THE VALUATION OF GMWB VARIABLE ANNUITIES UNDER ALTERNATIVE FUND DISTRIBUTIONS AND POLICYHOLDER BEHAVIOURS

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Abstract. In this paper we present a dynamic programming algorithm for pricing variable annuities with Guaranteed Minimum Withdrawal Benefits (GMWB) under a general Lévy processes framework. The GMWB gives the policyholder the right to make periodical withdrawals from her policy account even when the value of this account is exhausted. Typically, the total amount guaranteed for withdrawals coincides with her initial investment, providing then a protection against downside market risk. At each withdrawal date, the policyholder has to decide whether, and how much, to withdraw, or to surrender the contract. We show how different policyholder's withdrawal behaviours can be modelled. We perform a sensitivity analysis comparing the numerical results obtained for different contractual and market parameters, policyholder behaviours, and different types of Lévy processes.

1. Introduction

Variable annuities are very flexible life insurance contracts that package several types of options and guarantees, at the policyholder's discretion. Typically, a lump sum premium is paid at contract inception and is invested in one or more mutual funds chosen by the policyholder among a range of alternative opportunities. Then this initial investment sets up a reference portfolio ('policy account') and each option or guarantee is financed by periodical deductions from the policy account value.

Key words and phrases. Variable annuities, GMWB, Dynamic approach, Lévy processes, Policyholder's behaviour.

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Guarantees are commonly referred to as GMxBs (Guaranteed Minimum Benefit of type 'x'), where 'x' stands for accumulation (A), death (D), income (I) or withdrawal (W). In particular, GMABs and GMDBs provide guarantees in the accumulation phase, prior to retirement, although sometimes the GMDB is offered also after retirement. In a GMIB, that consists of a (possibly indexed, or participating) deferred life annuity, the guarantee usually concerns the annuitized amount or the annuitization rate. However, GMABs and GMDBs can be found also in other types of life insurance contracts such as unit-linked or participating policies, and GMIBs become, after conversion, traditional life annuities. The GMWB, instead, is undoubtedly the most interesting feature of variable annuities. It is similar to an income drawdown, because it entitles the policyholder to make periodical withdrawals from her account, even when the account value is reduced to 0. This guarantee can cover a fixed period of time or can be lifelong. In the first case the policyholder is typically guaranteed her entire initial investment, that can be withdrawn within the given period of time. At the end of the withdrawal period any remaining fund in the reference portfolio is paid back to the policyholder. In the case of a lifelong GMWB the policyholder is also protected against the risk of underfunding due to high longevity. In order to distinguish fixed term from lifelong GMWBs, from now on the latter will be referred to as GLW.

When a variable annuity contains a GMWB (or GLW) rider, there is an amount, fixed or time-dependent, that the policyholder is entitled to withdraw at some specified dates (typically, annually or semiannually). Withdrawals below this fixed amount are allowed, while withdrawals above this amount, if permitted, are subject to a penalty. Then, the prediction of the policyholder behaviour is a key-element in the valuation of such guarantees. In particular, under the so called 'static' (or 'passive') approach, it is assumed that the policyholder withdraws exactly the amount contractually specified (see [32]). The 'dynamic' approach assumes instead that the policyholder chooses the amounts to withdraw according to some optimal policy. In-between these two approaches there is the 'mixed' one, coined by [3], that assumes a static behaviour with respect to the choice of the withdrawal amounts, but a dynamic one with respect to surrender decisions.

General information on variable annuity features can be found in [27], [2] and [1]. The market for variable annuities has been steadily growing in the past 20 years. However, sales fell during the recent financial crisis and many companies offering these products had to eventually exit the business as a result of poor, or lack of, hedging of the guarantees attached.

In this paper we present a dynamic programming algorithm aimed at pricing a variable annuity with a GMWB under the dynamic approach. This algorithm is general enough to allow for different withdrawal behaviours of the policyholder, so that, in particular, the static and the mixed approach can be accomodated as special cases. Variants and extensions of the basic GMWB contract can be easily dealt with. We overcome some well-known problems arising from assuming normality of the reference fund returns, as very often done in the literature, by putting ourselves in a general Lévy framework. This class of stochastic processes is flexible enough to allow for jumps and other desirable properties displayed by the empirical distribution of asset returns (such as fat tails and skewness) and is straightforward to implement. We present extensive numerical examples and compare the results obtained for different market and contractual parameters, policyholder behaviours, as well as for different types of Lévy processes.

The paper is structured as follows. In Section 2 we review the existing literature on GMWBs, focussing in particular on the dynamic approach. In Section 3 we describe the variable annuity contract and the discrete time framework adopted for the valuation. In Section 4 we briefly introduce and recall the main properties of Lévy processes. In Section 5 we develop the dynamic programming algorithm and in Section 6 we present the numerical results. Finally, Section 7 concludes the paper.

2. Review on the Literature on GMWBs

The pricing and hedging variable annuity contracts has attracted the interest of many academics and practitioners. This review focuses on GMWBs and GLWs and in no way claims to be exhaustive. We classify in Table 1 the papers of which we are aware according to the following features: type of benefit (GMWB, GLW), assumption on policyholder behaviour (static* - to be explained below - mixed or dynamic), and statistical assumption on the fund return distribution.

With the term *static** we extend the static behaviour described in the introduction by including any fixed withdrawal or surrender strategy. More precisely, in this class we include deterministic strategies (such as the static strategies discussed before), strategies based on the value of state variables (e.g. withdrawal or surrender behaviours based on the moneyness of the guarantees) that is, in the language of stochastic processes, adapted strategies, and also randomization of such strategies. As opposed to the mixed or dynamic behaviour, the static* approach is not the result of an optimization process. We point out that the static* approach, frequently adopted by practitioners in the analysis of products and e.g. in profit testing exercises, is appealing, somewhat intuitive and straightforward to implement even under sophisticated assumptions on the evolution of the state (market, mortality, . . .) variables. However, it is undoubtedly hard to anticipate correctly the policyholder behaviour, e.g. to specify her policy as a function of the moneyness of the guarantees (see [26], [24] and [2] for the different factors influencing lapse rates). On the other hand, the dynamic approach overcomes this subjective side by taking a worst case scenario from the insurer's point of view, but is subject to the curse of dimensionality and hence very often forces to adopt a very simple setup. According to whether withdrawals are assumed to occur continuously or discretely, the dynamic approach is usually solved using respectively stochastic control and dynamic programming. Some papers (see [17], [14], [15] and [8]), although assuming discrete withdrawals, compute the contract values at any time, even between two withdrawal dates. For a review of the numerical methods used to compute the value of a general life insurance contract, see [7]. Looking at Table 1, one can see that few papers go beyond the assumption of normality for the fund returns. Notable exceptions are [15] and [20], where the jump diffusion model of Merton is considered, and [9], where regime switching type processes are used.

Paper	Rider	PH behaviour	Fund process
[12]	GMWB	static*	GBM
[32]	GMWB	static/dynamic	GBM
[8]	GMWB	static*/dynamic	GBM
[14]	GMWB	dynamic	GBM
[17]	GMWB	dynamic	GBM
[15]	GMWB	dynamic	GBM/Merton
[40]	GLW	static	SIR+SV
[34]	GMWB	static	SIR
[9]	GMWB	dynamic	RS
[35]	GLW	static	GBM
[3, 4]	GMWB/GLW	static/mixed	SIR+SV+SM
[25]	GLW	static*/mixed	SV
[36]	GLW	static	GBM+SM
[33]	GMWB	dynamic	GBM
[41]	GLW	dynamic	GBM
[18]	GMWB/GLW	static*/mixed	GBM
[20]	GMWB	dynamic	Merton
[19]	GMWB	dynamic	GBM

Table 1. GMWB=Guaranteed Minimum Withdrawal Benefit, GLW=Guaranteed Lifelong Withdrawal; GBM=Geometric Brownian Motion, Merton=Merton Jump Diffusion model, RS=Regime Switching, SIR=Stochastic Interest Rates, SV=Stochastic Volatility, SM=Stochastic Mortality.

In this paper, we show how an approach based on dynamic programming can accomodate any fund return distribution within the class of Lévy processes, allowing therefore a great variety of statistical features such as kurtosis and skewness.

3. Model Setup, Valuation and Policyholder Behaviours

Although there are different ways in which a GMWB can be arranged within a variable annuity contract, in what follows we focus on a specific case. Additional contract features can be easily introduced. At inception the policyholder pays a lump-sum premium that is invested in a well diversified mutual fund, then a reference portfolio backing the variable annuity is set up. The current value of this portfolio defines the first of two accounts which the policyholder is entitled to, called ‘personal account’. After that, the policyholder has the right to make periodical withdrawals, even if her personal account value is reduced to zero. Usually the total withdrawals guaranteed during the life of the contract amount to her whole initial investment. Then, in this case, the guarantee becomes effective if the reference portfolio is completely exhausted before the initial premium has been totally recouped. The second account, called ‘guarantee account’, gives the total amount of money that the policyholder is still guaranteed for withdrawals. The cost of the guarantee is financed by periodical deductions from the personal account value ('insurance fees'). The amount that the policyholder is entitled to withdraw at each available date is usually subject to a withdrawal level, fixed or time-dependent, over which some penalty is applied. At maturity, the policyholder (or her estate) receives the maximum between the balance of the personal account and the guarantee account.

When the contract contains a surrender option, the policyholder is allowed to terminate the contract before maturity. In this case she receives a cash amount, called surrender value, usually equal to the balance of the personal account with a proportional penalty if it is higher than the specified withdrawal level.

We observe that the introduction of mortality risk can be easily handled in the dynamic programming algorithm. We refer, more in detail, to the case in which the contract expires before maturity (and before surrender) if the insured dies, with the payment of a lump sum benefit specified in the contract, typically equal to the balance of one of the two accounts or to the maximum between them.

3.1. **Model and Valuation.** We now formalize what just described. Let A_t and W_t denote the time t guarantee account and personal account respectively, before any decision at t is made. Moreover, let S_t denote the unit price at time t of the reference fund, U the lump sum premium, T the maturity of the policy.

Assume that withdrawals are allowed only at times t_i, $i = 1, 2, \dots, N-1$, with $0 = t_0 < t_1 < \dots t_{N-1} < t_N = T$, where t_0 denotes the inception of the contract.[1] The return on the fund over (t_i, t_{i+1}), $i = 0, \dots, N-1$, is then

$$R_{t_i} = \frac{S_{t_{i+1}} \mathrm{e}^{q(t_{i+1}-t_i)}}{S_{t_i}} - 1,$$

where q is the dividend yield, assumed to be constant. Let θ_{t_i} denote the decision made at time t_i by the policyholder. In our case, θ_{t_i} is just the amount withdrawn at t_i, but examples involving other types of decisions could be considered. For the moment we think of θ_{t_i} as some element of a set of admissible decisions/amounts Θ_{t_i}, which can depend on the current value of the state variables A_{t_i} and W_{t_i}.

The personal account evolves according to the following equation:

$$W_{t_{i+1}} = \max\{W_{t_i} - \theta_{t_i}, 0\}\,(1 + R_{t_i})(1 - \varphi(t_{i+1} - t_i)), \tag{3.1}$$

where φ is the insurance fee, applied while the contract is still in force. Hence $W_{t_{i+1}}$ is determined by the current personal account value, the fund return and the withdrawn amount. Note that once W_t hits the value 0, it stays at this value thereafter. Withdrawals continue while the guarantee account is positive, even if the personal account is insufficient. The initial value is $W_0 = U$.[2]

[1] If t_0 coincides, instead, with the end of an accumulation period and U is the (possibly guaranteed) accumulation benefit, then t_0 could be included in the set of possible withdrawal times.

[2] For $i = 0$, we set conventionally $\theta_0 = \theta_{t_0} = 0$, so that W_{t_1} is determined only by the single premium U and the first period return R_{t_0}.

We denote by G the withdrawal level, which is typically equal to $\frac{A_0}{N}$, and assume that a proportional penalty at rate κ applies in case withdrawals exceed this level.[3] Moreover, when almost all the guarantee has been exploited resulting in a guarantee account less than G, the penalty applies also on withdrawals between A_t and G. Hence the cash flow (i.e., the net amount) paid to the policyholder at t_i, $i = 1, \ldots, N-1$, denoted by C_{t_i}, depends on the current withdrawal and state variables as follows:

$$\begin{aligned} C_{t_i} &= \begin{cases} \theta_{t_i} & \text{if } 0 \le \theta_{t_i} \le \min\{G, A_{t_i}\} \\ \min\{G, A_{t_i}\} + (1-\kappa)(\theta_{t_i} - \min\{G, A_{t_i}\}) & \text{if } \theta_{t_i} > \min\{G, A_{t_i}\} \end{cases} \\ &= \theta_{t_i} - \kappa \max\{\theta_{t_i} - \min\{G, A_{t_i}\}, 0\}. \end{aligned} \tag{3.2}$$

Summing up, any withdrawal in excess of the withdrawal level or the guarantee account is subject to the proportional penalty κ. For $i = N$ only, $C_{t_N} = C_T$ is specified separately, typically as a function of W_T, A_T. We assume, in particular, that $C_T = \max\{W_T, A_T\}$.[4]

In addition to the penalty on exceeding withdrawals, there could be a further penalization if some reset provision is in force. This means that the guarantee account can be reduced of an amount greater than the current withdrawal. Here we assume a reset provision corresponding to the so called 'pro rata adjustment' (see [33]), according to which the guarantee account evolves as follows:

$$A_{t_{i+1}} = \begin{cases} A_{t_i} - \theta_{t_i} & \text{if } 0 \le \theta_{t_i} \le \min\{G, A_{t_i}\} \\ \max\left\{\min\left\{A_{t_i} - \theta_{t_i}, A_{t_i}\left(1 - \frac{\theta_{t_i}}{W_{t_i}}\right)\right\}, 0\right\} & \text{if } \theta_{t_i} > \min\{G, A_{t_i}\} \end{cases} \tag{3.3}$$

with initial value $A_0 = U$. Hence, in case of exceeding withdrawals, the guarantee account is reduced by the greatest between the withdrawal amount and a proportion θ_{t_i}/W_{t_i} of the guarantee account itself. Note that equation (3.3) is well defined because, when $W_{t_i} = 0$, withdrawal amounts greater than $\min\{G, A_{t_i}\}$ are not allowed, see Sections 3.1.1-3.1.3.[5]

Now, let π denote a possible sequence of (withdrawals) decisions, i.e. $\pi = (\theta_{t_1}, \theta_{t_2}, \ldots, \theta_{t_{N-1}})$ with $\theta_{t_i} \in \Theta_{t_i}$. The initial value of the cash flows generated from holding the GMWB variable annuity and adopting the sequence of decisions π is given by:

$$V_0^\pi = E\left[\sum_{i=1}^{N} \mathrm{e}^{-r\,t_i}\, C_{t_i}\right], \tag{3.4}$$

where E denotes the expectation taken under a suitable risk-adjusted measure and r is the (assumed constant) risk-free rate. Finally, the no-arbitrage

[3] The extension to a time dependent withdrawal level, fee or penalty rate is straightforward.

[4] Different type of contracts, or more general frameworks, can be represented with a similar scheme, by adding state variables and their state equations. See [7] for a general description of life insurance contracts based on accounts and state variables.

[5] An alternative to (3.3) is to reset $A_{t_{i+1}}$ to $\max\{\min\{A_{t_i} - \theta_{t_i}, W_{t_i} - \theta_{t_i}\}, 0\}$ in case of exceeding withdrawals, see [32].

value of the variable annuity is given by:

$$V_0 = \sup_{\pi} V_0^{\pi}, \tag{3.5}$$

where the supremum is taken over all sequences $\pi = \left(\theta_{t_1}, \theta_{t_2}, \ldots, \theta_{t_{N-1}}\right)$ of withdrawal decisions satisfying the constraint $\theta_{t_i} \in \Theta_{t_i}$, and where the personal account satisfies (3.1) and the guarantee account (3.3). This approach assumes that the policyholder behaves rationally and acts so as to maximize the expected present value of all the cash flows generated by the GMWB variable annuity. This problem can be solved using the classical dynamic programming algorithm, as explained in Section 5. In the rest of this section, we specify the set of admissible decisions under the alternative approaches (dynamic, static and mixed) described in Section 1 and resulting from the assumptions concerning the policyholder behaviour. Moreover, when possible, we simplify the guarantee account dynamics and cash-flow (equations (3.3) and (3.2)).

3.1.1. *Dynamic Withdrawals.* At any date t_i, $i = 1, \ldots, N-1$, the policyholder can always choose to withdraw any amount θ_{t_i} up to $\min\{G, A_{t_i}\}$, whatever is the personal account value W_{t_i}. Moreover, if $W_{t_i} > \min\{G, A_{t_i}\}$, the policyholder is allowed to withdraw even more, subject to the penalties described before, until the personal account is completely exhausted. This last case, i.e. $\theta_{t_i} = W_{t_i} \geq \min\{G, A_{t_i}\}$, corresponds to the contract surrender. Hence the admissible decisions set is the interval

$$\Theta_{t_i} = \left[0, \max\left\{W_{t_i}, \min\{G, A_{t_i}\}\right\}\right].$$

3.1.2. *Static Withdrawals.* The policyholder is constrained to withdraw the amount G, provided this is lower than the guarantee account, or the guarantee account otherwise. This behaviour is obtained by setting the set of decisions at t_i as the singleton

$$\Theta_{t_i} = \{\min\{G, A_{t_i}\}\}.$$

The accounts and cash-flow are still defined by (3.1), (3.3) and (3.2). In particular, the guarantee account dynamics and cash-flow simplify as follows:

$$A_{t_{i+1}} = A_{t_i} - \min\{A_{t_i}, G\} = \max\{A_{t_i} - G, 0\}, \; C_{t_i} = \min\{A_{t_i}, G\}.$$

More generally, the static* approach defined in Section 2 corresponds to fixing a sequence π of withdrawal decisions, and the value of the contract is then V_0^{π}.

As shown in [32], under the static approach the variable annuity contract can be decomposed (in the case $t_i = i$ and $G = W_0/T$) into an immediate annuity with instalment G and maturity T and a Quanto-Asian put option corresponding to the guarantee of receiving at maturity the policyholder account net of the last instalment, if positive. More specifically, the pay-off of the put option is

$$W_0 \frac{1}{Y_T} \max\{1 - \overline{Y}_T, 0\},$$

where

$$Y_t = S_t^{-1}(1-\varphi)^{-t}, \qquad \overline{Y}_T = \frac{1}{T}\sum_{t=1}^{T} Y_t.$$

3.1.3. *Mixed (Static+Surrender).* As in the static approach, the policyholder is behaving passively with respect to partial withdrawals, but can choose to surrender in a dynamic fashion, see [3]. Recall that, in case of surrender, the policyholder receives the personal account value, net of the penalty if it exceeds the minimum between the guarantee account and the withdrawal level. The decision set at t_i would then be the two-points set

$$\Theta_{t_i} = \{\min\{A_{t_i}, G\}, W_{t_i}\},$$

with the cash flow and the personal account equations (3.2) and (3.1) unchanged. The first row of equation (3.3) would instead change as the guarantee account is set at zero immediately after surrender, regardless of the value of W_{t_i}. However, the surrender decision is never optimal if $W_{t_i} \leq \min\{A_{t_i}, G\}$ so that, taking this optimality condition into account, we restrict the decision set to

$$\Theta_{t_i} = \begin{cases} \{\min\{A_{t_i}, G\}\} & \text{if } W_{t_i} \leq \min\{A_{t_i}, G\} \\ \{\min\{A_{t_i}, G\}, W_{t_i}\} & \text{if } W_{t_i} > \min\{A_{t_i}, G\} \end{cases},$$

and specialize the guarantee account dynamics as follows:

$$A_{t_{i+1}} = \begin{cases} \max\{A_{t_i} - G, 0\} & \text{if } \theta_{t_i} = \min\{A_{t_i}, G\} \\ 0 & \text{if } \theta_{t_i} > \min\{A_{t_i}, G\} \end{cases}.$$

3.2. **Examples.** To understand the difference between the three approaches introduced in Sections 3.1.1, 3.1.2 and 3.1.3, we briefly consider some examples illustrating the policyholder possible withdrawal choices at any given date and the corresponding cash-flows, penalties and change in the guarantee account value. To simplify the notation, fix the attention on a given withdrawal date t_i with $0 < i < N$ and drop the dependence on the date t_i on the various symbols.

Table 2 contains the possible ranking among the guarantee account A, the guaranteed withdrawal level G, the personal account W, and the corresponding withdrawal decision set Θ in the three approaches.

Assume that the guaranteed annual withdrawal level is $G = 10$, which may correspond to a single premium $U = 100$ and maturity $T = 10$ years. Suppose that the current value of the guarantee account is $A = 40$ (before a decision is made), which may be the result of constant withdrawals equal to G in the previous 6 years. The guarantee account after the decision is made is $A_+ = A_{t_{i+1}}$, and is given by (3.3). A 5% penalty applies in case of withdrawals in excess of the guaranteed amount $\min\{A, G\}$.

If $W = 70$ (the fund return has been positive and has accrued, net of fees and withdrawals, the initial investment), then the policyholder can withdraw any amount in the interval $[0, 70]$ (dynamic), withdraw 10 or 70 (mixed), or just 10 (static). Any withdrawal above 40 will kill the guarantee so the policyholder will only be entitled to withdrawals from the personal account, provided this is positive. Consider the following examples of decisions:

- $\theta = 70$ (dynamic or mixed). The contract is terminated and the net amount paid to the policyholder is $10 + 95\%(70 - 10) = 67$.
- $\theta = 50$ (dynamic). The guarantee account value becomes 0 and no further guarantees of withdrawals are hold by the policyholder.

The personal account depends on the return earned and the fees subtracted from the remaining fund $70 - 50 = 20$. The net amount paid to the policyholder is $10 + 95\%(50 - 10) = 48$.

- $\theta = 30$ (dynamic). The guarantee account value is $A_+ = \min\{40 - 30, 40(1 - 42.86\%)\} = 10$, where $42.86\% = \frac{\theta}{W}$, so that there is guarantee of future withdrawals of 10. The net amount paid to the policyholder is $10 + 95\%(30 - 10) = 29$.
- $\theta = 10$ (dynamic, mixed or static). The guarantee account value is $A_+ = 40 - 10 = 30$, so that there is guarantee of future withdrawals of 30. The net amount paid to the policyholder is 10, so no penalty is applied.

As a second example suppose that $W = 30$ (the fund return has been not sufficient to compensate the fees). The policyholder can choose any amount in the interval $[0, 30]$ (dynamic), withdraw 10 or 30 (mixed), or just 10 (static).

- $\theta = 30$ (dynamic or mixed). The contract is terminated. The net amount paid to the policyholder is $10 + 95\%(30 - 10) = 29$. Note that the pro rata rule (3.3) implies that the new guarantee account value is $A_+ = \min\{40 - 30, 40(1 - 100\%)\} = 0$.
- $\theta = 10$ (dynamic, mixed or static). The guarantee account value is $A_+ = 40 - 10 = 30$. The net amount paid to the policyholder is 10.

As a final example, suppose that $W = 0$ (the fund has suffered huge losses and, due to withdrawals and fees, its value has hit the level 0). The policyholder can choose any amount in the interval $[0, 10]$ (dynamic), or just withdraw 10 (static and mixed). If $\theta = 10$, the guarantee account value is $A_+ = 40 - 10 = 30$ and the net amount paid to the policyholder is 10.

<table>
<tr><th></th><th colspan="3">Θ</th></tr>
<tr><th></th><th>Dynamic</th><th>Mixed</th><th>Static</th></tr>
<tr><td>$A \geq G \geq W$</td><td>$[0, G]$</td><td>$\{G\}$</td><td rowspan="3">$\{G\}$</td></tr>
<tr><td>$W \geq A \geq G$</td><td rowspan="4">$[0, W]$</td><td rowspan="2">$\{W, G\}$</td></tr>
<tr><td>$A \geq W \geq G$</td></tr>
<tr><td>$W \geq G \geq A$</td><td rowspan="2">$\{W, A\}$</td><td rowspan="3">$\{A\}$</td></tr>
<tr><td>$G \geq W \geq A$</td></tr>
<tr><td>$G \geq A \geq W$</td><td>$[0, A]$</td><td>$\{A\}$</td></tr>
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TABLE 2. The set of possible withdrawal decisions for different policyholder behaviours.

3.3. **Fair Pricing and Comparison.** Recall that the cost of the guarantee is charged to the policyholder through the application of the proportional insurance fee rate φ to the personal account. Hence the contract is fairly priced if and only if its initial value V_0, computed under any of the approaches introduced in 3.1.1-3.1.3, coincides with the initial premium U. Then, the fair fee rate φ^* can be defined as a solution of the following equation:

$$V_0(\varphi) = U \tag{3.6}$$

where, with a slight abuse of notation, we explicitly indicate that V_0 is a function of the proportional fee rate φ.

Denote now by V_0^{dynamic}, V_0^{static} and V_0^{mixed} the initial values of the contract, for a given fee, under each of the assumptions in 3.1.1-3.1.3, and by φ^{dynamic}, φ^{static} and φ^{mixed} the corresponding fair fees. It is clear that

$$V_0^{\text{static}} \leq V_0^{\text{mixed}} \leq V_0^{\text{dynamic}}, \qquad \varphi^{\text{static}} \leq \varphi^{\text{mixed}} \leq \varphi^{\text{dynamic}}.$$

The spread $V_0^{\text{mixed}} - V_0^{\text{static}}$ can be interpreted as the extra cost, in terms of single premium, required to add to the 'static' contract (the policyholder can only withdraw the amount G and receive the remaining personal account at maturity) the surrender option, while the spread $V_0^{\text{dynamic}} - V_0^{\text{mixed}}$ can be seen as the extra cost required to add the possibility to withdraw any amount up to the maximum admitted level described in Section 3.1.1. Similar interpretations apply to the corresponding spreads computed in terms of fair fees.

4. Lévy Processes Framework

In order to model the fund value, we start with a stochastic process $(X_t)_{t\geq 0}$, with $X_0 = 0$, defined on the basic probability space equipped with the risk neutral measure introduced in the previous section. We assume that X_t is a Lévy process, that is X_t has right-continuous with left limits paths, $X_s - X_t$ is independent of $(X_u)_{0\leq u\leq t}$ and is distributed as X_{s-t}, for $0 \leq t < s$. For a comprehensive description of Lévy processes, their properties and applications we refer to [16] and [38]. Lévy processes are a combination of a linear drift, a Brownian motion, and a jump process. A Lévy process (X_t) is determined by its characteristic function

$$\Phi_t(u) := E\left[\mathrm{e}^{iuX_t}\right] = [\Phi_1(u)]^t$$

and, in particular, all moments of X_t can be numerically recovered from the knowledge of Φ_t, when they are not available in closed form. If Φ_t is integrable, then X_t has density given by:

$$f_t(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{e}^{-izx}\Phi_t(z)\mathrm{d}z.$$

We model the reference portfolio value S_t as an exponential Lévy process:

$$S_t = S_0 \mathrm{e}^{(r-q+d)t+X_t},$$

where q is the dividend yield and $d = -\frac{1}{t}\ln\Phi_t(-i) = -\ln\Phi_1(-i)$ represents the adjustment so that $(S_t\,\mathrm{e}^{-(r-q)t})$ is a martingale under the risk-neutral measure.

In the numerical experiments we consider the following examples of Lévy processes, commonly used in finance applications, although we could in principle use any exponential Lévy model to represent the fund dynamics.

(1) Geometric Brownian Motion (GBM)

$$\Phi_t(u) = \exp\left(iu\mu t - \frac{1}{2}\,\sigma^2 u^2 t\right),$$

with $\mu \in \mathbb{R}$, $\sigma > 0$.

(2) Merton Jump Diffusion (MJD)

$$\Phi_t(u) = \exp\left(iu\mu t - \frac{1}{2}\,\sigma^2 u^2 t + \lambda t\left(\mathrm{e}^{ium - \frac{1}{2}\,c^2 u^2} - 1\right)\right),$$

with μ, $m \in \mathbb{R}$, σ, c, $\lambda > 0$.

(3) Variance-Gamma (VG)

$$\Phi_t(u) = \exp\left(-\frac{t}{\nu}\ln\left(1 - iu\mu\nu + \frac{1}{2}\,\sigma^2 u^2 \nu\right)\right),$$

with $\mu \in \mathbb{R}$, σ, $\nu > 0$.

(4) Carr, Geman, Madan, Yor (CGMY)

$$\Phi_t(u) = \exp\left(c\,\Gamma(-y)\,t\,[(m - iu)^y - m^y + (g + iu)^y - g^y]\right),$$

with $g, m \geq 0$, $c > 0$, $y < 2$ and Γ is the gamma function.

Example (1) is a pure diffusion, without jump component. A jump component in the equity returns is introduced by [31] through a compound Poisson process, leading to Example (2), where λ denotes the jump intensity and m and c are the mean and standard deviation of the log jump sizes, assumed to be normally distributed. This jump component produces a finite number of jumps within any finite time interval, i.e., the process exhibits finite activity, allowing to capture rare and large events such as market crashes or corporate defaults. However, market prices can also experience very frequent jumps of different sizes within any finite time interval. This property is captured by infinite activity processes, e.g. by Example (3), which is a pure jump process with infinite activity and paths of finite variation. The Variance-Gamma process was first introduced by [30] and [29], and then extended by [28]. Here, in Example (3), we refer to this latter extension. In particular, the VG process can be seen as a Brownian motion with constant drift μ and volatility σ,[6] with a stochastic time change defined through a gamma process with unit mean rate and variance rate ν. Alternatively, the VG can also be seen as the difference between two independent gamma processes with suitable parameters. This process has a lot of desirable properties consistent with empirical evidence; it allows, e.g., to control skewness and kurtosis of the return distribution and to correct some biases in option pricing implied by the [11] model. A further generalization of the VG process is the [13] model (CGMY), given by Example (4), that allows for both a diffusion and a jump component. Moreover, it can be suitably parametrized in order to capture finite or infinite activity as well as finite or infinite variation.

Lévy processes have been employed extensively in the mathematical finance literature, see [16]. In the context of the pricing of life insurance guarantees, Lévy processes have been used e.g. by [6] and [23]. In the context of variable annuities not including a GMWB, a risk management application has been investigated in [37].

[6]The original model by [30] was instead without drift.

5. Dynamic programming algorithm

The value of the GMWB is found by implementing the following standard dynamic programming algorithm for discrete stochastic control problems (see e.g. [10] and [39]). As we act in a Markovian framework, for each t_i, $i = 1, \dots, N$, and each value of the guarantee account A_{t_i} and personal account W_{t_i}, we denote the no-arbitrage value at date t_i of the variable annuity as $V(t_i, A_{t_i}, W_{t_i})$. The initial value of the GMWB, that is the solution of (3.5), is found by solving the Bellman recursive equation, which proceeds backward in time for $i = N-1, \dots, 1$:

$$
\begin{aligned}
V(t_i, A_{t_i}, W_{t_i}) &= \sup_{\theta \in \Theta_{t_i}} E\left[C_{t_i} + \mathrm{e}^{-r(t_{i+1}-t_i)} V(t_{i+1}, A_{t_{i+1}}, W_{t_{i+1}}) \middle| A_{t_i}, W_{t_i} \right], \\
V(t_N, A_{t_N}, W_{t_N}) &= \max\{A_{t_N}, W_{t_N}\}.
\end{aligned}
$$

Note that the equations for the cash-flow and the accounts are given by (3.1)-(3.3). The initial value of the contract is then found by computing

$$
V_0 = E\left[\mathrm{e}^{-r(t_1-t_0)} V(t_1, A_{t_1}, W_{t_1}) \middle| A_{t_0} = W_{t_0} = U \right].
$$

The execution of the algorithm requires a discretization over the state variables W and A and interpolation of the value function over the resulting grid in order to compute the expectation (see for instance [22]). As the density of the 1 year log return can be straightforwardly computed through Fourier inversion, the expectation can be calculated via numerical integration.

5.1. **Algorithm.**

We outline the algorithm employed to value a GMWB variable annuity under the dynamic approach. The valuation under the alternative approaches described in 3.1.2 and 3.1.3 requires minor and obvious modifications.

Step 0. For each $i = 0, \dots, N$, discretize the state space $[0, A_0]$ for A_{t_i} and $[0, \infty)$ for W_{t_i}:

$$
\mathbb{A} = \{a_1, \dots, a_H\},\ 0 = a_1 < a_2 < \dots < a_H = A_0,
$$

$$
\mathbb{W} = \{w_1, \dots, w_L\},\ 0 = w_1 < w_2 < \dots < w_L.
$$

Step 1. Start at $t_N = T$ by setting $V(t_N, a_h, w_l) = \max\{a_h, w_l\}$ for each $(a_h, w_l) \in \mathbb{A} \times \mathbb{W}$.

Step 2. Proceed backwards: for $i = N-1, \dots, 1$

I - interpolate the $H \cdot L$ triplets $(a_h, w_l, V(t_{i+1}, a_h, w_l))$, $h = 1, \dots, H$ and $k = 1, \dots, L$, to construct the function $\widetilde{V}(t_{i+1}, a, w)$ for $0 \le a \le A_0$ and $w \ge 0$;

II - for each $(a_h, w_l) \in \mathbb{A} \times \mathbb{W}$ compute

$$
V(t_i, a_h, w_l) = \sup_{\theta \in \Theta_{t_i}} \left\{ C_{t_i} + \mathrm{e}^{-r(t_{i+1}-t_i)} \int_{-\infty}^{\infty} \widetilde{V}\left(t_{i+1}, \widetilde{a}, \widetilde{b}\right) f_1(z) \mathrm{d}z \right\},
$$

where

$$C_{t_i} = \theta - \kappa \max\left\{\theta - \min\{G, a_h\}, 0\right\},$$

$$\widetilde{a} = \begin{cases} a_h - \theta & \text{if } \theta \le \min\{a_h, G\} \\ \max\{\min\{a_h - \theta, a_h(1 - \frac{\theta}{w_l})\}, 0\} & \text{if } \theta > \min\{a_h, G\} \end{cases},$$

$$\widetilde{b} = \max\{w_l - \theta, 0\}\mathrm{e}^{(r-q+d)(t_{i+1}-t_i)+z}(1 - \varphi(t_{i+1} - t_i)).$$

Computing the sup in II requires discretization of the control set Θ_{t_i} to select the supremum.

Step 3. The value of the contract at inception is

$$V_0 = \mathrm{e}^{-r\,t_1} \int_{-\infty}^{\infty} \widetilde{V}\left(t_1, U, U\mathrm{e}^{(r-q+d)t_1+z}(1 - \varphi\, t_1)\right) f_1(z)\mathrm{d}z.$$

Note that f_1 and d have been introduced in Section 4 and can be computed before implementing the above algorithm. In particular, the density f_1 is obtained through inversion of the characteristic function Φ_1 (see [5]) and then the constant d can be calculated via numerical integration. Similarly, numerical integration (e.g. simple trapezoidal rule or Gauss quadrature) can be used to compute the integrals in Step 2.II and Step 3 of the algorithm.

6. Numerical results

We fit the four models introduced in Section 4 to option prices on the S&P 500 observed on 31 December 2012, using maturity specific interest rates and dividend yields. We consider both call and put options for maturities up to 2 years, and discard options too far in or out of the money. When not available in closed form, plain vanilla option prices in a Lévy framework can be computed easily using Fourier inversion techniques, see for instance [21]. The fitting results in the parameter estimates are contained in Table 3, together with other key statistics of the 1 year log return.[7] The densities of the 1 year log return for the different estimated models are displayed in Figure 1.[8]

model	GBM	Merton	VG	CGMY
	$\sigma = 0.1361$	$\sigma = 0.1114$	$\sigma = 0.1301$	$c = 0.6817$
		$\lambda = 0.5282$	$\mu = -0.3150$	$g = 18.0293$
		$m = -0.1825$	$\nu = 0.1753$	$m = 57.6250$
		$c = 0.1094$		$y = 0.8000$
volatility (%)	13.61	21.58	18.53	15.59
skewness	0	2.1783	-0.7430	-0.3156
kurtosis	3	9.9050	3.9237	3.2743

Table 3. Parameters of the Lévy processes obtained by calibration to S&P 500 option prices.

[7]Moments related to Lévy processes can be directly computed using cumulants, see [16].

[8]In the CGMY model we fix $y = 0.8$, implying a finite variation, infinite activity process.

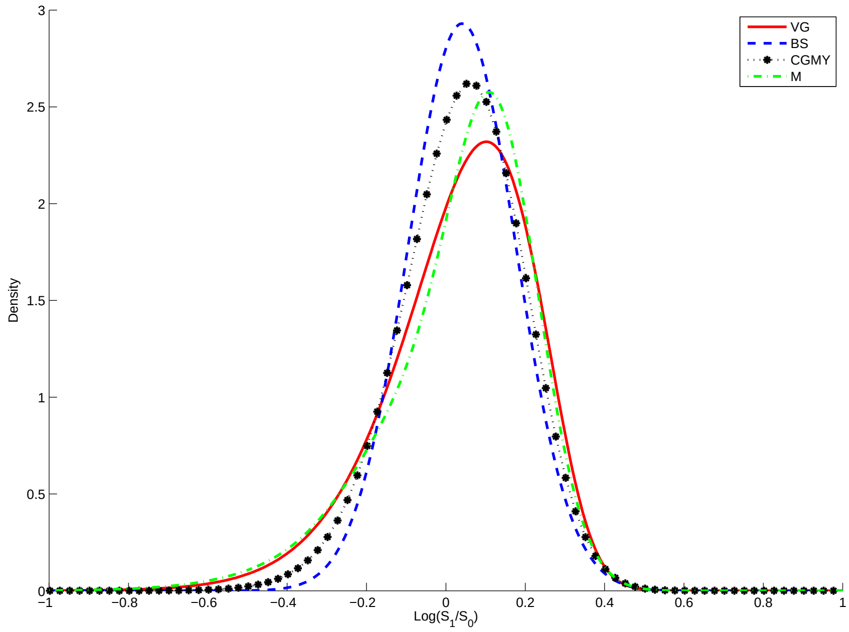

FIGURE 1. Density of 1 year log return.

From Figure 1 and Table 3 one can observe that the calibration leads to notable differences among the four models here considered. Nevertheless, the simplest and the most sophisticated model, namely GBM and CGMY, are relatively close to each other, both in terms of moments and numerical results (see also Tables 4-6 and Figures 2-3). On the other hand, the Merton and VG models, although they differ in terms of skewness and kurtosis, produce comparable results that are always much higher than those obtained with the GBM and CGMY. This can be attributed to the heavier tail displayed by the Merton and VG models, so that the guarantees implicit in the GMWB are underpriced by models that are not able to capture extreme movements in the fund process.

A comparative static analysis is performed for the contract value and fair fees for the different models and contract parameters, market interest rate and policyholder behaviour. If not otherwise mentioned, we use the following parameter values as *benchmark case*: $t_i = i$, $T = 20$, $r = 5\%$, $\kappa = 5\%$, $q = 0$, $U = 100$, $G = U/T$. In Table 4 we report the fair fee rates φ^{dynamic} and, in brackets, φ^{static}, in basis points, for different levels of the market interest rate r. Similar results are reported in Table 5, for different maturities T.

The fair fee rates φ^{dynamic} and φ^{static} decrease with r, as expected. Note that the fee required to compensate the option to act dynamically in the withdrawal and surrender decisions, given by the spread $\varphi^{\text{dynamic}} - \varphi^{\text{static}}$, decreases with r in each model. In particular, this spread ranges from 47-403 b.p when $r = 3\%$ to 1-7 b.p. when $r = 7\%$, thus confirming both the importance of the fund distribution tail and the high impact of the interest

r (%)	3	4	5	6	7
GBM	78	27	12	5	2
	(31)	(15)	(7)	(3)	(1)
Merton	469	129	46	27	16
	(66)	(40)	(25)	(15)	(9)
VG	454	116	42	24	14
	(63)	(38)	(23)	(14)	(8)
CGMY	183	44	22	11	6
	(43)	(24)	(13)	(7)	(3)

TABLE 4. φ^{dynamic} (φ^{static}) in b.p., for different risk-free interest rates.

T	10	15	20	25	30
GBM	50	24	12	7	4
	(32)	(14)	(7)	(4)	(3)
Merton	201	95	46	28	18
	(82)	(43)	(25)	(16)	(11)
VG	180	83	42	25	17
	(77)	(39)	(23)	(14)	(10)
CGMY	82	42	22	13	8
	(50)	(24)	(13)	(7)	(5)

TABLE 5. φ^{dynamic} (φ^{static}) in b.p., for different contract maturities.

rate on the cost of the flexibility added to a static contract by allowing a dynamic behaviour.

Similar findings can be seen in Table 5, as contracts with a longer maturity require a lower fee to be fair. As T increases, several effects on the contract value can be highlighted, and the overall impact is negative. Firstly, the insurance fee is applied over a longer period; secondly, the GMWB guarantee has a lower value since the market rate is higher than the minimum interest rate guaranteed on the personal account, which in our examples is 0% as $G = U/T$; finally, the guarantee is offered over a longer period, and this instead has a positive impact on the contract value.

In the previous tables we have not reported the values of the fair fee rate φ^{mixed} obtained under the mixed approach because, at least for the considered parameters, they always coincide with those obtained under the static approach, that is $\varphi^{\text{mixed}} = \varphi^{\text{static}}$. On one hand, this fact leads us to argue that the penalty applied on the portion of the personal account exceeding the withdrawal level G, equal to 5%, is high enough to discourage surrender. On the other hand, since the same penalty is applied in the dynamic approach for withdrawal amounts exceeding G, the spread $\varphi^{\text{dynamic}} - \varphi^{\text{mixed}} = \varphi^{\text{dynamic}} - \varphi^{\text{static}}$ seems to be completely (or mainly) attributable to the possibility of withdrawing amounts less than G. To verify if things change for levels of the penalty κ lower than 5%, in Table 6 we report the results obtained for φ^{dynamic}, φ^{mixed} (in braces) and φ^{static} (in round brackets) in all the three approaches.

κ (%)	0	1	2	3	4	5
GBM	56	19	12	12	12	12
	{30}	{7}	{7}	{7}	{7}	{7}
	(7)	(7)	(7)	(7)	(7)	(7)
Merton	195	142	103	75	57	46
	{94}	{55}	{34}	{25}	{25}	{25}
	(25)	(25)	(25)	(25)	(25)	(25)
VG	182	130	92	65	48	42
	{88}	{50}	{29}	{23}	{23}	{23}
	(23)	(23)	(23)	(23)	(23)	(23)
CGMY	100	56	30	22	22	22
	{51}	{19}	{13}	{13}	{13}	{13}
	(13)	(13)	(13)	(13)	(13)	(13)

TABLE 6. φ^{dynamic} $\{\varphi^{\text{mixed}}\}$ $(\varphi^{\text{static}})$ in b.p., for different penalties.

From the first two rows of Table 6 we notice that the penalties for exceeding withdrawals or surrender have a depressing effect on the fair fee rates, although this effect can be perceived only when the penalty rate is sufficiently low, expecially under the mixed approach (second row). For instance, with the GBM and the CGMY model the fee is constant for penalties greater or equal to 1% and 2% respectively, whereas in the Merton and VG models this happens starting from a penalty rate of 3%. The GBM and CGMY models produce less sensitive results. Under these models the fees become constant from penalty rates of 2-3% even under the dynamic approach (first line), thus confirming our previous conjecture that in these cases the greater flexibility allowed by a dynamic contract is worth only for withdrawal amounts less than G. Of course, under the static approach (third line) the penalty is never applied, so that the results are independent of κ. Finally, we notice that when the values under the mixed approach become penalty independent, i.e. for sufficiently high levels of κ, then they coincide with those obtained under the static approach.

To grasp visually the influence of the fee rate on the initial contract value under the different policyholder behaviours and Lévy processes here considered, in Figures 2 and 3 we plot V_0 against φ when the penalty rate is $\kappa = 0$. In particular, from Figure 2 one can capture the differences among the various approaches (for each price process), while from Figure 3 one can capture the differences induced by the various price processes (for each valuation approach). Recalling that $U = 100$, the intercept between the horizontal line at level 100 and the contract value gives the fair fee rate for each model and approach.

The (negative) impact of the fee rate on the initial contract value is particularly relevant in the static approach, under which the policyholder cannot exit the contract before maturity or withdraw high amounts (greater than G) in order to leave less money in the personal account and hence mitigate the fee effect.[9] When $\varphi = 0$ there is no difference between the static and the

[9]Recall that the fee is applied to the personal account value.

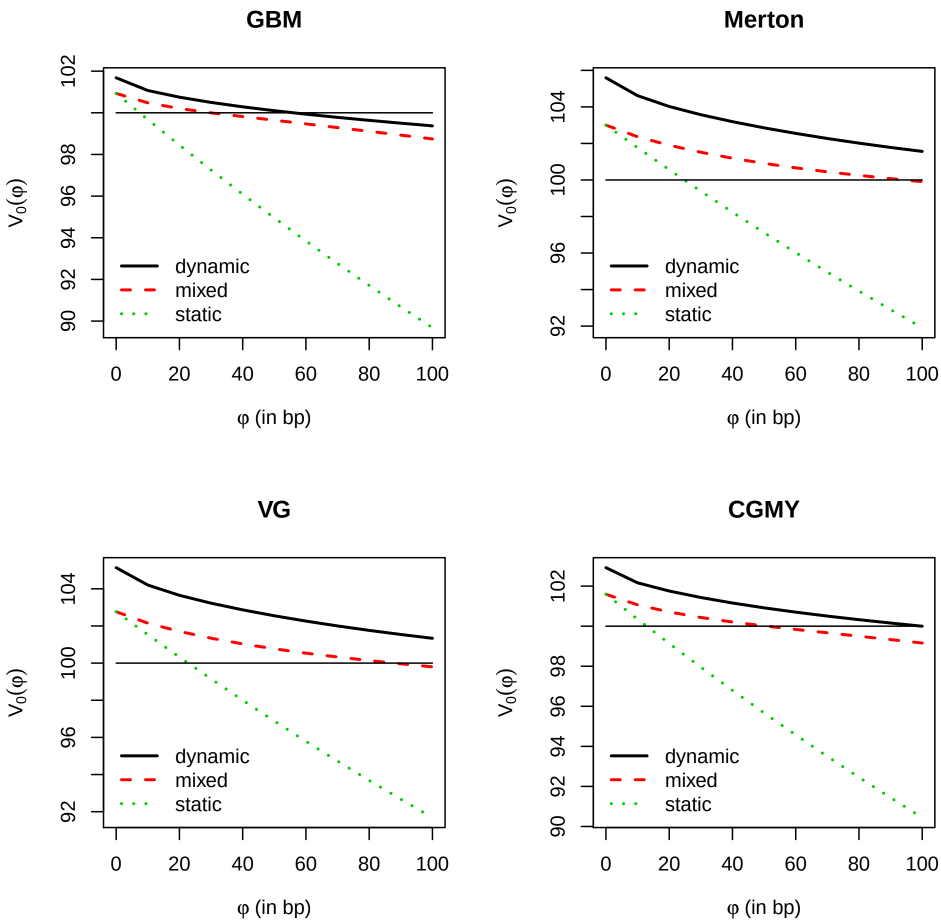

FIGURE 2. The initial contract value V_0 against the fee rate φ for each price process under the different policyholder behaviours.

mixed approach in all models, that is, it is never optimal to exit the contract before maturity, at least if one is not allowed to make partial withdrawals different from G. On one hand, since in this case the GMWB rider is offered at zero cost, we can argue that it is convenient to keep the guarantee alive as long as possible. On the other hand, the difference between the static (or mixed) and the dynamic approach is mainly attributable, as already remarked, to the possibility of withdrawing amounts less than G rather than higher. The differences among the various approaches, at least for low levels of the fee rate, are most remarkable under the Merton and VG processes, that always produce very close results.

7. CONCLUSIONS

In this paper we present a dynamic programming algorithm for the valuation of variable annuities with Guaranteed Minimum Withdrawal Benefits. A very crucial aspect underlying the valuation of such products is to predict how the policyholder behaves with respect to her withdrawal decisions. Our

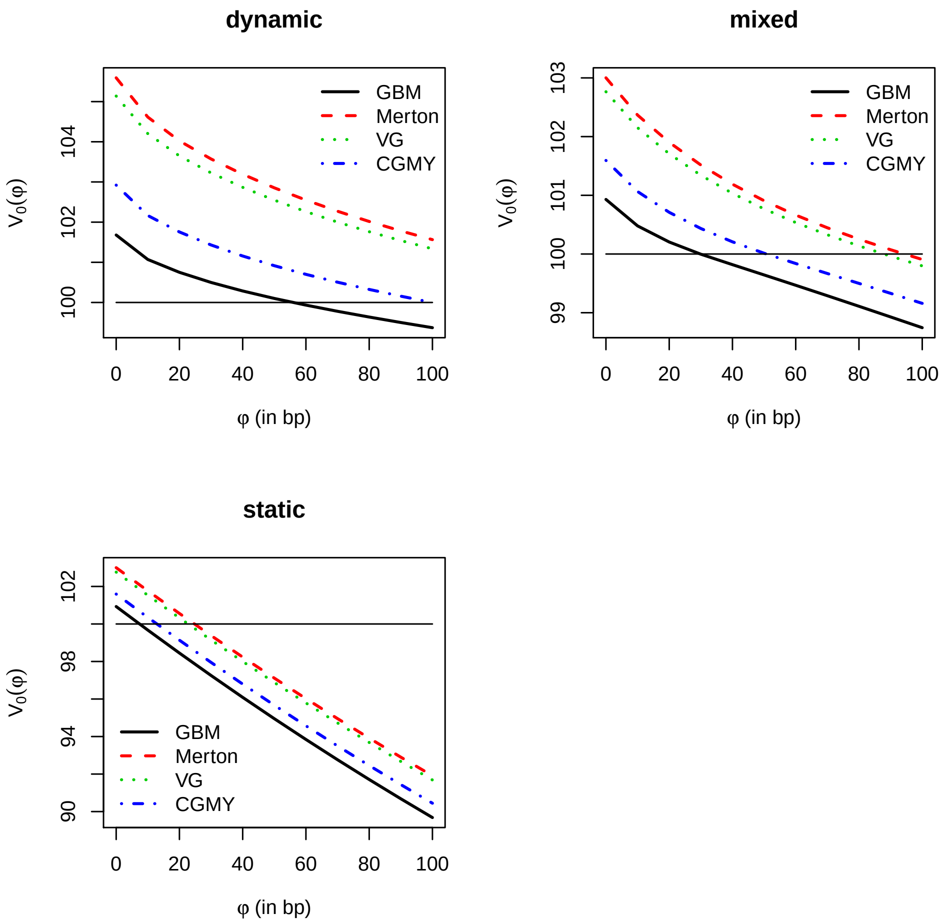

FIGURE 3. The initial contract value V_0 against the fee rate φ for each policyholder behaviour under the different price processes.

algorithm is general enough to encompass different policyholder behaviours, so that it is particularly suitable to meet different purposes of an insurance company (e.g., for pricing purposes it is reasonable to assume an approach based on the worst case scenario, while for realistic risk-management valuations an intermediate approach seems to be more appropriate). Moreover, the algorithm can be easily extended in order to include other policyholder decisions in addition to those concerning her withdrawal behaviour (e.g., switching between different reference portfolios, acquisition of new guarantees or cancellation of existing ones) or other contract features such as death benefits. Another important contribution of our paper with respect to the existing literature concerns the model assumptions governing the evolution of the reference portfolio. In this respect not only we go beyond the classical [11] model, but put ourselves in the general class of Lévy processes. In the numerical section we perform a sensitivity analysis choosing as examples four different types of Lévy processes. This analysis highlights the relevance of the specific assumption adopted in the valuation, i.e., the model risk, and in particular the fact that GMWB guarantees can be grossly underpriced by models that are not able to capture extreme movements in the fund process.

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