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# System Degeneracy and the Output Feedback Problem: Parametrisation of the Family of Degenerate Compensators

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**Abstract:** This paper develops in depth the concept of degeneracy associated with the standard output feedback configuration of a proper multivariable system. The motivation stems from the very important property that degenerate feedback gains may be used for the linearisation of the pole assignment map and hence for frequency assignment. In general, the objective is the characterisation and parametrisation of all feedback gains that may allow the asymptotic linearisation of the pole placement map, as well as the families of systems for which such parametrisations are non trivial. The paper reviews the Global Asymptotic Linearisation method associated with the core versions of determinantal pole assignment problems and defines the conditions which characterises degenerate solutions of different types. Using the theory of ordered minimal bases, we provide a parametrisation of special families of degenerate compensators according to their degree. Finally, the special properties of degenerate solutions that allow frequency assignment are considered.

*Keywords:* Linear multivariable systems; output feedback control; algebraic methods; geometric methods; pole assignment.

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## 1. INTRODUCTION

The construction of dynamic, or static output feedback controllers, that places the poles of a  $p$ -input,  $m$ -output,  $n$ -state multivariable system to arbitrary desired locations, was always a challenging problem in control theory of linear systems which is a multi-linear in the gain parameters. The pole assignment via output feedback problem belongs to the family of Determinantal Assignment Problems (DAP) which has been introduced in (Karcanias and Giannakopoulos, 1984) as a framework that unifies the study of all pole-zero frequency assignment problems of determinantal type. The multi-linear nature of DAP has been recently handled by a blow-up type methodology, using as a basis the notion of degenerate solution and which is known as Global Asymptotic Linearisation (Leventides and Karcanias, 1995, 1998). Under certain conditions, this methodology allows the computation of solutions of DAP. There are many challenging issues in the development of the DAP framework and amongst them are its ability to provide solutions even for non-generic cases, as well as providing approximate solutions (Leventides et al., 2014c), (Karcanias and Leventides, 2015) to the cases where generically there is no solution of the exact problem. The construction of such solutions requires the further development of the Global Linearisation methodology and an essential part of this is the parametrisation

of the degenerate solutions. The notion of degeneracy has been first introduced by (Brockett and Byrnes, 1981) and characterise the boundary cases for the compensation or the feedback configuration. Such solutions have the significant property that linearise asymptotically the multi-linear nature of the related frequency assignment map (i.e., the pole placement map in our case) and thus they become key instruments for developing Global Linearisation (Leventides and Karcanias, 1995, 1998). The classification of degenerate controllers and the parametrisation of such families play also an integral role for the application of the Global Linearisation methodology to special structure problems, such as decentralized control etc.

In (Leventides et al., 2014a,b), the authors have presented different implementations of the Global Linearisation methodology using homotopy continuation techniques, such as a predictor-corrector scheme and modified Newton methods, in order to be able to trace the solution fibre as far as possible from the locus of degenerate points and hence achieve solutions that assigns the desired pole polynomial with much better sensitivity properties regarding the closed-loop system. The current research is the algebraic part of the Global Asymptotic Linearisation methodology and its investigation is essential for the further development of the method. As mentioned before, the motivation stems from the very important property that

degenerate solutions may be used for the linearisation of the related frequency assignment map.

This paper extends the work which has started on the characterisation of the constant degenerate solutions in (Karcnias et al., 2013, 2014), based on the abstract DAP case. Here, an alternative approach on the original dynamic set-up is explored for the parametrisation of the degenerate solutions by focusing to the output feedback problem using the algebraic theory of minimal bases of rational vector spaces (Karcnias, 1996), (Forney, 1975).

The paper is structured as follows: Section 2, defines the problem and reviews the background theory of minimal bases and the tools that will be needed in the sequel. The family of all degenerate solutions for a given multivariable system is defined and a classification of the degenerate gains into two different types (regular and irregular) is provided together with the necessary and sufficient conditions for their existence. Section 3, introduces the procedure for the construction of such compensators together with the parametrisation of the corresponding families of degenerate solutions according to their degree. Finally, in Section 4, the main ingredients of the Global Asymptotic Linearisation method are revisited and the conditions under which the differential of the related frequency assignment map has certain properties that allow arbitrary frequency assignment are defined.

*Notation:* Throughout the paper the following notation is adopted: If  $\mathcal{F}$  is a field, or ring then  $\mathcal{F}^{m \times n}$  denotes the set of  $m \times n$  matrices over  $\mathcal{F}$ . If  $H$  is a map, then  $\mathcal{R}\{H\}, \mathcal{N}_r\{H\}, \mathcal{N}_l\{H\}$  denote the range, right, left null-spaces respectively and  $\rho\{H\}$  its rank.  $Q_{k,n}$  denotes the set of lexicographically ordered, strictly increasing sequences of  $k$  integers from the set  $\tilde{n} = \{1, 2, \dots, n\}$ .

## 2. PROBLEM STATEMENT AND BACKGROUND TOOLS

### 2.1 Definition of the Problem

Let consider linear systems described by  $S(A, B, C, D)$  state space descriptions with  $n$ -states,  $p$ -inputs and  $m$ -outputs, where  $(A, B)$  is controllable,  $(A, C)$  is observable, or by the transfer function matrix  $G(s) = C(sI - A)^{-1}B + D$  where the rank is  $\min(m, p)$ . In terms of left, right coprime Matrix Fraction Descriptions (MFD),  $G(s) \in \mathbb{R}^{m \times p}(s)$  may be represented as

$$G(s) = D_l(s)^{-1}N_l(s) = N_r(s)D_r(s)^{-1}$$

or by the associated composite system matrix

$$M(s) = \begin{bmatrix} D(s) \\ N(s) \end{bmatrix} \in \mathbb{R}^{(p+m) \times p}[s] \quad (1)$$

where,  $N_l(s), N_r(s) \in \mathbb{R}^{m \times p}[s]$ ,  $D_l(s) \in \mathbb{R}^{m \times m}[s]$  and  $D_r(s) \in \mathbb{R}^{p \times p}[s]$ . Throughout this paper we denote as

$$\mathcal{M} \triangleq \text{colsp}_{\mathbb{R}[s]} \{M(s)\}$$

the  $\mathbb{R}[s]$ -module which is uniquely defined by  $G(s)$ , it is a maximal Noetherian and its Forney dynamical indices are the controllability indices of any minimal realization  $S(A, B, C)$  of  $G(s)$ . For the standard feedback control scheme and assuming a right coprime MFD representation for the system and left MFD for the controller the following

frequency assignment problems from the DAP family (Karcnias and Giannakopoulos, 1984), are defined:

- I) The *Dynamic Output Feedback* (DOF) pole assignment problem is expressed via

$$\det \{H(s) \cdot M(s)\} = \det \left\{ [A_l(s), B_l(s)] \begin{bmatrix} D_r(s) \\ N_r(s) \end{bmatrix} \right\} = p(s)$$

- II) The *Static Output Feedback* (SOF) pole assignment problem is defined as:

$$\det \{H \cdot M(s)\} = \det \left\{ [I_p, K] \begin{bmatrix} D_r(s) \\ N_r(s) \end{bmatrix} \right\} = p(s)$$

where  $p(s)$  denote the prime pole polynomial to be assigned. We may now examine some special feedback gains for which the well formed nature of the feedback configuration is lost, that is the degenerate gains. Let us first introduce the following notions:

*Definition 1.* Any matrix of the type

$$H(s) = [A(s), B(s)] \in \mathbb{R}^{p \times (p+m)}[s] \quad (2)$$

where,  $A(s) \in \mathbb{R}^{p \times p}[s]$  and  $\text{rank}\{H(s)\} = p$ , will be a (dynamic) *generalized gain*. □

*Remark 2.* A special family of *generalized gains* exists when  $H(s) \equiv H$ , i.e. a constant matrix of the type

$$H = [A, B] \in \mathbb{R}^{p \times (p+m)} \quad (3)$$

where,  $A \in \mathbb{R}^{p \times p}$  and  $\text{rank}\{H\} = p$ . We define also, for any  $K \in \mathbb{R}^{p \times m}$ , as the composite gain of the feedback configuration, the matrix

$$\tilde{H} = [I_p, K] \in \mathbb{R}^{p \times (p+m)} \quad \square$$

Generalized gains may be further classified into *regular* and *irregular* type. In particular, if  $|A| \neq 0$ , then the gain will be called *regular*, otherwise (i.e  $|A| = 0$ ) it will be called *irregular*. It is well known that regular generalised gains correspond to standard bounded pre-compensation gains, given by  $K = A^{-1}B$ , whereas irregular generalised gains correspond to unbounded gains. We may now define:

*Definition 3.* Given a system described by the composite matrix  $M(s) \in \mathbb{R}^{(m+p) \times p}[s]$  and for the standard feedback configuration, we say that the generalised gain, as in (2), is a *Generalised Degenerate Gain (GDG)* if

$$\det \{H(s) \cdot M(s)\} \equiv 0 \quad (4)$$

□

*Remark 4.* A special family of GDGs is considered when  $H(s) \equiv H \in \mathbb{R}^{p \times (p+m)}$  is a constant real matrix, that is the *constant Generalised Degenerate Gains* defined as

$$\det \{H \cdot M(s)\} = \det \left\{ [A, B] \begin{bmatrix} D(s) \\ N(s) \end{bmatrix} \right\} \equiv 0 \quad (5)$$

□

We start off by investigating the conditions for existence of degenerate solutions.

*Proposition 5.* For the standard feedback configuration corresponding to a system described by  $M(s)$ , there exists a Generalised Degenerate Gain,  $H(s)$ , such that (4) is

satisfied, if and only if, there exist at least a polynomial vector  $\underline{m}(s) \in \text{colsp}_{\mathbb{R}[s]}(M(s))$ , such that

$$H(s) \cdot \underline{m}(s) \equiv 0 \quad (6)$$

**Proof.** Note that,  $\det\{H(s) \cdot M^*(s)\} \equiv 0$  holds true if and only if  $H(s)M^*(s)$  is rank deficient. Clearly then, there exists a unimodular matrix,  $Q(s)$ , such that

$$H(s)M^*(s)Q(s) = [T(s) \mid 0]$$

which  $Q(s)$  may be partitioned as

$$H(s)M^*(s) \begin{bmatrix} Q_1(s), q_1(s) \end{bmatrix} = [T(s) \mid 0]$$

it is clear that

$$H(s)M^*(s)q_1(s) \equiv 0$$

Therefore, the above implies that there exists a vector  $\underline{m}(s) = M^*(s)q_1(s) \in \mathcal{M}$  such that  $H(s) \cdot \underline{m}(s) = 0$  and this proves necessity. Sufficiency is obvious and this establishes the result.  $\square$

*Proposition 6.* Necessary and sufficient condition for the existence of a dynamic degenerate compensator is that

$$m \geq 1 \quad (7)$$

**Proof.** Since  $\underline{m}(s) \in \mathbb{R}^{(m+p) \times 1}$ , its left null-space has dimensions  $m + p - 1$  and thus  $H(s)$  exists if and only if  $m + p - 1 \geq p \Leftrightarrow m \geq 1$ .  $\square$

Let us now consider the special case when  $H(s) \equiv H \in \mathbb{R}^{p \times (p+m)}$ , and investigate the conditions for existence of constant GDGs. For the proof see (Karcaniyas et al., 2014),(Leventides and Karcaniyas, 1995).

*Proposition 7.* For the standard feedback configuration corresponding to a system described by  $M(s)$ , there exists a GDG  $H$ , if and only if, there exists a polynomial vector  $\underline{m}(s) \in \mathcal{M}$ , with degree  $\partial\{\underline{m}(s)\} = \delta$ , where

$$\underline{m}(s) = P_m \cdot e_{\delta(s)}, \quad P_m \in \mathbb{R}^{(m+p) \times (\delta+1)}$$

such that the following condition is satisfied

$$\rho\{P_m\} \leq m \quad (8)$$

where,  $P_m$  is the coefficient matrix of  $\underline{m}(s)$  and  $e_{\delta}(s) = [1, s, \dots, s^{\delta}]^t$  denotes the standard basis vector.  $\square$

Based on the result above the following sufficient conditions are implied. For the proof of the following theorem see (Karcaniyas et al., 2014).

*Theorem 8.* Let  $\underline{m}(s) \in \mathcal{M}$  with degree  $\partial\{\underline{m}(s)\} = \delta$ . A sufficient condition for the existence of a constant degenerate compensator is that

$$m \geq \delta + 1 \quad (9)$$

$\square$

Condition (6) implies that the search for degenerate solutions is reduced to finding  $\underline{m}(s) \in \mathcal{M}$  for which an  $H(s)$  exists. Every polynomial vector  $\underline{m}(s) \in \mathcal{M}$  for which there exists  $H(s) \in \mathbb{R}^{p \times (m+p)}[s]$ , with  $\rho\{H(s)\} = p$  such that (6) is satisfied will be called a *generator*, or *Degeneracy Progenitor* (DP) and the family of all such vectors will be referred to as the *Degeneracy Progenitors Family* (DPF) and will be denoted by  $\mathcal{D}$ .

*Remark 9.* Let a polynomial vector,  $\underline{m}(s) = P_m \cdot e_{\delta}(s) \in \text{colsp}_{\mathbb{R}[s]}\{M(s)\}$ , be a *generator* (or DP). The family of GDGs associated with the given DP is defined by the family of all  $p$ -dimensional subspaces of  $\mathcal{N}_{\ell}\{P_m\}$ .  $\square$

The family of all GDGs associated with a given  $\underline{m}(s) \in \mathcal{D}$  is denoted by  $\mathcal{L}_{\underline{m}}$  and will be called the  $\underline{m}(s)$ -GDG family (or  $\underline{m}(s)$ -GDGF). The set defined by

$$\langle \mathcal{L} \rangle = \{\mathcal{L}_{\underline{m}} : \forall \underline{m}(s) \in \mathcal{D}\}$$

is the set of all GDGs and will be called the *Gain Degeneracy Set* (GDS) of the system. In the following section, the necessary and sufficient conditions for the existence of different degree degenerate solutions and their parametrisation will be investigated in more detail. Let us before present some of the main tools that are required for the following analysis.

## 2.2 Background on the Theory of Minimal Bases

Clearly, the system composite polynomial MFD,  $M(s)$ , can always be factorized as

$$M(s) = M^*(s) \cdot Z(s) \quad (10)$$

where  $M^*(s)$  is an Ordered Minimal Basis (OMB) of the  $\mathbb{R}[s]$ -module  $\mathcal{M}$  and  $Z(s)$  is a greatest right divisor. Clearly, by (4) and (10) it is obvious that degenerate solutions are worked out by solving

$$\det\{H(s) \cdot M^*(s)\} \equiv 0 \quad (11)$$

Note also that,

$$\mathcal{M}^* \triangleq \text{colsp}_{\mathbb{R}[s]}\{M^*(s)\}$$

is a maximal Noetherian module as well. We shall also assume that

$$\mathcal{I}_{\mathcal{C}}(M) = \{(\delta_i, \rho_i) : 0 < \delta_1 < \dots < \delta_{\mu}, i \in \underline{\mu}\} \quad (12)$$

is the ordered representation of the set of controllability indices of the system and that  $M^*(s) \in \mathbb{R}^{(m+p) \times p}[s]$  is an *Ordered Minimal Basis* (OMB) matrix for the module  $\mathcal{M}$ , as shown below:

$$M^*(s) = [M_1(s); \dots; M_i(s); \dots; M_{\mu}(s)] \quad (13)$$

$$M_i(s) = [\underline{m}_1^i(s), \dots, \underline{m}_{\rho_i}^i(s)] \in \mathbb{R}^{(m+p) \times \rho_i}[s]$$

$$\partial\{\underline{m}_j^i(s)\} = \delta_i - 1 = \tilde{\delta}_i, \quad \forall j \in \{1, 2, \dots, \rho_i\}$$

Also, from the theory of minimal basis (Forney, 1975) we have the following result:

*Lemma 10.* Let us denote by  $\mathcal{N}_{\ell}^m \equiv \mathcal{N}_{\ell}\{\underline{m}(s)\}$  the left null-space of the polynomial vector  $\underline{m}(s)$ , with  $\deg\{\underline{m}(s)\} = \delta$  and  $\dim\{\mathcal{N}_{\ell}^m\} = p + m - 1$ . If we denote also by  $d_i, i = 1, 2, \dots, m + p - 1$  the minimal indices of  $\mathcal{N}_{\ell}^m$ , then

$$\sum_{i=1}^{p+m-1} d_i = \delta \quad (14)$$

$\square$

*Remark 11.* The indices  $\{d_i, i = 1, 2, \dots, p + m - 1\}$  define a partitioning of the number  $\delta$ .  $\square$

The properties of minimal bases have been examined in detail in (Karcaniyas et al., 2013, 2014), where new algebraic and geometric invariants have been introduced.

### 3. PARAMETRISATION OF DEGENERATE COMPENSATORS

Throughout this note it will be assumed that for a linear proper system  $G(s) \in \mathbb{R}^{m \times p}$  the triple  $(n, p, m)$  denotes the dimensions of the system, that is the number of states, inputs and outputs respectively.

#### 3.1 Parametrisation of Dynamic Degenerate Solutions

From Proposition (6), it is obvious that as long as  $m \geq 1$ , any vector in  $M^*(s)$  may be used as a generator of a dynamic degenerate compensator. The characterization and the parametrisation of the family of dynamic degenerate solutions is given by the following result. For the proof see (Karcianas et al., 2014).

*Theorem 12.* Let  $M^*(s) \in \mathbb{R}^{(m+p) \times p}[s]$  be an OMB, as seen in (13), with index set as in (12) and let,  $\delta_\nu$ , be the index such that

$$m \geq \delta_\nu + 1 \quad \text{and} \quad m < \delta_{\nu+1} + 1$$

then there always exists generators of dynamic degenerate compensators with maximal degree  $d = \delta_\nu$ . Furthermore, a family of such generators is defined parametrically by

$$\underline{m}(s) = \sum_{i=1}^{\nu} M_i(s) \cdot \underline{a}_i(s), \quad \partial\{\underline{a}_i(s)\} = \delta_\nu - \delta_i \quad (15)$$

□

Otherwise, if it is desired to find a constant degenerate compensator we have to use the vectors in  $M^*(s)$  with the least degree, let say  $\delta_1$ . The above observations readily leads to the characterization of constant degenerate solutions.

#### 3.2 Parametrisation of Constant Degenerate Solutions

A preliminary result characterising the *generators* (DPs) of constant degenerate compensators is given next.

*Proposition 13.* For a vector  $\underline{m}(s) \in \mathcal{M}$ , with  $\partial\{\underline{m}(s)\} = \delta$ , if  $\delta \leq m - 1$ , then  $\underline{m}(s)$  is a generator (DP).

**Proof.** Let  $\underline{m}(s) = P_m \cdot \underline{e}_\delta(s)$ ,  $P_m \in \mathbb{R}^{(m+p) \times (\delta+1)}$ . It has been shown that  $\underline{m}(s)$  is a DP, if  $\rho(P_m) \leq m$ ; however it is always true that  $\rho(P_m) \leq \delta + 1$  and thus, if  $\delta + 1 \leq m$  then automatically the rank condition is satisfied.

□

The above result establishes the existence of a certain family of generators (DPs) for which the rank condition,  $\rho(P_m) \leq m$ , is automatically satisfied due the dimensionality rather than the numerical values of  $\underline{m}(s)$ .

*Remark 14.* Stronger sufficient conditions may be derived if the partitioning of  $\delta$  to  $\{1, \dots, 1\}$  is not considered but other partitions are examined. However, such investigations are not vector independent and the properties of the vectors in  $M_1(s)$  as far as left null-space properties have to be examined.

□

The following corollary gives us an indication of how this particular family of degenerate solutions may be generated parametrically. For the proof see (Karcianas et al., 2014).

*Corollary 15.* Let  $M_1(s)$  be the submatrix of the minimal basis  $M^*(s)$  of  $\mathcal{M}$  with the least degree, let say  $\delta_1$ . If

$$m \geq \delta_1 + 1 \quad (16)$$

then, every vector of this form

$$\underline{m}(s) = M_1(s) \cdot \underline{a}, \quad \underline{a} \in \mathbb{R}^{\rho_1} \quad (17)$$

defines a generator for constant degenerate solutions (GDGs).

#### 3.3 Characterisation of a Special Family of Generators

A special family of generators for which the condition  $\delta \leq m$  is satisfied, may be referred to as *Simple DPs* (SDP) whereas those DPs for which  $\delta > m - 1$  will be referred to as *non-simple DPs* (CDP). A system having at least a SDP will be called a *simple system*, whereas those having all DPs non-simple will be referred to as a *non-simple system*. Obviously we have:

*Proposition 16.* If  $\delta_1$  is the smallest controllability index, then the system is simple, iff  $\delta_1 \leq m$ .

□

The following Theorem summarizes the properties of the set of simple generators (SDPs). For the proof see (Karcianas et al., 2014).

*Theorem 17.* Let  $\mathcal{I}_c$  be the set of controllability indices, as in (12), of a simple system having  $m$ -outputs and let  $\delta_i$  be the index such that

$$\delta_i = \max\{\delta_j : \delta_j \leq m\}$$

If  $M(s) = [M_1(s); \dots; M_i(s); \dots; M_\mu(s)]$  is an OMB of  $\mathcal{M}$  defined by conditions (13), then the following properties hold true:

- (i) Every vector of the partial OMB

$$\bar{M}(s) = [M_1(s); \dots; M_i(s); \dots; M_\mu(s)]$$

associated with the partial set of indices

$$\mathcal{I}_c^i = \{(\delta_1, \rho_1), \dots, (\delta_i, \rho_i)\}$$

is a simple generator (DP).

- (ii) The family of all simple DPs of the system, denoted by  $\mathcal{D}_s$  is the set which is defined parametrically by:

$$\begin{aligned} \mathcal{D}_s = \{ \underline{m}(s) : \underline{m}(s) = M_1(s) \cdot \underline{a}_1(s) + \dots + \\ + M_i(s) \cdot \underline{a}_i(s), \quad \forall \underline{a}_i(s) \in \mathbb{R}^{\rho_j}[s] \\ \text{and} \quad \partial\{\underline{a}_j(s)\} \leq m - \delta_j, \quad \forall j \in [1, \dots, i] \} \end{aligned}$$

- (iii) The set  $\mathcal{D}_s$  is a vector space over  $\mathbb{R}$  with dimension

$$\delta = \dim \mathcal{D}_s = \sum_{j=1}^i (m - \delta_j + 1) \rho_j$$

## 4. SELECTION OF DEGENERATE SOLUTIONS FOR FREQUENCY ASSIGNMENT

### 4.1 Review on the Global Asymptotic Linearisation Method for Linear Systems

Having examined the necessary conditions for their existence and the way we construct degenerate points of different degrees we may proceed to the crucial part of examining the asymptotic properties of the Pole Placement Map (PPM) close to a degenerate point. This will allow the selection of appropriate degenerate solutions shaping the

properties of the corresponding PPM. The extended PPM associated with the static problem is the map assigning  $H$  to the coefficient vector  $\underline{p}$  of the desired pole polynomial  $p(s)$ , is defined by:

$$\mathcal{F} : \mathbb{R}^{p \times (p+m)} \rightarrow \mathbb{R}^{n+1} : \mathcal{F}(H) = \underline{p} \quad (18)$$

The PPM  $\mathcal{F}$ , as shown in (18), is a well defined polynomial map, differentiable on the whole of its domain and has the property of homogeneity. Let now  $H_d$  be a degenerate compensator. The asymptotic properties of a degenerate compensator can be examined by calculating the differential of  $\mathcal{F}$ ,  $D(\mathcal{F})$ , for  $H = H_d + \varepsilon H'$  evaluated at  $H_d$ , which will be denoted for simplicity as  $DF_{H_d}$ . Following (Leventides and Karcianas, 1995) we may state:

*Theorem 18.* : Let  $H_d \in \{H\}_d$  be a degenerate compensator. Then, the differential of the frequency assignment map  $\mathcal{F}$  at  $H_d(s)$  is given by:

$$DF_{H_d} = \sum (b_{ij} \cdot p_{ij}) \quad (19)$$

where  $p_{ij}$  is the determinant of the polynomial matrix  $D_{ij}$  having the same rows with the matrix  $HM(s)$  apart from the  $i$ -th which is replaced by the  $j$ -th row of  $M(s)$ .  $\square$

It is well known that a multivariable system has the arbitrary pole assignment property if the map  $\mathcal{F}$  is onto (or surjective) (Wang, 1992). We will show that the answer to this fundamental problem is reduced to exploring when there exists degenerate solutions that permit the application of the global linearisation technique and hence the placement of an arbitrary characteristic polynomial. A  $p \times (m+p)$  matrix is a degenerate compensator if  $\mathcal{F}(H_d) = 0$ . Let us use the following to define an important class of compensators.

*Definition 19.* A matrix  $H$ , whose Jacobian  $DF_H$  is of full rank (i.e. is an onto linear map), will be a *full compensator* for the plant  $M(s)$ .  $\square$

The importance of degenerate compensators that are also full is due to the following:

*Theorem 20.* For a proper multivariable system, if there exists a full degenerate compensator  $H_d$ , such that the differential of the PPM  $\mathcal{F}$  evaluated at  $H_d$  is onto, then any polynomial of degree  $n$  may be assigned via some static compensator.  $\square$

In order to prove that the differential of the PPM has full rank we will use a relaxed formulation of the Dominant Morphism Theorem. For the proof see (Borel, 1991, Theorem 17.3).

*Lemma 21.* (Dominant Morphism Theorem). If  $\mathbb{F}$  is an algebraic map between two algebraic varieties  $X, Y$ , such that  $\dim\{X\} \geq \dim\{Y\}$  then, there exists

$$\underline{x} \in X : \text{rank}\{D(\mathbb{F})_{\underline{x}}\} = \dim\{Y\}$$

if and only if, the map  $\mathbb{F}$  is (almost) onto.  $\square$

Therefore, in order to show that the differential of  $\mathcal{F}$  at  $H_d$  has full rank it is sufficient to prove that the corresponding map is onto. The following proposition suggests:

*Corollary 22.* The map  $\mathcal{F}$  introduced in (18) is onto as long as there is a degenerate and full compensator  $H_d$ .

**Proof.** By the Inverse Function Theorem,  $\mathcal{F}$  maps a neighbourhood  $H$  into a neighbourhood of 0 in  $\mathbb{R}^{n+1}$ . Since  $\mathcal{F}$  is homogeneous, then is onto.  $\square$

Thus, it is clear that for a system  $M(s)$  the arbitrary pole assignment property is linked directly with the existence of degenerate compensators which conditions and parametrisation has been discussed in this paper.

#### 4.2 Characterisation of the Regular Degeneracy Progenitors

In this section we are going to examine the properties of the generators  $\underline{m}(s) \in \mathcal{D}$ , which allow the presence of special properties in the associated family of degenerate gains,  $\mathcal{L}_{\underline{m}}$ , that will guarantee the global asymptotic linearisation of the pole placement map and hence the assignment of the desired frequencies. Let us first consider some basic properties of the set  $\mathcal{L}_{\underline{m}}$  associated with a generator  $\underline{m}(s) \in \mathcal{D}$ .

*Definition 23.* Consider a generator,  $\underline{m}(s) = P_m \cdot \underline{e}_\delta(s) \in \mathcal{D}$  and let denote the left null-space of  $P_m$  by  $\mathcal{N}_\ell = \mathcal{N}_\ell\{P_m\}$ , where  $\tau = \dim\{\mathcal{N}_\ell\} \geq P_m$ . The  $\underline{m}(s) - GDG$  family,  $\mathcal{L}_{\underline{m}}$ , is defined as:

$$\mathcal{L}_{\underline{m}} = \{\forall H \in \mathbb{R}^{p \times (p+m)} : \text{rowspan}_{\mathbb{R}}\{H\} \in \mathcal{G}(p; \mathcal{N}_\ell)\}$$

where  $\mathcal{G}(p; \mathcal{N}_\ell)$  denotes the Grassmannian, i.e. the set of all  $p$ -dimensional subspaces in  $\mathcal{N}_\ell$ . Furthermore,  $\mathcal{N}_\ell$  will be called the *space generator* of  $\mathcal{L}_{\underline{m}}$ .  $\square$

The properties of  $\mathcal{L}_{\underline{m}}$  clearly depend on the properties of the generating space  $\mathcal{N}_\ell$ . The structure of the space  $\mathcal{N}_\ell$  is described by the row-Hermite form of  $\mathcal{N}_\ell$ ,  $N_H \in \mathbb{R}^{\tau \times (p+m)}$ , as described below:

$$N_H = \begin{bmatrix} 0 & \cdots & 0 & 1 & * & \cdots & * & 0 & * & \cdots & 0 & * & \cdots & * \\ 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & * & \cdots & 0 & * & \cdots & * \\ 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 1 & * & \cdots & 0 & * & \cdots & * \\ \vdots & \vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & 0 & 0 & 0 & \cdots & \cdots & 0 & 0 & \cdots & 0 & * & \cdots & * \\ \vdots & \vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & 0 & 0 & 0 & \cdots & \cdots & 0 & 0 & \cdots & 1 & * & \cdots & * \end{bmatrix}$$

where, we denote by  $n_i$  the standard basis column vectors. The  $0 < n_1 < \cdots < n_\tau$  and the elements denoted by  $*$  are uniquely defined. The set  $\{n_i, i = 1, 2, \dots, \tau\}$  are the *Hermite indices* of  $\mathcal{N}_\ell$  and will be referred to as the *Hermite Characteristic* of the generator  $\underline{m}(s)$ . The existence of regular type GDG in  $\mathcal{L}_{\underline{m}}$  is established by the following result which proof may be found in (Karcianas et al., 2014).

*Theorem 24.* Let  $\underline{m}(s) = P_m \cdot \underline{e}_\delta(s) \in \mathcal{D}$  and  $\{n_i, i = 1, 2, \dots, \tau\}$  be its Hermite Characteristic and let us partition  $P_m \equiv P$  as

$$P = \begin{bmatrix} P_1 \\ P_2 \end{bmatrix} \in \mathbb{R}^{(p+m) \times (\delta+1)}, P_1 \in \mathbb{R}^{p \times (\delta+1)} \quad (20)$$

The family  $\mathcal{L}_{\underline{m}}$  contains at least a regular gain  $H$ , if and only if, either of the following equivalent conditions hold true:

- (i)  $n_p = p$ ;
- (ii) If  $\mathcal{P} \triangleq \text{rowsp}_{\mathbb{R}}\{P_i\}$ ,  $i = 1, 2$ , then  $\mathcal{P}_1 \subseteq \mathcal{P}_2$ ;
- (iii)  $\text{rank}\{P_2\} = \text{rank}\{P\}$ .

□

The above result allows a parametrisation of all regular elements in the  $\mathcal{L}_{\underline{m}}$  family as shown below. For the proof see (Karcianas et al., 2014).

*Corollary 25.* Consider  $\underline{m}(s) = P_{\mathcal{L}_s}(s) \in \mathcal{D}$ , assume that  $n_p = p$ . If  $N_H$  is the Hermite basis matrix of  $\mathcal{N}_\ell$ , and we express  $N_H$  as:

$$N_H = \begin{bmatrix} I_p & B_H \\ 0 & C_H \end{bmatrix} \in \mathbb{R}^{\tau \times (m+p)}$$

then,  $H_H = [I_p, B_H]$ , is a particular regular element of  $\mathcal{L}_{\underline{m}}$  and is defined by:

$$\mathcal{L}_{\underline{m}}^r = \{H \in \mathcal{L}_{\underline{m}} : H = [Q_1, Q_1 B_H + Q_2 C_H]\} \quad (21)$$

for each  $Q_1 \in \mathbb{R}^{p \times p}$ ,  $Q_2 \in \mathbb{R}^{p \times (\tau-p)}$ , with  $|Q_1| \neq 0$ .

□

The above results provide a complete characterisation of all regular gains associated with a given DP-vector  $\underline{m}(s)$ . A general characterisation of those systems for which regular gains may be defined is examined now. We first note:

*Lemma 26.* (Kailath (1980)). If  $G(s) = N(s)D(s)^{-1}$  is a proper (strictly proper) transfer function matrix, then every column of  $N(s)$  has degree less than (less than or equal) to that of the corresponding column of  $D(s)$ .

□

Using the above Lemma the following result can be established. for the proof see (Karcianas et al., 2014).

*Proposition 27.* If the system is strictly proper, then for every generator element  $\underline{m}(s) \in \mathcal{D}$  all elements of  $\mathcal{L}_{\underline{m}}$  are irregular. The set  $\langle \mathcal{L} \rangle$  may contain regular elements for some generators  $\underline{m}(s)$  only when the system is proper.

□

## 5. CONCLUSION

In this note the notion of degeneracy and degenerate solutions for proper multivariable systems has been studied on the original dynamic set-up. It has been shown that such solutions are crucial for the linearisation of the related frequency assignment map and play an important role to the solvability of the output feedback pole assignment problem. Both the dynamic and the static versions have been examined and conditions for their existence are given. The parametrisation of different families of degenerate compensators according to their degree has been provided by using the theory of minimal bases. Then, by exploring the structure and the asymptotic properties of the PPM around degenerate points it has been shown that the existence of degenerate solutions guarantees that the related map is onto which is equivalent to say that the system has the arbitrary pole assignment property. Also, it has been proved that only proper systems contain regular-type, i.e. bounded compensation gains. Similar problems, appear as matrix completion or inverse eigenvalue problems, arise in the areas of systems identification and conformal quantum theory amongst others. The overall aim is to develop

stable numerical algorithms and expand further the global linearisation method into a powerful integrated tool which can be applied to solve any DAP for linear systems not necessarily generic.

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## REFERENCES

- Borel, A. (1991). *Linear Algebraic Groups*. Graduate Texts in Mathematics. Springer-Verlag.
- Brockett, R. and Byrnes, C. (1981). Multivariable Nyquist Criterion, Root Loci and Pole Placement: A Geometric Viewpoint. *IEEE Trans. Aut. Control*, AC-26, 271–283.
- Forney, G. (1975). Minimal Bases of Rational Vector Spaces. *SIAM J.*, 13, 493–520.
- Kailath, T. (1980). *Linear Systems*. Prentice Hall, NJ.
- Karcianas, N. (1996). Geometric Properties, Invariants and the Toeplitz Structure of Minimal Bases of Rational Vector Spaces. *IMA Journal of Control Theory Information*, 13, 1–11.
- Karcianas, N. and Giannakopoulos, C. (1984). Grassmann Invariants, Almost Zeros and the Determinantal Pole-zero Assignment Problems of Linear Systems. *Int. Journal of Control*, 40, 673–698.
- Karcianas, N. and Leventides, J. (2015). Solution of the Determinantal Assignment Problem using the Grassmann Matrices. *Int. Journal of Control*, 89, 352–367.
- Karcianas, N., Leventides, J., and Meintanis, I. (2013). Parameterisation of Degenerate Solutions of the Determinantal Assignment Problem. In *Proc. IFAC Symposium on Systems Structure and Control (SSSC2013)*, France.
- Karcianas, N., Leventides, J., and Meintanis, I. (2014). System Degeneracy and Constant Output Feedback: Parametrization of Systems and Families of Degenerate Gains. preprint S&CR/12-2014/NK-JL-IM 1/2, City University London.
- Leventides, J. and Karcianas, N. (1995). Global Asymptotic Linearisation of the Pole Placement Map: A Closed Form Solution of the Output Feedback Problem. *Automatica*, 31(9), 1303–1309.
- Leventides, J. and Karcianas, N. (1998). Dynamic Pole Assignment Using Global Blow-up Linearisation: Low Complexity Solution. *Journal of Optimization Theory and Applications*, 96(1), 57–86.
- Leventides, J., Karcianas, N., Meintanis, I., and Milonidis, E. (2014a). Reduced Sensitivity Solutions to Global Linearization for the Pole Assignment Map. In *19th IFAC World Congress*, 4727–4732. Cape Town, South Africa.
- Leventides, J., Meintanis, I., and Karcianas, N. (2014b). A Quasi-Newton Optimal Method for the Global Linearisation of the Output Feedback Pole Assignment. In *22nd Mediterranean Conf. of Control and Automation (MED14)*, 157–163. Palermo, Italy.
- Leventides, J., Petroulakis, G., and Karcianas, N. (2014c). The Approximate Determinantal Assignment Problem. *Linear Algebra and its Applications*, 461, 139–162.
- Wang, X. (1992). Pole placement by static output feedback. *J.Math. Sys. Estim. Control*, 2, 205.