A note on the envelope theorem

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Abstract. The purpose of this note is to discuss the envelope relationship between long run and short run cost functions. It compares the usually presented relationship with one of different form and implications, resulting from a simple production function and constant prices. It points out in particular that the tangency condition between the short and long run total cost functions does not necessarily hold always. The note also shows that a given value of the fixed factor might support in the long run a whole range of levels of output.

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1 Introduction

We wish to consider the relation between the short run and the long run cost functions in the context of two examples, in which the factors of production can assume any non-negative values. The first example gives rise to the usual textbook diagram while the second one does not. It is precisely the possibility of the second example that is the reason for this note. A main implication of the analysis is that the tangency condition between the short and long run total cost functions does not necessarily hold always. Also there is a given, optimal value of the fixed factor which in the long run will support all outputs beyond a particular level. Of course the quantity of the variable factor will adjust itself.

In the economics literature there have been some discussions of applications of a generalized envelope theorem (see for example Benveniste and Scheinkman (1979), Milgrom and Segal (2002), Mas-Colell, Whinston and Green, (1995)). On the other hand, in general in economics, and in particular in advanced textbooks, the envelope property is discussed in the context of equality between the tangents of short run and long run cost functions. Here we engage in a generalization of the envelope theorem where the possibility of a corner solution is also present.

2 Examples of the envelope theorem

We discuss the following two examples. The short run and long run total cost functions are denoted respectively by C_S^* and C_L^*.

Example 1.

We consider the simple model $Y = x_1^\alpha x_2^\beta$ where $\alpha,\ \beta > 0$ and $\alpha + \beta = 1$, and $x_1 \in \mathbb{R}_{\geq 0}$ the variable and $x_2 \in \mathbb{R}_{\geq 0}$ the fixed inputs in the short run. For the prices we assume $p_1,\ p_2 > 0$. We show below that the relation between the cost functions is the conventional one.

The short run

Given the value of x_2 we obtain the demand function $x_1 = \left(\frac{Y}{x_2^\beta}\right)^{1/\alpha}$, and the short run cost function, $C_S^* = p_1 \frac{Y^{1/\alpha}}{x_2^{\beta/\alpha}} + p_2 x_2$ which is rising and convex in Y.

The short run average and marginal cost functions are, respectively, $A_S^* = p_1 \frac{Y^{\beta/\alpha}}{x_2^{\beta/\alpha}} + p_2 \frac{x_2}{Y}$ and $M_S^* = p_1 \frac{1}{\alpha} \frac{Y^{\beta/\alpha}}{x_2^{\beta/\alpha}}$.

The long run

In order to obtain the long run cost function, C_L^*, where x_2 is also allowed to vary continuously, we can minimize C_S^* with respect to x_2. We have the first order condition $\frac{dC_S^*}{dx_2} = -p_1 \frac{\beta}{\alpha} \frac{Y^{1/\alpha}}{x_2^{(\beta/\alpha)+1}} + p_2 = 0$, and second order condition, $\frac{dC_S^{*2}}{dx_2^2} > 0$.

Solving the first order condition we obtain $x_2 = \left(\frac{p_1\ \beta}{\alpha\ p_2}\right)^\alpha Y$ and substituting into C_S^* we

obtain the long run cost function $C_S^{**} = C_L^* = \left[p_1\left(\frac{\alpha\, p_2}{p_1\, \beta}\right)^{\beta} + p_2\left(\frac{p_1\, \beta}{\alpha\, p_2}\right)^{\alpha}\right] Y = \left(\frac{p_1}{\alpha}\right)^{\alpha}\left(\frac{p_2}{\beta}\right)^{\beta} Y$,
where C_S^{**}=minimum C_S^*.

The function C_L^* can also be obtained from the cost minimization problem:
Minimize $C = p_1x_1 + p_2x_2$
Subject to

$$Y = x_1^{\alpha} x_2^{\beta},$$
$$x_1,\ x_2 \geq 0,$$

where $p_1,\ p_2,\ Y$ are fixed.

It is easy to see that the long run demand functions of the inputs are $x_1 = \left(\frac{\alpha\, p_2}{p_1\, \beta}\right)^{\beta} Y$ and $x_2 = \left(\frac{p_1\, \beta}{\alpha\, p_2}\right)^{\alpha} Y$, where the expression for x_2 is identical to the one that results from the condition $\frac{dC_S^*}{dx_2} = 0$. These demand functions imply, of course, the expression of C_L^* obtained above.

The long run average and marginal cost functions are:
$A_L^* = M_L^* = \left[p_1\left(\frac{\alpha\, p_2}{p_1\, \beta}\right)^{\beta} + p_2\left(\frac{p_1\, \beta}{\alpha\, p_2}\right)^{\alpha}\right] = \left(\frac{p_1}{\alpha}\right)^{\alpha}\left(\frac{p_2}{\beta}\right)^{\beta}$.

C_L^* is the envelope of the C_S^* curves and A_L^* that of the A_S^* ones. In both cases every point of the envelope curve corresponds to a point of a unique short run curve. This is the usual case when the fixed factor of production can vary continuously.

The tangency condition between the minimum of the C_S^* convex curves, given Y, and C_L^* follows from the fact that all functions are smooth and C_L^* is obtained from a minimization problem with an interior solution. This is looked at again in the Appendix.

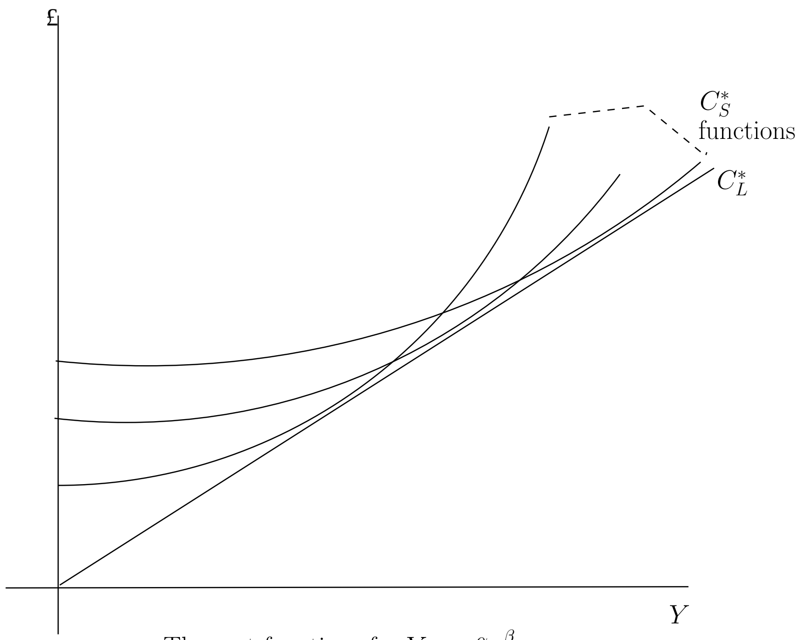
The connection between C_S^* and C_L^* is shown diagrammatically in Figure 1, where without loss of generality we have taken[1] $p_1,\ p_2 = 1$. The resulting relation between A_S^* and A_L^* is shown[2] in Figure 2. At the point of equality of the total cost curves we also have $M_S^* = M_L^*$. This follows from the fact that the marginal cost is the derivative of the total cost, and from the tangency condition between the C_S^* and the C_L^* curves. This equality holds precisely for that level of output. The tangency of the total curves implies the tangency of the average functions. We return to this in the appendix.

Now we wish to investigate the shape of the A_S^* curve. The first and second order derivatives of A_S^* with respect to Y are
$\frac{dA_S^*}{dY} = p_1\frac{\beta}{\alpha}\frac{Y^{(\beta/\alpha)-1}}{x_2^{\beta/\alpha}} - p_2\frac{x_2}{Y^2}$ and $\frac{d^2A_S^*}{dY^2} = p_1\frac{\beta}{\alpha}\left(\frac{\beta}{\alpha} - 1\right)\frac{Y^{(\beta/\alpha)-2}}{x_2^{\beta/\alpha}} + p_2\frac{2x_2}{Y^3}$.

The sign of $\frac{d^2A_S^*}{dY^2}$ is the same as that of $p_1\frac{\beta}{\alpha}\left(\frac{\beta}{\alpha} - 1\right)\frac{Y^{\beta/\alpha}}{x_2^{\beta/\alpha}} + p_2\frac{2x_2}{Y}$. It follows that for

[1] All figures are drawn under the assumption that $p_1 = p_2 = 1$.

[2] We note that for $\alpha+\beta > (<)1$, i.e. for the case of increasing (decreasing) returns to scale, the C_L^* curve will be concave (convex), and the A_L^* one will be decreasing (increasing). Also, in the case of increasing returns to scale the declining A_L^* curve will be convex.



The cost functions for $Y = x_1^{\alpha} x_2^{\beta}$
with $\alpha,\ \beta > 0,\ \ \alpha + \beta = 1$ and $p_1,\ p_2 = 1$.

Figure 1

$\dfrac{\beta}{\alpha} - 1 \geq 0$ the A_S^* curve is convex throughout.

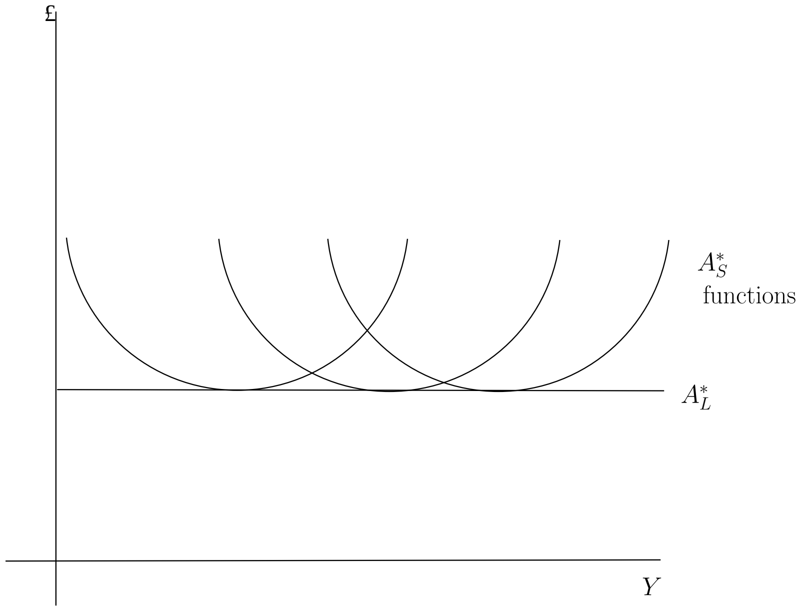
Next, we wish to investigate the case $\dfrac{\beta}{\alpha} - 1 < 0$. First we see what happens around the point of tangency of A_S^* with A_L^*. At that point we have $\dfrac{dA_S^*}{dY} = \dfrac{dA_L^*}{dY} = 0$ which implies $\dfrac{Y^{\alpha/\beta}}{x_2^{\alpha/\beta}} = \dfrac{\alpha}{\beta}\dfrac{p_2}{p_1}\dfrac{x_2}{Y}$, and substituting into the expression $p_1\dfrac{\beta}{\alpha}\left(\dfrac{\beta}{\alpha} - 1\right)\dfrac{Y^{\beta/\alpha}}{x_2^{\beta/\alpha}} + p_2\dfrac{2x_2}{Y}$ we get the equality $p_1\dfrac{\beta}{\alpha}\left(\dfrac{\beta}{\alpha} - 1\right)\dfrac{\alpha}{\beta}\dfrac{p_2}{p_1}\dfrac{x_2}{Y} + p_2\dfrac{2x_2}{Y} = \left(\dfrac{\beta}{\alpha} - 1\right)p_2\dfrac{x_2}{Y} + p_2\dfrac{2x_2}{Y}$; this is of the same sign as $\dfrac{\beta}{\alpha} + 1$ which is positive.

Therefore at the point of tangency with A_L^* the function A_S^* is convex.

Next we look at the total behaviour of the function $\dfrac{d^2 A_S^*}{dY^2}$. As noted above, its sign is determined by that of the expression $p_1\dfrac{\beta}{\alpha}\left(\dfrac{\beta}{\alpha} - 1\right)\dfrac{Y^{\beta/\alpha}}{x_2^{\beta/\alpha}} + p_2\dfrac{2x_2}{Y}$. Due to the fact that $\left(\dfrac{\beta}{\alpha} - 1\right) < 0$, for sufficient large Y it turns and stays concave. The concave part is beyond the point of tangency and it is of course rising to the right and falling to the left of this point.

Example 2.

The production function is now given by $Y = x_1 + 2x_2^{0.5}$, where, the non-negative, x_1 is



The average cost functions for $Y = x_1^\alpha x_2^\beta$
with $\alpha + \beta = 1$, $\dfrac{\beta}{\alpha} \geq 1$ and $p_1,\ p_2 = 1$.

Figure 2

the variable and x_2 the fixed inputs in the short run. The isoquants correspond to fixed Y and they have slope $dx_1 + x_2^{-0.5} dx_2 = 0$. They are are shown in Figure 3.

The short run

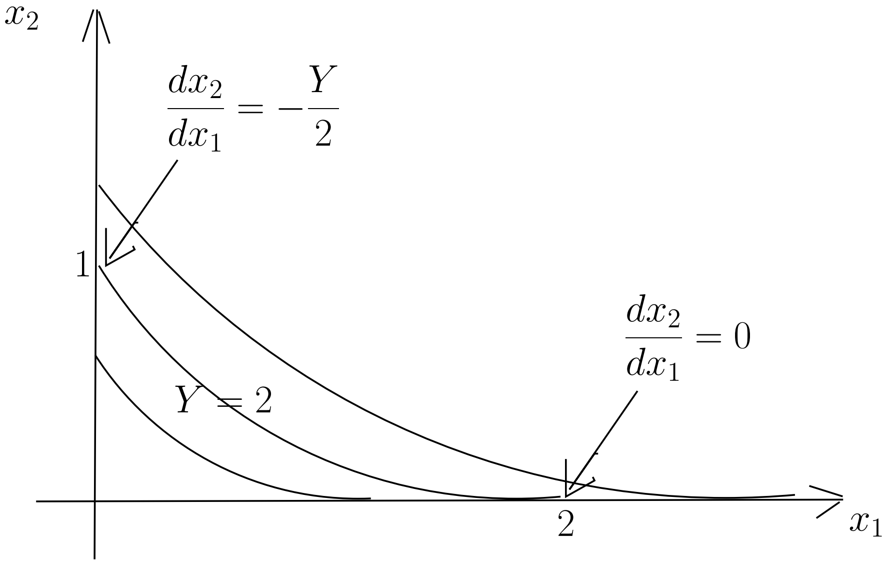
Replacing x_1 from the production function,the total, short run cost function is given by $C_S^* = (Y - 2x_2^{0.5})p_1 + p_2 x_2$ for $Y - 2x_2^{0.5} \geq 0$. The fact that in the production function the factors appear in an additive fashion implies that the short run cost function starts from a positive output $Y = 2x_2^{0.5}$, corresponding to $x_1 = 0$, with cost $C^* = p_2 \left(\dfrac{Y}{2}\right)^2$. As Y varies the curve $C^* = p_2 \left(\dfrac{Y}{2}\right)^2$ traces the short run cost for $x_1 = 0$.

It follows that A^*, the average of C^* is a straight line $A^* = p_2 \dfrac{Y}{4}$. As we increase x_2 we get a bigger Y and we start from a higher point on the convex curve C^*. Hence A^* will have bigger value. For $p_2 = 1$ it is the interrupted straight $A^* = \dfrac{Y}{4}$ in Figure 5.

We rewrite $C_S^* = p_1 Y + (p_2 x_2 - 2p_1 x_2^{0.5})$ and $A_S^* = p_1 + \dfrac{(p_2 x_2 - 2p_1 x_2^{0.5})}{Y}$.

Now, for each x_2 the C_S^* is a straight line with slope p_1 and if it was to be extended it would have an intercept corresponding to $Y = 0$. The C_S^* functions are shown in Figure 4 and the corresponding A_S^* ones in Figure 5, as will be explained below.

In order to calculate how the hypothetical intercept would vary with x_2 we calculate $\dfrac{d(p_2 x_2 - 2p_1 x_2^{0.5})}{dx_2} = p_2 - p_1 x_2^{-0.5}$. This says that the intercept would be increasing with



The isoquants of $Y = x_1 + 2x_2^{0.5}$.

Figure 3

$x_2 > \left(\frac{p_1}{p_2}\right)^2$. For $x_2 > \left(\frac{2p_1}{p_2}\right)^2$ it would be positive. It would be 0 for $x_2 = \left(\frac{2p_1}{p_2}\right)^2$ and negative for $x_2 < \left(\frac{2p_1}{p_2}\right)^2$. We note that there is no contradiction between the hypothetical intercept being negative and C_S^* positive.

The smallest such intercept is obtained for $x_2 = \left(\frac{p_1}{p_2}\right)^2$, for which of course $p_2x_2 - 2p_1x_2^{0.5} < 0$, and the straight line starting at that point with slope p_1 is tangential to the curve $C^* = p_2\left(\frac{Y}{2}\right)^2$. We see this as follows. The function C_S^* has slope p_1 and the one of C^* is $\frac{p_2Y}{2}$. The two are equal at $Y = \frac{2p_1}{p_2}$ which is the level of output at which the calculation of the derivative of C^* takes place.

For $p_1, p_2 = 1$ we have for C^* the value $Y = 2$ and the equation of the tangent at this point is $(TC)_S^* = Y - 1$.

The convexity of C^* means that for $0 \leq Y \leq \frac{2p_1}{p_2}$ the functions C_S^* merge eventually with those which start at a higher level of output. This is due to the fact that for these levels of output a straight line with slope p_1 meets the curve C_S^* twice. Eventually a larger quantity of the variable input will make up for the smaller quantity of the fixed factor. The convex curve $C^* = p_2\left(\frac{Y}{2}\right)^2$ traces the starting point of the short run fixed factor for various quantities x_2 and variable factor $x_1 = 0$.

We now observe that the average function $A_S^* = p_1 + \frac{p_2x_2 - 2p_1x_2^{0.5}}{Y}$ is either concave, a straight line or convex depending on whether the expression $p_2x_2 - 2p_1x_2^{0.5}$ is negative, zero or positive.

The graphs in Figure 4 and 5 have been drawn under the assumption that $p_1,\ p_2 = 1$. We have also included, as a point of reference, in dashed fashion the convex curve C^* and straight line A^* through the origin.

An explanation of Figure 4; $p_1, p_2 = 1$:

All short run cost functions C_S^* start from the convex curve C^*. For $p_2 = 1$ the convex curve $C^* = p_2\left(\frac{Y}{2}\right)^2$ becomes $C^* = \left(\frac{Y}{2}\right)^2$. Tangency between the curves C^* and C_L^*, the long run cost function, takes place at $x_2 = \left(\frac{p_1}{p_2}\right)^2$, i.e. at $x_2 = 1$, which gives $Y - 2x_2^{0.5} = 0$ that is $Y = 2$. This corresponds to $C^* = 1$ and $C_S^* = Y - 1$.

Consider the black straight lines which starts at points above the line through the origin. They do not merge with other lines but if they were to be extended backwards they would have intercepts above 0.

We now want to look at the relationship of certain derivatives. Consider the space $0 \leq Y < 2$. The graph shows that we do not have a tangency condition between the C_S^* functions and C^*. Let us look at the difference between the derivatives. The derivative of a C_S^* is 1. On the other hand the derivative of C^*, at the same Y, is $\frac{dC^*}{dY} = \frac{Y}{2}$. For $0 \leq Y < 2$ we have $\frac{C^*}{Y} = \frac{Y}{2} < 1$ and for $Y = 2$ the derivatives are equal.

Now for $0 \leq Y < 2$ the cost cannot be reduced further and therefore in this range of output C^*=C_L^*. Hence throughout $0 \leq Y < 2$ we have corner solutions of the relationship between the short run and the long run total cost curves. This corner solution will also manifest itself, as we see below, in the relationship between short run and long run average cost curves.

The long run

The envelope of C_S^* is the C_L^* curve, and the relation between the short run and the long average curves follows. First we show that the envelope of the short run cost functions is beyond the level of output $Y = \frac{2p_1}{p_2}$ a single $C_S^*(= C_S^{**})$ straight line, the lowest of all lying above it.

In order to calculate the C_L^* curve we solve directly the following problem with respect to x_1 and x_2.

Minimize $C = p_1x_1 + p_2x_2$

Subject to

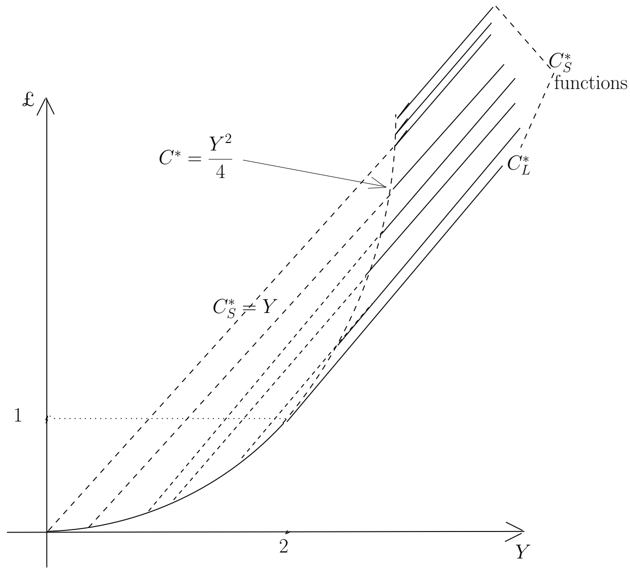
$$Y = x_1 + 2x_2^{0.5},$$
$$x_1,\ x_2 \geq 0.$$

Equivalently we can write

Minimize $C_S^* = p_1Y + (p_2x_2 - 2p_1x_2^{0.5})$

Subject to

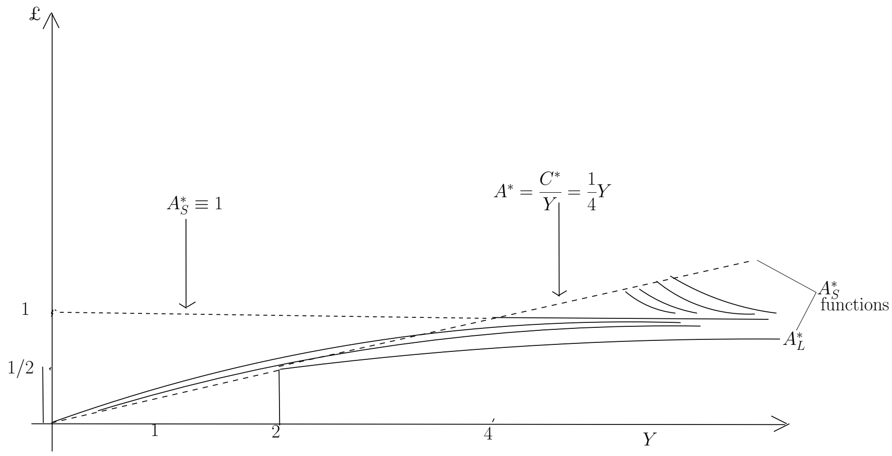
$$Y - 2x_2^{0.5} \geq 0,$$



The cost functions for $Y = x_1 + 2x_2^{0.5}$
and $p_1,\ p_2 = 1$.

$C_L^* = Y - 1$ for $Y \geq 2$ and $C_L^* = \frac{Y^2}{4}$ for $0 \leq Y \leq 2$.

Figure 4



The short run average cost functions for $Y = x_1 + 2x_2^{0.5}$,
with $p_1,\ p_2 = 1$.

Figure 5

$x_2 \geq 0.$

The minimization takes place with respect to x_2.

An interior solution requires $x_2 = \left(\dfrac{p_1}{p_2}\right)^2$ and $Y > \dfrac{2p_1}{p_2}$, and this can be extended to the corner $Y = \dfrac{2p_1}{p_2}$. The corresponding long run cost function will be $C_S^{**} = C_L^* = p_1 Y - p_1 \dfrac{p_1}{p_2}$.

The solution has to be extended to cover the corner solutions case $0 \leq Y < \dfrac{2p_1}{p_2}$ which will imply $Y = 2x_2^{0.5}$, i.e. $x_1 = 0$. The other possible corner with $x_2 = 0$ and $x_1 > 0$ is excluded as a solution because at these values the slope of the isoquant is zero which would require $p_1 = 0$. However the prices of both inputs are positive and therefore $p_1 > 0$.

For $0 \leq Y < \dfrac{2p_1}{p_2}$, the solution, $C_L^* = p_2 \left(\dfrac{Y}{2}\right)^2$ climbs along the vertical axis up to $C_L^* = \dfrac{p_1^2}{p_2}$, corresponding to $x_1 = 0$, $x_2 = \left(\dfrac{p_1}{p_2}\right)^2$. At the highest corner point the two branches of the C_L^* curve coincide. Substituting in $C_L^* = p_1 Y - p_1 \dfrac{p_1}{p_2}$ the value $Y = \dfrac{2p_1}{p_2}$ we obtain $C_L^* = \dfrac{p_1^2}{p_2}$ exactly as before.

C_L^* is the lowest of the C_S^* functions of Y, and A_L^* covers from below the A_S^* functions. C_L^* consists of a strictly convex segment corresponding to $C_L^* = p_2 \left(\dfrac{Y}{2}\right)^2$ and a straight line section from $C_L^* = p_1 Y - p_1 \dfrac{p_1}{p_2}$. It follows that the shape of the corresponding two sections of $A_L^* = \dfrac{C_L^*}{Y}$ will be respectively a straight and a concave one. Considering the C_L^* curve we observe that there is a given value of the fixed factor $x_2 = \left(\dfrac{p_1}{p_2}\right)^2$, i.e. a given C_S^*, which in the long run will support all outputs beyond $Y = \dfrac{2p_1}{p_2}$. For lower values of output C_L^* consists, for each level of a point of a different C_S^*, but there is no tangency condition. It follows that the usual textbook discussion is not typical.

<u>An explanation of Figure 5; $p_1, p_2 = 1$</u>:

For $p_2 = 1$ we have $A^* = \dfrac{Y}{4}$, shown by the interrupted straight line with slope of $\dfrac{1}{4}$.

At $Y = 2$ we have the values $C^* = 1$, $x_2 = 1$, $x_1 = 0$, $C_S^* = 1$, $A^* = A_S^* = \dfrac{1}{2}$. This explains the two numbers 2 and $\dfrac{1}{2}$.

Suppose now that we have $x_2 = 4$. Then $p_2 x_2 - 2p_1 x_2^{0.5} = 0$, i.e. $x_2 - 2x_2^{0.5} = 0$ for the case $p_1, p_2 = 1$ which we are examining. That is we are looking at the short run cost function $C_S^* = Y$ which goes through the origin and which implies $A_S^* = 1$. So this explains the number 1 on the graph. $A_S^* = 1$ intersects with A^* for $Y = 4$.

We want to calculate the slope of the $A_S^* = 1 + \dfrac{x_2 - 2x_2^{0.5}}{Y}$, in the area $0 \leq Y \leq 2$. The derivative is $\dfrac{dA_S^*}{dY} = \dfrac{2x_2^{0.5} - x_2}{Y^2}$. Now we are doing the calculation at $C^*(Y)$, i.e. at

$x_2 = \dfrac{Y^2}{4}$, and therefore we get $A^* = \dfrac{2x_2^{0.5} - x_2}{Y^2} = \dfrac{Y - Y/4}{Y^2} = \dfrac{1}{Y} - \dfrac{1}{4}$.
For $Y = 2$ the slopes of A^* and A_S^* are equal but for $Y < 2$ the slope of A_S^*, calculated where it starts is larger than the slope of A^*. This why for $Y < 2$ the curve A_S^* takes off above the interrupted straight line. We have again a corner solution and A^* in this area coincides with the segment of the interrupted straight. For $Y \geq 2$ the A_S^* curves start below A^* and there is minimum long run curve A_L^* which is itself a A_S^* curve.

For general $p_1,\ p_2$ the average function $A_S^* = p_1 + \dfrac{p_2x_2 - 2p_1x_2^{0.5}}{Y}$ is either concave, a straight line or convex depending on whether the expression $p_2x_2 - 2p_1x_2^{0.5}$, that is the intersection with the $Y = 0$ axis, is negative, zero or positive. That is $A_S^* = p_1 + \dfrac{<=> 0}{Y}$.
We return to Figure 5. The critical value is $x_2 = 4$ which as we have seen implies $A_S^* = 1$. Now, as $x_2 > 4$ increases the functions A_S^* start from a higher point on the $\dfrac{Y}{4}$ interrupted line, they are decreasing and convex. They never cross the flat line equal to 1 because $A_S^* = 1 + \dfrac{> 0}{Y}$ stays above 1 and it goes to it asymptotically.
Next we go to $x_2 < 4$. As x_2 increases the functions A_S^* start from a higher point on the $\dfrac{Y}{4}$ interrupted line, they are increasing and concave. They never cross the flat line equal to 1 because $A_S^* = 1 + \dfrac{< 0}{Y}$ stays below 1 and it approaches it asymptotically.
Now we want to consider more the position of the average curves A_S^* in the graph. We know that they start from the $\dfrac{Y}{4}$ interrupted line.

From Figure 4 we observe that for $Y < 2$ of the curve $C^* = \dfrac{Y^2}{4}$, as the starting output level Y increases, the short run C_S^* decreases for all common levels Y. On the other hand for starting for $Y > 2$, as the starting output level Y increases, the short run C_S^* increases. for all common levels Y. An C_S^* function starting from a $Y < 2$ will eventually merge with an C_S^* starting from some $Y > 2$. This is achieved through the accumulation of an appropriate quantity of the variable which is equivalent to a given level of the fixed factor.
Correspondingly in Figure 5, for $Y < 2$ the concave curves A_S^* are shifted up as they start from a lower point on the interrupted A^* line. On the other hand for $Y > 2$ the A_S^* curves are shifted up as they start from a higher point on the interrupted A^* line.

In conclusion the long run A_L consists of the interrupted straight $A^* = \dfrac{Y}{4}$ line up to $Y = 2$, and from then on from the minimum concave curve A_S^* coming out of $Y = 2$.

3 Concluding remarks

The discussion in this note has aimed to explain that the usual presentation, in the textbooks, of the envelope relationship between the short and long run cost functions is not always valid. It points out in particular that the tangency condition between the slopes of the short and long run total cost functions does not necessarily hold for all ranges of output. It is possible that in a particular such range a corner solution, with slopes that differ, is appropriate. The note also shows, as a possibility, that there might exist a given value of the fixed factor which could support in the long run a whole range of levels of output.

Appendix. Application of the envelope theorem in the examples

A.1. Example 1.

We consider the simple model $Y = x_1^\alpha x_2^\beta$ where $\alpha,\ \beta > 0$ and $\alpha + \beta = 1$, and $x_1 \in \mathbb{R}_{\geq 0}$ the variable and $x_2 \in \mathbb{R}_{\geq 0}$ the fixed inputs in the short run. For the prices we assume $p_1,\ p_2 > 0$. We show below that the relation between the cost functions is the conventional one.

We follow the conventional argument. The short run cost function is $C_S^* = p_1 \dfrac{Y^{1/\alpha}}{x_2^{\beta/\alpha}} + p_2 x_2$.

The long run cost function C_L^* is obtained by minimizing C_S^* with respect to x_2 and obtaining $C_L^* = C_S^{**}$. We have $C_L^* = p_1 \dfrac{Y^{1/\alpha}}{x_2(Y)^{\beta/\alpha}} + p_2 x_2(Y)$ where $x_2(Y) = \left(\dfrac{p_1\ \beta}{p_2\ \alpha}\right)^\alpha Y$.

We want to see how C_L^* varies with Y. The derivative is $\dfrac{dC_L^*}{dY} = p_1 \dfrac{1}{\alpha} \dfrac{Y^{\beta/\alpha}}{x_2(Y)^{\beta/\alpha}} + \dfrac{\partial C_L^*}{\partial x_2(Y)} x_2'(Y)$. However the term $\dfrac{\partial C_L^*}{\partial x_2(Y)}$ is equal to 0 because we must also satisfy the minimization of the C_S^* at $x_2(Y)$.

On the other hand we also have $\dfrac{dC_S^*}{dY} = p_1 \dfrac{1}{\alpha} \dfrac{Y^{\beta/\alpha}}{x_2^{\beta/\alpha}}$ and therefore for $x_2^* = x_2 = x_2(Y) = \left(\dfrac{p_1\ \beta}{p_2\ \alpha}\right)^\alpha Y$ we have $\dfrac{dC_L^*}{dY} = \dfrac{dC_S^*}{dY}$ and this is the envelope theorem. At this level of x_2^* the C_S^* and C_L^*, the two convex curves, have the same slope as the graph shows. More precisely C_L^* is straight line and C_S^* is strictly convex. Also by construction $C_S^* \geq C_L^*$. This justifies the graph in Figure 1.

The short run average and marginal cost functions are, respectively, $A_S^* = p_1 \dfrac{Y^{\beta/\alpha}}{x_2^{\beta/\alpha}} + p_2 \dfrac{x_2}{Y}$ and $M_S^* = p_1 \dfrac{1}{\alpha} \dfrac{Y^{\beta/\alpha}}{x_2^{\beta/\alpha}}$.

The tangency, for fixed Y, between minimum $A_S^* = \dfrac{C_S^*}{Y}$ and $A_L^* = \dfrac{C_L^*}{Y}$, shown in Figure 2, follows from the fact that $\left(\dfrac{dA_S^*}{dY} =\right) \dfrac{Y C_S^{*\prime} - C_S^*}{Y^2} = \dfrac{Y C_L^{*\prime} - C_L^*}{Y^2} \left(= \dfrac{dA_L^*}{dY}\right)$. The equality of the two expressions follows $C_S^{*\prime} = C_L^{*\prime}$ and $C_S^* = C_L^*$, i.e. from the tangency between the total curves.

A.2. Example 2.

The issue is to apply the idea of the envelope theorem to this example. Let us first confine ourselves to the interior set to which we also attach the limit point, i.e. let $Y \geq \dfrac{2p_1}{p_2}$.

The short run cost function is given by $C_S^* = (Y - 2x_2^{0.5})p_1 + p_2 x_2$ for $Y - 2x_2^{0.5} \geq 0$. In order to calculate the C_L^* curve we can minimize C_S^* in this area with respect to x_2. A solution requires $x_2 = \left(\frac{p_1}{p_2}\right)^2$ and $Y \geq \frac{2p_1}{p_2}$. That is every such Y can best be produced, i.e. at a minimum short run cost, if $x_2 = \left(\frac{p_1}{p_2}\right)^2$. We note that the optimal quantity of the fixed factor is the same for all $Y \geq \frac{2p_1}{p_2}$. Of course the quantity of x_1 itself will adjust to produce Y. The long run cost function will be (minimum $C_S^* =)C_S^{**} = C_L^* = p_1Y - p_1\frac{p_1}{p_2}$. The tangency between minimum $C_S^* = C_L^*$ works first at the specific point $Y = \frac{2p_1}{p_2}$. Beyond this point minimum C_S^* and C_L^* coincide in a straight line. So the average curves must also coincide. The question now arises what is the relationship between C_S^* and C_L^* for $0 < Y < \frac{2p_1}{p_2}$.

As in the graphs, we look specifically at $p_1, p_2 = 1$. Now C_L^* is made of the minimum of different C_S^* functions. The graph shows that the slopes of these C_S^* and C_L^* are different. The slope of the C_S^* functions is always equal to 1 but the slope of C_L^* is originally smaller and it climbs up to 1 at $Y = 2$. So, since for $0 \leq Y < 2$ we do not have an interior point solution, we do not also have an equality between these slopes.

For $Y - 2 < 0$, the slope 1 of C_S^* with respect to Y remains constant but the slope C_L^* is lower because there is still scope to adjust the quantity of the fixed factor.

Suppose on the other hand we were allowed to go into negative x_1 provided we kept $Y \geq 0$. Then we would get $C_L^* = p_1Y - p_1\frac{p_1}{p_2}$ and for $p_1 = p_2 = 1$ it would be $C_L^* = Y - 1$. We would also have minimum $C_S^* = Y - 1$ and the two curves and their slopes would coincide.

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