Zero Assignment Problem in RLC Networks

John Leventides * **Maria Livada** ** **Nicos Karcanias** **

* *University of Athens, Department of Economics, Section of Mathematics and Informatics, Pezmazoglou 8, Athens, Greece* (e-mail: ylevent@econ.uoa.gr)
** *City University London, Systems and Control Research Centre, School of Mathematics, Computer Science & Engineering, Northampton Square, London EC1V 0HB, United Kingdom* (e-mail: Maria.Livada.1@city.ac.uk, N.Karcanias@city.ac.uk)

Abstract: The paper deals with the problem of zero assignment in RLC networks by selection of appropriate values for the non- dynamical elements, the resistors. For a certain family of network redesign problems by the additive perturbations may be described as diagonal perturbations and such modifications are considered here. This problem belongs to the family of DAP problems (Determinantal Assignment Problem) and has common features with the pole assignment problem by decentralized output feedback and the zero assignment problem via structured additive perturbations. We demonstrate that the sufficient condition for generic zero assignment by selecting the resistors holds true. This condition is related to the rank of the differential of the related map and holds true generically when the degrees of freedom of the matrix of resistors exceeds the number of frequencies to be assigned ($n > p + q$). Using this result, we show through a generic example that the sufficient condition for the general zero assignment problems in RLC networks is satisfied and thus, zero assignment can be achieved via resistor determination.

Keywords: Algebraic Systems Theory, Network Theory, Systems Redesign.

1. INTRODUCTION

The problem of redesigning passive electrical networks (Karcanias and Leventides, 2010) requires the selection of alternative values both for the dynamic and non-dynamic elements within a fixed interconnection topology and/or alteration of the interconnection topology and with a possible evolution of the network by addition, or elimination of nodes/loops. The general redesign problem is far more complex than the problem considered here, which is defined by changing one type of elements and more specifically resistive elements without changing the respective topology of the network. The effect of such transformations on the Impedance/Admittance operator $W(s)$ are then considered and specifically the corresponding effects on the natural frequencies. The problem discussed here is within the general class of RLC network redesign problems and it will be shown that it can be reduced to a Determinantal Assignment Problem (DAP) discussed within the algebra-geometric framework for frequency assignment of Linear Systems (Karcanias and Giannakopoulos, 1984), (Leventides and Karcanias, 2009). The zero-assignment problem belongs to the general class of DAP (Determinantal Assignment Problem) (Humphreys, 1975),(Karcanias and Giannakopoulos, 1984),(Karcanias et al., 1988) which defines a unifying framework for the study for all frequency assignment problems in control theory with constant, dynamic centralized and decentralized control schemes (Karcanias and Giannakopoulos, 1984), (Wang, 1994a). The DAP framework relies on algebra, exterior algebra and new invariants associated with this framework (Karcanias and Giannakopoulos, 1984), (Karcanias and Giannakopoulos, 1989), (Karcanias et al., 1988) and it has been further developed by techniques to define solvability (Leventides, 2007), (Leventides and Karcanias, 2009), (Leventides and Karcanias, 1995b), (Karcanias and Leventides, 1996), (Wang, 1994a), (Wang, 1994b), (Leventides and Karcanias, 1995a) and (Karcanias and Leventides, 2015). The technique developed in (Leventides and Karcanias, 1995a), provides a powerful computational framework for computing exact solutions of DAP, when such solutions exist and the work in (Karcanias and Leventides, 2015) introduces a methodology for finding approximate solutions to DAP problems (even when exact solutions do not exist). The problem considered here has common features with the arbitrary pole assignment problem via constant decentralized output feedback (Leventides and Karcanias, 1995b), the zero assignment problem of matrix pencils by additive structured transformations (Leventides and Karcanias, 2009) and finally it is linked to work related with assigning frequencies via determinantal equations (Leventides, 2007). Mathematically, the problem is equivalent to solving a system of algebraic equations or to finding intersection of varieties (Fulton, 1998). Furthermore, it can be factored as a linear and a multi-linear problem, or as an intersection of a linear variety with a nonlinear projective variety. Its explicit structure leads to solutions, which do not employ conventional (typical) numerical analysis tools for this class of problems and it is based on the degenerate controllers (Leventides and Karcanias, 1995b). The methodology of "Global Linearisation" introduced in (Leventides and Karcanias, 1995a) relies on the selection of degenerate solutions (Brockett and Byrnes, 1981) and on the properties of the resulting pole placement map (Leventides, 2007). In general, when the differential of the defined related map (i.e. pole/ zero assignment map) has full rank at the degenerate controller, then the problem can be solved. This condition is satisfied generically for systems when the number of controller parameters exceeds the number of independent equations and can lead to a numerical procedure for the construction of solutions. However, this case is not considered here. The present approach for the study of the generic zero assignment, which is adopted here is based on the dominant morphism theorem (Humphreys, 1975), which relates to the onto properties of a complex rational or polynomial map. In fact, such a map is almost onto when there exists a point in the domain of the map, such that the differential at this point (a linear map) is onto. The exact problem can be tackled via degenerate controllers whenever there exist and are full, or via the Gröebner bases (Becker and Weispfenning, 1991),(Wait, 1979) of the equations defining the zero assignment.

2. THE W(S) OPERATOR AND THE IMPLICIT NETWORK DESCRIPTION

The general modelling for passive network provides a description of networks in terms of *symmetric* integral-differential operators (Karcanias and Leventides, 2010), (Leventides et al., 2014) the impedance and admittance models which are described in a general way by:

$$W(s) = s\mathbf{L} + s^{-1}\mathbf{C} + \mathbf{R} \tag{1}$$

where for the case of admittance we have that L is the matrix of A-type (capacitance) elements, C is the matrix of T-type (inductance) elements and R is the matrix of D-type (conductance) elements (for the case of impedance the reverse holds true). The operator W(s) is thus a common description of the Y(s) and Z(s) matrices and its properties will be investigated next. Clearly, the W(s) matrix is symmetric and the structure of L, C, R matrices characterizes the topology of A-, T- and D-type matrices associated with the network. Such matrices have a structure and properties that underpin the development of system theoretic framework based on network models. Using the general operator $W(s)$ we can provide an *Implicit Network Description* of the form:

$$\{p\mathbf{L} + p^{-1}\mathbf{C} + \mathbf{R}\} \cdot \mathrm{x} = 0 \tag{2}$$

where the vector x represents vertex voltages or loop currents. This network description has no inputs or outputs and it's characterized only by the general operator $W(s)$. An *Oriented Network Description* is defined when inputs and outputs are introduced, where for the oriented model we denote by $\underline{\mathrm{u}}$ the inputs and by $\underline{y}$ the outputs and it is described below:

$$\begin{cases} (p\mathbf{L} + p^{-1}\mathbf{C} + \mathbf{R}) \cdot \mathrm{x} = \mathbf{H} \cdot \mathrm{u} \\ y = \mathbf{E} \cdot \mathrm{x} \end{cases} \tag{3}$$

For this new implicit description we define $W^{-1}(s)$ as the *implicit transfer function* of network model and we define the explicit transfer function, or simply the transfer function as:

$$G(s) = \mathbf{E} \cdot W^{-1}(s) \cdot \mathbf{H} \tag{4}$$

$W^{-1}(s)$ as a rational function has a McMillan degree and this is defined as the *Implicit McMillan degree* δ_M of the network, whereas the *Explicit McMillan degree* δ is that defined on $G(s)$. A number of related properties will be discussed in Section 3.

Remark 1. An arbitrary network is well defined, that is $W(s) \neq 0$, if and only if its graph is connected (Karcanias and Leventides, 2010),(Leventides et al., 2014). □

Note that the operator $W(s)$ describes the dynamics of the network and of special interest are the properties of its zeros. In particular, we are interested how the topology and values of the different elements affect the natural frequencies of the network. The network re-engineering problem is then defined as defining changes in in the values and topologies of the different elements to produce a network with desirable natural frequencies. This is the general theme behind the current paper. A key question that arises is linking the McMillan degree δ_M of $W^{-1}(s)$ an RLC network to the rank properties of the matrices of the dynamical elements (inductances and capacitances). The maximum possible McMillan degree in an RLC network is achieved when the following conditions are satisfied:

Theorem 2. Let δ_m be the McMillan degree of $W^{-1}(s) = \left(s\mathbf{L} + \mathbf{R} + {}^1\!/_s\mathbf{C}\right)^{-1}$. Then the following are equivalent:

(a) $\delta_m = rank(\mathbf{L}) + rank(\mathbf{C})$

(b) $rank\left(\begin{bmatrix} \mathbf{R} & \mathbf{L} \\ \mathbf{L} & \mathbf{0} \end{bmatrix}\right) = n + rank(\mathbf{L})$ and

$rank\left(\begin{bmatrix} \mathbf{R} & \mathbf{C} \\ \mathbf{C} & \mathbf{0} \end{bmatrix}\right) = n + rank(\mathbf{C})$

Proof. Proof is given in detail in the technical report (see (Karcanias et al., 2015)). □

The zeros of the general operator $W(s)$ play a crucial role since they define the poles, natural frequencies of any RLC network having $W(s)$ as the implicit system operator. The assignment of these zeros by tuning the R, L, C elements and possibly changing their topology is an important problem that is considered here. Note that the W(s) operator can be re-written as:

$$W(s) = s\mathbf{L} + s^{-1}\mathbf{C} + \mathbf{R} = (s^2\mathbf{L} + s\mathbf{R} + \mathbf{C})/s = N(s)D^{-1}(s) \tag{5}$$

and the numerator of the above description $N(s)$ defines the zeros of $W(s)$.

3. PROBLEM DEFINITION AND PRELIMINARY ANALYSIS

3.1 Problem Definition

Given an arbitrary passive RLC network that is described by the general operator: $W(s) = (s^2\mathbf{L} + s\mathbf{R} + \mathbf{C})/s$ s , where R,L,C are symmetric matrices characterizing the topology of the network and the values of the corresponding elements, we need to determine a matrix of resistors R' such that if we add it to the network, then:

$$\det(s^2\mathbf{L} + s\mathbf{R} + \mathbf{C} + s\mathbf{R}') = \varphi(s) \quad (6)$$

where $\varphi(s)$ is the desired polynomial to be assigned. If R' is not diagonal then it is necessary to transform it into a diagonal matrix D , using the following lemma:

Lemma 3. Given a matrix R corresponding to a topology of resistors, there always exist matrices G, G^T which reduce R to a diagonal form and are strictly related to the topology of the given network. □

Remark 4. The transformations G, G^T are the graph incidence matrices. In particular $G^T \in \Re^{m\times n}$ is a matrix with i,$i = 1,...,m$ rows and j,$j = 1,...,n$ columns. Each row of the matrix corresponds to an element of the network, i.e. capacitor, inductance, resistor and each column corresponds to a loop or node of the given RLC network. Hence, an entry G_{ij} in the matrix is:

a. 1 if element i is present in loop / node j and the current i_j flows across the element i in the clockwise direction.
b. -1 if element i is present in loop / node j and the current i_j flows across the element i in the counter clockwise direction.
c. 0 if element i is not present in loop j.

If G^T denotes the incidence matrix for the matrices R, L, C then these matrices can be represented by:

$$\begin{aligned} \mathbf{R} &= G_R \cdot D_R \cdot G_R^T \\ \mathbf{L} &= G_L \cdot D_L \cdot G_L^T \\ \mathbf{C} &= G_C \cdot D_C \cdot G_C^T \end{aligned} \quad (7)$$

where D_C, D_R, D_L represent the diagonal matrices with entries the capacitors, resistors and inductances respectively in a given network. □

The above implies that instead of solving the equation (6), this is equivalent with solving the following equation as the determinant remains invariant.

$$\begin{aligned} &\det\left(s^2\mathbf{L} + s\mathbf{R} + \mathbf{C} + s\mathbf{R}^{'}\right) \\ &= \det\left(s^2\mathbf{L} + s\mathbf{R} + \mathbf{C} + s\mathbf{G^T}\cdot\mathbf{D}\cdot\mathbf{G}\right) \\ &= \det[\mathbf{G^T}\cdot(s^2\mathbf{G^{T^{-1}}}\mathbf{L}\mathbf{G^{-1}} + s\mathbf{G^{T^{-1}}}\mathbf{R}\mathbf{G^{-1}} \\ &\quad + \mathbf{G^{T^{-1}}}\mathbf{C}\mathbf{G^{-1}} + s\mathbf{D})\cdot\mathbf{G}] \\ &= \det\left(\mathbf{G^T}\right)\cdot(\det s^2\mathbf{G^{T^{-1}}}\mathbf{L}\mathbf{G^{-1}} + s\mathbf{G^{T^{-1}}}\mathbf{R}\mathbf{G^{-1}} \\ &\quad + \mathbf{G^{T^{-1}}}\mathbf{C}\mathbf{G^{-1}} + s\mathbf{D})\cdot\det\left(\mathbf{G}\right) \\ &= \lambda\cdot det\left(s^2\mathbf{L}' + \mathbf{sR}^{''} + \mathbf{C}' + \mathbf{sD}\right) \end{aligned} \quad (8)$$

3.2 The Determinantal Assignment Nature of the Problem

The study of Determinantal Assignment Problems (DAP) has been developed using tools from exterior algebra and classical algebraic geometry (Karcanias and Giannakopoulos, 1984), (Leventides, 2007), (Leventides and Karcanias, 1995b), (Leventides and Karcanias, 1995a), (Karcanias and Leventides, 2015). The starting point of our work is the problem of arbitrary assignment of frequencies via static compensation and the general formulation is given next. Given a polynomial matrix $H(s) \in \Re[s]^{(m\times p)\times p}$ investigate the solvability of the equation:

$$\det(K\cdot H(s)) = \varphi(s) \quad (9)$$

with respect to $K \in \Re[s]^{p\times(p+m)}$ where $\varphi(s)$ is an arbitrary polynomial of degree equal to the degree of $H(s)$.

Using the Binet-Cauchy Theorem (Marcus and Minc, 1964) eq. (9) can be formulated as follows:

$$C_p(K)\cdot C_p(H(s)) = \varphi(s) \quad (10)$$

Then the problem can be factored as a:

- Linear problem: Solve the following equation with respect to $\underline{x}$:
$$\underline{x}\cdot P = \underline{\varphi} \quad (11)$$
- Multi-linear problem: For a given $\underline{x}$ find a matrix such that:
$$\underline{x} = C_p(K) \quad (12)$$
which is an intersection of a linear variety, with the Grassmann set of all decomposable vectors (Karcanias and Giannakopoulos, 1984).

Applying the above approach to eq. (8) leads to:

$$\begin{aligned} &\det(s^2\mathbf{L}' + s\mathbf{R}'' + \mathbf{C}' + s\mathbf{D}) = \\ &= \det\left([\,\mathbf{I}\ \mathbf{D}\,]\cdot\begin{bmatrix} s^2\mathbf{L}' + s\mathbf{R}'' + \mathbf{C}' \\ s\mathbf{I}\end{bmatrix}\right) = \\ &= C_n\,[\,\mathbf{I}\ \mathbf{D}\,]\cdot C_n\begin{bmatrix} s^2\mathbf{L}' + s\mathbf{R}'' + \mathbf{C}' \\ s\mathbf{I}\end{bmatrix} = \varphi(s) \end{aligned} \quad (13)$$

In the following, we shall note here that $rank(\mathbf{D}) = n$, $rank(\mathbf{L}) = p$ and $rank(\mathbf{C}) = q$. It will be shown that to achieve complete frequency assignability the number of resistors that are added to the network should always exceed its implicit McMillan degree, i.e.$n > p + q$ also, the differential of the assigned map at a *degenerate point* (Leventides and Karcanias, 1995a) plays a vital role in the solvability of our problem as we will see in the next section.

Remark 5. For the DAP formulation of (9), or (13), a perturbation D will be called *degenerate* if (Leventides and Karcanias, 1995a), (Brockett and Byrnes, 1981):

$$\det\left([\,\mathbf{I}\ \mathbf{D}\,]\cdot\begin{bmatrix} s^2\mathbf{L}' + s\mathbf{R}'' + \mathbf{C}' \\ s\mathbf{I}\end{bmatrix}\right) \equiv 0 \quad (14)$$

□

It is now assumed that the matrix D is diagonal. If D is non-diagonal then we transform it by multiplying it with an appropriate matrix G, which is invertible (i.e $\det G \neq 0$, or G^{-1} exists) as it has been previously explained.

3.3 Differential of the Problem

The differential of the frequency assignment map P (Leventides and Karcanias, 2009), (Leventides, 2007), (Leventides and Karcanias, 1993) associated with our problem, plays a very important role in the determination of the onto properties of the map and it has thus a crucial role in the solvability of the problem. The differential can be calculated in many ways and for a general square polynomial matrix $A(s)$ the following results hold (Leventides and Karcanias, 2009).

Lemma 6. If $A(s)$ is a polynomial matrix, then:

$$\begin{aligned} &\det\left(A\left(s\right) + xB\left(s\right)\right) = \\ &\det\left(A\left(s\right)\right) + x\cdot trace\left[adj\left(A\left(s\right)\right)\cdot B\left(s\right)\right] + O\left(x^2\right) \end{aligned} \quad (15)$$

Corollary 7. If $adj\,(sA+B-\Lambda_0)=\underline{v}\,(s)\cdot\underline{g}^t\,(s)$ and $g_i\,(s)$, $v_i\,(s)$ are the coordinates of these vectors, then the differential at the degenerate point DF_{Λ_0} can be represented by the coefficient matrix of the polynomial vector:

$$(g_1(s)v_1(s),\ldots,g_n(s)v_n(s)) \tag{16}$$

□

Using the above established results we will now introduce the differential of an arbitrary RLC network that has a description given by the general operator $W(s)$.

Lemma 8. If P_t represents the frequency assignment map such that:

$$P_t:\mathbb{C}^n\to\mathbb{C}^{p+q+1} \tag{17}$$

then the differential of this map at arbitrary point D_0, can be shown to be of the form:

$$\begin{aligned} &DP_t|_{D_0}\,(B)= \\ &=Coef.Vector\left[trace\left(Adj\left(s^2L+s(R+D_0)+C\right)\cdot B\right)\right] \\ &=Coef.Vector\,(p_1\,(s)\cdot\beta_1+p_2\,(s)\cdot\beta_2+\ldots+p_n\,(s)\cdot\beta_n) \end{aligned} \tag{18}$$

□

The above results may now be used for the study of the Frequency Assignment by diagonal perturbations.

4. ARBITRARY FREQUENCY ASSIGNMENT VIA DIAGONAL PERTURBATIONS - GENERIC RESULTS AND CONSTRUCTION OF SOLUTIONS

In this section we present the main results of the paper. Using the *Dominant Morphism* Theorem (Borel, 1991) we will give a generic solution to the frequency assignment problem, by constructing a generic example and we will establish sufficient conditions for arbitrary frequency assignment under the diagonal perturbations. The construction of solutions may be achieved using various numerical methods. Gröebner Bases (Becker and Weispfenning, 1991), (Wait, 1979) techniques may be used as the computational method.

4.1 Dominant Morphism Theorem

To develop the Generic solution in the next section we will use the *Dominant Morphism* Theorem, which is illustrated below (Borel, 1991):

Proposition 9. (Dominant Morphism Theorem). If F is an algebraic map between two complex varieties X,Y such that $\dim X\geq\dim Y$ then: there exists x in X: $rankDF_x=\dim Y$ iff F is (almost) onto.

We shall note here that the dominant morphism theorem although proves the existence, is not appropriate for the construction of a solution. To construct the solutions we can utilize the usual methods based on the multi-linear/ determinantal formulation and then solving the set of algebraic equations using Gröebner bases methods.

4.2 Generic Solution

Let us consider the set:

$$S_{p,q}=(\mathbf{L},\mathbf{R},\mathbf{C}\in\mathbb{C}^{n\times n}:rank(\mathbf{L})=p,rank(\mathbf{C})=q,\ \text{where } \mathbf{R},\mathbf{L},\mathbf{C}\ \textit{symmetric})$$

For $t\in S_{p,q}$ consider the map: $P_t:\mathbb{C}^n\to\mathbb{C}^{p+q+1}$. This map, maps $D=diag\,(d_1,d_2,\ldots,d_n)$ to the coefficients of the powers of s $(p_0,p_1,\ldots,p_{p+q})$ of the determinant:

$$\begin{aligned} &\det\left(s^2\mathbf{L}+s(\mathbf{R}+\mathbf{D})+\mathbf{C}\right)= \\ &=\left(p_{p+q}\cdot s^{p+q}+\ldots+p_1\cdot s+p_0\right)\cdot s^{n-q} \end{aligned} \tag{19}$$

We will use the *Dominant Morphism Theorem* stated in Proposition 4.1. Now consider:

$$D_0=\begin{bmatrix} 1 & 0 & \cdots & & & & & & & & 0 \\ 0 & 2 & & & & & & & & & \vdots \\ \vdots & & 3 & & & & & & & & \\ & & & \ddots & & & & & & & \\ & & & & p & & & & & & \\ & & & & & \frac{1}{p+1} & & & & & \\ & & & & & & \frac{1}{p+2} & & & & \\ & & & & & & & \ddots & & & \\ & & & & & & & & \frac{1}{p+q} & & \\ & & & & & & & & & 1 & \\ 0 & & & & & & & & & \ddots & 0 \\ 0 & \cdots & & & & & & & & 0 & 1 \end{bmatrix} \tag{20}$$

Then the differential $DP_t|_{D_0}$ is an $n\times(p+q+1)$ matrix depending polynomially at the parameters of $t\in S_{p,q}$. Therefore, the set:

$$S^{'}=(t\in S_{p,q}:rank\left(DP_t|_{D_0}\right)=p+q+1) \tag{21}$$

is a Zarisky open subset of $S_{p,q}$. To prove that $S^{'}$ is non-void it is sufficient to demonstrate an example, such that: $DP_t|_{D_0}=p+q+1$.

4.3 Generic Example

Indeed consider the system t_0 with matrices $\mathbf{L_0},\mathbf{R_0},\mathbf{C_0}$ such that:

$s^2\mathbf{L_0} + s\mathbf{R_0} + \mathbf{C_0} =$

$$\begin{bmatrix} \underbrace{\left.\begin{matrix} s^2 & & & \\ & s^2 & & \\ & & \ddots & \\ & & & s^2 \end{matrix}\right\} p}_{p} & & \\ & \underbrace{\left.\begin{matrix} 1 & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{matrix}\right\} q}_{q} & \\ & & \underbrace{\left.\begin{matrix} 0 & & & \\ & 0 & & \\ & & \ddots & \\ & & & 0 \end{matrix}\right\} n-p-q}_{n-p-q} \end{bmatrix} \tag{22}$$

Then

$$s^2\mathbf{L_0} + s\mathbf{R_0} + \mathbf{C_0} + s\mathbf{D_0} = \begin{bmatrix} A_1 & & \\ & A_2 & \\ & & A_3 \end{bmatrix} \tag{23}$$

$$\text{where: } A_1 = \begin{bmatrix} \underbrace{\left.\begin{matrix} s^2+s & & & \\ & s^2+2s & & \\ & & \ddots & \\ & & & s^2+ps \end{matrix}\right\} p}_{p} \end{bmatrix},$$

$$A_2 = \begin{bmatrix} \underbrace{\left.\begin{matrix} \frac{1}{p+1}s+1 & & & \\ & \frac{1}{p+2}s+1 & & \\ & & \ddots & \\ & & & \frac{1}{p+q}s+1 \end{matrix}\right\} q}_{q} \end{bmatrix}$$

and

$$A_3 = \begin{bmatrix} \underbrace{\left.\begin{matrix} s & & & \\ & s & & \\ & & \ddots & \\ & & & s \end{matrix}\right\} n-p-q}_{n-p-q} \end{bmatrix}$$

and

$\det\left(\left(s^2\mathbf{L_0} + s(\mathbf{R_0}+\mathbf{D_0}) + \mathbf{C_0}\right)\right) =$

$= \dfrac{p!}{(p+q)!} \cdot (s+1)(s+2)\cdots(s+p+q) \cdot s^{n-q}$. Then $DP_t|_{D_0}$ contains the matrix:

$$\begin{bmatrix} & \underline{f}_0 \\ 0 & \underline{f}_1 \\ 0 & \underline{f}_2 \\ \vdots & \vdots \\ 0 & \underline{f}_{p+q} \end{bmatrix}$$

where $\underline{f}_0$ is the coefficient matrix of the polynomial:

$$f(s) = \frac{p!}{(p+q)!}(s+1)(s+2)\cdots(s+p+q)$$

and $\underline{f}_i$ is the coefficient matrix of the polynomial:

$$f_i(s) = \begin{cases} \dfrac{f(s)}{s+i}, 1 \le i \le p \\ i \cdot \dfrac{f(s)}{s+i}, \text{p+1} \le i \le p+q \end{cases}$$

.

For this matrix $DP_t|_{D_0}$ to have rank $(\mathrm{p}+\mathrm{q}+1)$ is equivalent for the matrix:

$$F = \begin{bmatrix} \underline{f}_1 \\ \underline{f}_2 \\ \vdots \\ \underline{f}_{p+q} \end{bmatrix}$$

to have rank: $(p+q)$. Indeed if we call V the $(p+q) \times (p+q)$ Vandermonde matrix:

$$V = \begin{bmatrix} 1 & 2^{p+q-1} & 3^{p+q-1} & \dots & (p+q)^{p+q-1} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & 2^2 & 3^2 & \dots & (p+q)^2 \\ 1 & 2 & 3 & \dots & p+q \\ 1 & 1 & 1 & \cdots & 1 \end{bmatrix}$$

then we have: $F \cdot V = Diag(\sigma_1, \sigma_2, \dots, \sigma_{p+q})$, where:

$$\sigma_i = \begin{cases} \dfrac{p!}{(p+1)!} \displaystyle\prod_{j=1, j\ne i}^{j=p+q} (i-j), 1 \le i \le p \\ \dfrac{p!}{(p+1)!} \cdot i \displaystyle\prod_{j=1, j\ne i}^{j=p+q} (i-j), \text{p+1} \le i \le p+q \end{cases}$$

As, the matrices V, $Diag(\sigma_1, \sigma_2, \dots, \sigma_{p+q})$ are invertible so is F, which means that F has rank $(p+q)$ and therefore $DP_t|_{D_0}$ has rank $(p+q+1)$.

4.4 The Main Result

Having established the example above we may state the main result:

Theorem 10. For a general element $t \in S_{p,q}$ the zero assignment map:

$$P_t : \mathbb{C}^{\mathrm{n}} \to \mathbb{C}^{\mathrm{p+q+1}}$$

is almost onto (i.e. the image of this map covers the whole $\mathbb{C}^{p+q+1}$, apart possibly from a set of measure zero).

Proof. Consider the subset $S^{'}$ of S_{p+q} defined as in section (4.3), i.e.:

$$S^{'} = \left\{ t \in S_{p,q} : rank\left(DP|_{D_0}\right) = p+q+1 \right\} \tag{24}$$

This is a Zarisky open subset of S_{p+q} and by the *dominant morphism* theorem $\forall t \in S^{'}$ the map:

$P_t : \mathbb{C}^{\mathrm{n}} \rightarrow \mathbb{C}^{\mathrm{p+q+1}}$ is almost onto. Since the network t_0 defined as previously has the property: $DP|_{D_0} = p+q+1$ it implies that $t_0 \in S^{'}$ and therefore $S^{'}$ is nonempty. Consequently, the subset of S_{p+q} such that P_t is not onto is a subset of $\left(S^{'}\right)^C$, which is contained in a proper sub-variety of S_{p+q}. This proves the theorem. A detailed proof can be found also in (Leventides et al., 2015).

Remark 11. The sufficient condition to obtain complete frequency assignability, i.e. $n > p+q$ arises from the fact that the zero assignment map:

$$P_t : \mathbb{C}^{\mathrm{n}} \rightarrow \mathbb{C}^{\mathrm{p+q+1}}$$

is almost onto when $n \geq p+q+1$. Consequently, $n > p+q$. This can be established from the Dominant Morphism Theorem and Theorem 10.

5. CONCLUSIONS

The problem of zero assignment for an RLC network with general operator $W(s) = s^2\mathbf{L} + s\mathbf{R} + \mathbf{C}$ has been considered in this paper. We only considered the case where non-dynamical elements, i.e. resistors were added to the network in order to have complete frequency assignability. The results that have been established are important and show that we can assign any frequency to a passive electrical network by adding resistors only as long as the number of resistors added exceeds the number of zeroes that need to be assigned (or the McMillan degree of the network) when the sufficient condition is met. We proved that the sufficient condition, i.e. the differential of the algebraic map DF_x has full rank (equals to n) and that happens in general when: $n > p+q$ and thus for every RLC network with that condition. The results here provide the means for studying problems of linear network redesign by modification of non-dynamical elements. The general network redesign problem may be still formulated in terms of the general network operator but more general transformations that the additive diagonal perturbations needs to be implemented. We shall note here that the dominant morphism theorem although proves the existence, is not appropriate for the construction of a solution. To construct the solutions we can utilize the usual methods based on the multi-linear/ determinantal formulation and then solving the set of algebraic equations using Gröebner bases methods.

REFERENCES

Becker, T. and Weispfenning, V. (1991). *Grö"ebner Bases: A Computational Approach to Commutative Algebra*. Springer- Verlag, New York.

Borel, A. (1991). *Linear Algebraic Groups*. GTM Springer- Verlag, New York.

Brockett, R. and Byrnes, C. (1981). Multivariable Nyquist Criterion, Root Loci and Pole Placement: A Geometric Viewpoint. *IEEE Trans. Aut. Control*, AC-26, 271–283.

Fulton (1998). *Intersection Theory*. Springer-Verlag, New York.

Humphreys, J. (1975). *Linear Algebraic Groups, Graduate Texts in Mathematics*. Springer- Verlag, New York.

Karcanias, N. and Giannakopoulos, C. (1984). Grassmann Invariants, Almost Zeros and the Determinantal Pole Assignment Problem of Linear Systems. *Int. J. Control*, 40, 673–698.

Karcanias, N. and Giannakopoulos, C. (1989). Necessary and Sufficient Conditions for Zero Assignment by Constant Squaring Down. *Linear Alg. Applications*, 415–446.

Karcanias, N., Laios, B., and Giannakopoulos, C. (1988). The Decentralized Determinantal Assignment Problem: Fixed and Almost Modes and Zeros. *Int. J. Control*, 48, 129–147.

Karcanias, N. and Leventides, J. (1996). Grassmann Invariants, Matrix Pencils and Linear System Properties. *Linear Alg. Applications*, 705–731.

Karcanias, N. and Leventides, J. (2010). Passive Electric Network Redesign and Matrix Pencils. Technical report, Control Engineering Centre, City University.

Karcanias, N. and Leventides, J. (2015). Solution of the Determinantal Assignment Problem Using the Grassmann Matrices. *International Journal of Control*.

Karcanias, N., Leventides, J., and Livada, M. (2015). Network Implicit Modeling and Redesign Issues. Technical report, Control Engineering Centre, City University, London. 05/15/1.

Leventides, J. (2007). Assigning Frequencies via Determinantal Equations: New Counterexamples and Invariants. *IEEE Transactions on Automatic Control*, 52(3).

Leventides, J. and Karcanias, N. (1993). The Pole Placement Map, Its Properties and Relationships to System Invariants. *IEEE Trans. on Auto Control*, 38(8), 1266–1270.

Leventides, J. and Karcanias, N. (1995a). Global Asymptotic Linearisation of the Pole Placement Map: A Closed Form Solution for the Constant Output Feedback Problem. *Automatica*, 31(9), 1303–1309.

Leventides, J. and Karcanias, N. (1995b). Sufficient Conditions for Arbitrary Pole Assignment by Constant Decentralized Output Feedback. *Math. Control Signals Syst.*, 8, 222–240.

Leventides, J. and Karcanias, N. (2009). Zero Assignment of Matrix Pencils by Additive Structured Transformations. *Linear Algebra and its Applications*, 431, 1380–1396.

Leventides, J., Livada, M., and Karcanias, N. (2014). Mcmillan Degree of Impedance, Admittance Functions of RLC Networks. In *MTNS 2014 Proceedings*.

Leventides, J., Livada, M., and Karcanias, N. (2015). Frequency Assignment of Passive Networks by Redesign. Technical report, Control Engineering Centre, City University, London. 07/5/15.

Marcus, M. and Minc, H. (1964). *A Survey of Matrix Theory and Matrix Inequalities*. Allyn and Bacon.

Wait, R. (1979). *The Numerical Solutions of Algebraic Equations*. Wiley, New York.

Wang, X. (1994a). Decentralised Pole Assignment and Product Grassmannians. *SIAM J. Control Optim.*, 855–875.

Wang, X. (1994b). On Output Feedback Via Grassmannians. *SIAM J. Control Optim.*, 926–935.