Breathers in the elliptic sine-Gordon model

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Abstract: We provide new expressions for the scattering amplitudes in the soliton-antisoliton sector of the elliptic sine-Gordon model in terms of cosets of the affine Weyl group corresponding to infinite products of q-deformed gamma functions. When relaxing the usual restriction on the coupling constants, the model contains additional bound states which admit an interpretation as breathers. These breather bound states are unavoidably accompanied by Tachyons. We compute the complete S-matrix describing the scattering of the breathers amonst themselves and with the soliton-antisoliton sector. We carry out various reductions of the model, one of them leading to a new type of theory, namely an elliptic version of the minimal $D_{n+1}^{(1)}$-affine Toda field theory.

1. Introduction

By investigating a $\mathbb{Z}_4$-symmetry of the particle wavefunctions for a soliton and an antisoliton, the elliptic sine-Gordon model was introduced originally by A.B. Zamolodchikov more than twenty years ago [1]. The S-matrix was found to correspond to the transfer matrix of Baxter's eight-vertex model [2, 3, 4]. The model has two free parameters ν and ℓ which were mutually restricted in [1] in order to avoid the presence of Tachyons. Thus, if it was not for that restriction, the model could be viewed as a generalization of the sine-Gordon model. Unfortunately, when constraining mutually the parameters, it corresponds in the trigonometric limit only to the soliton-antisoliton sector with no bound states. This means that the entire breather sector is absent.

The purpose of this paper is to investigate whether a meaningful breather sector of the model can be constructed. When relaxing the constraint on the parameters several poles in the soliton-antisoliton S-matrix amplitudes will move into the physical sheet. We will demonstrate below that some of them lie on the imaginary axis and therefore, with the appropriate sign of the residue, admit the usual interpretation as bound states corresponding to breathers. In addition, there are redundant poles which are of a tachyonic nature as they are in the physical sheet beyond the imaginary axis. In the context of integrable models solutions for scattering amplitudes with such properties have been discarded right away up to now. Nonetheless, Tachyons have emerged initially undesired in other areas and subsequently have been turned into virtues. For instance, recently there has been great activity in the context of string theory [5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16], where Tachyon condensates have turned out to be important. They also appear to be very useful in the context of cosmological considerations [17, 18, 19, 20, 21, 22, 23, 24, 25]. Here we want to adopt the lesson from string theory and cosmology and allow them to be present. The gain of this attitude is that we have additional poles at our disposal which admit the usual interpretation as bound states, which can be associated to breathers.

Our manuscript is organized as follows: In section 2 we assemble and derive various properties of q-deformed gamma functions and relate them to Jacobian elliptic functions. In section 3 we demonstrate how the affine Weyl group can be utilized to solve the central functional equations leading directly to an infinite product solution for the soliton-antisoliton backward scattering amplitude in terms of q-deformed gamma functions. In section 4 we discuss the soliton-antisoliton sector. Using a slightly modified procedure proposed originally by Karowski and Thun [26, 27], the soliton-breather amplitudes are constructed in section 5 and the breather sector is discussed in section 6. By specifying the parameters to certain values we reduce the model in section 7 to several models. Our conclusions are stated in section 8.

2. q-deformed gamma functions and Jacobian elliptic functions

Many of the well studied integrable quantum field theories which allow for backscattering, such as the sine-Gordon model, are known to possess a generic form for their scattering amplitudes which consists of infinite products of Euler's gamma functions

$$S(\theta) = f(\theta) \prod_{k=0}^{\infty} \prod_{i=1}^{p} \frac{\Gamma(\theta + k\alpha + x_i)}{\Gamma(\theta + k\alpha + y_i)} \qquad \text{for } \sum_{i=1}^{p} x_i = \sum_{i=1}^{p} y_i \ . \tag{2.1}$$

The rapidity dependent prefactor $f(\theta)$ is usually some finite product of ratios of trigonometric functions and the values of x_i, y_i, α, p are specific to each individual model. The constraint on the sums of the x_i, y_i is a necessary condition for the convergence of the infinite product (a proof of this can be found e.g. in [28]). Below we will describe how the structure (2.1) can be generalized very naturally by replacing in a controlled way the usual gamma functions by their q-deformed counterparts and the trigonometric functions in the prefactor by their elliptic version.

q-deformed quantities have turned out to be very useful objects as they allow for instance to carry out elegantly (semi)-classical limits when the deformation parameter is associated to Planck's constant. In the elliptic sine-Gordon model we have the two free parameters $\nu \in [0, \infty)$ and $\ell \in [0, 1]$ at our disposal. The former is the analogue of the coupling constant in the sine-Gordon model[1] and the latter is the modulus of the Jacobian elliptic functions. It is the ℓ which is associated most naturally to a deformation.

[1] This motivates to take ν to be positive, as negative values of it correspond to a regime in which according to the arguments of Coleman [29] the ground state of the theory is not bounded from below.

Accordingly, we define a deformation parameter q and its Jacobian imaginary transformed version, i.e. $\tau \to -1/\tau$, as

$$q = \exp(i\pi\tau), \qquad \hat{q} = \exp(-i\pi/\tau), \qquad \tau = iK_{1-\ell}/K_\ell \ . \tag{2.2}$$

We introduced here the quarter periods K_ℓ of the Jacobian elliptic functions depending on the parameter $\ell \in [0,1]$, defined in the usual way through the complete elliptic integrals $K_\ell = \int_0^{\pi/2}(1-\ell\sin^2\theta)^{-1/2}d\theta$. Recalling the well-known properties

$$\lim_{\ell\to 0} K_\ell = \lim_{\ell\to 1} K_{1-\ell} = \pi/2 \qquad \text{and} \qquad \lim_{\ell\to 0} K_{1-\ell} = \lim_{\ell\to 1} K_\ell \to \infty \ , \tag{2.3}$$

the definitions (2.2) obviously mean that the trigonometric limits correspond to the "classical" limit in the variables $\hat{q}, q$ as

$$\lim_{\ell\to 0} \equiv \lim_{\hat{q}\to 1} \qquad \text{and} \qquad \lim_{\ell\to 1} \equiv \lim_{q\to 1} \ . \tag{2.4}$$

It will turn out below that quantities in $\hat{q}$ will be most relevant for our purposes and therefore we state several identities directly in $\hat{q}$, rather than q, even when they hold for generic values. The most basic q-deformed objects one defines are q-deformed integers (numbers), for which we take the convention

$$[n]_{\hat{q}} := \frac{\hat{q}^n - \hat{q}^{-n}}{\hat{q} - \hat{q}^{-1}}. \tag{2.5}$$

They have the obvious properties

$$\lim_{\ell\to 0}[n]_{\hat{q}} = n, \tag{2.6}$$

$$\lim_{\ell\to 0} \frac{[n+m\tau]_{\hat{q}}}{[n'+m'\tau]_{\hat{q}}} = \begin{cases} 1 & \text{for} \quad m, m' \neq 0 \\ n/n' & \text{for} \quad m = m' = 0 \end{cases} \ . \tag{2.7}$$

With the motivation in mind mentioned at the beginning of this section we define a q-deformed version of Euler's gamma function

$$\Gamma_{\hat{q}}(x+1) := \prod_{n=1}^{\infty} \frac{[1+n]_{\hat{q}}^x [n]_{\hat{q}}}{[x+n]_{\hat{q}} [n]_{\hat{q}}^x} \ . \tag{2.8}$$

The crucial property of the function $\Gamma_{\hat{q}}$, which coins also its name, is

$$\lim_{\ell\to 0} \Gamma_{\hat{q}}(x+1) = \lim_{\hat{q}\to 1} \Gamma_{\hat{q}}(x+1) = \prod_{n=1}^{\infty} \frac{n}{n+x}\left(\frac{1+n}{n}\right)^x = \Gamma(x+1) \ . \tag{2.9}$$

We report now various properties of this function which will be useful below. We can relate deformations in q and $\hat{q}$ through

$$\frac{\hat{q}^{(x+\tau/2-1/2)^2}}{\hat{q}^{(y+\tau/2-1/2)^2}} \frac{\Gamma_{\hat{q}}(y)\Gamma_{\hat{q}}(1-y)}{\Gamma_{\hat{q}}(x)\Gamma_{\hat{q}}(1-x)} = \frac{\Gamma_q(-y/\tau)\Gamma_q(1+y/\tau)}{\Gamma_q(-x/\tau)\Gamma_q(1+x/\tau)} \ . \tag{2.10}$$

Frequently we have to shift the argument by integer values

$$\Gamma_{\hat{q}}(x+1) = \hat{q}^{x-1}[x]_{\hat{q}}\Gamma_{\hat{q}}(x) \ . \tag{2.11}$$

Relation (2.11) can be obtained directly from (2.8). As a consequence of this we also have

$$\Gamma_{\hat{q}}(x+m) = \Gamma_{\hat{q}}(x)\prod_{l=0}^{m-1}\hat{q}^{x+l-1}[x+l]_{\hat{q}} \qquad m\in\mathbb{Z} \tag{2.12}$$

$$\Gamma_{\hat{q}}(x) = \Gamma_{\hat{q}}(x-m)\prod_{l=0}^{m-1}\hat{q}^{x-l-2}[x-l-1]_{\hat{q}} \qquad m\in\mathbb{Z} \ . \tag{2.13}$$

Whereas (2.11)-(2.12) hold for generic q, the following identities are only valid for $\hat{q}$

$$\Gamma_{\hat{q}}(1/2-\tau/2)\Gamma_{\hat{q}}(1/2+\tau/2) = \ell^{1/4}\Gamma_{\hat{q}}(1/2)^2 \tag{2.14}$$

$$\frac{\Gamma_{\hat{q}}(x+2\tau)}{\Gamma_{\hat{q}}(y+2\tau)} = \frac{\Gamma_{\hat{q}}(x)}{\Gamma_{\hat{q}}(y)} \tag{2.15}$$

$$\prod_{i=1}^{p}\frac{\Gamma_{\hat{q}}(x_i)\Gamma_{\hat{q}}(x_i\pm\tau/2)}{\Gamma_{\hat{q}}(y_i)\Gamma_{\hat{q}}(y_i\pm\tau/2)} = \prod_{i=1}^{p}\frac{\Gamma_{\hat{q}^2}(x_i)}{\Gamma_{\hat{q}^2}(y_i)} \qquad \text{if} \quad \sum_{i=1}^{p}x_i=\sum_{i=1}^{p}y_i \tag{2.16}$$

$$\lim_{\hat{q}\to 1}\prod_{i=1}^{p}\frac{\Gamma_{\hat{q}}(x_i\pm\tau/2)}{\Gamma_{\hat{q}}(y_i\pm\tau/2)} = 1 \qquad \text{if} \quad \sum_{i=1}^{p}x_i=\sum_{i=1}^{p}y_i \tag{2.17}$$

$$\lim_{\hat{q}\to 1}\frac{1}{\ell^{1/4}}\Gamma_{\hat{q}}(\frac{x}{2K_\ell}\mp\frac{\tau}{2})\Gamma_{\hat{q}}(1-\frac{x}{2K_\ell}\pm\frac{\tau}{2}) = \pi \qquad \text{for} \quad x\neq 0 \tag{2.18}$$

The constraint on the sums in (2.16) and (2.17) was already encountered in (2.1). Most of these properties can be checked directly by means of the defining relation (2.8). We comment on some of the derivations below.

The singularity structure will be important for the physical applications. It follows from (2.8) that the $\Gamma_{\hat{q}}$-function has no zeros, but poles

$$\lim_{\theta\to\theta_{\Gamma,p}^{nm}=m\tau-n}\Gamma_{\hat{q}}(\theta+1)\to\infty \qquad \text{for} \quad m\in\mathbb{Z}, n\in\mathbb{N} \ . \tag{2.19}$$

Furthermore, we will employ the Jacobian elliptic functions, to generalize the prefactor $f(\theta)$ in (2.1), for which we use the common notation pq(z) with p,q $\in$ {s,c,d,n} (see e.g. [30] for standard properties). We derive important relations between the q-deformed gamma functions and the Jacobian elliptic sn-function

$$\mathrm{sn}(x) = \frac{1}{\ell^{\frac{1}{4}}}\frac{\Gamma_{\hat{q}}(\frac{x}{2K_\ell}\mp\frac{\tau}{2})\Gamma_{\hat{q}}(1-\frac{x}{2K_\ell}\pm\frac{\tau}{2})}{\Gamma_{\hat{q}}(\frac{x}{2K_\ell})\Gamma_{\hat{q}}(1-\frac{x}{2K_\ell})}, \tag{2.20}$$

$$= \frac{q^{\frac{1}{4}-\frac{ix}{2K_{1-\ell}}}}{i\ell^{\frac{1}{4}}}\frac{\Gamma_q(\frac{1}{2}+\frac{ix}{2K_{1-\ell}})\Gamma_q(\frac{1}{2}-\frac{ix}{2K_{1-\ell}})}{\Gamma_q(1-\frac{ix}{2K_{1-\ell}})\Gamma_q(\frac{ix}{2K_{1-\ell}})} \ . \tag{2.21}$$

These relations can be used to obtain some of the above mentioned expressions. For instance, recalling that $\mathrm{sn}(K_\ell)=1$, we obtain (2.14). With (2.8) we recover from this the well known identity $\mathrm{sn}(iK_{1-\ell}/2) = i/\ell^{1/4}$. The trigonometric limits

$$\lim_{\ell\to 0} \mathrm{sn}(x) = \lim_{\hat{q}\to 1} \mathrm{sn}(x) = \frac{\pi}{\Gamma(\frac{x}{\pi})\Gamma(1-\frac{x}{\pi})} = \sin(x) \tag{2.22}$$

$$\lim_{\ell\to 1} \mathrm{sn}(x) = \lim_{q\to 1} \mathrm{sn}(x) = \frac{1}{i}\frac{\Gamma(\frac{1}{2}+\frac{ix}{\pi})\Gamma(\frac{1}{2}-\frac{ix}{\pi})}{\Gamma(1-\frac{ix}{\pi})\Gamma(\frac{ix}{\pi})} = \tanh(x). \tag{2.23}$$

can be read off directly recalling (2.3), (2.9) and presuming that (2.18) holds. We recall the zeros and poles of the Jacobian elliptic $\mathrm{sn}(\theta)$-function, which in our conventions are located at

$$\text{zeros:} \qquad \theta^{lm}_{\mathrm{sn},0} = 2lK_\ell + i2mK_{1-\ell} \qquad l,m\in\mathbb{Z} \tag{2.24}$$

$$\text{poles:} \qquad \theta^{lm}_{\mathrm{sn},p} = 2lK_\ell + i(2m+1)K_{1-\ell} \qquad l,m\in\mathbb{Z}\ . \tag{2.25}$$

We have now assembled the main properties of the q-deformed functions which we shall use below.

3. Affine Weyl group and the unitarity/crossing relations

The functional relations of crossing, unitarity and bootstrap for the scattering amplitudes are usually solved by means of Fourier transformations, thus leading in general directly to integral representations for the desired quantities. When solving these integrals one ends up with infinite products over gamma functions of the type (2.1) for scattering amplitudes in non-diagonal theories or trigonometric functions when backscattering is absent. For the elliptic sine-Gordon model so far only the analogue of the integral representation was presented [1] in form of a discretized version. Instead of solving the discrete integrals in this function we provide here a systematic procedure which leads directly to product solutions by utilizing the affine Weyl group $\hat{\mathrm{W}}(\mathbf{g})$.

Let us first assemble the necessary mathematical tools and jargon. In general an affine Weyl reflection related to a simple root α_i, of a Lie algebra $\mathbf{g}$ [31] may be realized by the map

$$\sigma_{i,n}(x) = x - (x\cdot\alpha_i)\,\alpha_i + n\check{\alpha}_i\ , \tag{3.1}$$

where $\check{\alpha}_i$ denotes a coroot and n an arbitrary integer. In particular for $n=0$ one recovers the ordinary Weyl group $\mathrm{W}(\mathbf{g})$ and for $x=0$ the translation group on the coroot lattice. Hence the affine Weyl group may be thought of as a direct product

$$\hat{W}(\mathbf{g}) = W(\mathbf{g}) \otimes \check{\mathbf{T}}\ , \tag{3.2}$$

with $\check{\mathbf{T}}$ denoting translations on the coroot lattice. For the case $\mathbf{g} = A_1$ ($\hat{W}(\mathbf{g}) \sim D_\infty$ the infinite dihedral group) one has only one simple root and (3.1) becomes

$$\sigma_n(x) = n\alpha - x\ . \tag{3.3}$$

This group may be generated by two generators

$$\sigma_1(x) = \alpha - x \qquad \text{and} \qquad \sigma_0 = -x\quad , \tag{3.4}$$

having obviously the property $\sigma_1^2 = \sigma_0^2 = 1$. Defining then the transformation

$$\sigma := \sigma_1 \sigma_0 \tag{3.5}$$

one has

$$\sigma^n (x) = x + n\alpha \quad . \tag{3.6}$$

We denote by N_0, N_1 the number of times the generators σ_0, σ_1 occur in an arbitrary element of $\hat{W}(A_1)$. Then two types of subgroups $\left\{\sigma^{2n}, N_0\right\}, \left\{\sigma^{2n}, N_1\right\} \subset \hat{W}(A_1)$ are constituted by the elements with N_0 and N_1 even, respectively. The right cosets of $\left\{\sigma^{2n}, N_1\right\}$ may be divided into those with N_1 even $\left\{\sigma^{2n} I, N_1\right\}, \left\{\sigma^{2n}\sigma_0, N_1\right\}$ and N_1 odd $\left\{\sigma^{2n+1} I, N_1\right\}, \left\{\sigma^{2n}\sigma_1, N_1\right\}$. Similarly one may divide the right cosets of $\left\{\sigma^{2n}, N_0\right\}$ into those with N_0 even $\left\{\sigma^{2n} I, N_0\right\}, \left\{\sigma^{2n}\sigma_1, N_0\right\}$ and N_0 odd $\left\{\sigma^{2n+1} I, N_0\right\}, \left\{\sigma^{2n+1}\sigma_1, N_0\right\}$.

Assuming now that $\hat{W}(A_1)$ acts in the complex rapidity plane one may specify α and define the "unitarity" and "crossing" transformation on an arbitrary function $f(\theta)$ by

$$\sigma_0 f(\theta) = f(-\theta), \qquad \sigma_1 f(\theta) = f(i\pi - \theta) \ . \tag{3.7}$$

Then one obtains

$$\sigma^n f(\theta) = f(\theta + ni\pi) \ . \tag{3.8}$$

We have now provided all the tools to solve the key equations in this context. In [1] two functional relations for the soliton-antisoliton transmission amplitude (equations (2.10) and (3.8) therein), which we denote by $c(\theta)$, were derived from crossing, unitarity and the Yang-Baxter equations. Once this amplitude is known, some simple relations provided in [1] suffice to construct the remaining ones in the soliton-antisoliton sector. The equations to be solved are

$$c(\theta) = c(i\pi - \theta) \tag{3.9}$$

$$c(\theta)c(-\theta) = \frac{\mathrm{sn}^2(\pi/\nu)}{\mathrm{sn}^2(\pi/\nu) - \mathrm{sn}^2(i\theta/\nu)} \ . \tag{3.10}$$

We make now an ansatz by taking the ratio of right cosets in which N_0 is even and odd

$$c(\theta) = \kappa \frac{\left\{\sigma^{2n} I, N_0\right\}\left\{\sigma^{2n}\sigma_1, N_0\right\}}{\left\{\sigma^{2n+1} I, N_0\right\}\left\{\sigma^{2n+1}\sigma_1, N_0\right\}} = \kappa \prod_{k=1}^{\infty} \frac{\sigma^{2k}\rho(\theta)\,\sigma^{2k}\sigma_1\rho(\theta)}{\sigma^{2k+1}\rho(\theta)\,\sigma^{2k+1}\sigma_1\rho(\theta)}, \tag{3.11}$$

$$= \kappa \prod_{k=1}^{\infty} \frac{\rho[\theta + 2\pi i k]\rho[-\theta + 2\pi i(k+1/2)]}{\rho[\theta + 2\pi i(k+1/2)]\rho[-\theta + 2\pi i(k+1)]} \ . \tag{3.12}$$

Here κ is a constant and $\rho(\theta)$ an arbitrary function which remains to be fixed. One observes that the crossing relation (3.9) is solved by construction, whereas (3.10) requires that

$$\kappa^2 \rho(\theta + 2\pi i)\rho(2\pi i - \theta) = \frac{\Gamma_{\hat{q}}^2[-\frac{\tau}{2}]\Gamma_{\hat{q}}^2[1+\frac{\tau}{2}]\Gamma_{\hat{q}}[\hat{\theta}-\frac{\lambda}{2}]\Gamma_{\hat{q}}[1-\hat{\theta}+\frac{\lambda}{2}]\Gamma_{\hat{q}}[-\hat{\theta}-\frac{\lambda}{2}]\Gamma_{\hat{q}}[1+\hat{\theta}+\frac{\lambda}{2}]}{\Gamma_{\hat{q}}^2[-\frac{\lambda}{2}]\Gamma_{\hat{q}}^2[1+\frac{\lambda}{2}]\Gamma_{\hat{q}}[\hat{\theta}-\frac{\tau}{2}]\Gamma_{\hat{q}}[1-\hat{\theta}+\frac{\tau}{2}]\Gamma_{\hat{q}}[-\hat{\theta}-\frac{\tau}{2}]\Gamma_{\hat{q}}[1+\hat{\theta}+\frac{\tau}{2}]}.$$

We replaced here in (3.10) the sn- by $\Gamma_{\hat{q}}$-functions using some standard identities for Jacobian elliptic functions together with (2.20) and abbreviated for compactness

$$\lambda = -\pi/K_\ell \nu \qquad \text{and} \qquad \hat{\theta} = i\theta/2K_\ell \nu. \tag{3.13}$$

The problem of solving (3.9) and (3.10) has now been reduced to the much simpler task of fixing $\rho(\theta)$ and κ. We find

$$\kappa = \frac{\Gamma_{\hat{q}}[-\frac{\tau}{2}]\Gamma_{\hat{q}}[1+\frac{\tau}{2}]}{\Gamma_{\hat{q}}[-\frac{\lambda}{2}]\Gamma_{\hat{q}}[1+\frac{\lambda}{2}]} \qquad \text{and} \qquad \rho_{i,j}(\theta) = \frac{\rho_{n,i}(\theta)}{\rho_{d,j}(\theta)} \ , \tag{3.14}$$

where $\rho_{n,i}(\theta)$ and $\rho_{d,j}(\theta)$ could be any of the functions

$$\begin{aligned}
\rho_{n,1}(\theta+2\pi i) &= \Gamma_{\hat{q}}[\hat{\theta}-\tfrac{\lambda}{2}]\Gamma_{\hat{q}}[1-\hat{\theta}+\tfrac{\lambda}{2}] & \rho_{d,1}(\theta+2\pi i) &= \Gamma_{\hat{q}}[\hat{\theta}-\tfrac{\tau}{2}]\Gamma_{\hat{q}}[1-\hat{\theta}+\tfrac{\tau}{2}], \\
\rho_{n,2}(\theta+2\pi i) &= \Gamma_{\hat{q}}[\hat{\theta}-\tfrac{\lambda}{2}]\Gamma_{\hat{q}}[1+\hat{\theta}+\tfrac{\lambda}{2}] & \rho_{d,2}(\theta+2\pi i) &= \Gamma_{\hat{q}}[\hat{\theta}-\tfrac{\tau}{2}]\Gamma_{\hat{q}}[1+\hat{\theta}+\tfrac{\tau}{2}], \\
\rho_{n,3}(\theta+2\pi i) &= \Gamma_{\hat{q}}[-\hat{\theta}-\tfrac{\lambda}{2}]\Gamma_{\hat{q}}[1+\hat{\theta}+\tfrac{\lambda}{2}] & \rho_{d,3}(\theta+2\pi i) &= \Gamma_{\hat{q}}[-\hat{\theta}-\tfrac{\tau}{2}]\Gamma_{\hat{q}}[1+\hat{\theta}+\tfrac{\tau}{2}], \\
\rho_{n,4}(\theta+2\pi i) &= \Gamma_{\hat{q}}[-\hat{\theta}-\tfrac{\lambda}{2}]\Gamma_{\hat{q}}[1-\hat{\theta}+\tfrac{\lambda}{2}] & \rho_{d,4}(\theta+2\pi i) &= \Gamma_{\hat{q}}[-\hat{\theta}-\tfrac{\tau}{2}]\Gamma_{\hat{q}}[1-\hat{\theta}+\tfrac{\tau}{2}].
\end{aligned}$$

Hence, we notice that the solution of (3.9) and (3.10) is by no means unique and there exist additional ones to the one presented in [1] in form of a discrete integral representation. The function in the numerator $\rho_{n,i}(\theta)$ may be selected by the requirement that we would like to obtain the corresponding quantities of the sine-Gordon model in the trigonometric limit and $\rho_{d,j}(\theta)$ by discarding solutions which yield undesired poles. We end up with the choice $\rho_{4,4}(\theta) = \rho_{n,4}(\theta)/\rho_{d,4}(\theta)$, which after substitution into (3.12) yields.

$$\begin{aligned}
c(\theta) = {} & \frac{\Gamma_{\hat{q}}[1+\frac{\tau}{2}]\Gamma_{\hat{q}}[-\frac{\tau}{2}]}{\Gamma_{\hat{q}}[1+\frac{\lambda}{2}]\Gamma_{\hat{q}}[-\frac{\lambda}{2}]} \prod_{k=1}^{\infty} \frac{\Gamma_{\hat{q}}[\hat{\theta}-k\lambda]\Gamma_{\hat{q}}[1+\hat{\theta}-(k-1)\lambda]}{\Gamma_{\hat{q}}[-\hat{\theta}-k\lambda]\Gamma_{\hat{q}}[1-\hat{\theta}-(k-1)\lambda]} \\
& \times \frac{\Gamma_{\hat{q}}[-\hat{\theta}-(k-\frac{1}{2})\lambda]\Gamma_{\hat{q}}[1-\hat{\theta}-(k-\frac{3}{2})\lambda]\Gamma_{\hat{q}}[1+\hat{\theta}-k\lambda+\frac{\tau}{2}]}{\Gamma_{\hat{q}}[\hat{\theta}-(k+\frac{1}{2})\lambda]\Gamma_{\hat{q}}[1+\hat{\theta}-(k-\frac{1}{2})\lambda]\Gamma_{\hat{q}}[1-\hat{\theta}-(k-1)\lambda+\frac{\tau}{2}]} \\
& \times \frac{\Gamma_{\hat{q}}[\hat{\theta}-k\lambda-\frac{\tau}{2}]\Gamma_{\hat{q}}[-\hat{\theta}-(k-\frac{1}{2})\lambda-\frac{\tau}{2}]\Gamma_{\hat{q}}[1-\hat{\theta}-(k-\frac{1}{2})\lambda+\frac{\tau}{2}]}{\Gamma_{\hat{q}}[-\hat{\theta}-(k-1)\lambda-\frac{\tau}{2}]\Gamma_{\hat{q}}[\hat{\theta}-(k-\frac{1}{2})\lambda-\frac{\tau}{2}]\Gamma_{\hat{q}}[1+\hat{\theta}-(k-\frac{1}{2})\lambda+\frac{\tau}{2}]} .
\end{aligned} \tag{3.15}$$

We want to conclude this section with a general remark on the method provided to solve the functional relations (3.9) and (3.10). Of course we could have started right away with the ansatz (3.12) instead of introducing the affine Weyl group in the first place. However, this formulation automatically supplies a certain systematic. To illustrate this further we provide another example of some important functional relations occurring in the context of integrable models, that is Watson's equations, see e.g. [32], for the minimal form factors

$$F_{\min}(\theta+i\pi) = F_{\min}(i\pi-\theta) \qquad \text{and} \qquad F_{\min}(\theta) = F_{\min}(-\theta)S(\theta) \ . \tag{3.16}$$

Making here an ansatz by taking the ratio of right cosets in which N_1 is even and odd

$$F_{\min}(\theta) = \kappa \frac{\left\{\sigma^{2n}I, N_1\right\}\left\{\sigma^{2n}\sigma_0, N_1\right\}}{\left\{\sigma^{2n+1}I, N_1\right\}\left\{\sigma^{2n}\sigma_1, N_1\right\}} = \kappa \prod_{k=0}^{\infty} \frac{\sigma^{2k-2}\rho(\theta)\,\sigma^{2k}\sigma_0\rho(\theta)}{\sigma^{2k-1}\bar{\rho}(\theta)\,\sigma^{2k}\sigma_1\bar{\rho}(\theta)}, \tag{3.17}$$

we see that the first equation in (3.16) is solved by construction whereas the second requires that $S(\theta) = \rho(\theta)\,\sigma_1\bar{\rho}(\theta)/\sigma_0\rho(\theta)\,\sigma\bar{\rho}(\theta)$. Having a concrete expression for $S(\theta)$ one can now easily determine ρ and $\bar{\rho}$. For more complicated functional relations one may use groups of higher rank.

4. Soliton-antisoliton sector

Let us now discuss in more detail the elliptic sine-Gordon model. Scattering amplitudes are obtainable in general from the computation of matrix elements, but it is also well established that in an integrable (≡factorizable) theory they may be derived equivalently by analyzing the Zamolodchikov algebra. Its associativity corresponds to the Yang-Baxter equations. Its generators are thought of as particle creation operators and therefore internal symmetries of the model are respected by this algebra. Considering a theory with two particles which are conjugate to each other, say a soliton Z and an antisoliton $\bar{Z}$, one may demand a $\mathbb{Z}_4$-symmetry, that is one requires invariance under $Z \to \exp(i\pi/2)Z$, $\bar{Z} \to \exp(-i\pi/2)\bar{Z}$. The most general version of the Zamolodchikov algebra respecting this symmetry then reads [1]

$$Z(\theta_1)Z(\theta_2) = a(\theta_{12})Z(\theta_2)Z(\theta_1) + d(\theta_{12})\bar{Z}(\theta_2)\bar{Z}(\theta_1) , \tag{4.1}$$

$$Z(\theta_1)\bar{Z}(\theta_2) = b(\theta_{12})\bar{Z}(\theta_2)Z(\theta_1) + c(\theta_{12})Z(\theta_2)\bar{Z}(\theta_1) , \tag{4.2}$$

with rapidity difference $\theta_{12} = \theta_1 - \theta_2$. The charge conjugated relations also hold, that is $Z \leftrightarrow \bar{Z}$. In comparison with the more extensively studied sine-Gordon model the difference is the occurrence of the amplitude d in (4.1), i.e. the possibility that two solitons change into two antisolitons and vice versa. Invoking the Yang-Baxter equations, crossing and unitarity one finds the following solutions for the amplitudes

$$a(\theta) = \Phi(\theta) \prod_{k=0}^{\infty} \frac{\Gamma_{\hat{q}^2}[\hat{\theta} - (k+1)\lambda]\Gamma_{\hat{q}^2}[1+\hat{\theta} - k\lambda]\Gamma_{\hat{q}^2}[-\hat{\theta} - \frac{1+2k}{2}\lambda]\Gamma_{\hat{q}^2}[1-\hat{\theta} - \frac{1+2k}{2}\lambda]}{\Gamma_{\hat{q}^2}[-\hat{\theta} - (k+1)\lambda]\Gamma_{\hat{q}^2}[1-\hat{\theta} - k\lambda]\Gamma_{\hat{q}^2}[\hat{\theta} - \frac{1+2k}{2}\lambda]\Gamma_{\hat{q}^2}[1+\hat{\theta} - \frac{1+2k}{2}\lambda]} \tag{4.3}$$

$$b(\theta) = -\frac{\mathrm{sn}(i\theta/\nu)}{\mathrm{sn}(i\theta/\nu + \pi/\nu)} a(\theta) = \hat{b}(\theta)a(\theta), \tag{4.4}$$

$$c(\theta) = \frac{\mathrm{sn}(\pi/\nu)}{\mathrm{sn}(i\theta/\nu + \pi/\nu)} a(\theta) = \hat{c}(\theta)a(\theta), \tag{4.5}$$

$$d(\theta) = -\sqrt{\ell}\,\mathrm{sn}(i\theta/\nu)\,\mathrm{sn}(\pi/\nu)a(\theta) \;= \hat{d}(\theta)a(\theta), \tag{4.6}$$

$$\Phi(\theta) = \frac{\Gamma_{\hat{q}}[1+\frac{\tau}{2}]\Gamma_{\hat{q}}[-\frac{\tau}{2}]\Gamma_{\hat{q}}[1-\hat{\theta}+\frac{\lambda}{2}+\frac{\tau}{2}]\Gamma_{\hat{q}}[\hat{\theta}-\frac{\lambda}{2}-\frac{\tau}{2}]}{\Gamma_{\hat{q}}[1+\hat{\theta}+\frac{\tau}{2}]\Gamma_{\hat{q}}[-\hat{\theta}-\frac{\tau}{2}]\Gamma_{\hat{q}}[1+\frac{\lambda}{2}+\frac{\tau}{2}]\Gamma_{\hat{q}}[-\frac{\lambda}{2}-\frac{\tau}{2}]} . \tag{4.7}$$

Here we use as a common factor $a(\theta)$ rather than $c(\theta)$ for which we have explained in the previous section how to solve the key functional equations (3.9) and (3.10). We also used relation (2.11) to simplify (3.15) and refer the reader to [1] for the details on how to relate the different amplitudes to each other.

Our solutions (4.3)-(4.6) differ in form from the one presented in [1], where a discretized version of an integral representation was provided. Instead we derived here directly an infinite product representation in the previous section, which is a natural generalization of a very common version, of the form (2.1), used in the context of the sine-Gordon model. One of the advantages of our formulation is that it exhibits very explicitly the singularity structure. Furthermore, it is easier to handle under shifts of the argument as in integral representations such shifts will often be prohibited by convergence requirements.

4.1 Singularity structure

In order to extract the singularity structure for the amplitudes (4.3)-(4.6) we recall the relations (2.19), (2.24) and (2.25), which suffice to read off the poles and zeros of $a(\theta), b(\theta), c(\theta)$ and $d(\theta)$. We depict them most conveniently in a figure.

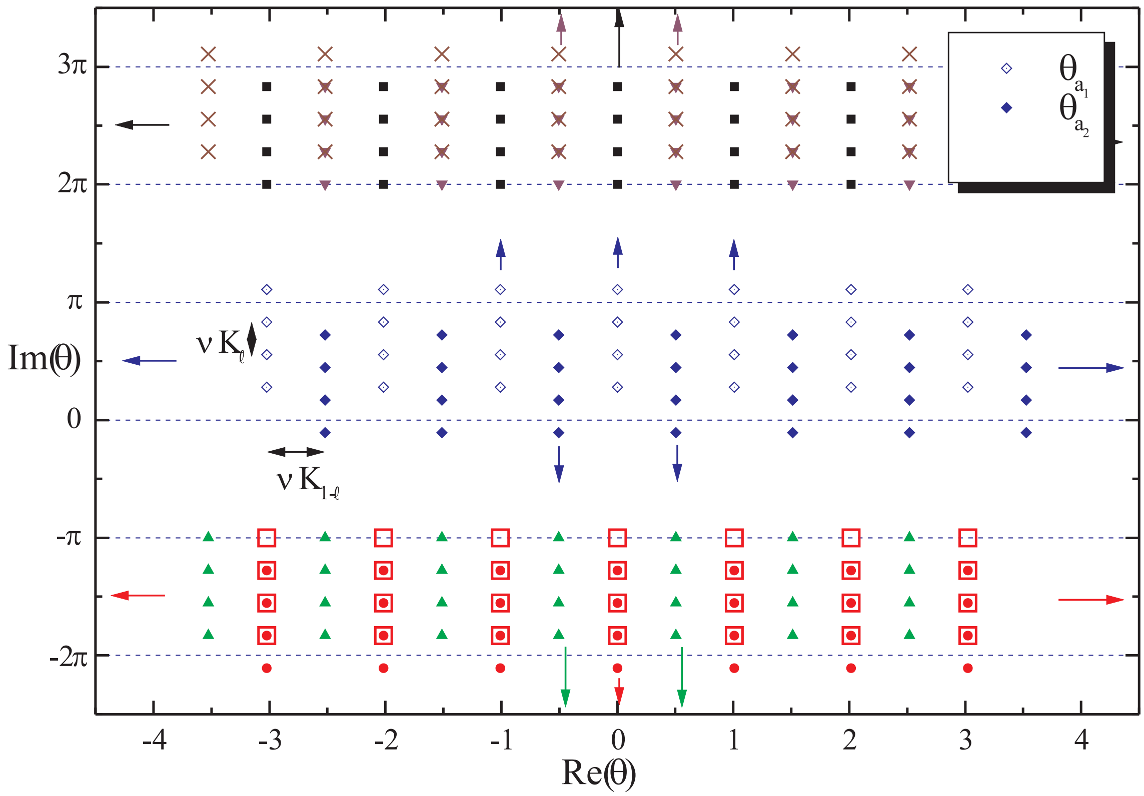

Figure 1: Poles of the soliton-soliton amplitude $a(\theta) = b(i\pi - \theta)$ for the concrete values $\nu = 0.25$, $\ell = 0.344$, that is $\nu K_{1-\ell} \sim 0.504$ and $\nu K_\ell \sim 0.435$. The arrows indicate in which directions the "strings" of poles are extended for growing integer values n or m.

Explicitly, we just report here the poles which have potentially a chance to be situated inside the physical sheet, i.e. $0 < \operatorname{Im}\theta < \pi$. We take $l, m \in \mathbb{Z}, n \in \mathbb{N}$ and associate always two sets of poles $\theta^{nm}_{a_1,p}$ and $\theta^{nm}_{a_2,p}$ to $a(\theta)$, $\theta^{nm}_{b_1,p}$ and $\theta^{nm}_{b_2,p}$ to $b(\theta)$ etc.

$$
\begin{array}{ll}
\theta^{nm}_{a_1,p} = 2m\nu K_{1-\ell} + i2n\nu K_\ell, & \theta^{nm}_{a_2,p} = (2m+1)\nu K_{1-\ell} + i(\pi - 2n\nu K_\ell), \\
\theta^{lm}_{b_1,p} = 2m\nu K_{1-\ell} + i(\pi - 2l\nu K_\ell), & \theta^{lm}_{b_2,p} = (2m+1)\nu K_{1-\ell} + i2l\nu K_\ell, \\
\theta^{lm}_{c_1,p} = 2m\nu K_{1-\ell} + i2l\nu K_\ell, & \theta^{lm}_{c_2,p} = 2m\nu K_{1-\ell} + i(\pi - 2l\nu K_\ell), \\
\theta^{lm}_{d_1,p} = (2m+1)\nu K_{1-\ell} + i2l\nu K_\ell, & \theta^{lm}_{d_2,p} = (2m+1)\nu K_{1-\ell} + i(\pi - 2n\nu K_\ell).
\end{array}
\tag{4.8}
$$

One readily sees from (4.8) that if one restricts the parameter $\nu \geq \pi/2K_\ell$ all poles move out of the physical sheet into the non-physical one, where they can be interpreted in principle as resonances, i.e. unstable particles. This was already stated in [1] where the choice $\nu \geq \pi/2K_\ell$ was made in order to avoid the occurrence of tachyonic states. In fact, even in that regime there are Tachyons present, since the poles in the non-physical sheet with negative real part can not be explained on the basis of the Breit-Wigner formula. Their occurence can be avoided by an additional breaking of parity (see discussion in [33]). The restriction on the parameters makes the model somewhat unattractive as this limitation eliminates the analogue of the entire breather sector which is present in the sine-Gordon model, such also that in the trigonometric limit one only obtains the soliton-antisoliton sector of that model, instead of a theory with a richer particle content. For this reason, the arguments outlined in the introduction and the fact that the constraint does not yield any Tachyon free theory anyhow, we relax here the restriction on ν. The the poles

$$\theta_{b_1,p}^{n0} = \theta_{c_2,p}^{n0} \qquad \text{for} \;\; 0 < n < n_{\max} = [\pi/2\nu K_\ell], n \in \mathbb{N} \tag{4.9}$$

are located on the imaginary axis in the physical sheet and are therefore candidates for the analogue of the n^{th}-breather bound states in the sine-Gordon model. We indicate here the integer part of x by $[x]$. In other words, there are at most $n_{\max} - 1$ breathers for fixed ν and ℓ. The price one pays for the occurrence of these new particles in the $\mathbb{Z}_4$-model is that one unavoidably also introduces additional Tachyons into the model as the poles always emerge in "strings". It remains to be established whether the poles (4.9) may really be associated to a breather type behaviour.

5. Soliton-breather amplitudes

Once the solitonic sector of a theory is constructed there is a well defined bootstrap procedure proposed by Karowski and Thun [26, 27], which allows to complete the theory and to construct also the breather sector. There are numerous solutions known for a non-trivial soliton sector, since for instance for all affine Toda field theories with purely imaginary coupling constant the Yang-Baxter equations can be solved by representations of the corresponding quantum group [34, 35, 36]. Nonetheless, the completion of the models has not been carried out for very many cases [37, 38, 39]. In reverse, one should note that there exist many scalar theories which do not possess a known solitonic counterpart. One reason for that is, that the procedure [26, 27] is not reversible and one can in general not construct the solitonic sector from the breather sector alone. Drawing a loose analogy to group theory one may think of the breather sector as a subgroup, which of course does not contain the information of a larger group in which it might be embedded. It is very desirable to complete the picture, tie up the loose ends and construct the respective missing sectors.

To start the construction, one first of all has to find creation operators for the particles corresponding to the bound state poles in the soliton sector. We presume here that the particles related to the poles (4.9) are breathers and borrow some intuition from the classical theory, exploiting the fact that a breather is an oscillatory object made out of a superposition of a soliton and an antisoliton. For the sine-Gordon model this prescription was used in [27] to define the n^{th}-breather particle creation operator. Even though we do not have a classical counterpart for the elliptic sine-Gordon model, we follow the same approach here and define the auxiliary state

$$Z_n(\theta_1,\theta_2) := \frac{1}{\sqrt{2}}\left[Z(\theta_1)\bar{Z}(\theta_2) + (-1)^n \bar{Z}(\theta_1)Z(\theta_2)\right] \ . \tag{5.1}$$

Obviously, this state has properties of the classical sine-Gordon breather being chargeless and having parity $(-1)^n$. Choosing thereafter the rapidities such that the state (5.1) is on-shell, we can speak of a breather bound state

$$\lim_{(p_1+p_2)^2\to m_{b_n}^2} Z_n(\theta_1,\theta_2) \equiv \lim_{\theta_{12}\to\theta+\theta_{12}^{b_n}} Z_n(\theta_1,\theta_2) = Z_n(\theta) \ . \tag{5.2}$$

Here $\theta_{12}^{b_n}$ is the fusing angle related to the poles in the soliton-antisoliton scattering amplitudes. To simplify notation we abbreviate in what follows $a_{ij} = a(\theta_{ij})$, $b_{ij} = b(\theta_{ij})$, etc. With the help of the braiding relations (4.1) and (4.2) we compute the scattering amplitude between a soliton and the auxiliary breather states (5.1)

$$\begin{aligned} Z_n(\theta_1,\theta_2)Z(\theta_3) = \frac{1}{\sqrt{2}}\big[& a_{13}b_{23}Z(\theta_3)Z(\theta_1)\bar{Z}(\theta_2) + (-1)^n b_{13}a_{23}Z(\theta_3)\bar{Z}(\theta_1)Z(\theta_2) \\ & + c_{13}c_{23}Z(\theta_3)\bar{Z}(\theta_1)Z(\theta_2) + (-1)^n d_{13}d_{23}Z(\theta_3)Z(\theta_1)\bar{Z}(\theta_2) \\ & + b_{13}c_{23}\bar{Z}(\theta_3)Z(\theta_1)Z(\theta_2) + (-1)^n c_{13}a_{23}\bar{Z}(\theta_3)Z(\theta_1)Z(\theta_2) \\ & + d_{13}b_{23}\bar{Z}(\theta_3)\bar{Z}(\theta_1)\bar{Z}(\theta_2) + (-1)^n a_{13}d_{23}\bar{Z}(\theta_3)\bar{Z}(\theta_1)\bar{Z}(\theta_2)\big] \ . \end{aligned} \tag{5.3}$$

Going now on-shell in the sense as defined in (5.2), splitting the fusing angle into $\theta_1 = \theta_0 - \theta_{21}^n/2$, $\theta_2 = \theta_0 + \theta_{21}^n/2$ and denoting $\theta_{03} = \theta$ the following identities can be shown to hold

$$b_{13}c_{23} + (-1)^n c_{13}a_{23} = 0 \qquad \text{for} \quad \theta_{21}^n = i(\pi - 2n\nu K_\ell) \tag{5.4}$$

$$d_{13}b_{23} + (-1)^n a_{13}d_{23} = 0 \qquad \text{for} \quad \theta_{21}^n = i(\pi - 2n\nu K_\ell) \tag{5.5}$$

$$b_{13}a_{23} + (-1)^n c_{13}c_{23} = a_{13}b_{23} + (-1)^n d_{13}d_{23} \qquad \text{for} \quad \theta_{21}^n = i(\pi - 2n\nu K_\ell) \ . \tag{5.6}$$

Therefore the braiding relation (5.3) reduces to a diagonal scattering process, i.e. there is no backscattering amplitude. We obtain after some algebra

$$Z_n(\theta_0)Z(\theta_3) = S_{b_n s}(\theta)Z(\theta_3)Z_n(\theta_0) \ , \tag{5.7}$$

with

$$\begin{aligned} S_{b_n s}(\theta) &= a_{13}b_{23} + (-1)^n d_{13}d_{23} & (5.8) \\ &= \frac{\operatorname{sn}(i\theta/\nu - \pi/2\nu + nK_\ell)}{\operatorname{sn}(i\theta/\nu + \pi/2\nu + nK_\ell)}[\ell \operatorname{sn}^2\frac{\pi}{\nu}\operatorname{sn}^2\left(\frac{i\theta}{\nu} + \frac{\pi}{2\nu} + nK_\ell\right) - 1]a_{13}a_{23} & (5.9) \end{aligned}$$

where

$$\begin{aligned} a_{13}a_{23} = \Phi_{13}\Phi_{23} & \frac{\Gamma_{\hat{q}^2}[1+\hat{\theta}+\frac{\lambda}{4}-\frac{n}{2}]\Gamma_{\hat{q}^2}[-\hat{\theta}-\frac{\lambda}{4}-\frac{n}{2}]\Gamma_{\hat{q}^2}[-\hat{\theta}+\frac{\lambda}{4}+\frac{n}{2}]\Gamma_{\hat{q}^2}[\hat{\theta}+\frac{\lambda}{4}-\frac{n}{2}]}{\Gamma_{\hat{q}^2}[1-\hat{\theta}+\frac{\lambda}{4}-\frac{n}{2}]\Gamma_{\hat{q}^2}[\hat{\theta}-\frac{\lambda}{4}-\frac{n}{2}]\Gamma_{\hat{q}^2}[\hat{\theta}+\frac{\lambda}{4}+\frac{n}{2}]\Gamma_{\hat{q}^2}[-\hat{\theta}+\frac{\lambda}{4}-\frac{n}{2}]} \\ \times \prod_{k=0}^{\infty} & \frac{[\hat{\theta}-\frac{n}{2}+\frac{\lambda}{4}-k\lambda]_{\hat{q}^2}[-\hat{\theta}+\frac{n}{2}-\frac{\lambda}{4}-k\lambda]_{\hat{q}^2}[\hat{\theta}+\frac{n}{2}+\frac{\lambda}{4}-k\lambda]_{\hat{q}^2}[-\hat{\theta}-\frac{n}{2}-\frac{\lambda}{4}-k\lambda]_{\hat{q}^2}}{[-\hat{\theta}-\frac{n}{2}+\frac{\lambda}{4}-k\lambda]_{\hat{q}^2}[\hat{\theta}+\frac{n}{2}-\frac{\lambda}{4}-k\lambda]_{\hat{q}^2}[-\hat{\theta}+\frac{n}{2}+\frac{\lambda}{4}-k\lambda]_{\hat{q}^2}[\hat{\theta}-\frac{n}{2}-\frac{\lambda}{4}-k\lambda]_{\hat{q}^2}} \\ \times \prod_{l=1}^{n-1} & \frac{[\hat{\theta}-\frac{n}{2}+\frac{\lambda}{4}-k\lambda+l]_{\hat{q}^2}^2[-\hat{\theta}+\frac{n}{2}-\frac{\lambda}{4}-k\lambda-l]_{\hat{q}^2}^2}{[-\hat{\theta}-\frac{n}{2}+\frac{\lambda}{4}-k\lambda+l]_{\hat{q}^2}^2[\hat{\theta}+\frac{n}{2}-\frac{\lambda}{4}-k\lambda-l]_{\hat{q}^2}^2} \ . \end{aligned} \tag{5.10}$$

Similarly, we compute the braiding of $Z_n(\theta_1,\theta_2)\bar{Z}(\theta_3)$ which leads us to

$$S_{b_n\bar{s}}(\theta) = S_{b_n s}(\theta)\ . \tag{5.11}$$

Denoting the anti-particle always by an overbar, we deduce from (5.11) that the breathers are self-conjugate due to the general relation $S_{ab} = S_{\bar{a}\bar{b}}$, that is $\bar{b}_n = b_n$.

The matrix $S_{b_n s}(\theta)$ contains various types of poles. i) simple and double poles inside the physical sheet beyond the imaginary axis, which are redundant from our point of view as they are of a tachyonic nature, ii) double poles on the imaginary axis which can be explained by the usual "box diagrams" corresponding to the Coleman-Thun mechanism [40], iii) simple poles in the non-physical sheet which can be interpreted as unstable particles (see e.g. [33] and references therein) and iv) one simple pole on the imaginary axis inside the physical sheet at $\theta = i\pi/2 + in\nu K_\ell$ which is associated to a soliton produced as an n^{th}-breathers-soliton bound state.

6. Breather-breather amplitudes

We now proceed similarly as in the previous section and compute the scattering amplitudes of the breathers amongst themselves. As mentioned, this is a very interesting sector as, under certain circumstances which will be specified below, it usually closes independently from the remaining sectors of the model. Similarly as in (5.7) we exchange now two of the auxiliary states (5.1)

$$Z_n(\theta_1,\theta_2)Z_m(\theta_3,\theta_4) = S_{b_n b_m}(\theta_1,\theta_2,\theta_3,\theta_4)Z_m(\theta_3,\theta_4)Z_n(\theta_1,\theta_2)\ . \tag{6.1}$$

This is a somewhat cumbersome computation and we will not report here the analogue expression of (5.3), since it involves 64 terms, each one of them consisting of a product of four amplitudes and four creation operators. As in the previous section, in the next step we have to go on-shell by specifying the fusing angles as in (5.2). We choose for this purpose $\theta_1 = \theta_0 - \theta_{21}^n/2$, $\theta_2 = \theta_0 + \theta_{21}^n/2$, $\theta_3 = \theta_0' - \theta_{43}^m/2$, $\theta_4 = \theta_0' + \theta_{43}^m/2$ with $\theta_{43}^m = i(\pi - 2m\nu K_\ell)$, $\theta_{21}^n = i(\pi - 2n\nu K_\ell)$, $\theta = \theta_0 - \theta_0'$ such that

$$\theta_{13} = \theta + i(n-m)\nu K_\ell, \quad \theta_{14} = \theta - \pi i + i(n+m)\nu K_\ell, \tag{6.2}$$

$$\theta_{24} = \theta + i(m-n)\nu K_\ell, \quad \theta_{23} = \theta + \pi i - i(n+m)\nu K_\ell. \tag{6.3}$$

For this choice of the fusing angles there are several on-shell identities of the type (5.4)-(5.6). Extracting here the common factors of a_{ij} we compute

$$\begin{aligned}
&(-1)^m\ (\hat{b}_{24}\,\hat{c}_{13}\,\hat{c}_{14} + \hat{b}_{14}\,\hat{d}_{23}\,\hat{d}_{24}) + (-1)^n\ (\hat{b}_{13}\,\hat{c}_{14}\,\hat{c}_{24} + \hat{b}_{14}\,\hat{d}_{13}\,\hat{d}_{23}) \\
&+\hat{c}_{13}\,\hat{c}_{14}\,\hat{c}_{23}\,\hat{c}_{24} + \hat{b}_{14}\,\hat{b}_{23} + (-1)^{m+n}\,(\hat{b}_{14}\,\hat{b}_{23}\,\hat{d}_{13}\,\hat{d}_{24} + \hat{b}_{13}\,\hat{b}_{24}\,\hat{c}_{14}\,\hat{c}_{23}) = \\
&(-1)^m\ (\hat{b}_{23}\,\hat{d}_{13}\,\hat{d}_{14} + \hat{b}_{13}\,\hat{c}_{23}\,\hat{c}_{24}) + (-1)^n\ (\hat{b}_{24}\,\hat{c}_{13}\,\hat{c}_{23} + \hat{b}_{23}\,\hat{d}_{14}\,\hat{d}_{24}) \\
&+\hat{d}_{13}\,\hat{d}_{14}\,\hat{d}_{23}\,\hat{d}_{24} + \hat{b}_{13}\,\hat{b}_{24} + (-1)^{m+n}\ (\hat{d}_{14}\,\hat{d}_{23} + \hat{c}_{13}\,\hat{c}_{24})
\end{aligned} \tag{6.4}$$

and

$$
\begin{aligned}
&(-1)^m\ (\hat{b}_{13}\,\hat{b}_{24}\,\hat{d}_{14}+\ \hat{d}_{13}\,\hat{d}_{23}\,\hat{d}_{24})+(-1)^n\ (\hat{c}_{13}\,\hat{c}_{24}\,\hat{d}_{14}+\ \hat{d}_{23})\\
&+\hat{b}_{13}\,\hat{c}_{23}\,\hat{c}_{24}\,\hat{d}_{14}+\hat{b}_{23}\,\hat{d}_{13}+(-1)^{m+n}\ (\hat{b}_{23}\,\hat{d}_{24}+\ \hat{b}_{24}\,\hat{c}_{13}\,\hat{c}_{23}\,\hat{d}_{14})=0\ , \qquad (6.5)\\
&(-1)^m\ (\hat{b}_{13}\,\hat{b}_{14}\,\hat{c}_{24}+\ \hat{c}_{14}\,\hat{d}_{13}\,\hat{d}_{23})+(-1)^n\,(\,\hat{b}_{14}\,\hat{b}_{24}\,\hat{c}_{13}+\ \hat{c}_{14}\,\hat{d}_{23}\,\hat{d}_{24})\\
&+\hat{b}_{23}\,\hat{c}_{14}\,\hat{d}_{13}\,\hat{d}_{24}+\hat{b}_{13}\,\hat{b}_{14}\,\hat{b}_{24}\,\hat{c}_{23}+(-1)^{m+n}\ (\hat{b}_{23}\,\hat{c}_{14}+\ \hat{b}_{14}\,\hat{c}_{13}\,\hat{c}_{23}\,\hat{c}_{24})=0\ . \qquad (6.6)
\end{aligned}
$$

The relations (6.5) and (6.6) lead to a cancellation of the backscattering terms in an analogous fashion as in (5.3). With (6.4) we indeed end up with a diagonal scattering matrix

$$
Z_n(\theta_0)Z_m(\theta_0')=S_{b_nb_m}(\theta)Z_m(\theta_0')Z_n(\theta_0) \qquad (6.7)
$$

where

$$
\begin{aligned}
S_{b_nb_m}(\theta)\ =&\ a_{13}a_{14}\,a_{23}\ a_{24}\Big[\hat{c}_{13}\,\hat{c}_{14}\,\hat{c}_{23}\,\hat{c}_{24}+\hat{b}_{14}\,\hat{b}_{23}+(-1)^{n+m}(\hat{b}_{14}\,\hat{b}_{23}\,\hat{d}_{13}\,\hat{d}_{24}+\hat{b}_{13}\,\hat{b}_{24}\,\hat{c}_{14}\,\hat{c}_{23})\\
&+(-1)^m(\hat{b}_{24}\,\hat{c}_{13}\,\hat{c}_{14}+\hat{b}_{14}\hat{d}_{23}\,\hat{d}_{24})+(-1)^n(\hat{b}_{14}\,\hat{d}_{13}\,\hat{d}_{23}+\hat{b}_{13}\,\hat{c}_{14}\,\hat{c}_{24})\Big] \qquad (6.8)\\
=&\left[1-\ell\mathrm{sn}^2\frac{\pi}{\nu}\mathrm{sn}^2\left(\frac{i\theta}{\nu}+(n+m)K_\ell\right)\right]\left[1-\ell\mathrm{sn}^2\frac{\pi}{\nu}\mathrm{sn}^2\left(\frac{i\theta}{\nu}+(n+m)K_\ell+\frac{\pi}{\nu}\right)\right]\\
&\times\frac{\mathrm{sn}(i\theta/\nu-\pi/\nu+(n+m)K_\ell)}{\mathrm{sn}(i\theta/\nu+\pi/\nu+(n+m)K_\ell)}a_{13}a_{14}\,a_{23}\ a_{24} \qquad (6.9)
\end{aligned}
$$

and

$$
\begin{aligned}
a_{13}a_{14}\,a_{23}\ a_{24}\ =&\ \Phi_{13}\Phi_{14}\Phi_{23}\Phi_{24}\frac{\Gamma_{\hat{q}^2}(1+\frac{m}{2}+\frac{n}{2}+\hat{\theta}+\frac{\lambda}{2})\,\Gamma_{\hat{q}^2}(\frac{-m}{2}-\frac{n}{2}-\hat{\theta}-\frac{\lambda}{2})}{\Gamma_{\hat{q}^2}(1+\frac{m}{2}+\frac{n}{2}-\hat{\theta}+\frac{\lambda}{2})\Gamma_{\hat{q}^2}(\frac{-m}{2}-\frac{n}{2}+\hat{\theta}-\frac{\lambda}{2})} \qquad (6.10)\\
&\times\prod_{k=1}^{\infty}\prod_{l=1}^{n-1}\frac{[\frac{m}{2}+\frac{n}{2}-l-\hat{\theta}-k\,\lambda+\lambda]_{\hat{q}^2}[\frac{-m}{2}-\frac{n}{2}+l+\hat{\theta}-k\,\lambda]_{\hat{q}^2}}{[\frac{m}{2}+\frac{n}{2}-l+\hat{\theta}-k\,\lambda+\lambda]_{\hat{q}^2}[\frac{-m}{2}-\frac{n}{2}+l-\hat{\theta}-k\,\lambda]_{\hat{q}^2}}\\
&\times\prod_{k=0}^{\infty}\prod_{l=1}^{n-1}\frac{[\frac{m}{2}+\frac{n}{2}-l+\hat{\theta}-\frac{\lambda}{2}-k\,\lambda]_{\hat{q}^2}[-\frac{m}{2}-\frac{n}{2}+l-\hat{\theta}-\frac{\lambda}{2}-k\,\lambda]_{\hat{q}^2}}{[\frac{m}{2}+\frac{n}{2}-l-\hat{\theta}-\frac{\lambda}{2}-k\,\lambda]_{\hat{q}^2}[-\frac{m}{2}-\frac{n}{2}+l+\hat{\theta}-\frac{\lambda}{2}-k\,\lambda]_{\hat{q}^2}}\\
&\times\prod_{k=1}^{\infty}\prod_{l=0}^{m-1}\frac{[\frac{m}{2}+\frac{n}{2}-l-\hat{\theta}-k\,\lambda+\lambda]_{\hat{q}^2}[-\frac{m}{2}-\frac{n}{2}+l+\hat{\theta}-k\,\lambda]_{\hat{q}^2}}{[\frac{m}{2}+\frac{n}{2}-l+\hat{\theta}-k\,\lambda+\lambda]_{\hat{q}^2}[-\frac{m}{2}-\frac{n}{2}+l-\hat{\theta}-k\,\lambda]_{\hat{q}^2}}\\
&\times\prod_{k=0}^{\infty}\prod_{l=0}^{m-1}\frac{[\frac{m}{2}+\frac{n}{2}-l+\hat{\theta}-\frac{\lambda}{2}-k\,\lambda]_{\hat{q}^2}[-\frac{m}{2}-\frac{n}{2}+l-\hat{\theta}-\frac{\lambda}{2}-k\,\lambda]_{\hat{q}^2}}{[\frac{m}{2}+\frac{n}{2}-l-\hat{\theta}-\frac{\lambda}{2}-k\,\lambda]_{\hat{q}^2}[-\frac{m}{2}-\frac{n}{2}+l+\hat{\theta}-\frac{\lambda}{2}-k\,\lambda]_{\hat{q}^2}}.
\end{aligned}
$$

The latter expression (6.10) is tailored to make contact to the expressions in the literature corresponding to the trigonometric limit.

The matrix $S_{b_nb_m}(\theta)$ also exhibits several types of poles. i) simple and double poles inside the physical sheet beyond the imaginary axis, ii) double poles located on the imaginary axis, iii) simple poles in the non-physical sheet and iv) one simple pole on the imaginary axis inside the physical sheet at $\theta=\theta_b=i\nu(n+m)K_\ell$ which is related to the fusing process of two breathers $b_n+b_m\to b_{n+m}$. To be really sure that this pole admits such an interpretation, we have to establish that the imaginary part of the residue is strictly positive, i.e.

$$-i \lim_{\theta \to \theta_b} (\theta - \theta_b) S_{b_n b_m}(\theta) > 0 \ . \tag{6.11}$$

Since the scattering process is parity invariant, we have $S_{b_n b_m} = S_{b_m b_n}$, such that we can choose without loss of generality $n \geq m$. With this choice we compute

$$\begin{aligned}
&\operatorname*{Res}_{\theta \to \theta_b} S_{b_n b_m}(\theta) = i(-1)^{n+m} \frac{2K_{1-\ell}\nu}{\pi} \sinh\left[\pi \frac{K_\ell}{K_{1-\ell}}(n+m)\right] \left(1 - \ell \mathrm{sn}^4 \frac{\pi}{\nu}\right) \hat{q}^{-2(n+m)(n+m+\lambda+1)} \\
&\times \Phi\left[i\pi\right] \Phi\left[2in\nu K_\ell\right] \Phi\left[2in\nu K_\ell\right] \Phi\left[2i(n+m)\nu K_\ell - i\pi\right] \prod_{l=1}^{n-1} \frac{[n+m-l]_{\hat{q}^2}}{[-l]_{\hat{q}^2}} \prod_{l=1}^{m-1} \frac{[n+m-l]_{\hat{q}^2}}{[-l]_{\hat{q}^2}} \\
&\times \prod_{k=1}^{\infty} \Bigg(\frac{[n-k\,\lambda]_{\hat{q}^2}[-n-k\,\lambda]_{\hat{q}^2}[n+\frac{\lambda}{2}-k\,\lambda]_{\hat{q}^2}[-n+\frac{\lambda}{2}-k\,\lambda]_{\hat{q}^2}[n+m-k\,\lambda]_{\hat{q}^2}}{[m-k\,\lambda]_{\hat{q}^2}[-m-k\,\lambda]_{\hat{q}^2}[m+\frac{\lambda}{2}-k\,\lambda]_{\hat{q}^2}[-m+\frac{\lambda}{2}-k\,\lambda]_{\hat{q}^2}[n+m+\frac{\lambda}{2}-k\,\lambda]_{\hat{q}^2}} \\
&\times \frac{[-n-m-k\,\lambda]_{\hat{q}^2}[\frac{\lambda}{2}-k\,\lambda]^2_{\hat{q}^2}}{[-n-m+\frac{\lambda}{2}-k\,\lambda]_{\hat{q}^2}[-k\,\lambda]^2_{\hat{q}^2}} \prod_{l=1}^{m-1} \frac{[n+m-l-k\,\lambda]^2_{\hat{q}^2}[l-n-m-k\,\lambda]^2_{\hat{q}^2}}{[-l-k\,\lambda]^2_{\hat{q}^2}[l-k\,\lambda]^2_{\hat{q}^2}} \\
&\times \frac{[\frac{\lambda}{2}+n+m-l-k\,\lambda]^2_{\hat{q}^2}[\frac{\lambda}{2}-n-m+l-k\,\lambda]^2_{\hat{q}^2}}{[\frac{\lambda}{2}-l-k\,\lambda]^2_{\hat{q}^2}[\frac{\lambda}{2}+l-k\,\lambda]^2_{\hat{q}^2}} \Bigg)
\end{aligned} \tag{6.12}$$

The first line in (6.12) equals $i(-1)^{n+m}\kappa$ with $\kappa \in \mathbb{R}^+$. Noting that the functions Φ with the above arguments are positive real numbers, the second line in (6.12) is $(-1)^{n+m}\kappa'$ with $\kappa' \in \mathbb{R}^+$. Recalling finally that $n+m < -\lambda/2$ we deduce that $\prod_{k=1}^{\infty}(\ ) \in \mathbb{R}^+$ such that (6.11) is indeed satisfied.

Due to the factorizability of the theory this fusing process can be associated in the usual fashion to a bootstrap equation. For consistency the following equations have to be satisfied

$$S_{lb_{n+m}}(\theta) = S_{lb_n}(\theta + i\nu m K_\ell) S_{lb_m}(\theta - i\nu n K_\ell) \quad \text{for } l \in \{b_k, s, \bar{s}\} \ ; k, m+n < n_{\max} \ . \tag{6.13}$$

With some algebra we verified (6.13) for the amplitudes derived above (5.9) and (6.9).

7. Reductions of the $\mathbb{Z}_4$-model

The elliptic sine-Gordon model can be considered as a master theory, which contains many other theories as submodels. By choosing various specific values for the two free parameters of the model ℓ and ν, one obtains these different types of theories. For the different choices we have the following interrelations

$$
\begin{array}{ccccc}
\boxed{\text{elliptic sine-Gordon}} & \overset{1/\nu\to 2nK_\ell/\pi+2imK_{1-\ell}/\pi}{- - - - - - - - \longrightarrow} & \boxed{\text{elliptic } D_{n+1}^{(1)}\text{-affine Toda theory}} & & \\
 & & \swarrow & & \\
| & m\neq 0, \ell\to 0 & & | & \\
| & \downarrow & & | & \\
\ell\to 0 & \boxed{\text{free theory}} & & m=0, \ell\to 0 & \qquad (7.1)\\
| & \uparrow & & | & \\
\downarrow & 1/\nu\to i\infty & & \downarrow & \\
 & \nearrow & & & \\
\boxed{\text{sine-Gordon}} & \underset{1/\nu\to n}{- - - - - - - - \longrightarrow} & \boxed{\text{minimal } D_{n+1}^{(1)}\text{-affine Toda theory}} & &
\end{array}
$$

Let us discuss this schematic diagram in more detail:

7.1 Trigonometric limits

As was already stated in [1], when carrying out one of the trigonometric limit $\ell\to 0$ for the $\mathbb{Z}_4$-model for the amplitudes (4.3)-(4.6), one recovers the soliton sector of the sine-Gordon model. In our formulation this can be seen directly, as we only have to use the relations (2.9) and (2.22) for the $\Gamma_{\hat{q}}$- and sn-functions and note that $\lim_{\ell\to 0}\Phi = 1$. We have employed here the somewhat generalized conventions of the sine-Gordon model formulation in [32]. To make contact with the infinite product representation as presented in the literature (see equations (2.16) and appendix C in [32]), we also have to use the identity

$$
\prod_{k=0}^{\infty}\frac{\Gamma^2(\frac{2k}{\nu}+\alpha)}{\Gamma(\frac{2k}{\nu}+\alpha+\gamma)\Gamma(\frac{2k}{\nu}+\alpha-\gamma)} = \prod_{k=0}^{\infty}\frac{\Gamma^2(\frac{k\nu}{2}+\frac{\alpha\nu}{2})}{\Gamma(\frac{k\nu}{2}+\frac{\alpha\nu}{2}+\frac{\gamma\nu}{2})\Gamma(\frac{k\nu}{2}+\frac{\alpha\nu}{2}-\frac{\gamma\nu}{2})}\ , \qquad (7.2)
$$

after the limit $\ell\to 0$ is performed. As we demonstrated above, when relaxing the constraint $\nu\geq\pi/2K_\ell$, the $\mathbb{Z}_4$- model exhibits also a breather sector. It is easily seen that the corresponding amplitudes can also be obtained in that limit. The expressions for the soliton-breather amplitude (5.9) and (5.10) are tailored in such a way that we obtain the corresponding amplitude in the sine-Gordon model (see equation (20) in [27]) upon the use of (2.9), (2.6) and

$$
\frac{\sin(x\pi/\lambda)}{\sin(y\pi/\lambda)} = \prod_{k=1}^{\infty}\frac{(x-k\lambda+\lambda)(-x-k\lambda)}{(y-k\lambda+\lambda)(-y-k\lambda)}. \qquad (7.3)
$$

Similarly, we recover the breather-breather amplitude of the sine-Gordon model (see equation (22) in [27]) using in addition to (7.3) also

$$
\frac{\cos(x\pi/\lambda)}{\cos(y\pi/\lambda)} = \prod_{k=0}^{\infty}\frac{(y-k\lambda-\frac{\lambda}{2})(-y-k\lambda-\frac{\lambda}{2})}{(x-k\lambda-\frac{\lambda}{2})(-x-k\lambda-\frac{\lambda}{2})}. \qquad (7.4)
$$

There is of course the other trigonometric limit $\ell\to 1$, which one could in principle compute by exploiting the relations between the q and $\hat{q}$-deformed quantities mentioned in section 2. However, just by considering the pole structure (4.8) and recalling (2.3), we see that there are no poles left inside the physical sheet which could produce a bound state, such that this theory will only possess a soliton sector. We will not consider this case here.

Whereas these limits served essentially only as a consistency check, the next one will lead to a new type of theory.

7.2 Diagonal limit

For the sine-Gordon model it is well known [41, 42] that in the limit $\nu \to 1/n$ the backscattering amplitude vanishes and one obtains a diagonal S-matrix which can be identified with a minimal $D^{(1)}_{n+1}$- affine Toda field theory. We observe, that there is an analogue to this behaviour in the $\mathbb{Z}_4$-model as also in this case the backscattering amplitudes vanish in the limit

$$1/\nu \to 1/\nu_{n,m} = (2nK_\ell + i2mK_{1-\ell})/\pi \ . \tag{7.5}$$

We find

$$\lim_{\nu \to \nu_{n,m}} c(\theta) = 0 \qquad \text{for } n,m \in \mathbb{Z}, \tag{7.6}$$

$$\lim_{\nu \to \nu_{n,m}} d(\theta) = 0 \qquad \text{for } n,m \in \mathbb{Z}. \tag{7.7}$$

For the remaining amplitudes (4.3) and (4.4) in the soliton sector we compute for this limit

$$\lim_{\nu \to \nu_{n,m}} b(\theta) = (-1)^{n+1} a_d(\theta) \qquad \text{for } n,m \in \mathbb{Z}, \tag{7.8}$$

$$\lim_{\nu \to \nu_{n,m}} a(\theta) = a_d(\theta) \qquad \text{for } n,m \in \mathbb{Z} \ . \tag{7.9}$$

where

$$a_d(\theta) = \prod_{k=0}^{\infty} \prod_{l=0}^{n-1} \frac{[-\frac{\theta}{i\pi}(n+m\tau) + (2k+1)n + l]_{\hat{q}^2} [\frac{\theta}{i\pi}(n+m\tau) + (2k+1)n - l]_{\hat{q}^2}}{[\frac{\theta}{i\pi}(n+m\tau) + (2k+1)n + l]_{\hat{q}^2} [-\frac{\theta}{i\pi}(n+m\tau) + (2k+1)n - l]_{\hat{q}^2}} \tag{7.10}$$
$$\times \frac{\Gamma_{\hat{q}}[1+\frac{\tau}{2}]\Gamma_{\hat{q}}[-\frac{\tau}{2}]\Gamma_{\hat{q}}[1-n+(\frac{1}{2}-m)\tau + \frac{\theta}{i\pi}(n+m\tau)]\Gamma_{\hat{q}}[n+(m-\frac{1}{2})\tau - \frac{\theta}{i\pi}(n+m\tau)]}{\Gamma_{\hat{q}}[1-n+(\frac{1}{2}-m)\tau]\Gamma_{\hat{q}}[n+(m-\frac{1}{2})\tau]\Gamma_{\hat{q}}[1+\frac{\tau}{2} - \frac{\theta}{i\pi}(n+m\tau)]\Gamma_{\hat{q}}[-\frac{\tau}{2} + \frac{\theta}{i\pi}(n+m\tau)]}$$

This is obtained when replacing in (4.3)

$$\lim_{\nu \to \nu_{n,m}} \hat{\theta} \to -\frac{\theta}{i\pi}(n+m\tau) \quad \text{and} \quad \lim_{\nu \to \nu_{n,m}} \lambda \to -2(n+m\tau), \tag{7.11}$$

and exploiting the property (2.15) thereafter. Of course one may also carry out the limit (7.5) for the soliton-breather and for the breather-breather amplitude. We do not report those expressions here as they are quite obvious, unlike in (7.10), where several cancellations could be carried out and the infinite product in the $\Gamma_{\hat{q}}$ could be turned into products in $[\ ]_{\hat{q}^2}$. The scattering matrix obtained in this way belongs to a new type of theory. One should mention that the case $m \neq 0$, which complexifies the coupling constant is most likely only of a formal nature, but we expect the case $m = 0$ to be a meaningful theory. However, this requires more analysis especially since there are additional Tachyons present.

Let us now carry out the limit $\ell \to 0$ and verify that the above mentioned diagram (7.1) is indeed commutative. For the cases $m = 0, n \geq 3$ we reproduce the entire scattering matrix of the minimal $D_{n+1}^{(1)}$-affine Toda field theory. In particular when carrying out the limit in (7.10) we find, upon the use of (7.3)

$$\lim_{\hat{q}\to 1, m=0} a_d(\theta) = \prod_{k=0}^{\infty}\prod_{l=0}^{n-1} \frac{[n(-x+2k+1)+l][n(x+2k+1)-l]}{[n(x+2k+1)+l][n(-x+2k+1)-l]} \tag{7.12}$$
$$= \prod_{l=0}^{n-1} \frac{\sinh\frac{1}{2}\left(\theta + \frac{i\pi l}{n}\right)}{\sinh\frac{1}{2}\left(\theta - \frac{i\pi l}{n}\right)},$$

which coincides with the $S_{n+1,n+1} = S_{n,n}$-amplitudes in the minimal $D_{n+1}^{(1)}$-affine Toda field theory. For the cases $m = 0, n = 1, 2$ we reproduce the minimal $A_1^{(1)} \otimes A_1^{(1)}, A_3^{(1)}$-affine Toda field theories. We remark that these correspondences hold up to a change of statistics, that is some amplitudes are only recovered up to a factor of -1, which changes bosonic to fermionic statistics or vice versa. These facts make it natural to call the above theories elliptic versions of the associated limiting theory. As we mentioned above, taking the breather sector alone constitutes a consistent theory in itself. For $\hat{q} = 1, m = 0$ we have minimal $A_{2n-2}^{(2)} \subset D_{n+1}^{(1)}$-affine Toda field theory, a property which transcends also into the elliptic version. A final comment is related to the center of the diagram. We find $\lim_{\hat{q}\to 1, m\neq 0} a_d(\theta) = 1$ simply due to the property (2.7).

8. Conclusion

We demonstrated that when one relaxes the constraint on the coupling constants, one can construct a consistent breather sector for the elliptic sine-Gordon model. The scattering of the breathers amongst themselves and with the soliton sector satisfies a bootstrap equation related to the fusing of two breathers to a third. For the formulation of the scattering amplitudes we used as natural objects q-deformed functions. Roughly speaking one replaces in the amplitudes in the soliton-antisoliton sector the infinite products of Euler's gamma functions by a q-deformed version and in the infinite product in the breather sector integers by q-deformed ones. This tailors the models automatically in a form which allows to carry out various limits. Instead of carrying out the bootstrap analysis one could alternatively take a spin chain as a starting point and use a method based on the algebraic Bethe ansatz, pursued for instance in [43], to compute the breather S-matrix amplitudes. It would be interesting to compare that approach with our findings.

In the diagonal limit we obtain an interesting new theory, which can be viewed as an elliptic generalization of the minimal $D_{n+1}^{(1)}$-affine Toda field theory. In [44] we proposed a procedure which also lead to elliptic generalizations of theories whose scattering amplitudes can be expressed in terms of trigonometric functions. The theories obtained in that fashion were, however, of a quite different nature. The procedure in [44] works strictly on the principle that there are no redundant poles present in the amplitudes, such that as a difference the string of tachyonic states which was encountered here was confined to the non-physical sheet where they can be viewed as unstable particles. Further investigation is needed to clarify more the interpretation of these models and in particular to establish whether they posses a meaningful conformal limit [45].

From a mathematical point of view it will be interesting to generalize the method of section 3 to affine Weyl groups of higher rank.

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