.5 part (ii) it now follows that, under the assumption that $\|A\| = 1$, condition $\rho(A) < 1$ is equivalent to the observability of the pair (A,W), or equivalently the observability of the pair $(A, I_n - A^t A)$, as required. Note that under the assumption that $\|A\| = 1 \Rightarrow \rho(A) \leq 1$ and that the pair (A,W) is observable, it follows from Proposition 3.5(i) that A is free from eigenvalues on the unit circle (because any such eigenvalue would be unobservable through W). Hence $\rho(A) < 1$ and A is Hurwitz. $\square$

Remark 3.2***:*** The condition for strong asymptotic stability w.s. given in Corollary 3.3 can be explained as follows: Assume that $\|A\| = 1 \Rightarrow \rho(A) \leq 1$ and that the pair $(A, I_n - A^t A)$ is observable (or equivalently that (A,W) is observable). Then from Proposition 3.5(i), A is free from eigenvalues on the unit circle (because any such eigenvalue would be unobservable through W). Hence $\rho(A) < 1$ and A is Hurwitz. This, together with the equality $\|A\| = 1$ shows that $\Sigma_d(A)$ is strongly asymptotically stable w.s. but not strongly asymptotically stable s.s..

For the system $\Sigma_d(A)$, the state-matrix A^k maps vectors x_0 to vectors x_k in k-transition steps (k consecutive linear maps through A), according to the matrix equation $x_k = A^k x_0$.
Assume that $\|A\| = 1$ and recall that in the proof of Proposition 3.5 the integer $\kappa(A,k)$ was defined as the number of singular values of A^k which are less than one (the remaining $n - \kappa(A,k)$ singular values being equal to one). The state-space $\mathcal{R}^n$ can be decomposed as a direct sum

$$\mathcal{R}^n = \mathcal{X}_c^k \oplus \mathcal{X}_i^k,$$

where $\mathcal{X}_c^k$ is the column span of the right singular vectors of A^k corresponding to the $\kappa(A,k)$ singular values which are less than one and $\mathcal{X}_i^k$ is the column span of the remaining $n-\kappa(A,k)$ right singular vectors of A^k which correspond to the singular values of A^k which are equal to one.

Thus, the dimension of the maximal subspace of $\mathcal{R}^n$ on which the restriction of A^k defines an isometry (and hence A^k is strictly contractive for any other vector in $\mathcal{R}^n$) is $n-\kappa(A,k)$. Since (from the Proof of Proposition 3.5) we have $I_n-(A^t)^kA^k=\Gamma_o^t(A,k)\Gamma_o(A,k)$, we conclude that $\mathcal{X}_i^k=\mathcal{N}_r(\Gamma_o(A,k))$. Since $\mathcal{N}_r(\Gamma_o(A,k+1))\subseteq\mathcal{N}_r(\Gamma_o(A,k))$, the dimension of this maximal subspace cannot increase as k increases and we have $n-\kappa(A,k)\geq n-\kappa(A,k+1)$, as claimed in Proposition 3.5. This Proposition also says that as k increases, the dimension of $\mathcal{X}_i^k$ cannot become less than a minimum value equal to r, the number of eigenvalues of A on the unit circle, and that this value is reached within the first n transition steps (linear maps through A). This property is an immediate consequence of the fact that the sequence of subspaces $\{\mathcal{X}_i^k=\mathcal{N}_r(\Gamma_o(A,k)),\ k\in\mathcal{N}\}$, converges, after at most n steps, to $\mathcal{X}_i^n=\mathcal{N}_r(\Gamma_o(A,n))$, the unobservable subspace of the pair (A,W). If A is a Schur matrix ($\rho(A)<1$ or $r=0$), A^n is strictly contractive for every non-trivial input direction and in this case $A^k\to 0$ as $k\to\infty$. We formalize the main arguments of this remark via the following Proposition.

Proposition 3.6*: Let $x_0\in\mathcal{R}^n$ with $\|x_0\|=1$, $A\in\mathcal{R}^{n\times n}$ with $\|A\|=1$ and k be a positive integer. Then, the following three statements are equivalent:*
(i) $x_0\in\mathcal{N}_r[I_n-(A^t)^kA^k]$,
(ii) $\|A^kx_0\|=\|x_0\|=1$,
(iii) $x_0\in\mathcal{N}_r[\Gamma_o(A,k)]$.
Moreover, in this case we also have:

$$\|A^kx_0\|=\|A^{k-1}x_0\|=\|Ax_0\|=\|x_0\|=1, \tag{3}$$

and

$$x_0\in\mathcal{N}_r[\Gamma_o(A,k)]\subseteq\mathcal{N}_r[\Gamma_o(A,k-1)]\subseteq\ldots\subseteq\mathcal{N}_r[\Gamma_o(A,1)], \tag{4}$$

where $\Gamma_o(A,k)$ is defined in Proposition 3.5*.*

Proof: (i) $\Rightarrow$ (ii): $x_0\in\mathcal{N}_r[I_n-(A^t)^kA^k]$ implies $x_0^t(A^t)^kA^kx_0=x_0^tx_0=1$, which in turn implies that $\|A^kx_0\|=\|x_0\|=1$. (i) $\Rightarrow$ (ii): $\|A^kx_0\|=\|x_0\|=1$ implies $x_0^t[I_n-(A^t)^kA^k]x_0=0$. Since $\|A\|=1$, $\|A^k\|\leq\|A\|^k=1$ and the matrix $I_n-(A^t)^kA^k$ is positive semi-definite. Thus $x_0^t[I_n-(A^t)^kA^k]x_0=0$ implies $[I_n-(A^t)^kA^k]x_0=0$ or $x_0\in\mathcal{N}_r[I_n-(A^t)^kA^k]$. (i) $\Leftrightarrow$ (iii): Follows from the identity $I_n-(A^t)^kA^k=\Gamma_o(A,k)\Gamma_o^t(A,k)$ and the fact that $\|A^k\|\leq 1$. To show (3) note that for any $x_0\in\mathcal{R}^n$ with $\|x_0\|=1$, such that $\|A^kx_0\|=\|x_0\|=1$ and any $i=1,2,\ldots,k-1$ we have

$$1=\|A^kx_0\|=\|A^{k-i}(A^ix_0)\|\leq\|A^{k-i}\|\|A^ix_0\|\leq\|A\|^{k-i}\|A^ix_0\|=\|A^ix_0\|\leq\|A\|^i\|x_0\|=\|x_0\|=1$$

and hence $\|A^ix_0\|=1$ for each $i=1,2,\ldots,k-1$. This, together with the assumed relations $\|A^kx_0\|=\|x_0\|=1$ proves (3). Finally, on noting that every row of $\Gamma_o(A,i-1)$ is also a row of $\Gamma_o(A,i)$, we have that $\mathcal{N}_r[\Gamma_o(A,i)]\subseteq\mathcal{N}_r[\Gamma_o(A,i-1)]$ and (4) follows. □

Part (ii) of the following Proposition is based on [5] and gives an alternative simplified proof of the condition $\|A\|=\rho(A)$ relative to the original proof given in [26]. As discussed above, this condition is important in the distinction between different notions of strong stability and is related to the dimensionality of the maximal subspace of $\mathcal{R}^n$ on which the restriction of A^k defines an *isometry.*

***Proposition** 3.7:(i) Let $A \in \mathcal{R}^{n\times n}$ such that $\|A\| = 1$, and $x_0 \in \mathcal{N}_r[I_n-(A^t)^n A^n]$ such that $\|x_0\| = 1$. Then $A^i x_0 \in \mathcal{N}_r[I_n - A^t A]$ for every integer $i \geq 0$.*
Hence, if $\Gamma_c(A,n) \triangleq [x_0\ Ax_0\ \ldots\ A^{n-1}x_0]$, then $\mathcal{R}[\Gamma_c(A,n)]$ is a subspace of $\mathcal{N}_r[I_n - A^t A]$.
(ii) The restriction of the linear transformation A defined as:

$$A|_{\mathcal{R}[\Gamma_c(A,n)]} : \mathcal{R}^n \rightarrow \mathcal{R}[\Gamma_c(A,n)] \subseteq \mathcal{R}^n$$

is orthogonal and hence $\rho(A|_{\mathcal{R}[\Gamma_c(A,n)]}) = \rho(A) = 1$. In particular, $\|A^n\| = 1$ if and only if $\rho(A) = 1$.

Proof: (i) Let $x_0 \in \mathcal{N}_r[I_n - (A^t)^n A^n]$ with $\|x_0\| = 1$. Then $x_0 \in \mathcal{N}_r[\Gamma_o(A,n)]$ (from Proposition 3.6) and hence $WA^i x_0 = 0$, $i = 0,1,\ldots,n-1$ (recall that W is a square root of $I_n - A^t A$). Thus $W^t W A^{i-1} x_0 = (I_n - A^t A)A^i x_0 = 0$, $i = 0,1,2,\ldots,n-1$, or equivalently $A^i x_0 \in \mathcal{N}_r[I_n - A^t A]$ for each $i = 0,1,2,\ldots,n-1$. Since (using the Cayley-Hamilton theorem) every A^i, $i \geq 0$, can be expressed as a linear combination of the matrices $\{I_n, A, A^2, \ldots, A^{n-1}\}$, condition $A^i x_0 \in \mathcal{N}_r[I_n - A^t A]$ can be generalized for every $i \geq 0$. Thus each column of $\Gamma_c(A,n)$ is contained in $\mathcal{N}_r[I_n - A^t A]$ and hence $\mathcal{R}(\Gamma_c(A,n)) \subseteq \mathcal{N}_r[I_n - A^t A]$. (ii) Using Proposition 3.6, equation (3), we have $\|A(A^i x_0)\| = \|A^{i+1}x_0\| = \|x_0\| = 1$, $i = 0,1,\ldots,n-1$, and hence, the transformation under A of every generating vector of $\mathcal{R}[\Gamma_c(A,n)]$ is an isometry. This means that the map under A of any linear combination of the columns of $\Gamma_c(A,n)$ is also an isometry: Take an arbitrary linear combination $\Gamma_c(A,n)\theta$, $\theta \in \mathcal{R}^n$. Consider also the matrix

$$B = \Gamma_c^t(A,n)\Gamma_c(A,n) - \Gamma_c^t(A,n)A^t A\Gamma_c(A,n) = \Gamma_c^t(A,n)(I - A^t A)\Gamma_c(A,n)$$

The matrix B is symmetric and positive semi-definite since $\|A\| = 1$. Moreover, the (i,i)-th entry of B is:

$$B_{ii} = x_0^t (A^t)^{i-1}(I - A^t A)A^{i-1}x_0 = \|A^{i-1}x_0\|^2 - \|A^i x_0\|^2 = 0, \quad i = 1,2,\ldots,n$$

and hence $B = 0$. This implies that

$$\theta^t \Gamma_c^t(A,n)\Gamma_c(A,n)(I_n - A^t A)\Gamma_c(A,n)\theta = 0 \Rightarrow \|A\Gamma_c(A,n)\theta\| = \|\Gamma_c(A,n)\theta\|$$

as required. This means that the linear map defined by the restriction of A on $\mathcal{R}[\Gamma_c(A,n)]$ is orthogonal and hence all eigenvalues of $A|_{\mathcal{R}[\Gamma_c(A,n)]}$ have modulus equal to one. Thus $\rho(A) \geq 1$; however, since $\rho(A) \leq \|A\|$ and $\|A\| = 1$, it follows that $\rho(A) = 1$. The above argument shows that for any matrix $A \in \mathcal{R}^{n\times n}$ with $\|A\| = 1$, $\|A^n\| = 1 \Rightarrow \rho(A) = 1$. The reverse implication follows easily from the series of inequalities and equalities $1 = \rho(A) = \rho(A^n) \leq \|A^n\| \leq \|A\|^n = 1$ which implies that $\|A^n\| = 1$. $\square$

Table 3.1 below summarizes the necessary and sufficient conditions for each stability notion for the continuous and discrete-time case.

	Continuous-time: $\dot{x} = Ax$	Discrete-time: $x_{k+1} = Ax_k$
Lyapunov stability	$\mathrm{Re}(\lambda_i(A)) \leq 0$ for all i, simple Jordan structure for any $\lambda_i(A)$ on $j\omega$-axis	$\rho(A) \leq 1$, simple Jordan structure for any $\lambda_i(A)$ with $\lvert\lambda_i(A)\rvert = 1$
Asymptotic stability	$\mathrm{Re}(\lambda_i(A)) < 0$ for all i	$\rho(A) < 1$
Strong Lyapunov stability	$A + A^t \leq 0$	$\lVert A\rVert \leq 1$
Strong asymptotic stability (w.s.)	$A + A^t \leq 0$ and $\mathrm{Re}(\lambda_i(A)) < 0$, or $A + A^t \leq 0$ and $(A, A + A^t)$ obs.	$\lVert A\rVert \leq 1$ and $\rho(A) < 1$, or $\lVert A\rVert \leq 1$ and $(A, I - A^tA)$ obs.
Strong asymptotic stability (s.s.)	$A + A^t < 0$	$\lVert A\rVert < 1$

Table 1: Summary of stability conditions

We conclude the section by establishing a relation between strong asymptotic stability s.s. in the two domains (discrete and continuous-time). This result relies on standard properties of the bilinear transformation [22], and is potentially useful because it can be used to translate strong stability properties across the two domains.

Proposition 3.8: *Consider the autonomous LTI discrete and continuous systems $\Sigma_d(A)$ and $S_c(\hat{A})$, respectively, where $-1 \notin \lambda(A)$ and*

$$\hat{A} = (A - I)(A + I)^{-1}$$

Then,
(i) $\Sigma_d(A)$ is strongly asymptotically stable s.s. if and only if $S_c(\hat{A})$ is strongly asymptotically stable s.s..
(ii) $\Sigma_d(A)$ is strongly asymptotically stable w.s. if and only if $S_c(\hat{A})$ is strongly asymptotically stable w.s.; and
(iii) $\Sigma_d(A)$ is strongly Lyapunov stable if and only if $S_c(\hat{A})$ is strongly Lyapunov stable.

Proof: Part (i) follows from the following sequence of equivalent statements:

$$\begin{aligned}
S_c(\hat{A}) \text{ is strongly as. stable (s.s.)} &\Leftrightarrow (I - A)(I + A)^{-1} + (I + A^t)^{-1}(I - A^t) > 0 \\
&\Leftrightarrow \ (I + A^t)^{-1}\{(I - A^t)(I + A) \\
&\qquad + (I + A^t)(I - A)\}(I + A)^{-1} > 0 \\
&\Leftrightarrow \ (I + A^t)^{-1}\{2I - 2A^tA\}(I + A)^{-1} > 0 \\
&\Leftrightarrow \ A^tA < I \\
&\Leftrightarrow \ \lVert A\rVert < 1 \\
&\Leftrightarrow \ \Sigma_d(A) \text{ is strongly as. stable (s.s.)}
\end{aligned}$$

An almost identical sequence of arguments shows that:

$$(A - I)(I + A)^{-1} + (I + A^t)^{-1}(A^t - I) \leq 0 \Leftrightarrow \lVert A\rVert \leq 1 \tag{5}$$

proving part (iii). Finally, part (ii) follows from part (iii) and the fact that under the bilinear transformations the eigenvalues of A and $\hat{A}$ are related as:

$$\lambda_i(\hat{A}) = \frac{\lambda_i(A) - 1}{\lambda_i(A) + 1}, \ \ i = 1, 2, \ldots, n$$

Thus, for each $i = 1, 2, \ldots, n$,

$$\mathrm{Re}(\lambda_i(A)) < 0 \quad \Leftrightarrow \quad |\lambda_i(\hat{A})| < 1$$

and hence A is asymptotically stable if and only if $\hat{A}$ is Hurwitz. $\square$

4. Strong and Asymptotic Stability: Exact and approximate equivalence

In the previous section, two notions of "strong asymptotic stability" were introduced (w.s. and s.s.), each being a stronger notion than the classical notion of "asymptotic stability", and hence the set of systems which are strongly asymptotically stable (in either sense) is a strict subset of the set of all asymptotically stable systems. In this section we attempt to characterize the set of systems $\Sigma_d(A)$ for which the two notions are "equivalent" or "almost equivalent".

Remark 4.1: *Throughout this section and for the remaining parts of the paper we simplify our nomenclature by taking "strong stability" to mean "strong asymptotic stability in the strict sense (s.s)".*

It follows from Proposition 3.1 that the two notions of strong and asymptotic stability coincide precisely for those systems $\Sigma_d(A)$ for which $\rho(A) = \|A\|$, i.e. those systems for which the state-matrix is "radial" [7], [21]. References [7], [5], [26], [21] give various characterizations of the structure of radial matrices. We summarize the main results in the following Theorem:

Theorem 4.1 [7], [5], [26], [21]: *The matrix $A \in \mathcal{R}^{n\times n}$ is radial if and only if one of the following four equivalent conditions is satisfied:*

(i) *The matrix $\rho(A)^2 I_n - A^t A$ is positive semi-definite.*

(ii) *A is unitarily similar to a matrix of the form diag(Λ, B) where*

$$\Lambda = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_s \end{pmatrix}, \quad B = \begin{pmatrix} \lambda_{s+1} & & 0 \\ & \ddots & \\ (B_{ij}) & & \lambda_n \end{pmatrix}$$

in which the eigenvalues of A are ordered as

$$|\lambda_1| = |\lambda_2| = \ldots = |\lambda_s| > |\lambda_{s+1}| \geq \ldots \geq |\lambda_n|$$

and $\rho(A)^2 I_{n-s} - B^t B$ is positive semi-definite.

(iii) *$\|A^k\| = \|A\|^k$ for all integers $k \geq 1$.*

(iv) *There exists $\epsilon_R > 0$, such that for each $q \in \mathcal{R}$, the fact that $|q| < \epsilon_R$ implies that $\rho(A - qI_n) = \|A - qI\|$.*

Proof: For parts (i) and (ii) see [7]. Part (iii) follows from the following series of inequalities:

$$\rho(A)^k = \rho(A^k) \leq \|A^k\| \leq \|A\|^k, \quad k = 1, 2, \ldots$$

If A is radial all inequalities in the above expression must be equalities and hence $\|A^k\| = \|A\|^k$ for all $k > 0$. Conversely, if $\|A^k\| = \|A\|^k$ for all $k > 0$, we have:

$$\rho(A) = \lim_{k\to\infty} \|A^k\|^{1/k} = \|A\|$$

and hence A is radial. Actually, it can be shown that the condition given in this part can be simplified to $\|A^n\| = \|A\|^n$, where $A \in \mathcal{R}^{n\times n}$ (see previous section or Proposition 4.1 below). Finally, for part (iv), which shows that "radiality" is not a "pointwise" property, see [21]. □

As the analysis of the last section has shown, condition (iii) of Theorem 4.1 can be relaxed as follows:

Proposition 4.1: *$A \in \mathcal{R}^{n\times n}$ is a radial matrix if and only if $\|A^n\| = \|A\|^n$.*

Proof: The original proof of this result was given in [26] and subsequently simplified by [5]. See also Proposition 3.7 part (ii) for a similar proof based on [5]. □

Corollary 4.1: *If A is normal then it is also radial; hence in this case $\Sigma_d(A)$ is strongly stable if and only if $\Sigma_d(A)$ is asymptotically stable.*

Proof: Since A is normal, it is unitarily similar to a diagonal matrix (e.g. via its spectral decomposition) and hence A is radial (see Theorem 4.1 part(ii)). Thus $\Sigma_d(A)$ is strongly stable if and only if $\|A\| < 1$, or equivalently if and only if $\rho(A) < 1$ (i.e. if and only if A is Hurwitz). □

How closely related are the two sets of normal and radial matrices? It follows from Theorem 4.1 part (ii) that if $A \in \mathcal{R}^{n\times n}$ is radial and $s \geq n-1$ then A is normal (here s is the multiplicity of the eigenvalues of A with modulus equal to the spectral radius of A); in particular the two notions of "radiality" and "normality" are equivalent if $n = 2$ [7]. As $n - s$ increases, the class of normal matrices is much broader than the class of radial matrices. For a detailed discussion and examples, see [7].

Next, we investigate briefly the property of strong stability in terms of measures of eigen-frame skewness and departure from normality of the state matrix. More specifically, we investigate under what conditions the two notions of strong and asymptotic stability are "almost" or "approximately" equivalent.

Proposition 4.2: *Consider the system $\Sigma_d(A)$ and assume that A is diagonalisable so that $A = W\Lambda W^{-1}$ with $\Lambda = diag(\Lambda)$. Then a sufficient condition for strong stability of $\Sigma_d(A)$ is that $\kappa(W)\rho(A) < 1$, where $\kappa(W) = \|W\|\|W^{-1}\|$.*

Proof: $\Sigma_d(A)$ is strongly stable if and only if $\|A\| < 1$, or equivalently $\|W\Lambda W^{-1}\| < 1$. Since $\|W\Lambda W^{-1}\| \leq \|W\|\|W^{-1}\|\|\Lambda\|$ and $\|\Lambda\| = \rho(A)$ when Λ is diagonal, a sufficient condition for strong stability is $\kappa(W)\rho(A) < 1$ as claimed. □

Remark 4.2: If A is normal, $\kappa(W) = 1$ and the sufficient condition for strong stability given by Proposition 4.2 above reduced to $\rho(A) < 1$, i.e. asymptotic stability of $\Sigma_d(A)$. In this case, this is actually both a sufficient and necessary condition. Note also that if the eigen-frame of A is "almost orthogonal" (so that A is "approximately normal"), $\kappa(W) = 1 + \epsilon$ for some small $\epsilon > 0$, and hence strong stability of $\Sigma_d(A)$ is guaranteed if $\rho(A) < \frac{1}{1+\epsilon}$, which restricts the set of Hurwitz matrices only marginally.

Alternatively, perform a Schur transformation on the state-matrix of the form $A = UTU^*$, where U is unitary and T is upper-triangular. Under this transformation, A and T have the same strong stability properties (since the spectral norm is unitarily invariant). The diagonal elements of T are the eigenvalues of A and hence have modulus less than one, if A is asymptotically stable. Decompose $T = D + N$, where D is diagonal and N is strictly upper-triangular. In general, the decomposition $U^*AU = D + N$ is not unique, so let $\mathcal{S}$ represent the set of all such N. The non-normality of A can be measured by Henrici's departure from normality [10] in terms of an arbitrary matrix norm:

$$\delta(A, \|\cdot\|) := \delta(A) = \inf_{N \in \mathcal{S}} \|N\| \tag{6}$$

We can now obtain the following sufficient condition for strong stability:

Proposition 4.3: *Given $A \in \mathcal{R}^{n\times n}$, consider the Schur decomposition of A, $U^*AU = D + N$, where U is unitary, D is diagonal and N is strictly upper triangular and let $\delta(A, \|\cdot\|) = \delta_2(A)$ be defined as in equation (6) above, in which the indicated norm is chosen as the spectral norm. Then A is strongly stable if $\rho(A) < 1 - \delta_2(A)$.*

Proof: Since the spectral norm is unitarily invariant:

$$\|A\| = \|U^*AU\| = \|D + N\| \leq \|D\| + \|N\| = \rho(A) + \|N\|$$

Note that this applies for every Schur decomposition of A (parametrised by $N \in \mathcal{S}$), while $\|D\| = \rho(A)$ is independent of the choice of N. Taking the infimum of the right hand side of this inequality over $\mathcal{S}$ gives the required result. □

Remark 4.3: While any N derived from an arbitrary Schur decomposition may be used to derive a sufficient condition for strong stability, clearly the optimal choice above provides the sharpest bound, although it is not obvious how to calculate the minimum-norm N. This is in contrast to the Frobenius-norm case, where $\|N\|_F$ is independent of the particular Schur form [12] and

$$\delta_F(A) = \left(\|A\|_F^2 - \sum_i |\lambda_i|^2\right)^{\frac{1}{2}} \leq \left(\frac{n^3 - n}{12}\right)^{\frac{1}{4}} \|A^tA - AA^t\|_F^{\frac{1}{2}}.$$

It is also interesting to note that Henrichi's measure of departure from normality can be used to derive spectral norm bounds of the form [12]:

$$\begin{aligned} \|A^k\| &\leq \sum_{i=0}^{n-1} \binom{k}{i} \rho(A)^{k-i} \delta_2(A)^i, \quad && \rho(A) > 0 \\ &\leq \delta_2(A)^k, && \rho(A) = 0 \text{ and } k < n \end{aligned}$$

For additional issues related to transient response peak/energy characteristics see [13], [14], [29], [33].

In the last part of this section we investigate the effect of similarity transformations on the strong stability property. Since the eigenvalues (and spectral radius) of a matrix A are invariant under similarity transformation, so are the asymptotic stability properties of $\Sigma_d(A)$, i.e., for any non-singular matrix T the systems $\Sigma_d(A)$ and $\Sigma_d(TAT^{-1})$ have identical asymptotic stability properties. In contrast, the spectral norm is not invariant under a similarity transformation T, except from the special case where T is orthogonal. In conclusion we have the following result:

Proposition 4.4: *Strong stability is invariant under orthogonal state-space transformations, i.e. for an arbitrary orthogonal matrix U, $\Sigma_d(A)$ is strongly stable if and only if $\Sigma_d(UAU^t)$ is strongly stable.*

Proof: Follows from the fact that the spectral norm is unitarily invariant, i.e. $\|UAU^t\| = \|A\|$. □

It should be noted that strong stability only makes sense for physical system representations, i.e. representations in which the states represent physical variables, and hence the strong stability properties of a system are expected to vary under arbitrary coordinate transformations. In fact, as is shown in the next few paragraphs, if a system is asymptotically stable, there is always a state-space transformation defining a coordinate frame in which the system is strongly stable.

For $A \in \mathcal{C}^{n\times n}$ and any $p \in [1,\infty]$ we have [8]:

$$\rho(A) = \inf_{X\in\mathcal{C}^{n\times n},\, \det(X)\neq 0} \|XAX^{-1}\|_p$$

where $\|\cdot\|_p$ denotes the matrix norm induced by the l_p vector norm in $\mathcal{C}^n$. In the special case when $A \in \mathcal{R}^{n\times n}$ and $p = 2$ (but not otherwise, see [8]) we also have:

$$\rho(A) = \inf_{X\in\mathcal{R}^{n\times n},\, \det(X)\neq 0} \|XAX^{-1}\|_2 := \inf_{X\in\mathcal{R}^{n\times n},\, \det(X)\neq 0} \|XAX^{-1}\| \tag{7}$$

which implies the following Proposition:

Proposition 4.5: *Let $A \in \mathcal{R}^{n\times n}$. For each asymptotically stable autonomous LTI discrete-time system $\Sigma_d(A)$ there exists a (real) similarity transformation matrix X, such that the system $\Sigma_d(XAX^{-1})$ is strongly stable.*

Proof: Since A is asymptotically stable $\rho(A) < 1$, Thus equation (7) implies that there exists $X \in \mathcal{R}^{n\times n}$ such that $\|XAX^{-1}\| < 1$ and hence $\Sigma_d(XAX^{-1})$ is strongly stable. □

A specific similarity transformation X such that $\Sigma_d(XAX^{-1})$ is strongly stable is a balancing transformation [27], [28]. Assume that A is Hurwitz and define any two matrices B and C such that the system $\Sigma_d(A,B,C) : x_{k+1} = Ax_k + Bu_k,\ y_k = Cx_k$ is minimal. Then, there is always a state-space transformation X, such that

$$A \to XAX^{-1} = \hat{A},\ \ B \to XB := \hat{B},\ \ C \to CX^{-1} = \hat{C}$$

such that $\Sigma_d(\hat{A},\hat{B},\hat{C})$ is balanced, i.e. there exists a diagonal positive-definite matrix Σ which is the unique solution of the discrete Lyapunov equations:

$$A\Lambda A^t - \Lambda = -BB^t \quad \text{and} \quad A^t\Lambda A - \Lambda = -C^tC$$

It may be shown [27], [28] that $\|\hat{A}\| \leq 1$. Further, if Λ has distinct diagonal entries, then $\|\hat{A}\| < 1$, so that $\Sigma_d(\hat{A})$ is strongly stable. This condition can be enforced for almost every choice of B and C.

5. Application in Systems and Control: Strong stabilization under state and output feedback

In this section, as application of the previous sections, we consider strong stabilization problems under state feedback, output injection and output feedback. Recall that throughout the section strong stability is taken to mean strong asymptotic stability s.s..

The three static strong stabilisation problems under consideration are defined as follows:

- P.1 *State-feedback strong stabilization:* Given a matrix pair (A, B) with $A \in \mathcal{R}^{n\times n}$ and $B \in \mathcal{R}^{n\times m}$, find a state-feedback matrix $F \in \mathcal{R}^{m\times n}$ such that the matrix $A + BF$ is strongly stable.
- P.2 *Output injection strong stabilization:* Given a matrix pair (A, C) with $A \in \mathcal{R}^{n\times n}$ and $C \in \mathcal{R}^{p\times n}$, find an output injection matrix $H \in \mathcal{R}^{n\times p}$ such that the matrix $A + HC$ is strongly stable.
- P.3 *Output feedback strong stabilization:* Given a matrix triplet (A, B, C) with $A \in \mathcal{R}^{n\times n}$, $B \in \mathcal{R}^{n\times m}$ and $C \in \mathcal{R}^{p\times n}$ find an output feedback matrix $F \in \mathcal{R}^{m\times p}$ such that the matrix $A + BFC$ is strongly stable.

The main objective of the work is to establish necessary and sufficient conditions of strong stabilization (for each problem type) and parametrize the set of all strongly state-feedback (resp. output injection, output feedback) matrices.

Before presenting detailed solutions to these three static-feedback problems, it is first shown that dynamic output feedback does not offer any additional flexibility to strong stabilisation. We consider the feedback configuration shown in Figure 1, which is used for the study of dynamic stabilization problems. We make the following definition:

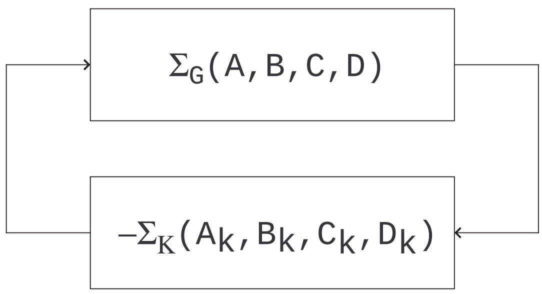

Figure 1: Feedback Configuration

Definition 5.1: *Given a system $\Sigma_G(A, B, C, D)$ and a dynamic compensator $\Sigma_K(A_k, B_k, C_k, D_k)$ in the feedback configuration of Figure 1, we say that Σ_K is a strong stabilizer of Σ_G if: (i) The feedback system is well-posed, i.e.* $\det(I + D\hat{D}) \neq 0$*, and (ii) the natural state-space realization of the closed-loop system (Σ_G, Σ_K) is strongly stable.* □

Remark 5.1: Note that strong stability of the feedback system (Σ_G, Σ_k) implies asymptotic stability and hence is an internal stability condition of the feedback system. □

The following result says that the static and dynamic strong output feedback stabilization problems are essentially equivalent. The assumption that the direct feed-through term of Σ_G is zero involves no loss of generality and can be easily removed, if required.

Proposition 5.1: *(i) The system $\Sigma_G(A, B, C, D)$ is strongly stabilizable by output dynamic feedback if and only if it strongly stabilizable by static output feedback. (ii) If $\Sigma_G(A, B, C, D)$ is strongly stabilizable by output static feedback, then it is also strongly stabilizable by a dynamic output feedback of arbitrary state dimension.*

Proof: Part (i): Necessity is obvious since the set of static controllers is a subset of the set of dynamic controllers. To prove sufficiency, assume that the dynamic controller $K(s)$ with state space realization:

$$\Sigma_K(A, B, C, D): \ \xi_{k+1} = \hat{A}\xi_k + \hat{B}u_k, \ \ u_k = -\hat{C}\xi_k - \hat{D}y_k$$

is a strong stabilizer of $\Sigma_G(A, B, C, D)$. Then the natural state-space realization of the closed-loop system is:

$$\begin{pmatrix} x_{k+1} \\ \xi_{k+1} \end{pmatrix} = \begin{pmatrix} A - B\hat{D}C & -B\hat{C} \\ \hat{B}C & \hat{A} \end{pmatrix} \begin{pmatrix} x_k \\ \xi_k \end{pmatrix} := A_c \begin{pmatrix} x_k \\ \xi_k \end{pmatrix}$$

Since by assumption Σ_K is a strong stabilizer, A_c is strongly stable, i.e. $\|A_c\| < 1$. This implies that $\|A - B\hat{D}C\| < 1$ and hence $-\hat{D}$ is a static strong stabilizer. For part (ii) note that if $\hat{D}$ exists such that $\|A - B\hat{D}C\| < 1$, then it is always possible to choose $\hat{A}$, $\hat{B}$ and $\hat{C}$ (of sufficiently small norms) so that $\|A_c\| < 1$. □

It is clear from the last proposition that strong stabilization is essentially a static feedback property and there is no need to consider dynamics. In the remaining parts of the section we turn our attention to the three static strong stabilization problems [P.1]-[P.3] defined above.

The solution of the state feedback problem is based on the theory of Linear Matrix Inequalities and is given next.

Proposition 5.2: *[30], [31] Let matrices $A \in \mathcal{R}^{n\times n}$, $B \in \mathcal{R}^{n\times m}$ and $C \in \mathcal{R}^{p\times n}$ be given and suppose that B has full column rank and that C has full row rank. Then the following two statements are equivalent:*

(i) There exists a matrix F such that $\|A + BFC\| < 1$ (i.e. $\Sigma_d(A, B, C)$ is strongly stabilizable under output feedback).

(ii) The following two conditions hold: $B^{\perp}(I - AA^t)B^{\perp t} > 0$ and $C^{t\perp}(I - A^tA)C^{t\perp t} > 0$.

If the above statements hold, then all matrices F such that $\|A + BFC\| < 1$ are given by:

$$F = -(B^t\Phi B)^{-1}B^t\Phi AC^t(CC^t)^{-1} + (B^t\Phi B)^{-1/2}L\Psi^{1/2}$$

where L is an arbitrary matrix such that $\|L\| < 1$ and

$$\Phi = (I - AA^t + AC^t(CC^t)^{-1}CA^t)^{-1}$$
$$\Psi = (CC^t)^{-1} - (CC^t)^{-1}CA^t(\Phi - \Phi B(B^t\Phi B)^{-1}B^t\Phi)AC^t(CC^t)^{-1}$$

We also have the following Corollary which applies to strong stabilization under state feedback and output injection (Clearly the two problems are dual of each other so solving the one will solve automatically the other).

Corollary 5.1: *Let matrices $A \in \mathcal{R}^{n\times n}$ and $B \in \mathcal{R}^{n\times m}$ be given and suppose that B has full column rank. Then the following two statements are equivalent:*

(i) There exists a matrix F such that $\|A+BF\| < 1$ (i.e. $\Sigma_d(A,B,C)$ is strongly stabilizable under state-feedback).

(ii) The following condition holds: $B^{\perp}(I - AA^t)B^{\perp t} > 0$.

If the above statements hold, then all matrices F satisfying $\|A+BF\| < 1$ are given by:

$$F = -(B^tB)^{-1}B^tA + (B^tB)^{-1/2}L\Psi^{1/2}$$

where L is an arbitrary matrix such that $\|L\| < 1$ and $\Psi = I - A^tA + A^tB(B^tB)^{-1}B^tA$.

Proof: Follows by specialising the result of Proposition 5.2 above. □

6. Conclusions

In this work three notions of "strong stability" have been defined for autonomous, linear, time-invariant, discrete-time state-space descriptions, which generalize parallel notions defined for continuous-time systems [9], [17], [18]. Necessary and sufficient conditions have been derived for each type of strong stability and the class of systems for which strong and asymptotic stability are equivalent notions have been identified. The invariance of the strong stability property under orthogonal transformations has been shown and links between the skewness of the eigen-frame of the state matrix and the violation of strong stability property have been obtained. Relations between strong stability in the discrete and continuous domains have been derived. Finally, the preservation or violation of strong stability has been studied under arbitrary coordinate transformations, balancing transformations and state/output feedback stabilizing transformations.

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