Stochastics: An International Journal of Probability and Stochastic Processes
Vol. 00, No. 00, January 2009, 1–27

RESEARCH ARTICLE

Finite Time Non-Ruin Probability for Erlang Claim Inter-arrivals and Continuous Inter-dependent Claim Amounts

Zvetan G. Ignatov[a] and Vladimir K. Kaishev[b]*

[a] *Faculty of Economics and Business Administration, Sofia University "St Kliment Ohridski", 125 Tsarigradsko Shosse Blv., bl.3, Sofia, 1113, Bulgaria*;
[b] *Faculty of Actuarial Science and Insurance, Cass Business School, City University, 106 Bunhill Row, EC1Y 8TZ London, UK*

(*Received 00 Month 200x; in final form 00 Month 200x*)

A closed form expression, in terms of some functions which we call exponential Appell polynomials, for the probability of non-ruin of an insurance company, in a finite-time interval is derived, assuming independent, non-identically Erlang distributed claim inter-arrival times, $\tau_i \sim \text{Erlang}(g_i, \lambda_i), i = 1, 2, \ldots$, any continuous joint distribution of the claim amounts and any non-negative, non-decreasing real function, representing its premium income. In the special case when $\tau_i \sim \text{Erlang}(g_i, \lambda), i = 1, 2, \ldots$ it is shown that our main result yields a formula for the probability of non-ruin expressed in terms of the classical Appell polynomials. We give another special case of our non-ruin probability formula for $\tau_i \sim \text{Erlang}(1, \lambda_i), i = 1, 2, \ldots$, i.e., when the inter-arrival times are non-identically exponentially distributed and also show that it coincides with the formula for Poisson claim arrivals, given in [18], when $\tau_i \sim \text{Erlang}(1, \lambda), i = 1, 2, \ldots$. The main result is extended further to a risk model in which inter-arrival times are dependent random variables, obtained by randomizing the Erlang shape or/and rate parameters. We give also some useful auxiliary results which characterize and express explicitly (and recurrently) the exponential Appell polynomials which appear in our finite time non-ruin probability formulae.

Keywords: finite-time (non-) ruin probability; risk process; Erlang claim inter-arrival times; dependent claim amounts; exponential Appell polynomials; divided difference;

AMS Subject Classification: Primary 60K30; Secondary 60K99;

1. Introduction

Evaluating the probability, $P(x)$, that the path of a stochastic process, S_t does not cross a curved boundary, $h(t)$, before time x, is known as the first crossing of a curved upper boundary problem. First crossing problems arise in insurance, finance, queuing and storage and have attracted a lot of attention in the corresponding research communities. In the context of risk theory, the process, $R_t = h(t) - S_t$, is known as the risk process of an insurance company, where S_t, models the arrival of consecutive claims up to time t, the deterministic function $h(t)$, represents the aggregate premium income up to time t and $P(x)$ is interpreted as the probability of survival (non-ruin) of the company within the finite time interval $[0, x]$, $x > 0$. In classical ruin theory, S_t is assumed a compound Poisson process and $h(t) = u + ct$, where $u > 0$ is the initial reserve of the company, and c is the positive premium rate.

*Corresponding author. Email: v.kaishev@city.ac.uk

ISSN: 1744-2508 print/ISSN 1744-2516 online

DOI: 10.1080/1744250YYxxxxxxxx
http://www.informaworld.com

Since the seminal paper [26] where the (classical) risk model was first considered, huge volume of applied probability literature has been devoted to various ruin-theoretic problems related to estimating ruin probabilities and first crossing time distributions under various definitions of the process S_t and the boundary $h(t)$. To mention only a few of the contributions in this strand of literature we refer to the papers [2, 7, 13, 27–29] and more recently, to [12, 19, 23, 33] and [32]. The reader is referred to the books [16] and [3] where more ruin probability results and references can be found.

Another stream of literature on ruin probability is devoted to the so called Sparre Andersen risk model in which claim amounts and the premium income are as in the classical case but the Poisson assumption for the claim arrivals is released, assuming that claim inter-arrival times are independent and identically distributed random variables with generic distribution F. Ruin probabilistic results for the special case $F \sim \text{Erlang}(2, \lambda)$ in the Sparre Andersen model have been obtained in [5, 6, 8–10, 25, 31] and in [24] and [14, 15], in the case when claim inter-arrival times have distribution $F \sim \text{Erlang}(n, \lambda)$. In the latter case, [11] derive expressions for the density of the time to ruin in the special case of independent identically exponentially distributed claim amounts. Some research has also been performed beyond the Sparre Andersen assumption of independence of the times between consecutive claim arrivals. Thus, risk models in which an appropriate dependence structure is imposed on the claim inter-arrival times and claim sizes, has been considered in [1], assuming the premium income function, $h(t) = u + ct$, and also in [4].

Despite the great attention which ruin probabilities have received, finding closed form expressions for $P(x)$ has in general proved a difficult task. Such expressions involving generalized Appell polynomials have been obtained in [29] in the case when, $h(t)$ is a non-decreasing premium income function, claims arrive according to a Poisson process and claim amounts are assumed integer valued, independent and identically distributed random variables. Closed form expressions for $P(x)$, involving classical Appell polynomials have been derived in [17, 18] and [20] in a more general risk model, assuming, any non-decreasing real-valued function $h(t)$, Poisson claim arrivals and any integer-valued or continuous joint distribution for the claim sizes, thus allowing them to be dependent.

In this paper, we consider a reasonably general risk model, in which claim inter-arrival times are assumed independent, non-identically Erlang distributed random variables with arbitrary shape and rate parameters, claim amounts may be dependent, with any continuous joint distribution and the premium income function $h(t)$ is any non-negative non-decreasing real function. Our main result is a closed form expression of the non ruin probability in terms of a new class of functions which we call *exponential Appell polynomials.* We extend further the generality of the risk model and incorporate dependence between consecutive claim inter-arrival times, by appropriately randomizing the Erlang shape, and/or rate parameters and give the ruin probability in this case as well.

The precise formulation of the risk model considered in the paper is as follows. The aggregate claim amount to the insurance company is modelled by the increasing pure-jump process

$$S_t = \sum_{i=1}^{N_t} W_i,$$

where $W_1, W_2, \ldots,$ is a sequence of positive random variables, representing the sizes of consecutive claims and N_t is a process, counting the number of claims up to time

t ($S_t = 0$ when $N_t = 0$). We will denote $N_t = \#\{i : \tau_1 + \ldots + \tau_i \leq t\}$, where $\#$ is the cardinality of the set $\{\cdot\}$ and $\tau_1, \tau_2, \ldots$ are the consecutive inter-arrival times of the claims. We will also assume that the sequence $W_1, W_2, \ldots$ is independent of $\tau_1, \tau_2, \ldots$. The random variables $W_1, W_2, \ldots$ may be dependent with any continuous joint distribution with joint probability density function, $\psi_{W_1,\ldots,W_k}(w_1, \ldots, w_k)$.

We further assume that the claim inter-arrival times τ_i, $i = 1, 2, \ldots$ defining the process, $N(t)$, are independent, (non-identically) Erlang (g_i, λ_i) distributed random variables with shape parameter, $g_i > 0$ and rate parameter, $\lambda_i > 0$, i.e. $\tau_i \sim$ Erlang (g_i, λ_i), with density

$$f_{\tau_i}(t) = \frac{\lambda_i^{g_i} t^{g_i-1} \mathbf{e}^{-\lambda_i t}}{\Gamma(g_i)},$$

where g_i, $i = 1, 2, \ldots$ is a sequence of arbitrary positive integers and λ_i, $i = 1, 2, \ldots$ is a sequence of (possibly coincident) positive real numbers. In other words, we assume that the inter-arrival times τ_i, $i = 1, 2, ...$ have Gamma distributions with (positive) integer shape parameters g_i and scale parameters λ_i, $i = 1, 2, \ldots$.

Consider an upper boundary given by the non-decreasing, non-negative, real valued function $h(t)$ on $[0,\infty)$ such that $\lim_{t\to\infty} h(t) = +\infty$. The function $h(t)$, modelling the premium income up to time t, may be continuous or not. If $h(t)$ is discontinuous it will be assumed that $h^{-1}(y) = \inf\{z : h(z) \geq y\}$. Define the insurance risk process

$$R_t = h(t) - S_t$$

and denote by

$$T := \inf\{t : t > 0, R_t < 0\},$$

the time of the first crossing of the trajectory $t| \to S_t$ and the boundary $t| \to h(t)$. Let us consider the finite time interval $[0, x]$, $x > 0$, and denote by $P(T > x)$ the probability that the trajectory $t| \to S_t$ will not cross the boundary $t| \to h(t)$ in time x.

In what follows we will give an explicit expression for the probability of non-ruin $P(T > x)$, up to time x, assuming that the parameters g_i and λ_i are such that $\sum_{i=1}^{\infty} \frac{g_i}{\lambda_i} = \infty$, which is a sufficient condition for $\sum_{i=1}^{n} \tau_i \underset{n\to\infty}{\overset{a.c.}{\to}} +\infty$. The latter condition, is required since otherwise ruin may occur with probability one. We show that the probability of non-ruin, $P(T > x)$, is expressed in terms of a sequence of functions, $B_k(x)$, $k = 0, 1, 2, \ldots$ which obey a specific system of linear differential equations. As established by Lemmas A.5 and A.6, $B_k(x)$ is an exponential Appell polynomial. The latter is a linear combination of exponentials multiplied by classical Appell polynomials. We will also consider non-ruin probabilities in a model with dependent claim inter-arrival times in which dependence is introduced by randomizing the parameters of the Erlang distributed claim inter-arrival times.

The structure of the paper is as follows. In Section 2.1 we derive our main result, the closed form expression (4), for the probability of non-ruin, $P(T > x)$, in a risk model with independent non-identically Erlang distributed claim inter-arrival times. In Section 2.2 we give explicit formulae for $P(T > x)$ in the following special cases: 1) when $\lambda = \lambda_1 = \lambda_2 = \ldots$ and , $g_i, i = 1, 2, \ldots$ are arbitrary positive integers, (see Corollary 2.3); 2) when $1 = g_1 = g_2 = \ldots$ and $\lambda_i, i = 1, 2, \ldots$ are arbitrary, pairwise distinct positive real numbers, i.e., when claim inter-arrival times are non-identically exponentially distributed (see Corollary 2.4), and also; 3) when g_i are arbitrary positive integers and λ_i, $i = 1, 2, \ldots$ are arbitrary, pairwise distinct (see Corollary 2.5 and Lemma A.6). It is shown that the non-ruin probability formula (1), given in [18], for the case of Poisson claim arrivals is a special case of formula (19), for $1 = g_1 = g_2 = \ldots$, and $\lambda = \lambda_1 = \lambda_2 = \ldots$.

In Section 3.2 we introduce a risk model in which the claim inter-arrival times $\tau_1, \tau_2, \ldots$, are dependent random variables, obtained by randomizing the Erlang shape parameters $g_1, g_2, \ldots$ or/and the Erlang rate parameters $\lambda_1, \lambda_2, \ldots$, assuming they are random variables with appropriate joint distributions. We point out that, based on our main result, ruin probability formulae for these various dependent models are easily derived. As an illustration we give a formula for $P(T > x)$, in the special case when $\lambda = \lambda_1 = \lambda_2 = \ldots$ and the Erlang shape parameters are modelled by a sequence of integer valued positive random variables $G_1, G_2, \ldots$. (see Corollary 2.6).

In the Appendix we give some useful lemmas which are used in proving the results in Section 2 and establish some recurrent expressions and other important properties of the exponential Appell polynomials, $B_k(x)$, $k = 0, 1, 2, \ldots$.

2. The probability of survival under Erlang (g_i, λ_i) claim arrivals

In this section we present our main result for $P(T > x)$ assuming Erlang (g_i, λ_i) distributed claim inter-arrival times. We consider also several special cases of different choices of the Erlang parameters g_i, and λ_i, including their randomization under which claim inter-arrival times become dependent.

2.1 Main result

In order to prove our main result we start with representing the Erlang distributed inter-arrival times as sums of independent identically exponentially distributed random variables. For the purpose, we will need some auxiliary variables and functions. Let the integer-valued function $j(k)$, $k = 0, 1, 2, \ldots$, be such that

$$g_1 + \ldots + g_{j(k)} \leq k < g_1 + \ldots + g_{j(k)} + g_{j(k)+1} \tag{1}$$

so that

k	0	1	...	$g_1 - 1$	g_1	...	$g_1 + g_2 - 1$	$g_1 + g_2$	...	$g_1 + g_2 + g_3 - 1$	$g_1 + g_2 + g_3$	...
$j(k)$	0	0	...	0	1	...	1	2	...	2	3	...

Let $\tilde{\tau}_1, \tilde{\tau}_2, \ldots$ be a sequence of independent, exponentially distributed random variables with parameters $\theta_1, \theta_2, \ldots$ correspondingly, i.e. $\tilde{\tau}_i \sim \text{Exp}(\theta_i)$, such that $\theta_{k+1} = \lambda_{j(k)+1}$, $k = 0, 1, 2, \ldots$ and

$$(\tilde{\tau}_1 + \ldots + \tilde{\tau}_{g_1}, \tilde{\tau}_{g_1+1} + \ldots + \tilde{\tau}_{g_1+g_2}, \ldots) \stackrel{d}{=} (\tau_1, \tau_2, \ldots) .$$

Obviously, in this more refined representation of the Erlang claim arrivals in terms of sums of exponentials we have that

$$\theta_1, \ldots\theta_{g_1}, \theta_{g_1+1}, \ldots\theta_{g_1+g_2}, \ldots \equiv \underbrace{\lambda_1, \ldots\lambda_1}_{g_1}, \underbrace{\lambda_2, \ldots\lambda_2}_{g_2}, \ldots \tag{2}$$

noting that the λ_i-s may possibly coincide. In the sequel it will be convenient to use the notation $\tilde{\tau}_1^*, \tilde{\tau}_2^*, \ldots$ for the r.v.s $\tilde{\tau}_1, \tilde{\tau}_2, \ldots$, in the case when $\theta_{k+1} = \lambda_{j(k)+1} \equiv 1$, $k = 0, 1, 2, \ldots$.

Denote by $T_1 = \tau_1, T_2 = \tau_1 + \tau_2, \ldots$, the moments of claim arrivals and introduce the sequence of random variables $\tilde{T}_1 = \tilde{\tau}_1$, $\tilde{T}_2 = \tilde{\tau}_1 + \tilde{\tau}_2, \ldots$. Obviously, we can also write $T_i = \tilde{T}_{g_1+\ldots+g_i}$ $i = 1, 2, \ldots$. Let us also consider the partial sums, Y_i, $i = 1, 2, \ldots$ of the consecutive claim amounts, $Y_1 = W_1, Y_2 = W_1 + W_2, \ldots$ with probability density function

$$f_{Y_1,\ldots,Y_i}(y_1, \ldots, y_i) = \begin{cases} \varphi(y_1, \ldots, y_i), & \text{if } 0 \le y_1 \le \ldots \le y_i \\ 0 & \text{otherwise} \end{cases}, \tag{3}$$

where $\varphi(y_1, \ldots, y_i) \ge 0$ for $0 \le y_1 \le \ldots \le y_i$ and

$$\int \cdots \int_{0 \le y_1 \le \ldots \le y_i} \varphi(y_1, \ldots, y_i)\, dy_1 \ldots dy_i = 1.$$

We will also denote by $F_{Y_1,\ldots,Y_i}(y_1, \ldots, y_i)$, the cdf of $Y_1, \ldots, Y_i$. For brevity we will alternatively write $F(y_1, \ldots, y_i)$.

It can easily be seen that the joint density $\psi_{W_1,\ldots,W_i}(w_1, \ldots, w_i)$ of the claim amount random variables $W_1, \ldots, W_i$ can be expressed as

$$f_{Y_1,\ldots,Y_i}(y_1, \ldots, y_i) = \psi_{W_1,\ldots,W_i}(y_1, y_2 - y_1, \ldots, y_i - y_{i-1}).$$

It will be convenient to formulate and prove our main ruin probability result first in terms of the density $f_{Y_1,\ldots,Y_i}(y_1, \ldots, y_i)$ and then to restate it, in Corollary 2.2, in terms of the claim amount random variables $W_1, \ldots, W_i$. We will also need to introduce the non-decreasing sequence of variables $\tilde{Y}_1, \tilde{Y}_2, \ldots$, independent of $\tilde{\tau}_1, \tilde{\tau}_2, \ldots$. and such that $0 = \tilde{Y}_1 = \ldots = \tilde{Y}_{g_1-1} \le Y_1 = \tilde{Y}_{g_1} = \ldots = \tilde{Y}_{g_1+g_2-1} \le Y_2 = \tilde{Y}_{g_1+g_2} = \ldots = \tilde{Y}_{g_1+g_2+g_3-1} \le \ldots$. Our main result is given by the following theorem.

Theorem 2.1: *The probability of survival within a finite time x*

$$P(T > x) = e^{-\lambda_1 x} + \sum_{k=1}^{\infty} \int \cdots \int_{0 \le y_1 \le \ldots \le y_{j(k)} \le h(x)} B_k(x) f\left(y_1, \ldots, y_{j(k)}\right) dy_{j(k)} \cdots dy_1, \tag{4}$$

where,

$$0 < s = k + 1 - \left(g_1 + \ldots + g_{j(k)}\right), \tag{5}$$

and

$$B_k(x) = \lambda_{j(k-1)+1} e^{-\lambda_{j(k)+1} x} \int_{h^{-1}(y_{j(k)})}^{x} e^{\lambda_{j(k)+1} z} B_{k-1}(z) dz, k = 1, 2, \ldots \tag{6}$$

with $B_0(x) = e^{-\lambda_1 x}$.

Remark 1: Above and in what follows, $B_k(x)$ is an abbreviation for

$$B_k\Bigg(x; \underbrace{h^{-1}(0),\ldots,h^{-1}(0)}_{g_1-1}, \underbrace{h^{-1}(y_1),\ldots,h^{-1}(y_1)}_{g_2},\ldots,$$
$$\underbrace{h^{-1}(y_{j(k)-1}),\ldots,h^{-1}(y_{j(k)-1})}_{g_{j(k)}}, \underbrace{h^{-1}(y_{j(k)}),\ldots,h^{-1}(y_{j(k)})}_{s}\Bigg)$$

which stems from Lemmas A.1, A.2 and Corollary A.3, noting that (6) coincides with (A13) for $\nu_k = h^{-1}(y_{j(k)}), k = 1, 2, \ldots$. It will be convenient to use the two notations interchangeably.

Remark 2: Let us note that, as established by Lemmas A.5 and A.6, the functions $B_k(x), k = 0, 1, 2, \ldots$, are exponential Appell polynomials. Their numerical evaluation is facilitated by the results of Lemmas A.4-A.6 (see also Corollary 2.4).

Proof: By construction, the event $T > x$ can be expressed as

$$\begin{aligned}\{T > x\} &= \bigcap_{i=1}^{\infty}\left[\left\{h^{-1}(Y_i) < T_i\right\} \cup \{x < T_i\}\right] \\ &= \bigcap_{i=1}^{\infty}\left[\left\{h^{-1}\left(\tilde{Y}_{g_1+\ldots+g_i}\right) < \tilde{T}_{g_1+\ldots+g_i}\right\} \cup \left\{x < \tilde{T}_{g_1+\ldots+g_i}\right\}\right] \qquad (7)\end{aligned}$$

For the i-th event in (7) we have

$$\begin{aligned}&\left\{h^{-1}\left(\tilde{Y}_{g_1+\ldots+g_i}\right) < \tilde{T}_{g_1+\ldots+g_i}\right\} \cup \left\{x < \tilde{T}_{g_1+\ldots+g_i}\right\} \\ &\qquad \subseteq \left\{h^{-1}\left(\tilde{Y}_{g_1+\ldots+g_i}\right) < \tilde{T}_{g_1+\ldots+g_i+r}\right\} \cup \left\{x < \tilde{T}_{g_1+\ldots+g_i+r}\right\}\end{aligned}$$

for $r = 0, 1, ..., g_{i+1} - 1$, which is equivalent to

$$\left\{h^{-1}\left(\tilde{Y}_{g_1+\ldots+g_i}\right) < \tilde{T}_{g_1+\ldots+g_i}\right\} \cup \left\{x < \tilde{T}_{g_1+\ldots+g_i}\right\} \subseteq \left\{h^{-1}\left(\tilde{Y}_l\right) < \tilde{T}_l\right\} \cup \left\{x < \tilde{T}_l\right\}$$

for $g_1 + ... + g_i \le l < g_1 + ... + g_{i+1}$. Therefore, for any $i = 1, 2, ...$

$$\begin{aligned}&\left\{h^{-1}\left(\tilde{Y}_{g_1+\ldots+g_i}\right) < \tilde{T}_{g_1+\ldots+g_i}\right\} \cup \left\{x < \tilde{T}_{g_1+\ldots+g_i}\right\} \\ &\qquad \subseteq \bigcap_{l=g_1+\ldots+g_i}^{g_1+\ldots+g_{i+1}-1}\left[\left\{h^{-1}\left(\tilde{Y}_l\right) < \tilde{T}_l\right\} \cup \left\{x < \tilde{T}_l\right\}\right] \qquad (8)\end{aligned}$$

In addition, for $1 \le l < g_1, (g_1 \ne 1)$, we also have

$$\begin{aligned}\left\{h^{-1}\left(\tilde{Y}_l\right) < \tilde{T}_l\right\} \cup \left\{x < \tilde{T}_l\right\} &= \left\{h^{-1}(0) < \tilde{T}_l\right\} \cup \left\{x < \tilde{T}_l\right\} \\ &= \left\{0 < \tilde{T}_l\right\} \cup \left\{x < \tilde{T}_l\right\} = \Omega\end{aligned}$$

and hence

$$\bigcap_{l=1}^{g_1-1}\left[\left\{h^{-1}\left(\tilde{Y}_l\right) < \tilde{T}_l\right\} \cup \left\{x < \tilde{T}_l\right\}\right] = \Omega \qquad (9)$$

where Ω is the sure event. Thus, from (7), (8) and (9) we obtain

$$\{T > x\} = \bigcap_{l=1}^{\infty} \left[\left\{ h^{-1}\left(\tilde{Y}_l\right) < \tilde{T}_l \right\} \cup \left\{ x < \tilde{T}_l \right\} \right]. \tag{10}$$

Let us consider the (complete) set of events $C_k = \left\{\tilde{T}_k \le x\right\} \cap \left\{\tilde{T}_{k+1} > x\right\}$, $k = 0, 1, 2, ...$, where $x > 0$ and $\tilde{T}_0 = 0$. For $k = 0$, we obviously have $\left\{\tilde{T}_0 \le x\right\} \cap \left\{\tilde{T}_1 > x\right\} \equiv \left\{\tilde{T}_1 > x\right\}$. Note that the events C_k, $k = 0, 1, ...$ are mutually exclusive and that $\cup_{k=0}^{\infty} C_k = \Omega$. Hence, from (10) we have

$$\begin{aligned} P(T > x) &= P\left(\bigcap_{l=1}^{\infty} \left[\left\{ h^{-1}\left(\tilde{Y}_l\right) < \tilde{T}_l \right\} \cup \left\{ x < \tilde{T}_l \right\} \right] \cap (\cup_{k=0}^{\infty} C_k) \right) \\ &= \sum_{k=0}^{\infty} P\left(\bigcap_{l=1}^{\infty} \left[\left\{ h^{-1}\left(\tilde{Y}_l\right) < \tilde{T}_l \right\} \cup \left\{ x < \tilde{T}_l \right\} \right] \cap C_k \right). \end{aligned} \tag{11}$$

The event in (11) can be expressed as

$$\begin{aligned} &\left(\bigcap_{l=1}^{k} \left[\left\{ h^{-1}\left(\tilde{Y}_l\right) < \tilde{T}_l \right\} \cup \left\{ x < \tilde{T}_l \right\} \right] \cap C_k \right) \\ &\qquad \cap \left(\bigcap_{l=k+1}^{\infty} \left[\left\{ h^{-1}\left(\tilde{Y}_l\right) < \tilde{T}_l \right\} \cup \left\{ x < \tilde{T}_l \right\} \right] \cap C_k \right) \\ &= \left(\bigcap_{l=1}^{k} \left[\left(\left\{ h^{-1}\left(\tilde{Y}_l\right) < \tilde{T}_l \right\} \cap C_k \right) \cup \left(\left\{ x < \tilde{T}_l \right\} \cap C_k \right) \right] \right) \\ &\qquad \cap \left(\bigcap_{l=k+1}^{\infty} \left[\left(\left\{ h^{-1}\left(\tilde{Y}_l\right) < \tilde{T}_l \right\} \cap C_k \right) \cup \left(\left\{ x < \tilde{T}_l \right\} \cap C_k \right) \right] \right) \end{aligned} \tag{12}$$

Now, taking into consideration the facts that $\left\{x < \tilde{T}_l\right\} \cap \left\{\tilde{T}_k \le x\right\} \equiv \emptyset$, for $l = 1, \dots, k$, that $\left\{x < \tilde{T}_l\right\} \cap C_k \subset C_k$, for $l = k+1, \dots, \infty$, since $\left\{x < \tilde{T}_l\right\} \supset \left\{\tilde{T}_{k+1} > x\right\}$, and also that $\left(\bigcap_{l=k+1}^{\infty} \left\{ h^{-1}\left(\tilde{Y}_l\right) < \tilde{T}_l \right\} \cap C_k \right) \cup C_k \equiv C_k$ we can rewrite (12) as

$$\begin{aligned} \left(\bigcap_{l=1}^{\infty} \left[\left\{ h^{-1}\left(\tilde{Y}_l\right) < \tilde{T}_l \right\} \cup \left\{ x < \tilde{T}_l \right\} \right] \right) \cap C_k &= \left(\bigcap_{l=1}^{k} \left[\left\{ h^{-1}\left(\tilde{Y}_l\right) < \tilde{T}_l \right\} \cap C_k \right] \right) \cap C_k \\ &= \left(\bigcap_{l=1}^{k} \left\{ h^{-1}\left(\tilde{Y}_l\right) < \tilde{T}_l \right\} \right) \cap C_k. \end{aligned} \tag{13}$$

In view of (13), (11) can now be rewritten as

$$
\begin{aligned}
P(T > x) &= \sum_{k=0}^{\infty} P\left(\bigcap_{l=1}^{k} \left\{ h^{-1}\left(\tilde{Y}_l\right) < \tilde{T}_l \right\} \cap C_k \right) \\
&= \sum_{k=0}^{\infty} P\left(\bigcap_{l=1}^{k} \left\{ h^{-1}\left(\tilde{Y}_l\right) < \tilde{T}_l \right\} \cap \left\{ \tilde{T}_k \le x \right\} \cap \left\{ \tilde{T}_{k+1} > x \right\} \right) \\
&= \sum_{k=0}^{\infty} P\left(\left\{ h^{-1}\left(\tilde{Y}_1\right) < \tilde{T}_1 \le x \right\} \cap \ldots \cap \left\{ h^{-1}\left(\tilde{Y}_k\right) < \tilde{T}_k \le x \right\} \cap \left\{ \tilde{T}_{k+1} > x \right\} \right) \\
&= \sum_{k=0}^{\infty} \mathbb{E}\left[\int_{h^{-1}(\tilde{Y}_1)}^{x} \cdots \int_{h^{-1}(\tilde{Y}_k)}^{x} \int_{x}^{\infty} f_{\tilde{T}_1,\ldots\tilde{T}_{k+1}}(t_1,\ldots,t_{k+1}) dt_{k+1} \ldots dt_1 \right], \qquad (14)
\end{aligned}
$$

where the expectation $\mathbb{E}[\cdot]$ is with respect to the random variables, $\tilde{Y}_1,\ldots,\tilde{Y}_k$ and $f_{\tilde{T}_1,\ldots,\tilde{T}_{k+1}}(t_1,\ldots,t_{k+1})$ is the joint density of $\tilde{T}_1,\ldots,\tilde{T}_{k+1}$. It can easily be seen that the random vector $\tilde{\boldsymbol{T}} = \left(\tilde{T}_1,\ldots,\tilde{T}_{k+1}\right)'$ coincides in distribution with the random vector $\boldsymbol{B}_{k+1}\tilde{\tau}^*$, where $\tilde{\tau}^* = \left(\tilde{\tau}_1^*,\ldots,\tilde{\tau}_{k+1}^*\right)'$, and $\boldsymbol{B}_{k+1}$ is a $(k+1)\times(k+1)$ dimensional matrix, i.e., $\boldsymbol{B}_{k+1}\tilde{\tau}^* \stackrel{d}{=} \tilde{\boldsymbol{T}}$. Recall that $s = k+1-g_1-\ldots-g_{j(k)}$ and it is not difficult to see that $1 \le s \le g_{j(k)+1}$. From the definition of s, we have that $k+1 = g_1+\ldots+g_{j(k)}+s$, which we will use frequently in the sequel. The matrix $\boldsymbol{B}_{k+1}$ is then given in a block-matrix form as

$$
\boldsymbol{B}_{k+1} \equiv \begin{pmatrix} \boldsymbol{b}_{1,1} & \cdots & \boldsymbol{b}_{1,j(k)} & \boldsymbol{b}_{1,j(k)+1} \\ \vdots & \ddots & \vdots & \vdots \\ \boldsymbol{b}_{j(k),1} & \cdots & \boldsymbol{b}_{j(k),j(k)} & \boldsymbol{b}_{j(k),j(k)+1} \\ \boldsymbol{b}_{j(k)+1,1} & \cdots & \boldsymbol{b}_{j(k)+1,j(k)} & \boldsymbol{b}_{j(k)+1,j(k)+1} \end{pmatrix},
$$

where $\boldsymbol{b}_{m,n}$ is a $g_m \times g_n$ matrix for $m,n = 1,\ldots,j(k)$, with all entries equal to $\frac{1}{\lambda_n}$ if $m > n$, all entries equal to zero if $m < n$, and where $\boldsymbol{b}_{n,n}$ is a lower triangular matrix with all elements in the lower triangle equal to $\frac{1}{\lambda_n}$ if $m = n$. The matrixes $\boldsymbol{b}_{j(k)+1,n}$ and $\boldsymbol{b}_{m,j(k)+1}$, $m,n = 1,\ldots,j(k)$, have dimensions correspondingly, $s \times g_n$ and $g_m \times s$. All the entries of $\boldsymbol{b}_{j(k)+1,n}$ $n = 1,\ldots,j(k)$, are equal to $\frac{1}{\lambda_n}$, whereas all the entries of $\boldsymbol{b}_{m,j(k)+1}$, $m = 1,\ldots,j(k)$ are zero. The matrix $\boldsymbol{b}_{j(k)+1,j(k)+1}$ is a lower triangular matrix of dimension $s \times s$ with all entries in the lower triangle equal to $\frac{1}{\lambda_{j(k)+1}}$. Then, it is not difficult to see that

$$
f_{\tilde{T}_1,\ldots,\tilde{T}_{k+1}}(t_1,\ldots,t_{k+1}) = \begin{cases} e^{-\mathbf{1}\cdot\boldsymbol{B}_{k+1}^{-1}\cdot\boldsymbol{t}} \left|\det \boldsymbol{B}_{k+1}^{-1}\right| & \text{if } 0 \le t_1 \le t_2 \le \ldots \le t_{k+1} \\ 0 & \text{otherwise...} \end{cases}, \qquad (15)
$$

where, $\mathbf{1} = \Big(\underbrace{1,\ldots,1}_{k+1}\Big)$, $\boldsymbol{t} = (t_1,\ldots,t_{k+1})'$, $()'$ stands for transposition, and $\det \boldsymbol{B}_{k+1}^{-1}$ denotes the determinant of the inverse of $\boldsymbol{B}_{k+1}$. It can also be directly verified that the inverse matrix, $\boldsymbol{B}_{k+1}^{-1}$, is an incomplete, lower triangular matrix, with non-zero elements only at the main and next lower diagonals, given as

$$\tilde{b}_{l,l} = \begin{pmatrix} \lambda_1 & \text{if} & 1 \le l \le g_1 \\ \lambda_2 & \text{if} & g_1+1 \le l \le g_1+g_2 \\ \vdots & \vdots & \vdots \\ \lambda_{j(k)+1} & \text{if} & g_1+\cdots+g_{j(k)}+1 \le l \le g_1+\cdots+g_{j(k)+1} \end{pmatrix},$$

$$\tilde{b}_{l+1,l} = \begin{pmatrix} -\lambda_1 & \text{if} & 1 \le l \le g_1-1 \\ -\lambda_2 & \text{if} & g_1 \le l \le g_1+g_2-1 \\ \vdots & \vdots & \vdots \\ -\lambda_{j(k)+1} & \text{if} & g_1+\cdots+g_{j(k)} \le l \le g_1+\cdots+g_{j(k)+1}-1 \end{pmatrix} \quad (16)$$

and with all other elements equal to zero. In view of (15) and (16), and taking the expectation, (14) becomes

$$P(T>x) = e^{-\lambda_1 x} + \sum_{k=1}^{\infty} \int \cdots \int_{0 \le \tilde{y}_1 \le \ldots \le \tilde{y}_k \le h(x)} \int_{h^{-1}(\tilde{y}_1)}^{x} \cdots \int_{h^{-1}(\tilde{y}_k)}^{x} \int_{x}^{+\infty} \lambda_1^{g_1} \ldots \lambda_j^{g_{j(k)}} \lambda_{j(k)+1}^{s}$$
$$\exp\left[-\left\{\lambda_1 t_{g_1} + \lambda_2 \left(t_{g_1+g_2} - t_{g_1}\right) + \ldots + \right.\right.$$
$$\left.\left.\lambda_{j(k)} \left(t_{g_1+\ldots+g_{j(k)}} - t_{g_1+\ldots+g_{j(k)-1}}\right) + \lambda_{j(k)+1} \left(t_{g_1+\ldots+g_{j(k)}+s} - t_{g_1+\ldots+g_{j(k)}}\right)\right\}\right]$$
$$dt_{k+1} \ldots dt_1 dF_{\tilde{Y}_1,\ldots,\tilde{Y}_k}\left(\tilde{y}_1,\ldots,\tilde{y}_k\right) \quad (17)$$

It can be seen that, the sequence of random variables $\tilde{Y}_1, \tilde{Y}_2, \ldots$ is independent of $\tilde{T}_1, \tilde{T}_2, \ldots$, and is non-decreasing, as required with respect to the random variables $Y_1, Y_2, \ldots$ in equality (2) of [18],that $dF_{\tilde{Y}_1,\ldots,\tilde{Y}_k}\left(\tilde{y}_1,\ldots,\tilde{y}_k\right) = dF_{Y_1,\ldots,Y_{j(k)}}\left(y_1,\ldots,y_{j(k)}\right)$ and hence, that (17) can be rewritten as

$$P(T>x) = e^{-\lambda_1 x} + \sum_{k=1}^{\infty} \int \cdots \int_{0 \le y_1 \le \ldots \le y_{j(k)} \le h(x)} \underbrace{\int_{h^{-1}(0)}^{t_2} \cdots \int_{h^{-1}(0)}^{t_{g_1}}}_{g_1-1} \underbrace{\int_{h^{-1}(y_1)}^{t_{g_1+1}} \cdots \int_{h^{-1}(y_1)}^{t_{g_1+g_2}}}_{g_2} \cdots$$
$$\underbrace{\int_{h^{-1}(y_{j(k)-1})}^{t_{g_1+\cdots+g_{j(k)-1}+1}} \cdots \int_{h^{-1}(y_{j(k)-1})}^{t_{g_1+\cdots+g_{j(k)}}}}_{g_{j(k)}} \underbrace{\int_{h^{-1}(y_{j(k)})}^{t_{g_1+\cdots+g_{j(k)}+1}} \cdots \int_{h^{-1}(y_{j(k)})}^{t_{g_1+\cdots+g_{j(k)}+s-1}}}_{s-1} \int_{h^{-1}(y_{j(k)})}^{x} \int_{x}^{+\infty}$$
$$\lambda_1^{g_1} \ldots \lambda_{j(k)}^{g_{j(k)}} \lambda_{j(k)+1}^{s} \exp\left[-\left\{\lambda_1 t_{g_1} + \lambda_2 \left(t_{g_1+g_2} - t_{g_1}\right) + \ldots + \lambda_{j(k)} \left(t_{g_1+\ldots+g_{j(k)}} - t_{g_1+\ldots+g_{j(k)-1}}\right)\right.\right.$$
$$\left.\left.+\lambda_{j(k)+1}\left(t_{g_1+\ldots+g_{j(k)}+s} - t_{g_1+\ldots+g_{j(k)}}\right)\right\}\right] dt_{k+1} \ldots dt_1 dF_{Y_1,\ldots,Y_{j(k)}}\left(y_1,\ldots,y_{j(k)}\right). \quad (18)$$

Results (4) and (6) now follow from (18), Lemmas A.1 and A.2 and Corollary A.3, noting that the multivariate integral in (18), with respect to the variables $t_1, t_2, \ldots t_{k+1}$, coincides with that in (A3) and (6) coincides with (A13) for $\nu_k = h^{-1}(y_{j(k)}), k = 1, 2, \ldots$. □

In Corollary 2.2, we give a useful restatement of our main result, in terms of the joint density, $\psi_{W_1,\ldots,W_k}\left(w_1,\ldots,w_k\right)$, of the individual claim amount random variables, $W_1,\ldots,W_k$, $k = 1, 2\ldots$, noting that $\psi_{W_1,\ldots,W_k}\left(w_1,\ldots,w_k\right) = f_{Y_1,\ldots,Y_k}\left(w_1, w_1+w_2,\ldots,w_1+\ldots+w_k\right)$.

Corollary 2.2: *The probability of survival within finite time x*

$$P(T > x) =$$

$$\sum_{r=0}^{g_1-1} e^{-\lambda_1 x}\lambda_1^r \frac{x^r}{r!} + \sum_{i=1}^{\infty} \sum_{l=g_1+\ldots+g_i}^{g_1+\ldots+g_{i+1}-1} \int\cdots\int_{0\le w_1+\ldots+w_i\le h(x)} B_l\Bigg(x; \underbrace{h^{-1}(0),\ldots,h^{-1}(0)}_{g_1-1},$$

$$\underbrace{h^{-1}(w_1),\ldots,h^{-1}(w_1)}_{g_2},\ldots,\underbrace{h^{-1}(w_1+\ldots+w_{i-1}),\ldots,h^{-1}(w_1+\ldots+w_{i-1})}_{g_i},$$

$$\underbrace{h^{-1}(w_1+\ldots+w_i),\ldots,h^{-1}(w_1+\ldots+w_i)}_{l-(g_1+\ldots+g_i)+1}\Bigg)\psi_{W_1,\ldots,W_i}(w_1,\ldots,w_i)\,dw_i\ldots dw_1.$$

2.2 Special cases

Let us now consider several corollaries of Theorem 2.1, for particular choices of the Erlang model parameters, g_i and λ_i, $i = 1, 2, \ldots$. In the special case when $\lambda_i = \lambda$, i.e., $\tau_i \sim \text{Erlang}(g_i, \lambda)$, we have

Corollary 2.3: *The probability of survival within finite time x*

$$P(T > x) = e^{-\lambda x}\Bigg(1 + \sum_{l=1}^{g_1-1}\lambda^l A_l\Bigg(x; \underbrace{h^{-1}(0),\ldots,h^{-1}(0)}_{l}\Bigg)$$

$$+\sum_{i=1}^{\infty}\sum_{l=g_1+\ldots+g_i}^{g_1+\ldots+g_{i+1}-1}\lambda^l\int_0^{h(x)}dy_1\int_{y_1}^{h(x)}dy_2\ldots\int_{y_{i-1}}^{h(x)}A_l\Bigg(x;\underbrace{h^{-1}(0),\ldots,h^{-1}(0)}_{g_1-1},$$

$$\underbrace{h^{-1}(y_1),\ldots,h^{-1}(y_1)}_{g_2},\ldots,\underbrace{h^{-1}(y_{i-1}),\ldots,h^{-1}(y_{i-1})}_{g_i},$$

$$\underbrace{h^{-1}(y_i),\ldots,h^{-1}(y_i)}_{l-(g_1+\ldots+g_i)+1}\Bigg) f(y_1,\ldots,y_i)\,dy_i\Bigg), \tag{19}$$

where $y_0 \equiv 0$, the first sum vanishes if $g_1 = 1$ and $A_l(x; v_1, ..., v_l)$, $l = 1, 2, ...$ are the classical Appell polynomials $A_l(x)$ of degree l with a coefficient in front of x^l equal to $1/l!$, defined by

$$\begin{aligned} A_0(x) &= 1 \\ A_l'(x) &= A_{l-1}(x) \\ A_l(v_l) &= 0, l = 1, 2, \ldots \end{aligned} \tag{20}$$

Proof: The result follows from Theorem 2.1, noting that the multiple integral with respect to $t_1,\ldots,t_{k+1}$, in (18) takes then the simpler form

$$\underbrace{\int_{h^{-1}(0)}^{t_2}\cdots\int_{h^{-1}(0)}^{t_{g_1}}}_{g_1-1}\underbrace{\int_{h^{-1}(y_1)}^{t_{g_1+1}}\cdots\int_{h^{-1}(y_1)}^{t_{g_1+g_2}}}_{g_2}\cdots\underbrace{\int_{h^{-1}(y_{j(k)-1})}^{t_{g_1+\ldots+g_{j(k)-1}+1}}\cdots\int_{h^{-1}(y_{j(k)-1})}^{t_{g_1+\ldots+g_{j(k)}}}}_{g_{j(k)}}$$

$$\underbrace{\int_{h^{-1}(y_{j(k)})}^{t_{g_1+\ldots+g_{j(k)}+1}}\cdots\int_{h^{-1}(y_{j(k)})}^{t_{g_1+\ldots+g_{j(k)}+s-1}}}_{s-1}\int_{h^{-1}(y_{j(k)})}^{x}\int_{x}^{+\infty}\lambda^{k+1}e^{-\lambda t_{k+1}}dt_{k+1}\ldots dt_1$$

$$=\lambda^k e^{-\lambda x}\underbrace{\int_{h^{-1}(0)}^{t_2}\cdots\int_{h^{-1}(0)}^{t_{g_1}}}_{g_1-1}\underbrace{\int_{h^{-1}(y_1)}^{t_{g_1+1}}\cdots\int_{h^{-1}(y_1)}^{t_{g_1+g_2}}}_{g_2}\cdots$$

$$\underbrace{\int_{h^{-1}(y_{j(k)-1})}^{t_{g_1+\ldots+g_{j(k)-1}+1}}\cdots\int_{h^{-1}(y_{j(k)-1})}^{t_{g_1+\ldots+g_{j(k)}}}}_{g_{j(k)}}\underbrace{\int_{h^{-1}(y_{j(k)})}^{t_{g_1+\ldots+g_{j(k)}+1}}\cdots\int_{h^{-1}(y_{j(k)})}^{t_{g_1+\ldots+g_{j(k)}+s-1}}}_{s-1}\int_{h^{-1}(y_{j(k)})}^{x}dt_k\ldots dt_1$$

$$=\lambda^k e^{-\lambda x}A_k\left(x;\underbrace{h^{-1}(0),...,h^{-1}(0)}_{g_1-1},\underbrace{h^{-1}(y_1),...,h^{-1}(y_1)}_{g_2},...,\underbrace{h^{-1}\left(y_{j(k)}\right),...,h^{-1}\left(y_{j(k)}\right)}_{s-1},h^{-1}\left(y_{j(k)}\right)\right),$$

where the $A_k(x)$'s in the right-hand side of the last equality are Appell polynomials defined as in (20). Hence,

$$P(T>x)=$$

$$e^{-\lambda x}+\sum_{k=1}^{\infty}\int\cdots\int_{0\le y_1\le\ldots\le y_{j(k)}\le h(x)}\lambda^k e^{-\lambda x}A_k\left(x;\underbrace{h^{-1}(0),...,h^{-1}(0)}_{g_1-1},\underbrace{h^{-1}(y_1),...,h^{-1}(y_1)}_{g_2},...,\right.$$

$$\left.\underbrace{h^{-1}\left(y_{j(k)}\right),...,h^{-1}\left(y_{j(k)}\right)}_{s}\right)f\left(y_1,...,y_{j(k)}\right)dy_{j(k)}\cdots dy_1,$$

which is directly seen to admit the form (19). □

Remark 3: It can be directly verified that in the special case when $g_i=1$, $i=1,2,\ldots$, formula (19) coincides with formula (1) given in [18] for the case of Poisson claim arrivals.

Let us now consider the special case in which $g_i=1$, $i=1,2,\ldots$ and $\lambda_i, i=1,2,\ldots$ are pairwise distinct real numbers. In the latter case it will be convenient for us to change notation for the parameters λ_i, and denote them by μ_i, $i=1,\ldots.$, i.e., $\tau_i\sim\mathrm{Exp}(\mu_i)$. From Theorem 2.1 and Lemma A.4, with $\nu_k=h^{-1}(y_k)$, $k=1,2,\ldots$. it follows that

Corollary 2.4: *The probability of non-ruin, $P(T>x)$, within finite time x is*

$$P(T>x)=e^{-\mu_1 x}+\sum_{k=1}^{\infty}\int\cdots\int_{0\le y_1\le\ldots\le y_k\le h(x)}B_k\left(x;h^{-1}(y_1),\ldots,h^{-1}(y_k)\right)$$

$$f\left(y_1,\ldots,y_k\right)dy_k\cdots dy_1,$$

where

$$B_k(x) = \sum_{i=1}^{k+1} c_i(k) e^{-\mu_i x} \ , \ k = 0, 1, 2, \ldots,$$

$c_i(k)$ *are appropriate constants, for which the following recurrence relations hold*

$$c_i(k+1) = \begin{cases} \frac{\mu_{k+1}}{\mu_{k+2}-\mu_i} c_i(k), & \text{for } i = 1, 2, ..., k+1 \\ -\sum_{r=1}^{k+1} e^{(\mu_{k+2}-\mu_r)h^{-1}(y_{k+1})} \frac{\mu_{k+1}}{\mu_{k+2}-\mu_r} c_r(k) & \text{for} i = k+2 \end{cases},$$

$k = 0, 1, 2, \ldots,$ *and* $c_1(0) = 1$.

Remark 4: Alternatively and explicitly, for $B_k(x)$, by induction, one has

$$B_k(x) = \mu_1 \ldots \mu_k \sum_{(r_1,\ldots,r_k)\in \underbrace{\{1,2\} \times \ldots \times \{1,2\}}_{k}} \left(\prod_{m=1}^{k} H\left(m, r_m\right) \right)$$

$$\exp\left\{ -x \left[\prod_{i=1}^{k} (2-r_i)\,\mu_1 + \sum_{j=1}^{k-1} \left[(r_j - 1) \prod_{i=j+1}^{k} (2 - r_i) \right] \mu_{j+1} + (r_k - 1)\,\mu_{k+1} \right] \right\},$$

where

$$H\left(m, r_m\right) = (2 - r_m) \frac{1}{\mu_{m+1} - a\left(m, r_1, \ldots, r_{m-1}\right)} + (r_m - 1) \frac{-\exp\left\{ \left(\mu_{m+1} - a\left(m, r_1, \ldots, r_{m-1}\right)\right) h^{-1}\left(y_m\right) \right\}}{\mu_{m+1} - a\left(m, r_1, \ldots, r_{m-1}\right)},$$

$$a\left(m, r_1, \ldots, r_{m-1}\right) = \left(\prod_{i=1}^{m-1} (2 - r_i) \right) \mu_1 + \sum_{j=1}^{m-1} \left((r_j - 1) \prod_{i=j+1}^{m-1} (2 - r_i) \right) \mu_{j+1},$$

and where $\sum_{\emptyset} = 0$, $\prod_{\emptyset} = 1$, *and* $\emptyset$ *is the empty set.*

Let us finally consider the reasonably general special case of the initially stated Erlang claim arrival model in which g_i, $i = 1, 2, ...$ are arbitrary positive integers and the Erlang rate parameters, $\lambda_i, i = 1, 2, ...$ are as in (2), but are assumed positive, pair-wise distinct real numbers. In this case, for the probability of survival within a finite time x, from Theorem 2.1, Corollary A.3 and Lemma A.6, we have

Corollary 2.5: *The probability of non-ruin,* $P(T > x)$, *within finite time* x *is*

$$P(T > x) = e^{-\lambda_1 x} + \sum_{k=1}^{\infty} \int \cdots \int_{0 \le y_1 \le \ldots \le y_{j(k)} \le h(x)} B_k\left(x; \nu_1, \ldots, \nu_k\right) f\left(y_1, \ldots, y_{j(k)}\right) dy_{j(k)} \cdots dy_1,$$

where, $B_k\left(x; \nu_1, \ldots, \nu_k\right)$, *coincides with the expression (A25), given by Lemma*

A.6, for

$$\begin{aligned}
&\nu_1 = \ldots = \nu_{g_1-1} = h^{-1}(0);\\
&\nu_{g_1} = \ldots = \nu_{g_1+g_2-1} = h^{-1}(y_1);\\
&\ldots\\
&\nu_{g_1+\ldots+g_{j(k)-1}} = \ldots = \nu_{g_1+\ldots+g_{j(k)}-1} = h^{-1}\left(y_{j(k)-1}\right);\\
&\nu_{g_1+\ldots+g_{j(k)}} = \ldots = \nu_{g_1+\ldots+g_{j(k)}+s-1} \equiv \nu_k = h^{-1}\left(y_{j(k)}\right),
\end{aligned}$$

with, $0 < s = k + 1 - \left(g_1 + \ldots + g_{j(k)}\right)$.

Expression (A25), of Lemma A.6, does not involve integration and facilitates the exact numerical computation of the functions, $B_k(x)$, and hence of the non-ruin probability, $P(T > x)$, given by Corollary 2.5. Expression (A25) is recurrent, with respect to the terms, $R(.)$ and involves divided differences of a simple power function and classical Appell polynomials, both of which can easily be computed recurrently. For properties of divided differences and their numerical evaluation, we refer to [30]. For an elegant recurrent expression for computing classical Appell polynomials see e.g., Lemma 4 of [17]. Further details of how the recurrence (A25) and also the non-ruin probability, $P(T > x)$, can be computed using the *Mathematica* system will appear elsewhere.

2.3 Dependent claim inter-arrival times.

Let us note here that our main result given by formula (4), can be generalized further to cover the case of dependent claim inter-arrival times. In view of the generality of formula (4), dependence can be introduced in various ways, in particular, by randomizing the set of shape parameters $g_1, g_2, ...$ or/and by randomizing the set of rate parameters $\lambda_1, \lambda_2, \ldots$In other words, we can assume that the inter-arrival times τ_i, $i = 1, 2, \ldots$ are either Erlang (G_i, λ_i), or Erlang (g_i, Λ_i), or Erlang (G_i, Λ_i) distributed, where $G_1, G_2, \ldots$.is a sequence of positive integer valued random variables with a sequence of joint probability mass functions

$$p_{g_1,\ldots,g_l} = P\left(G_1 = g_1, \ldots, G_l = g_l\right) \text{ for } g_1 \geq 1, \ldots, g_l \geq 1,\ l = 1, 2, \ldots,$$

and $\Lambda_1, \Lambda_2, \ldots$ are continuous (dependent) random variables with the sequence of marginal joint densities

$$f_{\Lambda_1,\ldots,\Lambda_l}\left(\lambda_1, \ldots \lambda_l\right) \text{ for } \lambda_1 > 0, \ldots, \lambda_l > 0,\ l = 1, 2, \ldots.$$

Clearly, in the case of Erlang (G_i, λ_i), the consecutive claims $W_1, W_2, ...$ arrive with inter-arrival times $\tau_1 = \tilde{\tau}_1 + \ldots + \tilde{\tau}_{G_1}$ and $\tau_2 = \tilde{\tau}_{G_1+1} + \ldots + \tilde{\tau}_{G_1+G_2}, \ldots$ which are dependent random variables. In particular, one can see that τ_1, τ_2 are dependent with covariance

$$\text{Cov}\left(\tau_1, \tau_2\right) = \text{Cov}\left(G_1, G_2\right)/\lambda_1\lambda_2$$

and correlation

$$\begin{aligned}
\text{Corr}\left(\tau_1, \tau_2\right) &= \text{Cov}\left(G_1, G_2\right)/\left(\sqrt{\text{Var}\left(G_1\right) + E\left(G_1\right)}\sqrt{\text{Var}\left(G_2\right) + E\left(G_2\right)}\right)\\
&\leq \text{Corr}\left(G_1, G_2\right).
\end{aligned}$$

In the case of Erlang (g_i, Λ_i) claim inter-arrival times $\tau_1 = \tilde{\tau}_1 + \ldots + \tilde{\tau}_{g_1}$, $\tau_2 = \tilde{\tau}_{g_1+1} + \ldots + \tilde{\tau}_{g_1+g_2}, \ldots$, it can be seen that $\tau_1, \tau_2, \ldots$ are dependent random variables as well. In particular, it is easy to establish that τ_1, τ_2 are dependent with covariance

$$\mathrm{Cov}\,(\tau_1, \tau_2) = g_1 g_2 \mathrm{Cov}\left(\Lambda_1^{-1}, \Lambda_2^{-1}\right)$$

and correlation

$$\mathrm{Corr}\,(\tau_1, \tau_2) = \sqrt{g_1 g_2}\,\mathrm{Corr}\left(\Lambda_1^{-1}, \Lambda_2^{-1}\right).$$

In principle, a large class of multivariate discrete distributions can be used to introduce dependence in our risk model through, e.g. the Dirichlet-compound multinomial distribution (see [21], p.80), the multivariate logarithmic series distribution (see [21], p.158), and the multivariate Pólya-Eggenberger distributions (see [21], p.200), subject to appropriate 'zeros-truncation' (as described in [21], p.21). As an example we will give the 'zeros-truncated' multinomial distribution ($\mathrm{MD_{ZT}}$) of [20]. The joint probability mass function of the $\mathrm{MD_{ZT}}$ distribution with parameters m and d_1, ..., d_l is defined as

$$P\left(G_1 = g_1, \ldots, G_l = g_l\right) = \frac{m!}{(g_1-1)! \ldots (g_l-1)!\,(m+l-g_1-\ldots-g_l)!} d_1^{g_1-1} \ldots d_l^{g_l-1} \left(1-d_1-\ldots-d_l\right)^{m+l-g_1-\ldots-g_l}$$

for $g_i \geq 1$, $i = 1, 2, \ldots, l$, $l = 1, 2, \ldots$, positive integers, $g_1 + \ldots + g_l \leq m + l$ and $P\left(G_1 = g_1, \ldots, G_l = g_l\right) = 0$ otherwise, where $m \geq 1$ is a positive integer and $d_i \in \mathbb{R}_+$, $i = 1, \ldots, l$ are such that $d_1 + \ldots + d_l < 1$, $l = 1, 2, \ldots$. In the case of Erlang (g_i, Λ_i), claim inter-arrivals, there is also abundance of joint distributions for the random variables $\Lambda_1, \Lambda_2, \ldots$ to choose from (see [22]) . It is worth noting that various copula models, can also be used to construct the dependent joint distribution of $\Lambda_1, \Lambda_2, \ldots$. It is not difficult to see that formulae for $P(T > x)$, for the models of possibly dependent claim inter-arrival times, introduced in this section, can be easily obtained applying the formula of total probability, with respect to the set $G_1, G_2, \ldots$ or/and to the set $\Lambda_1, \Lambda_2, \ldots$. To illustrate this, next, we give a straightforward generalization of formula (19), assuming that the inter-arrival times τ_i, $i = 1, 2, \ldots$ have Erlang (G_i, λ) distribution, where $\lambda = \lambda_1 = \lambda_2 = \ldots$.

Corollary 2.6: *The probability of survival within a finite time x under the assumption that consecutive claims $W_1, W_2, \ldots$ arrive with inter-arrival times $\tau_1 = \tilde{\tau}_1 + \ldots + \tilde{\tau}_{G_1}$, $\tau_2 = \tilde{\tau}_{G_1+1} + \ldots + \tilde{\tau}_{G_1+G_2}, \ldots$, is given as*

$$P(T > x) = e^{-\lambda x}\left(1 + \sum_{k=1}^{\infty} \sum_{r=1}^{k} \sum_{(g_1,\ldots,g_{r+1}) \in \mathcal{G}_k(r)} p_{g_1,\ldots,g_{r+1}} D_k\left(x, h(x), g_1, \ldots, g_r\right)\right),$$

where $\mathcal{G}_k(r) = \{(g_1, \ldots, g_{r+1}) : g_1 + \ldots + g_r \leq k < g_1 + \ldots + g_r + g_{r+1}\}$, *and*

$$D_k\left(x, h(x), g_1, \ldots, g_r\right) = \lambda^k \int_{0\leq y_1\leq\ldots\leq y_r\leq h(x)} \cdots \int A_k\Bigg(x; \underbrace{h^{-1}(0), \ldots, h^{-1}(0)}_{g_1-1}, \underbrace{h^{-1}(y_1), \ldots, h^{-1}(y_1)}_{g_2},$$

$$\ldots, \underbrace{h^{-1}(y_{r-1}), \ldots, h^{-1}(y_{r-1})}_{g_r}, \underbrace{h^{-1}(y_r), \ldots, h^{-1}(y_r)}_{s}\Bigg) dF\left(y_1, \ldots, y_r\right),$$

with $s = k + 1 - (g_1 + \ldots + g_r)$, *so that* $1 \leq s \leq g_{r+1}$.

Acknowledgements

The authors would like to thank Dimitrina Dimitrova for carefully reading previous versions of the paper and suggesting various notational improvements and corrections. Early versions of this work have been presented at the 12th International Congress on Insurance: Mathematics and Economics 2008, the International Workshop on Applied Probability (IWAP) 2008 and the 2b) or not 2b) 2009 Conference in honor of Professor Hans U. Gerber. The authors thank the participants of these conferences for useful feedback. The authors also thank the editor and the associate editor for their comments, and the anonymous referee for her/his thorough review and constructive suggestions, which helped to improve the paper.

Appendix A.

In what follows, we are going to prove five auxiliary lemmas, which introduce a class of exponential polynomials and establish some of their important properties, needed in order to prove our main non-ruin probability formula, given by Theorem 2.1, and facilitate its numerical evaluation.

Lemma A.1: *Let* $g_1, g_2, \ldots$ *be a sequence of positive integers and let* $\lambda_1, \lambda_2, \ldots$ *be a sequence of positive real numbers. Based on these two sequences, let us define the sequence of functions* $B_0(x), B_1(x), \ldots, B_k(x)$, *for* $x > 0$ *as*

$$B_0(x) = e^{-\lambda_1 x} \tag{A1}$$

$$B_1(x) = \begin{cases} \frac{\lambda_1}{\lambda_2-\lambda_1} e^{-\lambda_1 x} - \frac{\lambda_1}{\lambda_2-\lambda_1} e^{(\lambda_2-\lambda_1)h^{-1}(y_1)} e^{-\lambda_2 x}, & \text{for } g_1 = 1 \text{ and } \lambda_1 \neq \lambda_2 \\ \lambda_1 x e^{-\lambda_1 x} & \text{for} \{g_1 = 1 \text{ and } \lambda_1 = \lambda_2\} \text{ or } g_1 \geq 2 \end{cases} \tag{A2}$$

and for $k = 2, 3, \ldots,$ *as*

$$B_k(x) = \underbrace{\int_{h^{-1}(0)}^{t_2} \cdots \int_{h^{-1}(0)}^{t_{g_1}}}_{g_1-1} \underbrace{\int_{h^{-1}(y_1)}^{t_{g_1+1}} \cdots \int_{h^{-1}(y_1)}^{t_{g_1+g_2}}}_{g_2} \cdots \underbrace{\int_{h^{-1}(y_{j(k)-1})}^{t_{g_1+\ldots+g_{j(k)-1}+1}} \cdots \int_{h^{-1}(y_{j(k)-1})}^{t_{g_1+\ldots+g_{j(k)}}}}_{g_{j(k)}}$$

$$\underbrace{\int_{h^{-1}(y_{j(k)})}^{t_{g_1+\ldots+g_{j(k)}+1}} \cdots \int_{h^{-1}(y_{j(k)})}^{t_{g_1+\ldots+g_{j(k)}+s-1}}}_{s-1} \int_{h^{-1}(y_{j(k)})}^{x} \int_{x}^{+\infty} \lambda_1^{g_1} \ldots \lambda_j^{g_{j(k)}} \lambda_{j(k)+1}^{s} \exp\Big[$$

$$- \left\{ \lambda_1 t_{g_1} + \lambda_2 \left(t_{g_1+g_2} - t_{g_1}\right) + \ldots + \lambda_{j(k)} \left(t_{g_1+\ldots+g_{j(k)}} - t_{g_1+\ldots+g_{j(k)-1}}\right) \right.$$
$$\left. + \lambda_{j(k)+1} \left(t_{g_1+\ldots+g_{j(k)}+s} - t_{g_1+\ldots+g_{j(k)}}\right) \right\} \Big] dt_{k+1} \ldots dt_1, \quad \text{(A3)}$$

where the index function, $j(k)$*, is defined in (1),* $0 < s = k - \left(g_1 + \ldots + g_{j(k)}\right) + 1$ *(as given in (5)) and* $y_1, y_2, \ldots$ *are realizations of the random variables* $Y_1, Y_2, \ldots,$ *defined in (3) (see Section 2.1). This sequence of functions obeys the following system of linear differential equations*

$$\begin{aligned} B_0'(x) &= -\lambda_1 e^{-\lambda_1 x} \\ B_k'(x) &= -\lambda_{j(k)+1} B_k(x) + \lambda_{j(k-1)+1} B_{k-1}(x), \quad \textit{for } k = 1, 2, \ldots, \end{aligned} \quad \text{(A4)}$$

with initial conditions

$$\begin{aligned} &B_0(0) = 1, B_1\left(h^{-1}(0)\right) = 0, \ldots, B_{g_1-1}\left(h^{-1}(0)\right) = 0, \\ &B_{g_1}\left(h^{-1}(y_1)\right) = 0, B_k\left(h^{-1}\left(y_{j(k)}\right)\right) = 0, \ldots, \ \textit{for } k = g_1+1, g_1+2, \ldots. \end{aligned} \quad \text{(A5)}$$

Proof: By means of direct differentiation it can be verified that the functions $B_0(x)$ and $B_1(x)$ for $g_1 = 1$, given by (A1) and (A2) respectively, obey the differential equations

$$\begin{aligned} B_0'(x) &= -\lambda_1 e^{-\lambda_1 x} \\ B_1'(x) &= -\lambda_2 B_1(x) + \lambda_1 B_0(x), \end{aligned}$$

and for $g_1 \geq 2$, obey the equations

$$\begin{aligned} B_0'(x) &= -\lambda_1 e^{-\lambda_1 x} \\ B_1'(x) &= -\lambda_1 B_1(x) + \lambda_1 B_0(x). \end{aligned}$$

The function, $B_k(x)$, defined in (A3), is a well defined $(k+1)-$variate integral, for $g_1 \geq 2$, with limits of integration depending on the variable x. It will be convenient for us to change the notation for the parameters λ_i, and denote them by μ_i, $i = 1, \ldots.$in the case when λ_i, $i = 1, 2, \ldots$ are pairwise-distinct and $1 = g_1 = g_2 = \ldots$ In this case, consider $B_k(x)$ for any fixed $k \geq 2$. We have

$$\begin{aligned}
B_k(x) &= \int_{h^{-1}(y_1)}^{t_2} dt_1 ... \int_{h^{-1}(y_{k-1})}^{t_k} dt_{k-1} \int_{h^{-1}(y_k)}^{x} dt_k \int_{x}^{+\infty} dt_{k+1} \left(\mu_1 ... \mu_{k+1}\right. \\
&\quad \left.\exp\left[-\left\{\mu_1 t_1 + \mu_2 (t_2 - t_1) + ... + \mu_k (t_k - t_{k-1}) + \mu_{k+1}(t_{k+1} - t_k)\right\}\right]\right) \\
&= \int_{h^{-1}(y_1)}^{t_2} dt_1 ... \int_{h^{-1}(y_{k-1})}^{t_k} dt_{k-1} \int_{h^{-1}(y_k)}^{x} dt_k \int_{x}^{+\infty} dt_{k+1} \\
&\quad \left(\mu_{k+1} e^{-\mu_{k+1} t_{k+1}} \mu_1 ... \mu_k \exp\left[-\left\{\mu_1 t_1 + \mu_2 (t_2 - t_1) + ... + \mu_k (t_k - t_{k-1}) - \mu_{k+1} t_k\right\}\right]\right) \\
&= e^{-\mu_{k+1} x} \int_{h^{-1}(y_1)}^{t_2} dt_1 ... \int_{h^{-1}(y_{k-1})}^{t_k} dt_{k-1} \int_{h^{-1}(y_k)}^{x} dt_k \\
&\quad \left(\mu_1 ... \mu_k \exp\left[-\left\{\mu_1 t_1 + \mu_2 (t_2 - t_1) + ... + \mu_k (t_k - t_{k-1}) - \mu_{k+1} t_k\right\}\right]\right) \qquad \text{(A6)}
\end{aligned}$$

from where, denoting the multiple integral on the right-hand side by $I_k(x)$, we have

$$B_k(x) = e^{-\mu_{k+1} x} I_k(x) \ . \qquad \text{(A7)}$$

One sees that the derivative of $I_k(x)$ is given by

$$\begin{aligned}
\frac{dI_k(x)}{dx} &= \int_{h^{-1}(y_1)}^{t_2} dt_1 ... \int_{h^{-1}(y_{k-2})}^{t_{k-1}} dt_{k-2} \int_{h^{-1}(y_{k-1})}^{x} dt_{k-1} \\
&\quad \left(\mu_1 ... \mu_k \exp\left[-\left\{\mu_1 t_1 + \mu_2 (t_2 - t_1) + ... + \mu_{k-1} (t_{k-1} - t_{k-2}) + \mu_k (x - t_{k-1}) - \mu_{k+1} x\right\}\right]\right) \\
&= \mu_k e^{(\mu_{k+1} - \mu_k) x} I_{k-1}(x). \qquad \text{(A8)}
\end{aligned}$$

Differentiating both sides of (A7) and using (A8), we obtain

$$\begin{aligned}
\frac{dB_k(x)}{dx} &= -\mu_{k+1} e^{-\mu_{k+1} x} I_k(x) + \mu_k e^{-\mu_{k+1} x} e^{\mu_{k+1} x - \mu_k x} I_{k-1}(x) \\
&= -\mu_{k+1} B_k(x) + \mu_k B_{k-1}(x). \qquad \text{(A9)}
\end{aligned}$$

Let us now consider the general case in which, $g_1, g_2, ...$ is an arbitrary sequence of positive integers and λ_i, $i = 1, 2, ...$ is a sequence of positive, possibly coincident, real-valued intensities. In order to consider this general case, we need to pass to the limit in the integrand function in (A6), with respect to the parameters μ_i, $i = 1, 2, ...$, as $\mu_1 \to \lambda_1, ..., \mu_{g_1} \to \lambda_1$, $\mu_{g_1+1} \to \lambda_2, \ldots, \mu_{g_1+g_2} \to \lambda_2, \ldots, \mu_{g_1+\ldots+g_{j(k)}} \to \lambda_{j(k)}, \ldots, \mu_{k+1} \to \lambda_{j(k)+1}$. It is not difficult to establish that the latter limit exists and the limit of the integral in (A6) exists as well. We have

$$\begin{aligned}
&\lim \mu_1 \ldots \mu_{k+1} e^{-\{\mu_1 t_1 + \mu_2 (t_2 - t_1) + \ldots + \mu_k (t_k - t_{k-1}) + \mu_{k+1}(t_{k+1} - t_k)\}} \\
&= \lambda_1^{g_1} \ldots \lambda_{j(k)}^{g_{j(k)}} \lambda_{j(k)+1}^{s} \exp\left[-\left\{\lambda_1 t_{g_1} + \lambda_2 \left(t_{g_1+g_2} - t_{g_1}\right) + \ldots\right.\right. \\
&+ \lambda_{j(k)} \left(t_{g_1+\ldots+g_{j(k)}} - t_{g_1+\ldots+g_{j(k)-1}}\right) \\
&\left.\left.+ \lambda_{j(k)+1} \left(t_{g_1+\ldots+g_{j(k)}+s} - t_{g_1+\ldots+g_{j(k)}}\right)\right\}\right] \qquad \text{(A10)}
\end{aligned}$$

where s is defined as in (5). Hence, in view of (A10), it can be seen that the integral in (A6), i.e., $B_k(x)$ also has a limit which admits the representation (A3) and obeys the system of equations (A4) which is established similarly, by passing to the limit in (A9), as $\mu_1 \to \lambda_1, ..., \mu_{g_1} \to \lambda_1$, $\mu_{g_1+1} \to \lambda_2, \ldots, \mu_{g_1+g_2} \to \lambda_2, \ldots, \mu_{g_1+\ldots+g_{j(k)}} \to$

$\lambda_{j(k)}, \ldots, \mu_{k+1} \to \lambda_{j(k)+1}$. The system of functions (A1), (A2) and (A3) obeys the initial conditions (A5) which follows since in this case the limits of integration with respect to t_k coincide and the multivariate integral in (A3) is zero. In the cases when $k = 0$ and $k = 1$ it is directly verified that the initial conditions hold. This completes the proof of Lemma A.1. □

We can now formulate the following lemma.

Lemma A.2: *The system of linear differential equations,*

$$\begin{aligned} B_0'(x) &= -\lambda_1 e^{-\lambda_1 x} \\ B_k'(x) &= -\lambda_{j(k)+1} B_k(x) + \lambda_{j(k-1)+1} B_{k-1}(x). \end{aligned} \tag{A11}$$

for $k = 1, 2, \ldots$ *with initial conditions*

$$B_0(0) = 1,\ B_k(\nu_k) = 0,\ k = 1, 2, ... \tag{A12}$$

has a unique solution, given by the following sequence of functions

$$B_k(x) = \lambda_{j(k-1)+1} e^{-\lambda_{j(k)+1} x} \int_{\nu_k}^{x} e^{\lambda_{j(k)+1} z} B_{k-1}(z) dz\ ,\ k = 1, 2, \ldots \tag{A13}$$

where $B_0(x) = e^{-\lambda_1 x}$ *and* ν_k, $k = 1, 2, \ldots$ *is a non-decreasing sequence of real numbers,* $0 \le \nu_1 \le \nu_2 \le \ldots$.

Proof: Let us differentiate with respect to x, each of the functions in the sequence $B_k(x)$, $k = 0, 1, 2, \ldots$, given by (A13). We have,

$$B_0'(x) = -\lambda_1 e^{-\lambda_1 x};$$

and

$$\frac{dB_k(x)}{dx} = -\lambda_{j(k)+1} B_k(x) + \lambda_{j(k-1)+1} B_{k-1}(x).$$

Moreover, $B_0(0) = 1$, and $B_k(\nu_k) = 0$ for $k = 1, 2, ...$ hence the asserted result holds true. □

Corollary A.3: *The sequence of functions* $B_k(x)$*, given by (A13) and the corresponding sequence* $B_k(x)$*, defined by (A1), (A2) and (A3) coincide.*

Proof: Denote, by $0 \le \nu_1 \le \nu_2 \le ...$, the sequence of real numbers

$$\underbrace{h^{-1}(0) \le \ldots \le h^{-1}(0)}_{g_1 - 1} \le \underbrace{h^{-1}(y_1) \le \ldots. \le h^{-1}(y_1)}_{g_2}, \ldots\ , \tag{A14}$$

correspondingly. The statement of Corollary A.3 follows from the uniqueness of the solution of the system (A11) with initial conditions (A12), Lemma A.2 and Lemma A.1, in which limits of integration are replaced according to (A14). □

Lemma A.4: *Let,* $1 = g_1 = g_2 = \ldots$ *and let the parameters* $\mu_1, \mu_2, \ldots$ *be pairwise distinct. Then*

$$B_k(x) = \sum_{i=1}^{k+1} c_i(k) e^{-\mu_i x}\ ,\ k = 0, 1, 2, \ldots \tag{A15}$$

where $c_i(k)$ are appropriate constants, for which the following recurrence relations hold

$$c_i(k+1) = \begin{cases} \frac{\mu_{k+1}}{\mu_{k+2}-\mu_i} c_i(k), & \textit{for } i = 1, 2, \dots, k+1 \\ -\sum_{r=1}^{k+1} e^{(\mu_{k+2}-\mu_r)\nu_{k+1}} \frac{\mu_{k+1}}{\mu_{k+2}-\mu_r} c_r(k) & \textit{fori} = k+2 \end{cases}, \quad \text{(A16)}$$

where $k = 0, 1, 2, \dots$ and $c_1(0) = 1$.

Proof: Using induction, we have $B_0(x) = e^{-\mu_1 x}$, hence $c_1(0) = 1$. Assume, (A15) holds for some $k \geq 1$. Then, from (A13) and the latter assumption we have

$$\begin{aligned} B_{k+1}(x) &= \mu_{k+1} e^{-\mu_{k+2} x} \int_{\nu_{k+1}}^{x} e^{\mu_{k+2} z} \sum_{i=1}^{k+1} c_i(k) e^{-\mu_i z} dz \\ &= \mu_{k+1} e^{-\mu_{k+2} x} \sum_{i=1}^{k+1} c_i(k) \left(\frac{\exp\left[(\mu_{k+2} - \mu_i)\, x\right]}{\mu_{k+2} - \mu_i} - \frac{\exp\left[(\mu_{k+2} - \mu_i)\, \nu_{k+1}\right]}{\mu_{k+2} - \mu_i} \right) \\ &= \sum_{i=1}^{k+1} e^{-\mu_i x} \frac{\mu_{k+1}}{\mu_{k+2} - \mu_i} c_i(k) - e^{-\mu_{k+2} x} \sum_{r=1}^{k+1} e^{(\mu_{k+2}-\mu_r)\nu_{k+1}} \frac{\mu_{k+1}}{\mu_{k+2} - \mu_r} c_r(k) \end{aligned} \quad \text{(A17)}$$

Comparing expression (A17) with (A15) for $k+1$, using (A16), we get the desired result. □

Let us now consider the problem of finding an expression for $B_k(x)$ in the general case of $\tau_i \sim \text{Erlang}\,(g_i, \lambda_i)$, where $g_1, g_2, \dots$is an arbitrary positive integer sequence and $\lambda_i, i = 1, 2, \dots$ is the sequence of possibly coincident positive, real intensities. One possible approach would be to proceed from the special case of $1 = g_1 = g_2 = \dots$ and μ_k, $k = 1, 2, \dots$ pairwise-distinct, by passing to the limit as $\mu_1 \to \lambda_1, ..., \mu_{g_1} \to \lambda_1$, $\mu_{g_1+1} \to \lambda_2, \dots, \mu_{g_1+g_2} \to \lambda_2, \dots$. However, we will follow a different approach, based on the decomposition of the Erlang random variables, τ_i as sums of Exponential random variables with parameters as in (2). Further more, we will consider an arbitrary sequence, $\theta_1, \theta_2, \dots$, for which (2) does not necessarily hold and will be aiming at identifying the distinct values within it. In this way, in $\theta_1, \theta_2, \dots$, we ignore the fact that the parameters g_i and λ_i are associated with one another through the Erlang claim inter-arrival model. Based on $\theta_1, \theta_2, \dots$, let us construct the sequence, $\mu_1, \mu_2, \dots$ of pairwise-distinct positive real numbers, according to the following rule: Set $\mu_1 \equiv \theta_1$, μ_2 to be the first number in the sequence $\theta_1, \theta_2, \dots$, which differs from μ_1, μ_3 to be the fist number in the sequence, $\theta_1, \theta_2, \dots$ which is different from μ_1 and μ_2 and so on. The sequence, obtained in this way may be either finite or infinite and we can use it to express the elements of the sequence, $\theta_1, \theta_2, \dots$ as

$$\theta_r = \mu_{i_r}, r = 1, 2, \dots, \quad \text{(A18)}$$

where $i_1 \equiv 1$, $i_r, r = 2, 3, \dots$ are some indexes for which the inequality $1 \leq i_r \leq r$, $r = 1, 2, \dots$ holds. Denote by $s(k+1)$ the number of pairwise-distinct indexes $i_1, i_2, \dots, i_k, i_{k+1}$. Obviously, $s(k+1) \leq k+1$. Based on (A18), let us introduce the index sets $m(k+1, j)$ as

$$m(k+1, j) = \{r - 1 : i_r = j,\ 1 \leq r \leq k+1\}, j = 1, 2, \dots, s(k+1)$$

for $k = 1, 2 \ldots$. Denote

$$l(k+1, j) = \#\left\{m(k+1, j)\right\}, j = 1, 2, \ldots, s(k+1)$$

for $k = 1, 2 \ldots$. We have $l(k+1, j) \geq 1$ and

$$\sum_{j=1}^{s(k+1)} l(k+1, j) = k+1$$

for $k = 1, 2 \ldots$. Denote

$$(m(k+1, j))_1 < \ldots < (m(k+1, j))_{l(k+1,j)}$$

the elements of the index set $m(k+1, j)$, ordered in a non-decreasing order. Finally, we define the set, $D(k, j, q)$, of ordered real values $d = (d_1, \ldots, d_q)$, as

$$D(k, j, q) =$$
$$\left\{\{d_1, \ldots, d_q\} : d_1 < \ldots < d_q; \{d_1, \ldots, d_q\} \subset \left\{\nu_{(m(k+1,j))_2}, \ldots, \nu_{(m(k+1,j))_{l(k+1,j)}}\right\}\right\}$$

where $q = 0, 1, 2, \ldots, l(k+1, j) - 1$ and $D(k, j, q)$ is empty when $l(k+1, j) = 1$ and/or $q = 0$.

The function $e^{-\mu x} A_q(x, d_1, \ldots, d_q)$, with parameters $\mu > 0$, $0 \leq d_1 \leq \cdots \leq d_q$ where $A_q(x, d_1, \ldots, d_q)$ is the classical Appell polynomial of degree q, defined as in (20), will be called exponential Appell monomial. A linear combination of such monomials with different parameters will be called exponential Appell polynomial. The following lemma establishes a characterization of the function $B_k(x)$ as an exponential Appell polynomial (see A20).

Lemma A.5: *The function,*

$$\tilde{B}_k(x) = \int_{\nu_1}^{t_2} \int_{\nu_2}^{t_3} \cdots \int_{\nu_{k-1}}^{t_k} \int_{\nu_k}^{x} \int_{x}^{+\infty} \mu_1 \mu_{i_2} \ldots \mu_{i_{k+1}}$$
$$\exp\left[-\left\{\mu_1 t_1 + \mu_{i_2}(t_2 - t_1) + \ldots + \mu_{i_k}(t_k - t_{k-1}) + \mu_{i_{k+1}}(t_{k+1} - t_k)\right\}\right]$$
$$dt_{k+1} \ldots dt_1, \qquad \text{(A19)}$$

constructed, using the first $k+1$ *terms,* $\theta_1 = \mu_1, \theta_2 = \mu_{i_2}, \ldots, \theta_{k+1} = \mu_{i_{k+1}}$*, of the sequence (A18) and the first* k *terms,* $\nu_1, \ldots, \nu_k$*, of a non-decreasing sequence of real numbers,* $0 \leq \nu_1 \leq \nu_1 \leq \ldots$*, can be expressed as*

$$\tilde{B}_k(x) = \sum_{j=1}^{s(k+1)} \sum_{q=0}^{l(k+1,j)-1} \sum_{d \in D(k,j,q)} c(k, j, q, d) e^{-\mu_j x} A_q(x, d)\ , \qquad \text{(A20)}$$

where $c(k, j, q, d)$ *are appropriate constants,* $A_q(x; d) \equiv A_q(x; d_1, \ldots, d_q)$ *is the classical Appell polynomial of degree,* q*, defined as in (20).*

Proof: We will apply induction with respect to the index k. For $k = 0$, obviously, from (A20) we see that $j = 1$, $q = 0$ and hence, $\tilde{B}_0(x) = e^{-\theta_1 x} = e^{-\mu_1 x}$, which is of the form as in (A20) with $c(0,1,0) \equiv 1$. For $k = 1$, and $\theta_1 = \mu_1, \theta_2 = \mu_2$ we have

$$\tilde{B}_1(x) = \frac{\mu_1}{\mu_2 - \mu_1} e^{-\mu_1 x} - \frac{\mu_1}{\mu_2 - \mu_1} e^{(\mu_2 - \mu_1)\nu_1} e^{-\mu_2 x},$$

and in this case

$$c(1,1,0) = \frac{\mu_1}{\mu_2 - \mu_1}, \text{ and } c(1,2,0) = \frac{-\mu_1}{\mu_2 - \mu_1} e^{(\mu_2 - \mu_1)\nu_1}.$$

For $\theta_1 = \mu_1, \theta_2 = \mu_1$ we have that

$$\tilde{B}_1(x) = \mu_1 (x - \nu_1) e^{-\mu_1 x},$$

and in this case

$$c(1,1,0) = 0, \text{ and } c(1,1,1,d) = \mu_1 \equiv \theta_1,$$

where $d = (d_1) = (\nu_1)$ and in $c(k,j,q)$ for $k = 1$, $j = 1,2$, $q = 0$ we have omitted the argument d, since the set $D(k,j,q)$ is empty for $q = 0$. In both cases, $\tilde{B}_1(x)$ has the form as in (A20). Assume, (A20) holds for some $k \geq 2$. Denote by $H(k)$ the set of all exponential Appell monomials which appear in (A20), i.e.,

$$H(k) = \left\{ e^{-\mu_j x} A_q(x;d); 1 \leq j \leq s(k+1), 0 \leq q \leq l(k+1,j) - 1, d \in D(k,j,q) \right\}.$$

In other words, $\tilde{B}_k(x)$ is a linear combination of elements of $H(k)$ in which some of the coefficients may be equal to zero. The elements of $H(k)$ can be uniquely defined by the index sets $m(k+1,j)$, $j = 1,\ldots,s(k+1)$. Therefore, we can alternatively denote $H(k)$ as $H(m(k+1,j), j = 1,\ldots,s(k+1)))$. Since, $m(k+1,j) \subset m(k+2,j))$ for every fixed j, hence $H(k) \equiv H(m(k+1,j), j = 1,\ldots,s(k+1))) \subset H(m(k+2,j), j = 1,\ldots,s(k+2))) \equiv H(k+1)$, i.e., $H(k) \subset H(k+1)$. For $\tilde{B}_k(x)$, by analogy with (A13) we have that

$$\tilde{B}_k(x) = \mu_{i_k} e^{-\mu_{i_{k+1}} x} \int_{\nu_k}^{x} e^{\mu_{i_{k+1}} z} \tilde{B}_{k-1}(z) dz \,. \tag{A21}$$

Hence, from (A21), for $k+1$ and the induction assumption that (A20) holds for some k we have that

$$\tilde{B}_{k+1}(x) = \mu_{i_{k+1}} e^{-\mu_{i_{k+2}} x} \sum_{j=1}^{s(k+1)} \sum_{q=0}^{l(k+1,j)-1} \sum_{d \in D(k,j,q)} c(k,j,q,d) \int_{\nu_{k+1}}^{x} e^{\left(\mu_{i_{k+2}} - \mu_j\right) z} A_q(z;d) dz. \tag{A22}$$

In order to show by induction that (A20) is valid for $k+1$, it suffices to show that, $\tilde{B}_{k+1}(x)$ is a linear combination of elements of $H(k+1) \equiv H(m(k+2,j), j = 1,\ldots,s(k+2)))$. Following this line of reasoning, for $i_{k+2} = s(k+1)+1 \equiv s(k+2)$, from (A22), integrating by parts and using the property of Appell polynomials,

$A'_q(x) = A_{q-1}(x)$, given in (20), we have

$$\tilde{B}_{k+1}(x) = \mu_{i_{k+1}} e^{-\mu_{i_{k+2}} x} \sum_{j=1}^{s(k+1)} \sum_{q=0}^{l(k+1,j)-1} \sum_{d \in D(k,j,q)} c(k,j,q,d)$$
$$\left(\sum_{t=0}^{q} \frac{(-1)^t}{\left(\mu_{i_{k+2}} - \mu_j\right)^{t+1}} A_{q-t}(x;d) e^{\left(\mu_{i_{k+2}} - \mu_j\right) x} \right.$$
$$\left. + \sum_{t=0}^{q} \frac{(-1)^{t+1}}{\left(\mu_{i_{k+2}} - \mu_j\right)^{t+1}} A_{q-t}\left(\nu_{k+1}; d\right) e^{\left(\mu_{i_{k+2}} - \mu_j\right)\nu_{k+1}} \right)$$
$$= \sum_{j=1}^{s(k+1)} \sum_{q=0}^{l(k+1,j)-1} \sum_{d \in D(k,j,q)} \mu_{i_{k+1}} c(k,j,q,d) \sum_{t=0}^{q} \frac{(-1)^t}{\left(\mu_{s(k+2)} - \mu_j\right)^{t+1}} e^{-\mu_j x} A_{q-t}(x;d)$$
$$+ \left(\sum_{j=1}^{s(k+1)} \sum_{q=0}^{l(k+1,j)-1} \sum_{d \in D(k,j,q)} \mu_{i_{k+1}} c(k,j,q,d) \right.$$
$$\left. \sum_{t=0}^{q} \frac{(-1)^{t+1}}{\left(\mu_{s(k+2)} - \mu_j\right)^{t+1}} A_{q-t}\left(\nu_{k+1}; d\right) e^{\left(\mu_{s(k+2)} - \mu_j\right)\nu_{k+1}} \right) e^{-\mu_{s(k+2)} x} A_0\left(x\right), \tag{A23}$$

where $A_{q-t}(x;d) \equiv A_{q-t}(x;d_1,\dots,d_q) \equiv A_{q-t}(x;d_1,\dots,d_{q-t})$ and $A_0(x) \equiv 1$. Since $i_{k+2} = s(k+1)+1 \equiv s(k+2)$, the index sets which define $\tilde{B}_{k+1}(x)$ are $m(k+2,j) = m(k+1,j), j = 1,\dots,s(k+1)$ and $m(k+2,s(k+2)) = \{k+2-1\}$. Hence, $H(k+1) = H(k) \cup \{e^{-\mu_{s(k+2)}x} A_0(x)\}$.

It can be directly verified that each exponential Appell monomial in (A23) is an element of $H(k+1)$. More precisely, the exponential Appell monomials

$$e^{-\mu_j x} A_{q-t}(x,d), 1 \le j \le s(k+1), 0 \le q \le l(k+1,j)-1, 0 \le t \le q, d \in D(k,j,q)$$

are elements of $H(k)$ whereas, $e^{-\mu_{s(k+2)}x} A_0(x)$ is not in $H(k)$ but is in $H(k+1)$. This completes the proof of the lemma in the case $i_{k+2} = s(k+1)+1$.

In the case $1 \le i_{k+2} = n \le s(k+1)$, we will briefly sketch the proof which is similar. From (A22), we have

$$\tilde{B}_{k+1}(x) =$$
$$\mu_{i_{k+1}} e^{-\mu_n x} \sum_{\substack{j=1 \\ j\neq n}}^{s(k+1)} \sum_{q=0}^{l(k+1,j)-1} \sum_{d\in D(k,j,q)} c(k,j,q,d) \int_{\nu_{k+1}}^{x} e^{(\mu_n-\mu_j)z} A_q(z;d)dz$$
$$+\mu_{i_{k+1}} e^{-\mu_n x} \sum_{q=0}^{l(k+1,n)-1} \sum_{d\in D(k,n,q)} c(k,n,q,d) \int_{\nu_{k+1}}^{x} e^{(\mu_n-\mu_n)z} A_q(z;d)dz$$
$$= \sum_{\substack{j=1 \\ j\neq n}}^{s(k+1)} \sum_{q=0}^{l(k+1,j)-1} \sum_{d\in D(k,j,q)} \mu_{i_{k+1}} c(k,j,q,d) \sum_{t=0}^{q} \frac{(-1)^t}{(\mu_n-\mu_j)^{t+1}} e^{-\mu_j x} A_{q-t}(x;d)$$
$$+ \left(\sum_{\substack{j=1 \\ j\neq n}}^{s(k+1)} \sum_{q=0}^{l(k+1,j)-1} \sum_{d\in D(k,j,q)} \mu_{i_{k+1}} c(k,j,q,d) \sum_{t=0}^{q} \frac{(-1)^{t+1}}{(\mu_n-\mu_j)^{t+1}} A_{q-t}\left(\nu_{k+1};d\right) e^{(\mu_n-\mu_j)\nu_{k+1}} \right) e^{-\mu_n x} A_0(x)$$
$$+ \sum_{q=0}^{l(k+1,n)-1} \sum_{d\in D(k,n,q)} \mu_{i_{k+1}} c(k,n,q,d) e^{-\mu_n x} A_{q+1}\left(x; d_1,\ldots,d_q,\nu_{k+1}\right) \,. \tag{A24}$$

Since $1 \le i_{k+2} = n \le s(k+1)$, the index sets which define $\tilde{B}_{k+1}(x)$ are $m(k+2,j) = m(k+1,j), j = 1,\ldots,s(k+1), j \neq n$ and $m(k+2,n) = m(k+1,n) \cup \{k+2-1\}$. Hence, we have

$$H(k+1) =$$
$$H(k) \cup \left\{ e^{-\mu_n x} A_{q+1}(x; d_1,\ldots,d_q,\nu_{k+1}), 0 \le q \le l(k+1,j)-1, (d_1,\ldots,d_q) \in D(k,j,q) \right\} .$$

It can be directly checked that each exponential Appell monomial in (A24) is an element of $H(k+1)$. More precisely, the exponential Appell monomials

$$e^{-\mu_n x} A_{q+1}(x; d_1,\ldots,d_q,\nu_{k+1}), 0 \le q \le l(k+1,j)-1, (d_1,\ldots,d_q) \in D(k,j,q)$$

which appear in the last term of (A24), are elements of $H(k+1)$ but not of $H(k)$. All other exponential Appell monomials in (A24) belong to $H(k)$. This completes the proof of Lemma A.5. □

Remark *A1***:** Let us note that the functions, $\tilde{B}_k(x)$, defined in (A19), coincide with $B_k(x)$, given by (A3), when ν_i, $i = 1,2,\ldots$ are defined as in Corollary 2.5 and μ_{i_k} are defined as in (A18) with θ_k given by (2).

Finally, let us give an expression for the functions, $B_k(x)$, defined in (A13) of Lemma A.2, under the initially stated Erlang claim arrivals model, given by (2) but assuming that the claim intensities λ_i, $i = 1,2,...$ are pair-wise distinct. The following lemma gives an expression for the exponential Appell polynomials, $B_k(x)$, in this reasonably general Erlang claim arrival model.

Lemma A.6: *For $k \geq g_1$*

$$B_k(x) = \lambda_1^{g_1}\lambda_2^{g_2}\ldots\lambda_{j(k)}^{g_{j(k)}}\lambda_{j(k)+1}^{s-1}\left(\frac{1}{(\lambda_2-\lambda_1)^{g_2}}\cdots\frac{1}{\left(\lambda_{j(k)}-\lambda_1\right)^{g_{j(k)}}}\frac{1}{\left(\lambda_{j(k)+1}-\lambda_1\right)^{s}}e^{-\lambda_1 x}\right.$$

$$\sum_{l=0}^{g_1-1}A_{g_1-1-l}\left(x;\nu_1,...,\nu_{g_1-1-l}\right)\left[\underset{g_2}{\frac{1}{\lambda_1-\lambda_2}},\ldots,\underset{g_{j(k)}}{\frac{1}{\lambda_1-\lambda_{j(k)}}},\underset{s}{\frac{1}{\lambda_1-\lambda_{j(k)+1}}}\right]z^{l+g_2+\ldots+g_{j(k)}+s-1}$$

$$-\frac{1}{(\lambda_3-\lambda_2)^{g_3}}\cdots\frac{1}{\left(\lambda_{j(k)}-\lambda_2\right)^{g_{j(k)}}}\frac{1}{\left(\lambda_{j(k)+1}-\lambda_2\right)^{s}}e^{-\lambda_2 x}$$

$$\sum_{l=0}^{g_2-1}e^{\lambda_2\nu_{g_1+l}}R\left(\nu_{g_1+l}\right)A_{g_2-1-l}\left(x;\nu_{g_1+l+1},\ldots,\nu_{g_1+g_2-1}\right)$$

$$\left[\underset{g_3}{\frac{1}{\lambda_2-\lambda_3}},\ldots,\underset{g_{j(k)}}{\frac{1}{\lambda_2-\lambda_{j(k)}}},\underset{s}{\frac{1}{\lambda_2-\lambda_{j(k)+1}}}\right]z^{l+g_3+\ldots+g_{j(k)}+s-1}+\ldots$$

$$+(-1)^{j(k)+1}\frac{1}{\left(\lambda_{j(k)+1}-\lambda_{j(k)}\right)^{s}}e^{-\lambda_{j(k)}x}\sum_{l=0}^{g_{j(k)}-1}e^{\lambda_{j(k)}\nu_{g_1+\ldots+g_{j(k)-1}+l}}R\left(\nu_{g_1+\ldots+g_{j(k)-1}+l}\right)$$

$$A_{g_{j(k)}-1-l}\left(x;\nu_{g_1+\ldots+g_{j(k)-1}+l+1},\ldots,\nu_{g_1+\ldots+g_{j(k)}-1}\right)\left[\underset{s}{\frac{1}{\lambda_{j(k)}-\lambda_{j(k)+1}}}\right]z^{l+s-1}-e^{-\lambda_{j(k)+1}x}$$

$$\sum_{l=0}^{s-1}e^{\lambda_{j(k)+1}\nu_{g_1+\ldots+g_{j(k)}+l}}R\left(\nu_{g_1+\ldots+g_{j(k)}+l}\right)$$

$$\left.A_{s-1-l}\left(x;\nu_{g_1+\ldots+g_{j(k)}+l+1},\ldots,\nu_{g_1+\ldots+g_{j(k)}+s-1}\right)\right) \tag{A25}$$

where s is defined as in (5) (i.e., $0 < s = k+1-\left(g_1+...+g_{j(k)}\right)$), $0 \leq \nu_1 \leq \nu_1 \leq \ldots$ is a non-decreasing sequence of real numbers, $\left[\underset{m_0}{t_0},\ldots,\underset{m_n}{t_n}\right]z^q$ is the $(m_0+\ldots+m_n-1)-th$ order divided difference of the power function z^q, for q, positive integer, and where, $R(\nu_i)$, for $i = g_1, g_1+1,\ldots$ is found from the equality

$$R\left(\nu_i\right)=\frac{B_i\left(\nu_i\right)}{\lambda_1^{g_1}\lambda_2^{g_2}\ldots\lambda_{j(i)}^{g_{j(i)}}\lambda_{j(i)+1}^{i-(g_1+\ldots+g_{j(i)})}}+e^{-\lambda_{j(i)+1}\nu_i}\sum_{l=0}^{i-(g_1+\ldots+g_{j(i)})}e^{\lambda_{j(i)+1}\nu_{g_1+\ldots+g_{j(i)}+l}}$$
$$R\left(\nu_{g_1+\ldots+g_{j(i)}+l}\right)A_{i-(g_1+\ldots+g_{j(i)})-l}\left(\nu_i;\nu_{g_1+\ldots+g_{j(i)}+l+1},\ldots,\nu_i\right),$$

assuming that, $R(\nu_m)$, $m = g_1, g_1+1,\ldots,i-1$ have already been computed.

Proof: We will first prove (A25) for $k = g_1$. Using the recurrent integral equation

(A13), we have

$$
\begin{aligned}
B_{g_1}(x) &= \lambda_{j(g_1-1)+1} e^{-\lambda_{j(g_1)+1}x} \int_{\nu_{g_1}}^{x} e^{\lambda_{j(g_1)+1}z} B_{g_1-1}(z)dz \\
&= \lambda_1 e^{-\lambda_2 x} \int_{\nu_{g_1}}^{x} e^{\lambda_2 z} \lambda_1^{g_1-1} A_{g_1-1}\left(z;\nu_1,\ldots,\nu_{g_1-1}\right) e^{-\lambda_1 z} dz \\
&= \lambda_1^{g_1} \left(\frac{1}{(\lambda_2-\lambda_1)} e^{-\lambda_1 x} \sum_{l=0}^{g_1-1} A_{g_1-1-l}\left(x;\nu_1,\ldots,\nu_{g_1-1-l}\right) \frac{1}{(\lambda_1-\lambda_2)^{l}} \right. \\
&\quad \left. - e^{-\lambda_2 x} \frac{1}{(\lambda_2-\lambda_1)} e^{-\lambda_1 \nu_{g_1}} \sum_{l=0}^{g_1-1} A_{g_1-1-l}\left(\nu_{g_1};\nu_1,\ldots,\nu_{g_1-1-l}\right) \frac{1}{(\lambda_1-\lambda_2)^{l}} e^{\lambda_2 \nu_{g_1}} \right) \\
&= \lambda_1^{g_1} \left(\frac{1}{(\lambda_2-\lambda_1)} e^{-\lambda_1 x} \sum_{l=0}^{g_1-1} A_{g_1-1-l}\left(x;\nu_1,\ldots,\nu_{g_1-1-l}\right) \left[\frac{1}{(\lambda_1-\lambda_2)} \right]_1 t^l - \right. \\
&\qquad \left. e^{-\lambda_2 x} R\left(\nu_{g_1}\right) e^{\lambda_2 \nu_{g_1}} \right).
\end{aligned}
$$

It is easily seen that the last equality is of the form (A25), for $k = g_1$. For $k = g_1+1$ and $g_2 > 1$ we have that $j\left(g_1\right) = j\left(g_1+1\right) = 1$ and applying again (A13), we can write

$$
\begin{aligned}
B_{g_1+1}(z) &= \lambda_2 e^{-\lambda_2 x} \int_{\nu_{g_1+1}}^{x} e^{\lambda_2 z} B_{g_1}(z)dz \\
&= \lambda_2 \lambda_1^{g_1} e^{-\lambda_2 x} \int_{\nu_{g_1+1}}^{x} e^{\lambda_2 z} \left(\frac{1}{(\lambda_2-\lambda_1)} e^{-\lambda_1 z} \sum_{l_1=0}^{g_1-1} A_{g_1-1-l_1}\left(z;\nu_1,\ldots,\nu_{g_1-1-l_1}\right) \right. \\
&\qquad \left. \frac{1}{(\lambda_1-\lambda_2)^{l_1}} - e^{-\lambda_2 z} R\left(\nu_{g_1}\right) e^{\lambda_2 \nu_{g_1}} \right) dz \\
&= \lambda_2 \lambda_1^{g_1} \left(\frac{1}{\lambda_2-\lambda_1} e^{-\lambda_2 x} \int_{\nu_{g_1+1}}^{x} e^{(\lambda_2-\lambda_1)z} \sum_{l_1=0}^{g_1-1} A_{g_1-1-l_1}\left(z;\nu_1,\ldots,\nu_{g_1-1-l_1}\right) \right. \\
&\qquad \left. \frac{1}{(\lambda_1-\lambda_2)^{l_1}} dz - \int_{\nu_{g_1+1}}^{x} R\left(\nu_{g_1}\right) e^{\lambda_2 \nu_{g_1}} dz \right) \\
&= \lambda_2 \lambda_1^{g_1} \left(\frac{1}{\lambda_2-\lambda_1} e^{-\lambda_2 x} \sum_{l_1=0}^{g_1-1} \int_{\nu_{g_1+1}}^{x} A_{g_1-1-l_1}\left(z;\nu_1,\ldots,\nu_{g_1-1-l_1}\right) \frac{1}{(\lambda_1-\lambda_2)^{l_1}} e^{(\lambda_2-\lambda_1)z} dz \right. \\
&\qquad \left. - e^{-\lambda_2 x} R\left(\nu_{g_1}\right) e^{\lambda_2 \nu_{g_1}} A_1\left(x;\nu_{g_1+1}\right) \right) \qquad \text{(A26)}
\end{aligned}
$$

After integrating by parts in (A26), we have

$$B_{g_1+1}(x) = \lambda_2 \lambda_1^{g_1} \left(\frac{1}{(\lambda_2 - \lambda_1)^2} e^{-\lambda_2 x} \left(\sum_{l_1=0}^{g_1-1} \sum_{l_2=0}^{g_1-1-l_1} A_{g_1-1-l_1-l_2}(z; \nu_1, \dots, \nu_{g_1-1-l_1-l_2}) \right. \right.$$

$$\left. \left. \frac{1}{(\lambda_1 - \lambda_2)^{l_1}} \frac{1}{(\lambda_1 - \lambda_2)^{l_2}} e^{(\lambda_2-\lambda_1)z} \right) \Big|_{z=\nu_{g_1+1}}^{x} - e^{-\lambda_2 x} R(\nu_{g_1}) e^{\lambda_2 \nu_{g_1}} A_1(x; \nu_{g_1+1}) \right)$$

$$= \lambda_2 \lambda_1^{g_1} \left(\frac{1}{(\lambda_2 - \lambda_1)^2} e^{-\lambda_1 x} \sum_{l_1=0}^{g_1-1} \sum_{l_2=0}^{g_1-1-l_1} A_{g_1-1-l_1-l_2}(x; \nu_1, \dots, \nu_{g_1-1-l_1-l_2}) \right.$$

$$\left. \frac{1}{(\lambda_1 - \lambda_2)^{l_1+l_2}} - e^{-\lambda_2 x} R(\nu_{g_1+1}) e^{\lambda_2 \nu_{g_1+1}} A_0 - e^{-\lambda_2 x} R(\nu_{g_1}) e^{\lambda_2 \nu_{g_1}} A_1(x; \nu_{g_1+1}) \right)$$

$$= \lambda_2 \lambda_1^{g_1} \left(\frac{1}{(\lambda_2 - \lambda_1)^2} e^{-\lambda_1 x} \sum_{l=0}^{g_1-1} A_{g_1-1-l}(x; \nu_1, \dots, \nu_{g_1-1-l}) \left[\frac{1}{\lambda_1 \underset{2}{-} \lambda_2} \right] z^{l+1} - \right.$$

$$\left. e^{-\lambda_2 x} R(\nu_{g_1+1}) e^{\lambda_2 \nu_{g_1+1}} A_0 - e^{-\lambda_2 x} R(\nu_{g_1}) e^{\lambda_2 \nu_{g_1}} A_1(x; \nu_{g_1+1}) \right), \tag{A27}$$

where

$$\left[\frac{1}{\lambda_1 \underset{2}{-} \lambda_2} \right] z^{l+1} = \sum_{l_1 \geq 0, l_2 \geq 0, l_1+l_2=l} \frac{1}{(\lambda_1 - \lambda_2)^{l_1+l_2}}$$

is the first order divided difference of the function z^{l+1} at the point $\frac{1}{\lambda_1 - \lambda_2}$ of multiplicity 2, (see e.g. [30], page 47, equalities (2.94) and (2.95)). As seen, equality (A27) is again of the form (A25). Iterating a similar integration leads to the announced formula (A25), for all $k \geq g_1$. □

References

[1] Albrecher, H. and Boxma, O. J. (2005). *On the discounted penalty function in a Markov-dependent risk model.* Insurance: Mathematics and Economics **37**, 650–672.

[2] Asmussen, S. (1998). *Subexponential asymptotics for stochastic processes: extremal behavior, stationary distributions and first passage probabilities.* The Annals of Applied Probability, **8**, 2, 354–374.

[3] Asmussen, S. (2000). *Ruin Probabilities.* World Scientific.

[4] Boudreault, M., Cossette, H., Landriault, D. and Marceau, E. (2006). *On a risk model with dependence between interclaim arrivals and claim sizes.* Scandinavian Actuarial Journal **5**, 265–285.

[5] Cheng, Y. and Tang, Q. (2003). *Moments of surplus before ruin and deficit at ruin in the Erlang(2) risk process.* North American Actuarial Journal **7**, 1, 1–12.

[6] Claramunt, M. M., Marmol, M. and Lacayo, R. (2005). *On the probability of reaching a barrier in an Erlang(2) risk process.* SORT **29**, (2), 235–248.

[7] Daniels, H. E. (1996). *Approximating the first crossing-time density for a curved boundary.* Bernoulli **2**, 2, 133–143.

[8] Dickson, D.C.M. (1998). *On a class of renewal risk process.* North American Actuarial Journal **2**, 3, 60–68.

[9] Dickson, D.C.M. and Hipp, C. (1998). *Ruin probabilities for Erlang(2) risk process.* Insurance: Mathematics and Economics **22**, 251–262.

[10] Dickson, D.C.M. and Hipp, C. (2001). *On the time to ruin for Erlang(2) risk process.* Insurance: Mathematics and Economics **29**, 333–344.

[11] Dickson, D.C.M., Hughes, B.D. and Lianzeng Z. (2005). *The density of the time to ruin for a Spare Andersen process with Erlang arrivals and exponential claims.* Scandinavian Actuarial Journal **5**, 358–376.

[12] Foss, S., Palmowski, Z. and S. Zachary (2005). *The probability of exceeding a high boundary on a random time interval for a heavy-tailed random walk.* The Annals of Applied Probability, **15**, 3, 1936–1957.

[13] Gaier, J., Grandits, P. and W. Schachermayer (2003). *Asymptotic ruin probabilities and optimal investment.* The Annals of Applied Probability, **13**, 3, 1054–1076.

[14] Gerber, H. U. and Shiu, E. S. W., (2003a). *Discussion of Yebin Cheng and Qihe Tang's "Moments of the surplus before ruin and the deficit at ruin".* North American Actuarial Journal, **7**, 3, 117–119.

[15] Gerber, H. U. and Shiu, E. S. W., (2003b). *Discussion of Yebin Cheng and Qihe Tang's "Moments of the surplus before ruin and the deficit at ruin".* North American Actuarial Journal, **7**, 4, 96–101.

[16] Grandell, I. (1991). *Aspects of Risk Theory.* Springer.

[17] Ignatov, Z. G. and Kaishev, V. K. (2000). *Two sided bounds for the finite time probability of ruin.* Scandinavian Actuarial Journal **1**, 46–62.

[18] Ignatov, Z. G. and Kaishev, V. K. (2004). *A finite time ruin probability formula for continuous claim severities.* Journal of Applied Probability **41**, 570–578.

[19] Ignatov, Z. G. and Kaishev, V. K. (2006). *On the infinite-horizon probability of (non)ruin for integer-valued claims.* Journal of Applied Probability **43**, 535–551.

[20] Ignatov, Z. G. Kaishev, V. K. and R. S. Krachunov (2001). *An improved finite-time ruin probability formula and its Mathematica implementation.* Insurance: Mathematics and Economics **29**, 375–386.

[21] Johnson, N. L., Kotz, S. and Balakrishnan, N. (1997). *Discrete Multivariate Distributions*, John Wiley, New York.

[22] Johnson, N. L. and Kotz, S. (1972). *Distributions in Statistics: Continuous Multivariate Distributions.* John Wiley, New York.

[23] Klüppelberg, C., Kyprianou, A. E., and R. A. Maller (2004). *Ruin probabilities and overshoots for general Lévy insurance risk processes.* The Annals of Applied Probability, **14**, 4, 1766–1801.

[24] Li, S. and Garrido, J. (2004). *On ruin for the Erlang(n) risk process.* Insurance: Mathematics and Economics **34**, 391–408.

[25] Li, S. and Yang, H. (2004). *Explicit expressions for the ruin probabilities of Erlang risk processes with Pareto individual claim distribution.* Insurance: Mathematics and Economics **34**, 391–408.

[26] Lundberg, F. (1903) *Approximerad framställning av sannolikehetsfunktionen, återförsäkering av kollektivrisker*, Almqvist & Wiksell, Uppsala.

[27] Mikosch, T. and Samorodnitsky, G. (2000). *Ruin probability with claims modeled by a stationary ergodic stable process.* The Annals of Probability, **28**, 4, 1814–1851.

[28] Paulsen, J. (2002). *On Cramer-like asymptotics for risk processes with stochastic return on investments.* The Annals of Applied Probability, **12**, 4, 1247–1260.

[29] Picard, P., and Lefèvre, C. (1997). *The probability of ruin in finite time with discrete claim size distribution.* Scand. Actuarial Journal, **1**, 58–69.

[30] Schumaker, L. L. (1981). *Spline Functions:Basic Theory.* John Wiley, New York.

[31] Sun, L. and Yang, H. (2004). *On the joint distributions of surplus immediately before ruin and the deficit at ruin for Erlang(2) risk processes.* Acta Mathematicae Applicatae Sinica **20**, 3, 495–506.

[32] Zacks, S (2007). *First exit times for ordinary and compound Poisson processes with non-linear boundaries.* Methodol. Comput. Appl. Probab. **9**, 359–375.

[33] Zwart, B., Borst, S. and K. Debicki (2005). *Subexponential asymptotics of hybrid fluid and ruin models.* The Annals of Applied Probability, **15**, 1A, 500–517.