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The derived category of representations of the special linear group of degree two over a finite field

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Declaration

The work presented in this thesis is based on investigations believed to be original and carried out at the School of Mathematics, Computer Science and Engineering, City University of London, in collaboration with Prof. Joseph Chuang. It has not been presented elsewhere for a degree, diploma, or similar qualification at any university or similar institution. I have clearly stated my contributions, in jointly-authored works, as well as referenced the contributions of other people working in the area. Powers of discretion are granted to the University Librarian to allow the thesis to be copied in whole or in part without further reference to the author. This permission covers only single copies made for study purposes, subject to normal conditions of acknowledgement.

Abstract

In this thesis, we study the modular representations of the special linear group of degree two over a finite field in defining characteristic. In particular, we study the automorphisms of derived category of representations. We have been able to obtain a new type of autoequivalence.

This autoequivalence has some uncommon features. It is more conveniently conceived and proved using the representation theory of its Brauer subgroup but at the same time it can be very neatly described, using a type of derived equivalence called perverse equivalence, in global settings.

Chapter 1

Introduction and Background

1.1 A brief account of the subject

Representation theory started in the mail exchange of Frobenius and Dedekind in 1896 with Frobenius using a group's mapping into the general linear group of a field to explain some properties Dedekind discovered in some subsets of matrices. Over time, other mathematicians like Burnside, Schur, Noether joined in the study of representations, developing the branch significantly. It started as a study of representations of groups, in which Maschke laid down the fundamental theorem of semisimplicity of group representations, which is always true over a field of characteristic 0. In this case, character theory develops and completely determines representations - the traces of the image of the group in the general linear group have such good properties that can determine a representation up to isomorphism when Maschke's theorem holds. One of the high points of representation theory is the success of Burnside's theorem in 1916, in which representation theory is used to prove group of certain orders cannot be simple, while no group theoretical approach had yet succeeded.

Brauer started the study of modular representation theory in 1935. It is to study properties of representations of groups when the characteristic of the underlying field is positive and when Maschke theorem fails. Although Brauer characters somewhat successfully generalise character theory, they fail to classify the isomorphism class of a representation, only the multiplicities of simples as composition factors can be decided. Brauer's study points out the general direction to express representations of groups of more complex structure by smaller subgroups. He contributes two main theorems, which are collectively known as Brauer correspondence. It links blocks of a group to blocks of its subgroups. These research results are an important part of this thesis,

which would be explained in detail in section 1.2.

Category theory was quickly adapted by representation theorists to formalise the study of representations since its introduction. In particular, for a certain algebra, all representations form a category that is being called its module category. The language of category theory allows a more systematic approach to maps between modules. For example, the introduction of the stable module category provides new developments on how representations of different algebras can be related. They have explored further relations between group representations and its subgroups.

In turn, the study of module categories enriched category theory. Examples include Freyd-Mitchell embedding theorem, stating that every small abelian category can be embedded as a full subcategory of a module category, and the discovery that the representation theory of wild representation type is undecidable, showing that representation theory actually involves the deepest connection in mathematics.

Later, homological algebra and algebraic geometry were introduced into the field and they quickly secured a place in representation theory. One of the particular achievements of this introduction is the construction of derived category. The derived category of a module category provides an area of study between the defining module category and the associated stable module category. For example, Broué's conjecture suggests a possible relation between representations of a group and its subgroups via Brauer correspondence.

Conjecture 1.1. *If a block of a group algebra has a defect group which is abelian, then such block is derived equivalent to its Brauer correspondent.*

Algebraic geometry is another way to study representation theory, as modules can be considered as sheaves, allowing research to use techniques in algebraic geometry. Perverse equivalence, which is a kind of derived equivalence, is inspired through perverse sheaves in algebraic geometry. These categories, equivalences and their relations will be formally introduced in section 1.3.

1.1.1 Notations and Conventions

We use the position of the algebra to indicate its side of the the action on the module. For example an A-module has A-action from the left. A module-B is a right B-module in the usual sense. An A-bimodule-B means there is an left A-action and right B-action on the module. In this article we compose functions and functors from right to left. The following is a general list of notations used:

- p is a prime number, $q = p^n$ a positive power of p;
- G,H are groups;
- $SL_2(q)$ is the special linear group of degree two over the field of $q = p^n$ elements;
- $\mathbb{F}$ is an algebraically closed field, of characteristic p;
- Bolded letters $\mathbb{A}$, $\mathbb{B}$, $\mathbb{b}$ for blocks in the later chapters; $\mathbb{S}$, $\mathbb{T}$ for sets of simples;
- M, N, U, V are modules, P, Q are projective modules, S, T are simple modules;
- Script letters are for categories. E, F for functors.
- $I_\bullet$, $J_\bullet$ are nested sets forming the filtration I and J;
- X, Y are cochain complexes and X^i the degree i^{th} entry of X.

Throughout the thesis G is a finite group of Lie Type. The characteristic of $\mathbb{F}$ is the defining characteristic of G and this prime number is denoted by p. A G-module means an $\mathbb{F}G$-module. We always assume all vector spaces are finite dimensional, direct sums are finite. Hom and $\otimes$ operations are widely used throughout the paper. We adopt the following convention for subscripts surrounding Hom and $\otimes$. Let U, V be an A-module for an algebra A,

1. $\mathrm{Hom}_A(U, V)$ means the set of A-module maps between U and V.
2. When there is no symbol beneath, then $\mathrm{Hom}(U, V)$ is the set of maps between U and V as vector spaces.
3. Let U, V be G-modules. We will treat $\mathrm{Hom}(U, V)$ and $U \otimes V$ as G-modules automatically. See group algebra section for details.

1.1.2 Structure of the thesis

The remaining part of chapter 1 is split into two main parts. The first part is to introduce representation theory, in particular for group algebras. We shall focus on the relation between representations of a group and those of its subgroup. That includes restriction and induction, block theory and Brauer correspondence. The second part is to introduce categories related to our thesis. That includes module category, stable module category, derived category and their properties. In particular we will focus on the equivalences of different types of categories and the relations between them. Finally we shall introduce perverse equivalence.

Chapter 2 is a detailed account of representations of special linear group of degree 2 over a finite field in their defining characteristic and their local subgroups, in this case Borel subgroups. We will find out their simple modules, introduce their projective modules if possible and explore extension groups between these modules.

Chapter 3 consists of the main idea and proof of the thesis, which utilise the methods introduced in chapter 1 and data in chapter 2 to obtain a perverse autoequivalence. We shall discuss the interesting properties at the end of this chapter.

Chapter 4 further develop the new perverse autoequivalence obtained in chapter 3 using a relatively new idea of poset perverse equivalence. This will lead to other new autoequivalences and some interesting observations.

Chapter 5 is dedicated to the smallest non-trivial example, the special linear group of degree 2 over a field of four elements.

Lastly there is an appendix to demonstrate how our perverse autoequivalence works in slightly larger groups. It hopes to give readers a point of reference upon reading the extension lemmas in chapter 2 and the proof of proposition 3.2 in chapter 3.

1.2 Background Part I: Representation

Let $\mathbb{F}$ be an algebraically closed field. Let A be a finite dimensional $\mathbb{F}$-algebra. A representation ρ of an $\mathbb{F}$-algebra A is an $\mathbb{F}$-algebra homomorphism

$$\rho : A \to \mathrm{End}(V)$$

where V is a vector space over a field $\mathbb{F}$ and $\mathrm{End}(V)$ is the endomorphism algebra of V. The map ρ induces a structure of A-module on V by left multiplication: $a.v = \rho(a)(v)$. Conversely, if V is an A-module, we define the map ρ by mapping $a \in A$ to the endomorphism of the underlying vector space of A given by left multiplying a. This gives a correspondence between representations of algebra A and A-modules. So for an algebra A it is enough to study A-module structure of vector spaces to understand representation theory of A. For G a finite group, a group representation: $G \to \mathrm{End}(V)$ can be extended linearly on $\mathbb{F}$, the underlying field of V and understood as an algebra representation:

$$\rho : \mathbb{F}G \to \mathrm{End}(V)$$

1.2.1 Modules

One of the principal results is the Krull-Schmidt theorem, which describes uniqueness of decomposition of modules:

Theorem 1.2. *(Krull-Schmidt) Let M be an A-module and*

$$M \cong U_1 \oplus U_2 \oplus ... \oplus U_r$$

$$M \cong V_1 \oplus V_2 \oplus ... \oplus V_s$$

where U, V are indecomposable A-modules, then $r = s$ and $U_i \cong V_i$ after suitable re-indexing.

Simple module is an important idea in studying representation theory.

Definition 1.3. *A simple A-module is a module with only zero module and itself as submodule. A semisimple A-module is a direct sum of simple A-modules.*

These modules are easier to study. To see that we have Schur's lemma:

Lemma 1.4. *Let S and T be two simple A-modules, then*

1. $\mathrm{Hom}(S, T) = 0$ *if S is not isomorphic to T.*
2. $\mathrm{End}(S)$ *is a division algebra; and*
3. *if further A is an $\mathbb{F}$-algebra with $\mathbb{F}$ algebraically closed, then* $\mathrm{End}(S) = \lambda \mathrm{Id}_S$, *where $\lambda \in \mathbb{F}$.*

To analyse the structure of a non-semisimple module we introduce the radical of an algebra.

Definition 1.5. *The radical of A, denoted by* $\operatorname{rad} A$, *consists of elements of A which annihilate every simple A-module. This is equal to*

1. *The unique minimal submodule of A whose quotient is semisimple*
2. *The intersection of all maximal submodules of A.*
3. *The maximal nilpotent ideal of A.*

The radical $\operatorname{rad} A$ help us to analyse the structure of an A-module by the following:

Proposition 1.6. *If M is an A-module then the following are equal*

1. $(\operatorname{rad} A).M$;

2. *The unique minimal submodule of M whose quotient is semisimple;*

3. *The intersection of all maximal submodules of U.*

Definition 1.7. *The module with these equivalent description is denoted by* $\operatorname{rad}(M)$. *The semisimple quotient* $M/\operatorname{rad}(M)$ *is denoted by head*(M).

From this proposition we can form a series of modules descending from M, by defining $\operatorname{rad}^n(M)$ recursively as $(\operatorname{rad} A)^n.M$. It follows that there is a descending chain of submodules

$$M = \operatorname{rad}^0(M) \supset \operatorname{rad}(M) \supset \operatorname{rad}^2(M) \supset \dots.$$

The minimal integer $l > 0$ with $\operatorname{rad}^l(M) = 0$ is called the radical length of M.

A similar construction can be done using submodules of an A-module M instead of quotients of M.

Proposition 1.8. *If M is an A-module the following are equal:*

1. *The set of m in M with $(rad\ A)u = 0$;*

2. *the largest semisimple submodule of M;*

3. *The sum of all the simple submodules of M.*

The submodule described is called the *socle* of M and written as $\operatorname{soc}(M)$. Considering iterative definition of $\operatorname{soc}^i(M) = \operatorname{soc}(M/\operatorname{soc}^{i-1}(M))$ and $\operatorname{soc}^0(M) = 0$, we formed a *socle series*

$$0 = \operatorname{soc}^0(M) \subset \operatorname{soc}^1(M) \subset ...$$

with *socle length* defined by the smallest l with $\operatorname{soc}^l(M) = M$. For finite-dimensional modules the radical length and socle length exists and coincides, which is called Loewy length.

One very natural example of an A-module is A itself. It is of particular importance.

Definition 1.9. *An A-module is free if it is a direct sum of copies of A (as A-module). An A-module is projective if it is a summand of a free module.*

Projective modules satisfy the following universal property. Some literature use this as its definition.

Proposition 1.10. *Let P be a projective A-module. Let M, N be A-modules with a surjective map $f : M \to N$ and $\pi : P \to N$. Then there exists a map $g : P \to M$ such that $fg = \pi$.*

Every A-module is a quotient of some projective A-module. The indecomposable projective modules are of particular importance.

Proposition 1.11. *Let P be a projective indecomposable A-module. Then the followings hold.*

1. *$P/rad(P)$ is simple.*
2. *P is a direct summand of A as an A-module.*
3. *There is a one-to-one correspondence between the set of isomorphism classes of simple A-modules and the set of isomorphism classes of indecomposable projective A-modules.*

Proof See Chapter 5, Theorem 3 of [Alperin11]. □

1.2.2 Group algebra

Given finite dimensional algebra A, the category of finitely generated (left) A-modules, denoted by A-mod, is an abelian category (see next section) with enough injectives and projectives. That is, any finitely generated module is a submodule of an injective module, and a quotient module of a projective module.

Definition 1.12. *Let A be an algebra. The* opposite algebra*, denoted by A^{op}, is an algebra with the same set of elements as A but with multiplication ab for $a, b \in A^{op}$ defined as ba in A.*

For any finite dimensional algebra A, there is a contravariant functor

$$\mathrm{Hom}(-, \mathbb{F}) : A\text{-mod} \to A^{op}\text{-mod}$$

which is an equivalence. We denote

$$A^* = \mathrm{Hom}(A, \mathbb{F})$$

as the $\mathbb{F}$-dual of A.

Definition 1.13. *An algebra A is self-injective if A is injective as an A-module.*

Since A, as an A-module is free and projective by definition, the definition of self-injective algebra is equivalent to saying that injective and projective modules coincide.

Definition 1.14. *A symmetric algebra A is an algebra with a linear map $\lambda : A \to k$ such that*

- $\ker(\lambda)$ *contains no non-zero left or right ideals of A.*
- $\lambda(ab) = \lambda(ba)$ *for all $a, b \in A$.*

Example 1.15. *A group algebra $\mathbb{F}G$ is a symmetric algebra where λ is given by*

$$\lambda(\sum_{g \in G} \alpha_g(g)) = \alpha_1.$$

Proposition 1.16. *An algebra A being symmetric is equivalent to any of the following:*

1. *$A \cong A^* = \mathrm{Hom}(A, \mathbb{F})$ as A-bimodule-A.*
2. *For any A-module M, there is a isomorphism*

$$\mathrm{Hom}_k(M, k) \cong \mathrm{Hom}_A(M, A).$$

   This isomorphism is natural, as functors from A-mod to A^{op}-mod.

3. *For a finitely generated projective A-module P and an arbitrary A-module M we have a natural isomorphism*

$$\mathrm{Hom}_A(P, M) \cong \mathrm{Hom}_A(M, P)^*$$

One consequences of the proposition is that a finitely generated projective module P of a symmetric algebra has the same top and socle, namely

$$\mathrm{soc}(P) \cong P/\mathrm{rad}(P),$$

by putting M as a simple module in the last statement.

The representation theory of group algebras relies on the study of groups. When the characteristic of the field $\mathbb{F}$ is zero or does not divide the order of G, Maschke's theorem states that the algebra $\mathbb{F}G$ is semisimple. Then the regular representation maps $\mathbb{F}G$ to a direct sum of simple matrix algebras as a consequence of Schur's lemma.

Thus any indecomposable module is simple and there are no non-zero maps between non-isomorphic simples. In this semisimple case, the associated character

$$\chi : G \to \mathbb{F}, \qquad \chi(g) = tr(\rho(g))$$

where tr denotes the trace of an endomorphism, uniquely determines ρ up to isomorphism. Since there are no non-trivial maps between non-isomorphic simples, the tools introduced in the rest of this chapter will not give further insight because maps between modules are direct sums of maps between simple modules. When the characteristic of the field $\mathbb{F}$ divides the order of G, however, Maschke's theorem fails and modules are not necessarily semisimple.

Any group algebra has a Hopf algebra structure, so we can define $\mathbb{F}G$-module structures on the tensor products and duals of $\mathbb{F}G$-modules.

Definition 1.17. *Let M, N be a left $\mathbb{F}G$-modules. We define*

1. *$M \otimes N$ is equipped with a left $\mathbb{F}G$-module structure with action given by $g(m \otimes n) = gm \otimes gn$.*

2. *$\mathrm{Hom}(M, N) \cong M^* \otimes N$ is equipped with a left $\mathbb{F}G$-module structure with action given by $g(\phi)(m) = g\phi(g^{-1}(m))$.*

Remark. We use comultiplication in Hopf algebra in defining the above tensor product as $\mathbb{F}G$-module. The antipode is further used in defining Hom-space as $\mathbb{F}G$-module

What the definition above does is to utilise the group structure to make new representations out of the known ones. Also the functors $- \otimes M$, $\mathrm{Hom}(M, -)$ and $\mathrm{Hom}(-, N)$ are all exact endofunctors of $\mathbb{F}G$-mod. The first two are covariant functors while the third is a contravariant functor. That is, given an exact sequence of modules $0 \to L' \to L \to L'' \to 0$ we have

$$0 \to L' \otimes M \to L \otimes M \to L'' \otimes M \to 0$$

$$0 \to \mathrm{Hom}(M, L') \to \mathrm{Hom}(M, L) \to \mathrm{Hom}(M, L'') \to 0$$

$$0 \to \mathrm{Hom}(L'', N) \to \mathrm{Hom}(L, N) \to \mathrm{Hom}(L', N) \to 0.$$

With groups involved, one natural question is the relation between representations of a group and those of its subgroups. We shall consider this through induction and restriction, and some correspondences between the representations.

1.2.3 Restriction and Induction; Green correspondence

Consider a group G and U a $\mathbb{F}G$-module. Let H be a subgroup of G. One way to obtain representations of H is to look at the module U as a vector space with (only) H-action. To do this we define:

Definition 1.18. *Let G be a group, let $H \leq G$ be a subgroup of G and U be a $\mathbb{F}G$-module. Define $U{\downarrow_H}$, the* restriction of U (from G) to H, *to be the $\mathbb{F}H$-module with the same underlying vector space and action by $\mathbb{F}H$.*

If instead we have a $\mathbb{F}H$-module V, there is a way to construct a $\mathbb{F}G$-module too.

Definition 1.19. *Let G be a group, let $H \leq G$ be a subgroup of G and V be a $\mathbb{F}H$-module. Define $V{\uparrow^G}$, the* induction of V (from H) to G, *as $\mathbb{F}G \otimes_{\mathbb{F}H} N$ with the action of G via $g(a \otimes v) = (ga) \otimes v$.*

One can check this G-action well-defined in the tensor space by considering $g(ah \otimes v - a \otimes hv) = gah \otimes v - ga \otimes hv$.

Remark. This is not the tensor product defined via Hopf algebra structure, where instead g acts diagonally.

Restriction and induction have very good properties. One that is crucial in this exposition is the fact that they are left and right adjoint to each other.

Lemma 1.20. *(Frobenius Reciprocity) Notation as in definition above, we have:*

1. $\mathrm{Hom}_{kG}(V{\uparrow^G}, U) \cong \mathrm{Hom}_{kH}(V, U{\downarrow_H})$.
2. $\mathrm{Hom}_{kG}(U, V{\uparrow^G}) \cong \mathrm{Hom}_{kH}(U{\downarrow_H}, V)$.

It is worth mentioning that if H is a normal subgroup of G there are extra properties that can be very useful. This is usually referred to as Clifford theory.

For a subgroup H, we can generalise the notion of free modules/projective modules to relatively H-free/projective modules. For our purpose, we only introduce the latter:

Definition 1.21. *Let U be a $\mathbb{F}G$-module and H be a subgroup of G. Then U is* relatively H-projective *if any of the following equivalent conditions hold:*

1. *If V is a G-module with $\varphi : V \to U$ a surjective homomorphism, then φ is split (surjective) whenever φ is split as $\mathbb{F}H$-homomorphism.*
2. *U is a direct summand of $(U{\downarrow_H}){\uparrow^G}$.*

A projective module is just a relatively 1-projective module, where 1 stands for the trivial subgroup. The following theorem suggests that p-subgroups affect projectivity:

Theorem 1.22. *If H is a subgroup of G containing a Sylow p-subgroup, then any $\mathbb{F}G$-module is H-projective.*

Theorem 1.23. *Let U be an indecomposable $\mathbb{F}G$-module. Then there exists a p-subgroup Q of G, unique up to conjugacy in G, such that U is relatively H-projective if and only if H contains a conjugate of Q.*

This subgroup Q is called a *vertex* of U; it is a p-group, as this theorem indicates. A vertex of a module measures how far an indecomposable module is away from being projective (a module has vertex 1 if and only if it is projective). Also, the concept of vertices plays a very important role in establishing a link between modules of a group and its subgroups. This relationship is explained by the Green correspondence. For further details see [Alperin11, Section 11]. In this thesis, we use a special case of such, namely when the Sylow p-subgroups P of G have 'trivial intersections'. That is, when $P \cap gPg^{-1}$ is either P or 1. When this holds, the Green correspondence reads

Theorem 1.24. *(Green correspondence for trivial intersections) Let G be a group with trivial intersection property and P be a Sylow p-subgroup and let $L = N_G(P)$. Then there is a one-to-one correspondence between the isomorphism classes of non-projective $\mathbb{F}G$-modules U and the isomorphism classes of non-projective $\mathbb{F}L$-modules V such that U and V have the same vertices, and*

$$U{\downarrow_L} \cong V \oplus Q$$

$$V{\uparrow^G} \cong U \oplus P$$

where P, Q are projective $\mathbb{F}G$ and $\mathbb{F}L$-modules respectively.

For details and proofs see [Alperin11, section 10].

1.2.4 Blocks and Brauer Correspondence

Theorem 1.25. *An algebra A has a unique decomposition into the direct sum of two-sided ideals*

$$A = A_1 + A_2 + ... + A_n$$

where each A_i is an indecomposable ideal.

These indecomposable ideals are called the *blocks* of A. Note that each ideal A_i above is also a unital algebra in its own right. Thus a block is also an algebra, with no non-trivial two-sided ideals by definition. Now consider an A-module M and let $A_iM = M_i$. Then one can decompose $M = M_1 + ... + M_n$ as direct sum of A-modules; where each M_i can also be regarded as an A_i-module. In particular, if M is an indecomposable A-module, there must exist i such that $A_iM = M$ and $A_jM = 0$ for all $j \neq i$. We say, in this case, that M is *lying* in the block A_i. To determine which block of A a certain indecomposable module is lying in, the following proposition states that it depends on the simple constituents of the module in question.

Proposition 1.26. *Let S and T be simple A-modules. Then S and T lie in the same block if and only if there is a sequence of simple A-modules*

$$S = T_0, T_1, ..., T_n = T$$

such that there is a non-split extension between T_i and T_{i+1} for all $i = 0, 1, \ldots, n-1$.

For group algebras, to determine its blocks one can consider $\mathbb{F}G$ as an $\mathbb{F}[G \times G]$-module with action given by $(g_1, g_2)a = g_1 a g_2^{-1}$. This gives the following:

Theorem 1.27. *If $\mathbb{B}$ is a block of $\mathbb{F}G$, then as $\mathbb{F}[G \times G]$-module, B has a vertex of the form*

$$\delta(D) = \{(d, d) | d \in D\}$$

where D is a p-subgroup of G.

This p-subgroup D in the theorem above is unique up to conjugacy in G and is called a *defect group* of $\mathbb{B}$. If D has order p^d, then the block $\mathbb{B}$ is also said to have *defect d*. Similar to the notion of vertex, defect groups measure how far a block $\mathbb{B}$ is from being semisimple.

Theorem 1.28. *Let $\mathbb{B}$ be a block of G with a defect group D, then any indecomposable G-module lying in $\mathbb{B}$ has a vertex contained in D.*

In fact, a defect group D can also be defined as a maximal vertex over all indecomposable $\mathbb{B}$-modules. Furthermore, the vertex of a simple module in a block is very restrictive, by Knörr's theorem [Knorr79]:

Theorem 1.29. *Let $\mathbb{B}$ be a block of group algebra $\mathbb{F}G$ with defect group D, S be a*

simple $\mathbb{B}$-module. Then S has a vertex Q such that

$$C_D(Q) \leq Q \leq D.$$

In particular, if D is abelian then we always have Q=D.

Remark. Knörr's theorem can be stated in a stronger form, namely Q is centric in $\mathscr{D}$ with respect to the fusion system of $\mathbb{B}$.

The correspondence of blocks from a group to its subgroups is more intriguing than their module counterparts. Let H be a subgroup of G, $\mathbb{b}$ and $\mathbb{B}$ be blocks of H and G respectively. We say $\mathbb{B}$ correspond to $\mathbb{b}$, denoted by $\mathbb{B} = \mathbb{b}^G$, if $\mathbb{B}$ is the only block of G such that $\mathbb{b}$ is a direct summand of $\mathbb{B}{\downarrow_{H\times H}}$ as $H \times H$-module. In particular, this correspondence is well-defined for a block $\mathbb{b}$ of a subgroup H with defect group D such that $C_G(D) \subset H$. We can establish Brauer's theorems:

Theorem 1.30. *Let D be a p-subgroup of G and $H \leq G$ be a subgroup containing $N_G(D)$. Then there is a one-to-one correspondence between the blocks of H with defect group D and those of G with defect group D.*

In the prospect of this thesis, D is a Sylow p-subgroup of G and $H = N_G(D)$, thus the one-to-one correspondence of blocks is guaranteed. The (special case) Green correspondence indicates the correspondence of $\mathbb{F}G$-modules and $\mathbb{F}H$-modules have some good properties, that is particularly useful when viewed in stable category, which we will see in the next section.

1.3 Background Part II: Categories and Equivalences

The notion of category was developed during the 1950's to unify different descriptions in different fields of mathematics. This tool has been applied quickly to representation theory by considering modules of an algebra as objects in a category. In this chapter, we will eventually focus on the case when the algebra is a symmetric algebra, in particular, the blocks of group algebras. These will form the basic language we use in later chapters.

Recall, as in the last section, all algebras are over $\mathbb{F}$ and finite dimensional.

1.3.1 Module category

Definition 1.31. *For an algebra A, The* category of (left) A-modules*, denoted by* A-mod*, is a category with*
Objects: Finitely generated A-modules.
Morphisms: A-module homomorphisms.
The set of morphisms from an object M to an object N in A-mod *is denoted by* $\mathrm{Hom}_A(M, N)$.

For any $\mathbb{F}$-algebra A, A-mod is an $\mathbb{F}$-linear abelian category. That is, we have the following properties:

1. $\mathrm{Hom}_A(M, N)$ is an $\mathbb{F}$-vector space.
2. Finite products of objects, i.e. direct sum of modules, exist.
3. The kernel and cokernel of a map exists.
4. Any monomorphism or epimorphism is normal (i.e. is a kernel or cokernel of some map).

Remark. A category with the first property is an $\mathbb{F}$-category, with the first two is an $\mathbb{F}$-linear category and the first three a $\mathbb{F}$-linear pre-abelian category.

In abelian categories, exact sequences arise naturally as the consequence of 4.

Definition 1.32. *For an abelian category $\mathscr{C}$, an* exact sequence *is a sequence of objects $C_1, ..., C_n$ with maps $f_i : C_i \to C_{i+1}$ such that $\mathrm{Im}(f_i) = \ker(f_{i+1})$ for $1 \leq i \leq n-1$.*
A short exact sequence *is an exact sequence of the form*

$$0 \to C_1 \to C_2 \to C_3 \to 0.$$

Definition 1.33. *Let $\mathscr{C}$ and $\mathscr{D}$ be two categories. A* functor *$F : \mathscr{C} \to \mathscr{D}$ maps an object $C \in \mathscr{C}$ to an object $F(C) \in \mathscr{D}$ and a morphism $f \in \mathscr{C}$ to $F(f) \in \mathscr{D}$, such that composition of morphisms are compatible in $\mathscr{C}$ and $\mathscr{D}$.*

A functor can be either covariant*: for $f : C_0 \to C_1$, $F(f)$ is a map from $F(C_0) \to F(C_1)$. Then the compatibility of morphisms is expressed as*

$$fg = h \text{ in } \mathscr{C} \Rightarrow F(f)F(g) = F(h) \text{ in } \mathscr{D}.$$

Or a functor can be contravariant*: for $f : C_0 \to C_1$, $F(f)$ is a map from $F(C_1) \to F(C_0)$.*

In this case the compatibility of morphisms is

$$fg = h \text{ in } \mathscr{C} \Rightarrow F(g)F(f) = F(h) \text{ in } \mathscr{D}\,.$$

Definition 1.34. *For a covariant functor $F : \mathscr{C} \to \mathscr{D}$ and every short exact sequence $0 \to C_1 \to C_2 \to C_3 \to 0$ in $\mathscr{C}$,*

1. *If $0 \to F(C_1) \to F(C_2) \to F(C_3) \to 0$ is always an exact sequence in $\mathscr{D}$ then we say the functor F is* exact.

2. *If $0 \to F(C_1) \to F(C_2) \to F(C_3)$ is always an exact sequence in $\mathscr{D}$ then we say the functor F is* left exact.

3. *If $F(C_1) \to F(C_2) \to F(C_3) \to 0$ is always an exact sequence in $\mathscr{D}$ then we say the functor F is* right exact.

For any left (resp. right) exact functors there exist right (resp. left) derived functors R^nF (resp. L^nF) for all $n > 0$ such that (for covariant functors)

$$0 \to F(C_1) \to F(C_2) \to F(C_3) \xrightarrow{\delta} R^1F(C_1) \to R^1F(C_2) \to R^1F(C_3) \xrightarrow{\delta} R^2F(C_1) \to ...$$

and

$$... \to L^2F(C_3) \xrightarrow{\delta} L^1F(C_1) \to L^1F(C_2) \to L^1F(C_3) \xrightarrow{\delta} F(C_1) \to F(C_2) \to F(C_3) \to 0$$

are exact sequences for every short exact sequence $0 \to C_1 \to C_2 \to C_3 \to 0$ with δ the (naturally occur) connecting homomorphisms.

Remark. To define exactness and derived functors of a contravariant functor, we exchange all C_1 and C_3 in the above definition.

Example 1.35. *Let A be an algebra, and let K be an A-module. Then $\mathrm{Hom}_A(K, -)$, defined by post-composition of maps, is a left exact covariant functor. Its n^{th} right-derived functor is denoted by $\mathrm{Ext}^n_A(K, -)$. In particular, for a short exact sequence of A-modules*

$$0 \to L \to M \to N \to 0$$

we have an exact sequence

$$0 \to \mathrm{Hom}_A(K, L) \to \mathrm{Hom}_A(K, M) \to \mathrm{Hom}_A(K, N) \to \mathrm{Ext}^1_A(K, L) \to \ldots.$$

These functors will be central to our construction. Next, we characterise algebras whose module categories are equivalent.

Definition 1.36. *We say two algebras A and B are* Morita equivalent *if they possess equivalent module categories. That is, there exist (covariant) functors $F : A\text{-mod} \to B\text{-mod}$ and $G : B\text{-mod} \to A\text{-mod}$ such that FG is naturally isomorphic to Id_B, the identity functor of B-mod and GF is naturally isomorphic to Id_A, the identity functor of A-mod.*

This equivalence was first studied by Kiiti Morita in 1958, and he gave a criterion for two algebras to be Morita equivalent.

Definition 1.37. *For any algebra A, a* finitely generated projective generator*, or* progenerator *P of A is a finitely generated projective module such that for any finitely generated A-module M there exists a surjective homomorphism $P^{\oplus i} \to M$ for some $i > 0$.*

Theorem 1.38. *Let A and B be two algebras. The following are equivalent.*

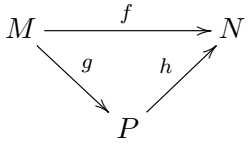
1. *A and B are Morita equivalent.*
2. *$A \cong \mathrm{End}_B(P_B)$, where P_B is a progenerator of B.*
3. *There exist an A-bimodule-B ${}_AM_B$ and a B-bimodule-A ${}_BN_A$ such that $M \otimes_B N \cong A$ as A-bimodule-A and $N \otimes_A M \cong B$ as B-bimodule-B.*

In particular, $M \otimes_B - : B\text{-mod} \to A\text{-mod}$ and $N \otimes_A - : A\text{-mod} \to B\text{-mod}$ are functors defining the Morita equivalence between A and B.

Morita equivalence is a quite restrictive condition. Especially it seldom happens that two non-isomorphic groups will have Morita equivalent group algebras over prime characteristic. One way to study relationship between group algebras and its subgroup algebras, is via the stable module category.

1.3.2 Stable module category

Definition 1.39. *For an A-module map $f : M \to N$, we say f* factors through projectives *if there is a projective A-module P such that there exist maps $g : P \to N$ and $h : M \to P$ such that $f = hg$. That is,*



commutes.

We denote the set of maps from M to N that factor through projectives as $\mathrm{Hom}_A^{Pr}(M,N)$.

Observe that the set of maps that factor through projectives is an ideal of A-module maps. Taking the quotient by this ideal gives the stable module category.

Definition 1.40. *The* stable module category *of A-modules, A-*$\underline{\mathrm{mod}}$ *is a category with*
Objects: Finitely generated A-modules.
Morphisms: $\mathrm{Hom}_A(M,N)/\mathrm{Hom}_A^{Pr}(M,N) = \underline{\mathrm{Hom}}_A(M,N)$

Any projective module P is isomorphic to 0 in the stable category, since the image of the identity morphism id_P is zero as it factors through P.

Since projective modules are zero objects, we cannot distinguish projectives in the stable module category. When considering equivalence in stable module categories there are no 'progenerators' to generate the other category as in Morita equivalence. However we can still define a type of stable equivalence similar to Morita equivalence.

Definition 1.41. *Suppose A and B are two algebras. If there exists an A-bimodule-B M and a B-bimodule-A N such that*

$$M \otimes_B N \cong A \oplus P \text{ as } A\text{-bimodule-}A$$

where P is a projective A-bimodule-A and

$$N \otimes_A M \cong B \oplus Q \text{ as } B\text{-bimodule-}B$$

where Q is a projective B-bimodule-B, then we say that A and B are stably equivalent of Morita type*. In such case, functors $M \otimes_B -$ and $N \otimes_A -$ induce mutually inverse equivalence B-*$\underline{\mathrm{mod}}$ $\to$ A-$\underline{\mathrm{mod}}$ *and A-*$\underline{\mathrm{mod}}$ $\to$ B-$\underline{\mathrm{mod}}$.

Asashiba claims in [Ashashiba99] that there are stable equivalences that cannot be described by this way. With this in mind, the advantage for stable equivalence of Morita type is then that two algebras of such equivalence are still similar enough, as manifested by Linckelmann's theorem[Linckelmann96].

Theorem 1.42 (Linckelmann's Theorem)**.** *Let A and B be two self-injective algebras with no simple projective summands. If A and B are stably equivalent of Morita type, and one of such equivalences sends simple A-modules to simple B-modules, then A and B are Morita equivalent.*

Okuyama has used this theorem to prove Broué's conjecture (Conjecture 1.1) for the blocks of $SL_2(p^n)$ [Okuyama97][Yoshii09].

Example 1.43. *Recall the Green correspondence in the case of trivial intersections (see 1.24), there is a one-to-one correspondence between the isomorphism classes of non-projective $\mathbb{F}G$-modules and the isomorphism classes of non-projective $\mathbb{F}H$-modules. Furthermore, induction and restriction induces such correspondences between non-projective $\mathbb{F}G$-modules and $\mathbb{F}H$-modules. It is worth to note that induction and restriction induce stable equivalence of Morita type in trivial intersection case, thus makes the correspondence functorial. That is, the map preserves morphisms between modules that correspond, given by*

$$\underline{\mathrm{Hom}}_{\mathbb{F}G}(U_1, U_2) \cong \underline{\mathrm{Hom}}_{\mathbb{F}H}(V_1, V_2)$$

where U_1, U_2 are $\mathbb{F}G$-modules and V_1, V_2 are $\mathbb{F}H$-modules.

Another main functor in the stable module category is the Heller functor. To start with, we define Ω as an A-bimodule-A.

Definition 1.44. *Let $\Omega = \ker(A \otimes A \to A)$ be the A-bimodule-A given by the kernel of the multiplication map. The* Heller functor *is the functor: $\Omega \otimes_A - : A\text{-mod} \to A\text{-mod}$.*

Proposition 1.45. *Let A be a symmetric algebra with no semisimple summands, then the functor $\Omega \otimes_A -$ induces a stable autoequivalence of Morita type of A-modules.*

Proof Note that we have an exact sequence

$$0 \to \Omega \to A \otimes A \to A \to 0,$$

by definition. Taking $\mathbb{F}$-duals we have

$$0 \to A \to A \otimes A \to \Omega^* \to 0$$

as $A \cong A^*$ as A-bimodule-A (Proposition 1.16). Apply functor $- \otimes_A \Omega^*$ to the first short exact sequence we have

$$0 \to \Omega \otimes_A \Omega^* \to A \otimes \Omega^* \to \Omega^* \to 0.$$

Note that the functor $- \otimes_A \Omega^*$ is exact since Ω^* is a projective A-module as Ω is

projective as module-A. Then now we have

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \Omega \otimes_A \Omega^* & \longrightarrow & A \otimes \Omega^* & \longrightarrow & \Omega^* & \longrightarrow & 0 \\ & & & & & & \| & & \\ 0 & \longrightarrow & A & \longrightarrow & A \otimes A & \longrightarrow & \Omega^* & \longrightarrow & 0 \end{array}$$

and notice that since $A \otimes \Omega^*$ and $A \otimes A$ are both projective A-bimodule-A, we have

$$(A \otimes \Omega^*) \oplus A \cong (\Omega \otimes_A \Omega^*) \oplus (A \otimes A)$$

as A-bimodule-A by Schanuel's lemma. Now consider the composition factors of $\Omega \otimes \Omega^*$ as A-bimodule-A. The only non-projective summand on the left hand side is A, hence $\Omega \otimes \Omega^*$ has to be isomorphic to a copy of A with other projective summands, by Krull-Schmidt theorem (Theorem 1.2). Thus we have established Ω and Ω^* induced a stable autoequivalence of Morita type, using the definition. □

Remark. The proposition should also be true when A is a self-injective algebra.

Now we consider how Heller functor applies to A-modules.

Definition 1.46. *Let M be an A-module. Define ΩM as the A-module without projective summands such that $\Omega M \oplus P \cong \Omega \otimes_A M$, where P is a projective A-module. The module ΩM is called the* Heller translate of M.

It is obvious that $\Omega M \cong \Omega \otimes_A M$ in the stable category of A-modules since all projectives are zero objects.

Proposition 1.47. *We have the short exact sequence of A-modules*

$$0 \to \Omega M \to P_M \to M \to 0$$

where P_M is the projective cover *of M (minimum projective module which surjects to M).*

Proof Consider the short exact sequence

$$0 \to \Omega \to A \otimes A \to A \to 0$$

of A-module-A. Tensor the sequence by $- \otimes_A M$ we have

$$0 \to \Omega \otimes_A M \to A \otimes M \to M \to 0$$

since A is free module-A. Note that $A \otimes M$ is projective and ΩM has no projective summands, we can obtain the short exact sequence required. □

The stable module category is a triangulated category - there are 'triangles' inside this type of category which mimics the role of exact sequence in an exact category, such as module category of group algebras. See section 1.3.4 for details.

With the Heller functor, we can evaluate $\mathrm{Ext}^n_A(M,N)$ using $\underline{\mathrm{Hom}}$-spaces when A is self-injective. Recall that Ext^1_A functor is the right derived functor of Hom_A. Consider the short exact sequence

$$0 \to \Omega M \xrightarrow{f} P \to M \to 0$$

in A-mod. Applying the contravariant functor $\mathrm{Hom}(-,N)$ yields a long exact sequence

$$0 \to \mathrm{Hom}_A(M,N) \to \mathrm{Hom}_A(P,N) \to \mathrm{Hom}_A(\Omega M,N) \to \mathrm{Ext}^1_A(M,N) \to \mathrm{Ext}^1_A(P,N) \to .$$

Since $\mathrm{Ext}^1_A(P,N) = 0$ and the image of $g \in \mathrm{Hom}_A(P,N)$ in $\mathrm{Hom}_A(\Omega M,N)$ is gf which can factor through a projective P, hence it is contained in $\mathrm{Hom}^{Pr}_A(\Omega M,N)$. On the other hand, consider a map $gh \in \mathrm{Hom}^{Pr}_A(\Omega M,N)$ such that $g : Q \to N$ and $h : \Omega M \to Q$ where Q is a projective module. Since Q is also injective, there exists a map p from P to Q with $h = pf$. Thus $gh = gpf$ is the image of the map $gp \in \mathrm{Hom}_A(P,N)$. Thus we have

$$\underline{\mathrm{Hom}}_A(\Omega M,N) \cong \mathrm{Ext}^1_A(M,N)$$

1.3.3 Derived category

The derived category is difficult to define and to do calculation in. However, it is a very important notion in category theory since the derived functors arise naturally. Many authors such as Rickard and Broué suggest that the derived category is the right place to consider representation theory. This is enhanced by the fact that one can generalise Morita's equivalence theorem in the derived category. At the same time a formulation is possible in the form of Broué's conjecture 1.1 to describe a relation between representations of a group and of its subgroups. In this section, we first define the homotopy category and then we have the derived category as the localisation of the former given by inverting quasi-isomorphism. After that we shall look at derived equivalence. First we define some terms and operations.

Definition 1.48. *Let $\mathscr{C}$ be an abelian category. A* cochain complex *X with objects in*

$\mathscr{C}$ is a collection of objects $(..., X^{-1}, X^0, X^1, ...)$ with a differential map (of degree 1) *$d^i_X : X^i \to X^{i+1}$ such that $d^i \circ d^{i-1} = 0$ for all i. The terms of X can be more precisely written as*

$$\cdots \to X^{-1} \xrightarrow{d^{-1}} X^0 \xrightarrow{d^0} X^1 \to \ldots.$$

A cochain map $f : X \to Y$ is a collection of maps f^i such that $f^i : X^i \to Y^i$ is a morphism of objects X^i and Y^i in $\mathscr{C}$ and each square in the following diagram commutes.

$$\begin{array}{ccccccccc} \ldots & \longrightarrow & X^0 & \xrightarrow{d^0_X} & X^1 & \xrightarrow{d^1_X} & X^2 & \longrightarrow & \ldots \\ & & \downarrow{\scriptstyle f^0} & & \downarrow{\scriptstyle f^1} & & \downarrow{\scriptstyle f^2} & & \\ \ldots & \longrightarrow & Y^0 & \xrightarrow{d^0_Y} & Y^1 & \xrightarrow{d^1_Y} & Y^2 & \longrightarrow & \ldots \end{array}$$

Remark. These are the type of complexes we shall use throughout the thesis, so we shall drop the word cochain. This is also because cochains complex and chain complex are essentially equivalent.

The condition $d \circ d = 0$ allows us to define homology.

Definition 1.49. *Let X be a complex. Define $H^n(X)$, the n^{th}* homology of *X by*

$$H^n(X) = \ker(d^n)/\operatorname{Im}(d^{n-1}).$$

Two distinguished classes of complex is of special interest:

Definition 1.50. *A complex X is* acyclic *if $H^n(X) = 0$ for all n; A complex Y is* contractible *if it is the direct sum of two-term complexes of the form*

$$... \to 0 \to Z \to Z \to 0 \to ...$$

where the map: $Z \to Z$ is the identity map.

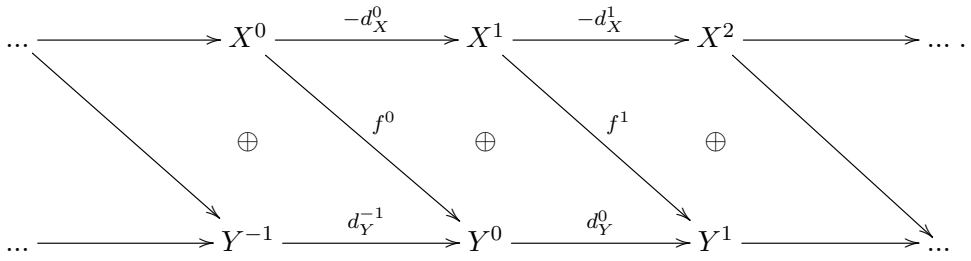
Remark. A contractible complex is acyclic (but not vice versa in general).

The following are some ways to generate new complexes with known complexes:

Definition 1.51. *A* left shift [1] *of X is a complex $X[1]$ defined by $X[1]^i = X^{i+1}$ and differential $d^i_{X[1]} = -d^{i+1}_X$; If $f : X \to Y$ is a chain map, then the* cone of f, *denoted $cone(f)$, is a complex with terms $X[1] \oplus Y$ (i.e. $cone(f)^n = X^{n+1} \oplus Y^n$) and differential map $\begin{pmatrix} -d^{n+1}_X & f \\ 0 & d^n_Y \end{pmatrix}$. Or pictorially, the complex $cone(f)$:*

$$... \longrightarrow cone(f)^{-1} \xrightarrow{d^{-1}_{cone(f)}} cone(f)^0 \xrightarrow{d^0_{cone(f)}} cone(f)^1 \longrightarrow ...$$

is equal to



It is easy to see that $d \circ d = 0$ in $cone(f)$. With complexes defined, one of the ways to describe a module M is to use projectives to approximate it.

Definition 1.52. *Let M be an A-module. A* projective resolution of M *is a complex*

$$P_M = \cdots \to P^2 \to P^1 \to P^0 \to 0$$

where P^i is in degree $-i$ for $i \geq 0$, and

$$H^n(P_M) \cong \begin{cases} M \text{ if } n = 0 \\ 0 \text{ otherwise.} \end{cases}$$

The following definition is inspired from topology.

Definition 1.53. *Two chain maps $f, g : X \to Y$ are* chain homotopic *if there exists a collection of degree -1 maps $h^i : X^i \to Y^{i-1}$ such that $f - g = dh + hd$. Or specifically on each degree,*

$$f^i - g^i = d_Y^{i-1} h^i + h^{i+1} d_X^i.$$

Two complexes X, Y are homotopy equivalent *if there exist chain maps $f' : X \to Y$ and $g' : Y \to X$ such that $f'g'$ is chain homotopic to the identity chain map id_Y of Y (i.e. Identity map on every term Y^i.) and $g'f'$ is chain homotopic to the identity chain map id_X of X.*

Definition 1.54. *Let $\mathscr{C}$ be an abelian category. Its homotopy category, denoted by $K(\mathscr{C})$, has*
Objects: Chain complexes with objects in $\mathscr{C}$.
Morphisms: Chain maps modulo all chain homotopies.
The set of morphisms from object X to Y in homotopy category is denoted by $\mathrm{Hom}_{K(\mathscr{C})}(X, Y)$.

Two complexes that are homotopy equivalent are isomorphic objects in the homotopy category, but this equivalence is not quite enough for our purpose. In particular,

we want a module being 'isomorphic' with its projective resolution. In that case the complexes are quasi-isomorphic: They have the same homology and there exist maps that induce isomorphisms in homology from one to the other, but there does not necessarily exist such a map in the other direction, thus failed to be homotopic equivalent.

Definition 1.55. *Let X and Y be two complexes of an abelian category $\mathscr{C}$. X and Y are said to be* quasi-isomorphic *if there exists a chain map $f : X \to Y$ which induces an isomorphism on their homologies. That is, for all n,*

$$f^n : H^n(X) \to H^n(Y)$$

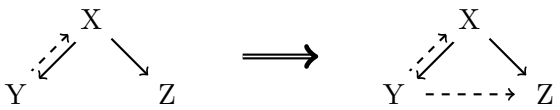
is an isomorphism. Such a map f is called a quasi-isomorphism.

Example 1.56. *A module M, regarded as a complex concentrated at degree zero, and a projective resolution of M are quasi-isomorphic. Two complexes that are homotopy equivalent are quasi-isomorphic.*

Now we can define the derived category using quasi-isomorphisms.

Definition 1.57. *Let $\mathscr{C}$ be an abelian category. Its derived category, denoted by $D(\mathscr{C})$, has*
Objects: Chain complexes with objects in $\mathscr{C}$.
Morphisms: Chain maps modulo all chain homotopies (as in homotopy category), added with formal inverses for all quasi-isomorphisms. The set of morphisms from object X to Y in derived category is denoted as $\mathrm{Hom}_{D(\mathscr{C})}(X, Y)$.

Note that adding the inverse for quasi-isomorphism means there are some non-obvious maps between complexes. For example let $f : X \to Y$ be a quasi-isomorphism and let $g : X \to Z$ be another chain map. Then $\mathrm{Hom}_{D(\mathscr{C})}(Y, Z)$ contains the composition $g \circ f^{-1} : Y \to Z$.



Definition 1.58. *Consider a complex X.*

1. *X is* right bounded *if $X^n = 0$ for $n >> 0$.*

2. *X is* left bounded *if $X^n = 0$ for $n << 0$.*

3. *X is* bounded *if it is both left and right bounded.*

We define $K^+(\mathscr{C})$, $K^-(\mathscr{C})$, $K^b(\mathscr{C})$, $D^+(\mathscr{C})$, $D^-(\mathscr{C})$, $D^b(\mathscr{C})$ by restricting the set of objects respectively in their respective categories.

Remark. Some authors define these categories by correspondingly bounded homologies. For example $D^b(\mathscr{C})$ is defined by complexes in $\mathscr{C}$ with non-zero homology in finitely many degrees. These definitions lead to categories equivalent to the one we have defined here.

The original abelian category $\mathscr{C}$ can be embedded into $K(\mathscr{C})$ and $D(\mathscr{C})$ by regarding an object in $\mathscr{C}$ as a complex concentrated in degree 0.

Now similar to the last section, we define derived equivalence. In the case of self-injective algebras, the equivalence of derived category and bounded derived category is essentially the same. For simplicity we consider bounded derived category.

Definition 1.59. *Let A and B be two algebras. If $D^b(A\text{-mod})$ is equivalent to $D^b(B\text{-mod})$ as triangulated categories (see section 1.3.4) then we say that A and B are derived equivalent.*

Notation. *When A is an algebra we use the shorthand $D(A)$ for $D(A\text{-mod})$, similarly for homotopy category $K(A)$ and all their bounded versions.*

The following generalisation of Morita theory and tilting theory are due to J. Rickard [Rickard89], which describes the condition for two algebras to be derived equivalent.

Definition 1.60. *A complex T is a* one-sided tilting complex *of an algebra A if it satisfies*

1. *T is a bounded complex with T^i being a finitely generated projective A-module for all i;*
2. $\mathrm{Hom}_{D^b(A)}(T, T[i]) = 0$ *for all $i \neq 0$;*
3. *The direct summands of T generate $K^b(A\text{-proj})$, the chain homotopy category of projective A-modules, as a triangulated category (see 1.3.4).*

Definition 1.61. *A complex X is a* two-sided tilting complex *of A-bimodule-B if*

1. *X is bounded.*
2. *When regarded as complex of left A-module, every term of X is finitely generated projective, and $B \cong \mathrm{End}_{D^b(A)}(X)$ as an algebra via the natural map.*

3. *When regarded as complex of right module-B, every term of X is finitely generated projective, and $A \cong \mathrm{End}_{D^b(B)}(X)$ as an algebra.*

4. *There exist a complex Y of B-bimodule-A such that*

$$X \otimes_B Y \cong A \oplus V$$

for A as a A-bimodule-A complex concentrated at degree 0 and V a contractible complex. Similarly for

$$Y \otimes_A X \cong B \oplus W.$$

Theorem 1.62. *The following are equivalent*

1. *A and B are derived equivalent.*

2. *$B \cong \mathrm{End}_{D^b(A)}(T)$, the endomorphism ring of a one-sided tilting complex T of A-modules.*

3. *$A \cong \mathrm{End}_{D^b(B)}(T')$, the endomorphism ring of a one-sided tilting complex T' of B-modules.*

4. *There exists a two-sided tilting complex X of A-bimodule-B.*

Further from the last condition, $X \otimes_B - : D^b(B) \to D^b(A)$ is an equivalence.

1.3.4 Triangulated category

Both the stable module category and the derived category we have introduced are examples of triangulated categories. The distinguished triangles in these category is playing a role like short exact sequences from their related abelian category, and their structure mimics those of long exact sequences.

Definition 1.63. *A* triangulated category *is an additive category $\mathscr{C}$ with*

1. *A* translation functor *$\Sigma : \mathscr{C} \to \mathscr{C}$ which is an autoequivalence.*

2. *A class of* distinguished triangles*: each of these consists of 3 objects X, Y, Z and morphisms $u : X \to Y$, $v : Y \to Z$, $w : Z \to \Sigma X$. Written as*

$$X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} \Sigma X$$

or simplified as

$$X \to Y \to Z \rightsquigarrow,$$

such that the following axioms are satisfied.

(a) For any object X there is a distinguished triangle

$$X \xrightarrow{id} X \to 0 \rightsquigarrow .$$

(b) For any morphism $u : X \to Y$ there is an object $Z = cone(u)$, a mapping cone of u that forms a distinguished triangle

$$X \xrightarrow{u} Y \to Z \rightsquigarrow .$$

(c) Any triangle isomorphic to a distinguished triangle is distinguished.

(d) Given two distinguished triangles

$$X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} \Sigma X; \qquad X' \xrightarrow{u'} Y' \xrightarrow{v'} Z' \xrightarrow{w'} \Sigma X'$$

and maps $f : X \to X'$ and $g : Y \to Y'$ such that $gu = u'f$. There exists a map $h : Z \to Z'$ such that all squares in the following diagram commute.

$$\begin{array}{ccccccc} X & \xrightarrow{u} & Y & \xrightarrow{v} & Z & \xrightarrow{w} & \Sigma X \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle g} & & \vdots{\scriptstyle h} & & \downarrow{\scriptstyle f[1]} \\ X' & \xrightarrow{u'} & Y' & \xrightarrow{v'} & Z' & \xrightarrow{w'} & X'[1] \end{array}$$

(e) If

$$X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} \Sigma X$$

is a distinguished triangle, then the rotated triangles

$$Y \xrightarrow{v} Z \xrightarrow{w} \Sigma X \xrightarrow{-\Sigma u} \Sigma Y;$$

$$\Sigma^{-1} Z \xrightarrow{-\Sigma^{-1} w} X \xrightarrow{u} Y \xrightarrow{v} Z$$

are distinguished triangles.

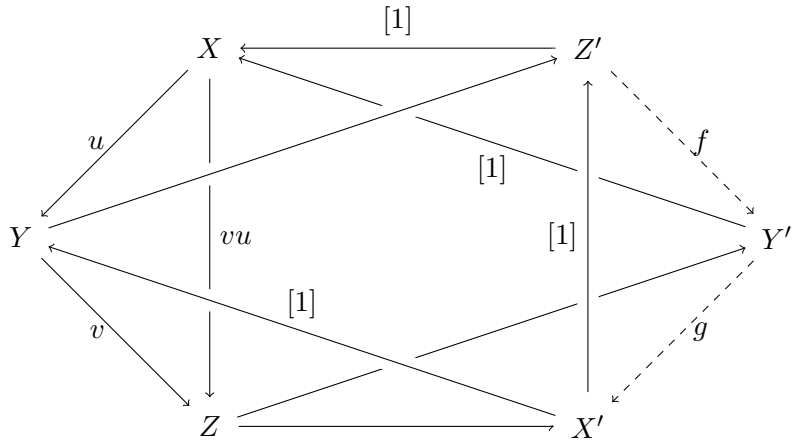
(f) (Octahedral axiom) Let $u : X \to Y$ and $v : Y \to Z$ be morphisms and $vu : X \to Z$ the composition of u and v. Denote by $Z' = cone(u)$, $Y' = cone(vu)$ and $X' = cone(v)$ as in (b). (These are well-defined by (d) and five-lemma).

Then there exists maps $f : Z' \to Y'$ and $g : Y' \to X'$ such that

$$\to Z' \to Y' \to X' \rightsquigarrow$$

is a distinguished triangle. Furthermore, all triangles and squares formed by f, g commutes.

The last axiom is called octahedral axiom because it is best depicted using an octahedron:



The axiom is equivalently saying that given u and v on the left (and hence all compositions and distinguished triangles) there exists f and g on the right to complete the octahedron, such that all squares and triangles either commute or are distinguished (determined by direction of arrows).

Example 1.64. *The stable module category of a self-injective algebra A is a triangulated category. The translation functor Σ is the inverse of Heller functor Ω^{-1} and pushout of A-module map $X \xrightarrow{f} Y$:*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & X & \longrightarrow & I(X) & \longrightarrow & \Omega^{-1}X & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle f} & & \downarrow & & \| & & \\ 0 & \longrightarrow & Y & \longrightarrow & Z & \longrightarrow & \Omega^{-1}X & \longrightarrow & 0 \end{array}$$

generates standard triangles

$$X \xrightarrow{f} Y \to Z \to \Sigma X \rightsquigarrow .$$

All distinguished triangles in A-<u>mod</u> can be generated by this way up to isomorphism

of triangles.

A derived category is also triangulated. The translation functor can be taken as the left shift [1] of complexes and distinguished triangles are obtained using the cone construction: for $X \xrightarrow{f} Y$, $cone(f)$ completes the triangle

$$X \xrightarrow{f} Y \xrightarrow{inj} cone(f) \xrightarrow{proj} X[1]$$

and all distinguished triangles arise in this way up to isomorphism of triangles.

Definition 1.65. *A set of objects $\mathscr{S}$* generates *a triangulated category, if every object in the triangulated category can be represented /constructed as iterated cones of maps and translations by objects in $\mathscr{S}$.*

Example 1.66. *The set of simple A-modules generates both A-$\underline{\text{mod}}$ and $D^b(A)$ as triangulated categories. This can be seen by the fact that for a short exact sequence $0 \to K \to L \to M \to 0$, the object L can be generated by the cone of the map $\Sigma^{-1}M \to K$. For $D^b(A)$, each A-module of a certain degree can be generated by the above, shifted to the correct degree using translations and connected using cones of maps.*

Definition 1.67. *A triangulated functor from (triangulated) category $\mathscr{D}$ to (triangulated category) $\mathscr{D}'$ is an additive functor such that it commutes with translation and preserves distinguished triangles. That is (for a covariant functor), a distinguished triangle in $\mathscr{D}$:*

$$X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} \Sigma_{\mathscr{D}} X$$

becomes

$$F(X) \xrightarrow{F(u)} F(Y) \xrightarrow{F(v)} F(Z) \xrightarrow{F(w)} \Sigma_{\mathscr{D}'} F(X)$$

a triangle in $\mathscr{D}'$.

One way to construct triangulated functors is to consider the quotient category of a triangulated category which respects its triangulation. First we define the correct kind of triangulated subcategories such that a quotient can be formed.

Definition 1.68. *A* thick (triangulated) subcategory *$\mathscr{S}$ of a triangulated category $\mathscr{D}$ is a full subcategory consisting of subsets of objects and morphisms of $\mathscr{D}$, such that*

1. *It is closed under translation Σ.*
2. *If two objects of $\mathscr{S}$ belong to a distinguished triangle of $\mathscr{D}$ so is the third object belongs to $\mathscr{S}$.*

3. *It is closed under direct summands, that is if $X \oplus Y$ is in $\mathscr{S}$ then both X and Y are in $\mathscr{S}$.*

If $\mathscr{S}$ is a thick subcategory, then one can form a category $\mathscr{D}/\mathscr{S}$, called the Verdier quotient. Just as one might expect all objects in the thick subcategory are zero in the quotient.

Example 1.69. *Passing from $K(A)$ to $D(A)$, where A is an algebra, is a Verdier quotient by the thick subcategory of acyclic complexes in $K(A)$.*

Definition 1.70. *An object (an complex) of $D^b(A)$ is perfect if it is quasi-isomorphic to a bounded complex of projective modules. The full subcategory of $D^b(A)$ whose objects are perfect complexes is denoted by $D^{pc}(A)$. It can be checked using properties of projective A-modules that $D^{pc}(A)$ is a thick subcategory.*

Theorem 1.71 (Rickard's theorem)**.** *Let A be a self-injective algebra. The quotient category $D^b(A)/D^{pc}(A)$ is equivalent naturally to A-$\underline{\text{mod}}$ as a triangulated category. That is, there exists an equivalence A-$\underline{\text{Mod}} \to D^b(A)/D^{pc}(A)$ the following square commutes:*

$$\begin{array}{ccc} A\text{-mod} & \hookrightarrow & D^b(A) \\ \downarrow & & \downarrow \\ A\text{-}\underline{\text{mod}} & \xleftrightarrow{\sim} & D^b(A)/D^{pc}(A) \end{array}$$

Using this, every derived equivalence $F : D^b(B) \to D^b(A)$ between two self-injective algebras induces a stable equivalence $\overline{F} : B\text{-}\underline{\text{mod}} \to A\text{-}\underline{\text{mod}}$. In particular $\overline{F}$ is of Morita type if F is induced by a two-sided tilting complex.

For a triangulated category we can construct its Grothendieck group as follows.

Definition 1.72. *Let $\mathscr{C}$ be a triangulated category. The Grothendieck group of $\mathscr{C}$, denoted by $\mathcal{K}(\mathscr{C})$, is the abelian group generated freely by every object in $\mathscr{C}$ modulo the following relation: If objects A, B, C in $\mathscr{C}$ form a distinguished triangle*

$$A \to B \to C \rightsquigarrow,$$

then we have the relation $[A] + [C] - [B] = 0$ in $\mathcal{K}(\mathscr{C})$, where $[A]$ is the group element generated by $[A]$.

When $\mathscr{C}$ is the derived category of A-modules, its Grothendieck group $\mathcal{K}(\mathscr{C})$ is freely generated by the set of isoclasses of simple A-modules, regarded as complexes

concentrated in degree 0. A derived autoequivalence of $\mathscr{C}$ always yield a group automorphism on $\mathcal{K}(\mathscr{C})$. When $\mathscr{C}$ is the stable module cateogry of a self-injective algebra A, its Grothendieck group $\mathcal{K}(A\text{-mod})$ can be regard as a further quotient of the Grothendieck group of its derived category, $\mathcal{K}(D^b(A))$, by the relation indicating the simple constituents, counted with multiplicity of a projective A-module add up to zero. For group algebra, this makes the Grothendieck group of the associated stable module category a finite group by the fact that Cartan matrix is of full rank. See Chapter 5 for example.

1.3.5 Perverse equivalence

Perverse equivalence is a type of derived equivalence that can be constructed by some combinatorial data. It has its origins from algebraic geometry - the construction of perverse sheaves. This tool is very recently developed to facilitate the description of some derived equivalences. However, it does not cover all types of derived equivalences, and composition of perverse equivalences might fail to be perverse. It is an open question whether any derived equivalence is a composition of perverse equivalences. Although it has a broad application to various type of categories, we shall only define the perverse equivalences for derived categories of abelian categories to simplify things and allow us bypass some technicalities (such as t-structures and hearts). Then we shall give some examples to explain perverse equivalences for module categories of symmetric algebras. We start by the notion of Serre subcategory of an abelian category:

Definition 1.73. *Let $\mathscr{C}$ be an abelian category and $\mathscr{D}$ be a full subcategory. $\mathscr{D}$ is a* Serre subcategory *if given any exact sequence $0 \to K \to L \to M \to 0$ in $\mathscr{C}$, $L \in \mathscr{D}$ if and only if $K, M \in \mathscr{D}$. Denote by $D^b_{\mathscr{D}}(\mathscr{C})$ the full subcategory of $D^b(\mathscr{C})$ of objects with cohomology in $\mathscr{D}$.*

Remark. It is easy to check (using definition 1.68) that $D^b_{\mathscr{D}}(\mathscr{C})$ is a thick subcategory of $\mathscr{C}$.

This is the right notion for quotient categories of abelian categories:

Definition 1.74. *Let $\mathscr{C}$ be an abelian category, let $\mathscr{D}$ be a Serre subcategory of $\mathscr{C}$, then we can define quotient category $\mathscr{C} / \mathscr{D}$ such that the objects of $\mathscr{C} / \mathscr{D}$ are those of $\mathscr{C}$ and the morphisms is the direct limit:*

$$\mathrm{Hom}_{\mathscr{C}/\mathscr{D}}(X, Y) = \lim \mathrm{Hom}_{\mathscr{C}}(X', Y/Y') \tag{1.1}$$

for subobjects $X' \subset X$ and $Y' \subset Y$ such that $X/X', Y' \in \mathscr{D}$. The quotient category constructed above is an abelian category.

Let $\mathscr{C}$ and $\mathscr{C}'$ be two abelian categories. Consider two filtrations

$$0 = \mathscr{C}_{-1} \subset \mathscr{C}_0 \subset \cdots \subset \mathscr{C}_r = \mathscr{C} \text{ and } 0 = \mathscr{C}'_{-1} \subset \mathscr{C}'_0 \subset \cdots \subset \mathscr{C}'_r = \mathscr{C}'$$

of Serre subcategories and a function $\pi : \{0, ..., r\} \to \mathbb{Z}$.

Definition 1.75. *An equivalence $F : D^b(\mathscr{C}) \to D^b(\mathscr{C}')$ is* perverse relative to $(\mathscr{C}_\bullet, \mathscr{C}'_\bullet, \pi)$ *if the following holds:*

- *F restricts to equivalences $D^b_{\mathscr{C}_i}(\mathscr{C}) \to D^b_{\mathscr{C}'_i}(\mathscr{C}')$.*
- *$F[-\pi(i)]$ induces equivalences $\mathscr{C}_i / \mathscr{C}_{i-1} \to \mathscr{C}'_i / \mathscr{C}'_{i-1}$.*

That is, with the natural embedding from $\mathscr{C}_i / \mathscr{C}_{i-1}$ to $D^b_{\mathscr{C}_i}(\mathscr{C})/D^b_{\mathscr{C}_{i-1}(\mathscr{C})}$ we have

$$\begin{array}{ccc} D^b_{\mathscr{C}_i}(\mathscr{C})/D^b_{\mathscr{C}_{i-1}(\mathscr{C})} & \xrightarrow{F} & D^b_{\mathscr{C}_i}(\mathscr{C})/D^b_{\mathscr{C}_{i-1}(\mathscr{C})} \\ \uparrow & & \uparrow \\ \mathscr{C}_i / \mathscr{C}_{i-1} & \xrightarrow{F[-\pi(i)]} & \mathscr{C}'_i / \mathscr{C}'_{i-1} \end{array}$$

(c.f. [Chuang, Rouquier13, Definition 2.53]).

Given an equivalence F perverse relative to $(\mathscr{C}_\bullet, \mathscr{C}'_\bullet, \pi)$, the filtration $\mathscr{C}'_\bullet$ is determined by $\mathscr{C}_\bullet$ and F via $\mathscr{C}'_i = \mathscr{C}' \cap F(D^b_{\mathscr{C}_i}(\mathscr{C}))$.

Proposition 1.76. *Let $F : D^b(\mathscr{C}) \to D^b(\mathscr{C}')$ be perverse relative to $(\mathscr{C}_\bullet, \mathscr{C}'_\bullet, \pi)$.*

1. *(reversibility) F^{-1} is perverse relative to $(\mathscr{C}'_\bullet, \mathscr{C}_\bullet, -\pi)$.*
2. *(composability) Let $F' : D^b(\mathscr{C}') \to D^b(\mathscr{C}'')$ be perverse relative to $(\mathscr{C}'_\bullet, \mathscr{C}''_\bullet, \pi')$, then $F' \circ F$ is perverse relative to $(\mathscr{C}_\bullet, \mathscr{C}''_\bullet, \pi + \pi')$.*
3. *(refineability) Let $\tilde{\mathscr{C}}_\bullet = (0 = \tilde{\mathscr{C}}_{-1} \subset \cdots \subset \tilde{\mathscr{C}}_{\tilde{r}})$ be a refinement of $\mathscr{C}_\bullet$. Define the weakly increasing map $f : \{0, \ldots, \tilde{r}\} \to \{0, \ldots, r\}$ such that $\tilde{\mathscr{C}}_\bullet$ collapses to $\mathscr{C}_\bullet$ under f (i.e. $\mathscr{C}_{f(i)-1} \subset \tilde{\mathscr{C}}_i \subset \mathscr{C}_{f(i)}$). Then F is perverse relative to $(\tilde{\mathscr{C}}_\bullet, \pi \circ f)$.*
4. *If $\pi = 0$ then F restricts to an equivalence $\mathscr{C} \to \mathscr{C}'$.*
5. *The information $(\mathscr{C}_\bullet, \pi)$ determine $\mathscr{C}'$ up to equivalence.*

Notation. *From 5, since $(\mathscr{C}_\bullet, \pi)$ determine $\mathscr{C}'$ we might sometimes simplify and say a perverse equivalence F is perverse relative to $(\mathscr{C}_\bullet, \pi)$*

Proof The first three can be read off directly from the definition. The fourth involves t-structures and hearts so we omit the proof here (see [Chuang, Rouquier13] for a proof). For the fifth, consider two maps with such information, composing one with the inverse of the other (made possible by 1 and 2) to obtain the result (by 4). □

When every object in an abelian category $\mathscr{C}$ has finite composition series, each object can be broken down to a collection of simple objects components via short exact sequences. Then, by definition, a Serre subcategory is generated by the collection of all simple objects inside it. Thus we can use filtration of simple objects to replace the filtration of Serre subcategories, making the description more concrete. (c.f. [Chuang, Rouquier13, 2.2.6])

Definition 1.77. *Let $\mathscr{C}$ and $\mathscr{D}$ be abelian categories with finite composition series. Let $\mathbb{S}$ be the set of non-isomorphic simple objects in $\mathscr{C}$. We say that an equivalence $F : D^b(\mathscr{C}) \xrightarrow{\sim} D^b(\mathscr{D})$ is perverse relative to $(\mathbb{S}_\bullet, \pi)$ when it is perverse relative to $(\mathscr{C}_\bullet, \pi)$ where $\mathbb{S}_\bullet$ is a filtration of isomorphism class of simple objects defined by $\mathscr{C}_\bullet$.*

Lemma 1.78. *Let $\mathscr{C}$, $\mathscr{D}$ be abelian categories with finite composition series,*

$$\mathbb{S}_\bullet = (\emptyset = \mathbb{S}_{-1} \subset \mathbb{S}_0 \subset \cdots \subset \mathbb{S}_r = \mathbb{S}) \text{ and } \mathbb{T}_\bullet = (\emptyset = \mathbb{T}_{-1} \subset \mathbb{T}_0 \subset \cdots \subset \mathbb{T}_r = \mathbb{T})$$

be filtrations of isomorphism class of simple objects on $\mathscr{C}$ and $\mathscr{D}$ respectively. Let $p : \{0, \ldots, r\} \to \mathbb{Z}$ be a function. An equivalence $F : D^b(\mathscr{C}) \xrightarrow{\sim} D^b(\mathscr{D})$ is perverse relative to $(\mathbb{S}_\bullet, \mathbb{T}_\bullet, p)$ if for every i the following holds.

- *Given $M \in \mathbb{S}_i - \mathbb{S}_{i-1}$, the composition factors of $H^r(F(M))$ are in $\mathbb{T}_{i-1}$ for $r \neq -p(i)$ and there is a filtration $L_1 \subset L_2 \subset H^{-p(i)}(I(M))$ such that the composition factors of L_1 and of $H^{-p(i)}(F(M))/L_2$ are in $\mathbb{T}_{i-1}$ and that of L_2/L_1 are in $\mathbb{T}_i - \mathbb{T}_{i-1}$.*
- *The map $M \to L_2/L_1$ induces a bijection $\mathbb{S}_i - \mathbb{S}_{i-1} \xrightarrow{\sim} \mathbb{T}_i - \mathbb{T}_{i-1}$, hence there is a bijection $\beta_F : \mathbb{S} \to \mathbb{T}$.*

For proof see [Chuang, Rouquier13, 2.64].

We put forward an important example of perverse equivalence of symmetric algebras. First we have to define:

Definition 1.79. *Let $\mathbb{S}' \subset \mathbb{S}$. Given $M \in A$-mod, $\phi_M : P_M \to M$ a projective cover. Denote by $M_{\mathbb{S}'}$ the largest quotient of P_M by a submodule of $\ker \phi_M$ such that all*

composition factors of the kernel of the induced map $M_{\mathbb{S}'} \to M$ are in $\mathbb{S}'$. Similarly for $M \to I_M$ be the injective hull. Denote by $M^{\mathbb{S}'}$ the largest submodule of I_M containing M such that all composition factors of $M^{\mathbb{S}'}/M$ are in $\mathbb{S}'$.

In this (very important) example, we first define a one-sided tilting complex, then we concern how the simple modules are being corresponded, Since the Serre subcategories are defined by subsets of simple objects as generators.

Example 1.80. *Let A be a symmetric algebra. Let S be a simple A-module, P_S be the projective cover of S. Take $\mathbb{S}'$ to be a subset of isomorphism classes of simple A-modules, We define X_S, a chain complex of projective A-modules depending on $\mathbb{S}'$ as follows.*

1. *If $S \in \mathbb{S}'$, we define*
$$X_S = (Q_S \xrightarrow{\alpha} P_S \to 0)$$
where α is a presentation of $S_{\mathbb{S}'}$, Q_S is in degree 0. Note that this forces all composition factors of $\mathrm{head}(Q_S)$ *not belong to $\mathbb{S}'$.*

2. *For $S \notin \mathbb{S}'$,*
$$X_S = (P_S \to 0),$$
where P_S is in degree 0.

Now consider
$$X_{\mathcal{I}} := \bigoplus_{S \in \mathbb{S}'} X_S.$$
It is easy to check this is a one-sided tilting complex (c.f. 1.60). Using 1.62, setting $B = End_{D^b(A)}(X_{\mathcal{I}})$ we have a functor
$$F : D^b(A) \xrightarrow{\sim} D^b(B)$$
inducing such equivalence. Denote by $\mathbb{T}$ the set of simple B-modules. We have a bijection between $\mathbb{S}$ and $\mathbb{T}$ and have $\mathbb{S}'$ correspond to $\mathbb{T}'$, a subset of $\mathbb{T}$. with $F(X_S) = P_T$, the projective cover of T as B-modules.

Note that $F(X_{\mathcal{I}}) = B$, consider $Hom(X_{\mathcal{I}}, S)$ for all $S \in \mathbb{S}$ we have
$$F(S) = \begin{cases} T[-1] & \text{if } S \in \mathbb{S}' \\ T^{\mathbb{T}'} & \text{otherwise} \end{cases} \quad \text{and} \quad F^{-1}(T) = \begin{cases} S[1] & \text{if } T \in \mathbb{T}' \\ S_{\mathbb{S}'} & \text{otherwise.} \end{cases}$$

Since the set of non-isomorphic simple modules generates the derived category as triangulated category and subsets of simple modules generate Serre subcategories, this correspondence of (simple stalk) complexes has equivalently defined the tilting by $X_{\mathcal{I}}$.

Remark. This example characterises elementary perverse equivalences for symmetric algebras. F is perverse relative to $(0 \subset \mathbb{S}' \subset \mathbb{S},\ 0 \subset \mathbb{T}' \subset \mathbb{T}, \varepsilon : \{0 \to 1; 1 \to 0\})$. See [Chuang, Rouquier13, 2.71] (shifted by 1 globally on perversity function).

Chapter 2

Representation theory of our groups

2.1 Introduction and Notation

In this chapter we lay down the detailed information of special linear group of finite fields of degree 2 and its block structure. From the viewpoint of local-global correspondent we refer it as the 'global' case. Except one semisimple block in $SL_2(q)$ modules, we have only one conjugacy class of defect group (Sylow p-subgroup) for other blocks. Using Brauer correspondent we can instead study the representation of the normaliser of Sylow p-subgroup, refer as 'local' modules. This information on global and local modules will be needed in constructing our derived equivalence. Although it styled as a autoequivalence of derived category globally, the proof nevertheless waded into the local group. In particular, we need the extensive use of a tensor functor by a local simple module, which we will define later.

Now recall p is a prime number, $n \geq 1$ is a natural number and $q = p^n$. Let $\mathbb{F}_q$ be the field of q elements (which is unique up to isomorphism). Particularly when we say Frobenius automorphism we mean the map $\sigma : \mathbb{F}_q \to \mathbb{F}_q$ sends an element x to x^p. $\mathbb{F} = \overline{\mathbb{F}}_q$ be the algebraic closure of $\mathbb{F}_q$ (with characteristic p). Let $G = SL_2(q)$ be the special linear group of finite field of q elements in degree 2. It is the collection of matrices of the form $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with $a, b, c, d \in \mathbb{F}_q$ and $ad - bc = 1$. Group operation is matrix multiplication. Let P be the set of the unipotent upper triangular matrices $\begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix}$. It forms a p-subgroup of G since its order is p^n. Furthermore it is a Sylow p-subgroup since G has order $(q+1)q(q-1)$. Note that P is an abelian group.

Let $H = N_G(P)$ be the normaliser of P in G. It is the subgroup of upper triangular matrices of the form $\begin{pmatrix} d^{-1} & * \\ 0 & d \end{pmatrix}$ in G, with $d \neq 0$.

Base-p numbers are very useful in labelling the related modules. For the rest of the chapter we will conveniently use the base-p presentation of integers (just like using 12 as 10+2 in base-10 setting) as defined below:

Definition 2.1. *For any integer a, $0 \leq a \leq q-1$, define $a_0, a_1, ..., a_{n-1}$ to be $n-1$ integers between 0 and $p-1$ inclusive such that*

$$a = \sum_{i=0}^{n-1} p^i a_i$$

is the base-p expression of a. On the other hand, an n-tuple of base-p digits $(a_0, a_1, ..., a_{n-1})$ uniquely determine an integer a between 0 and $q-1$.

2.2 Representations of G and H

We start by describing the simple $\mathbb{F}G$-modules. Let V be the natural two dimensional representation of $\mathbb{F}G$. More precisely for a vector $\begin{pmatrix} x \\ y \end{pmatrix}$ of V, the action of $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ of G is given by left multiplication.

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} ax + by \\ cx + dy \end{pmatrix}.$$

Denote by V^i the i^{th} symmetric power of V. (Note it is NOT the tensor product of i copies of V.) Frobenius automorphism σ acts on G by sending $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ to $\begin{pmatrix} a^p & b^p \\ c^p & d^p \end{pmatrix}$. Define the j^{th} Frobenius twist on V^i, $\sigma^j(V^i)$ to be the G-module whose underlying space is V^i but for $g \in G$, the action is defined as $g(v') = \sigma^j(g)(v)$ for all v.

Steinberg tensor product theorem describes all the simple modules of $SL_2(q)$.

Definition 2.2. *For $0 \leq a \leq q-1$ define*

$$S_a = \bigotimes_{i=0}^{n-1} \sigma^i(V^{a_i}). \tag{2.1}$$

Theorem 2.3. *S_a are simple for $0 \leq a \leq q-1$, mutually non-isomorphic. They form*

a complete set of mutually non-isomorphic simple $\mathbb{F}G$-modules.

Furthermore, S_{q-1} is (simple and) projective. In block theory of $SL_2(q)$, the Steinberg block is the block of defect zero containing S_{q-1}. The block itself is not very much of our concern since its Brauer correspondent is itself. However, we will use the fact that S_{q-1} is projective in some (strange) manner later.

For $p = 2$, all the remaining simple modules fall into one full defect block, the principal block $\mathbb{B}_0$. For odd primes they fall into two distinct full defect blocks. The principal block, $\mathbb{B}_0$, has all evenly numbered simple modules S_0, S_2,..., S_{q-2} and the non-principal block $\mathbb{B}_1$ consists of all oddly numbered simple modules S_1, S_3,..., S_{q-2}. To unify the description disregarding parity of primes,

Definition 2.4. *The direct sum of the full defect blocks of $\mathbb{F}G$ is denoted $\mathbb{B}$. Denote the complete set of non-isomorphic simple $\mathbb{B}$-modules by*

$$\mathbb{S} = \{S_a \mid 0 \leq a \leq q-2\}$$

Remark. $\mathbb{B}$ is the algebra such that $\mathbb{F}G = St \oplus \mathbb{B}$, where St is the Steinberg block.

The Sylow p-subgroup of G has trivial intersection [Alperin11], so the restriction functor from $\mathbb{F}G$-modules to $\mathbb{F}H$-modules induces a stable equivalence between blocks and their Brauer correspondents. Utilising this we define:

Definition 2.5. *For an integer a, $0 \leq a \leq q-1$, denote M_a the $\mathbb{F}H$-module $S_a{\downarrow_H}$ given by restricting the corresponding simple $\mathbb{F}G$-module.*

Now we discuss the representation theory of group H, all of whose block(s) is(are) the local Brauer correspondent(s) of the full defect block(s) of $\mathbb{F}G$. H as a group is $C_p^n \rtimes C_{q-1}$. It is quite easy to obtain its simple modules - they are all 1-dimensional. Let α be a generator of $\mathbb{F}_q$. Define U_i to be the 1-dimensional $\mathbb{F}H$-module on which $\begin{pmatrix} \alpha^{-1} & * \\ 0 & \alpha \end{pmatrix}$ acts on U_i by multiplying every vector by α^i. It is obvious that U_i is isomorphic to U_j if and only if $i \equiv j \pmod{q-1}$. Every simple $\mathbb{F}H$-module arises in this way.

Remark. Most literature (except Holloway in the list of referenced authors) define the simple $\mathbb{F}H$-modules U_i by having the matrix $\begin{pmatrix} \alpha^{-1} & * \\ 0 & \alpha \end{pmatrix}$ acts on U_i by multiplying α^{-i} instead. In other words the conventional U_i defined in other literature will be our U_{-i} instead. We use Holloway's convention to avoid many negative signs, for example, in the tables in the Appendix.

The Frobenius automorphism of G restricts to an automorphism to H. So it twists also $\mathbb{F}H$-modules. Simple calculation shows that $\sigma(U_i) \cong U_{pi}$.

The block structure of $\mathbb{F}H$ is similar to $\mathbb{B}$. Concretely, for $p = 2$, the whole group algebra $\mathbb{F}H$ forms a single block. For odd primes, $\mathbb{F}H$ decomposes into two blocks, the principal block containing evenly numbered simple modules U_0, U_2,..., U_{q-3} and non-principal block containing oddly numbered simple modules U_1, U_3,..., U_{q-2}.

Consider $U_i \otimes -$ as a functor from $\mathbb{F}H$-mod to itself. Since U_i is one-dimensional, the following conclusion can be easily checked:

1. $U_i \otimes U_j \cong U_{i+j}$;

2. $U_i \otimes -$ induces a Morita self-equivalence with inverse functor $U_{-i} \otimes -$;

3. The endo-functor on $\mathbb{F}H$-$\underline{\text{mod}}$ induced by $U_i \otimes -$ is exact.

Notation. *We omit the tensor product symbol from $U_i \otimes_{\mathbb{F}} -$ for convenience. For the rest of this paper we almost always treat U_i as a functor.*

Recall that G has 'trivial intersection' Sylow p-subgroups and $H \cong N_G(P)$. So using the Green correspondence 1.24 we have the following,

Lemma 2.6. *Let M and N be $\mathbb{F}G$-modules. Then*

$$\underline{\mathrm{Hom}}_{\mathbb{F}G}(M, N) \cong \underline{\mathrm{Hom}}_{\mathbb{F}H}(M{\downarrow_H}, N{\downarrow_H}).$$

Hence, $\mathbb{F}G$-$\underline{\text{mod}}$, $\mathbb{B}$-$\underline{\text{mod}}$ and $\mathbb{F}H$-$\underline{\text{mod}}$ are stably equivalent. This equivalence, since given by induction and restriction functors, is of Morita type.

The ultimate aim is to explore extensions in $\mathbb{F}G$-mod. A natural choice is to look at distinguished triangles in $D^b(\mathbb{F}G)$. However, it turns out to be extremely difficult. In fact, the piece of information in question is too cryptic in $\mathbb{F}G$-$\underline{\text{mod}}$. Luckily in $\mathbb{F}H$-$\underline{\text{mod}}$, we have the nice series of functors $U_i \otimes -$ to aid calculations which is enough for our job. In order to do so we consider the restriction of simple $\mathbb{F}G$-modules to $\mathbb{F}H$-modules. Using Steinberg tensor product theorem (c.f. 2.1) and the fact that the restriction of V^i is a uniserial $\mathbb{F}H$-module with 1-dimension components for $0 \leq i \leq n-1$ (see Chapter 5 of [Holloway01] for its proof), it is not hard to obtain the structure of these restrictions. They are indecomposable modules with a 'hypercuboid shape', see Appendix for details.

The remaining sections in this chapter consists of the needed work of tailoring the intrinsic structure of the related categories into useful lemmas and corollaries. These

calculation of possible extensions for all possible cases are highly combinatorial, but these cases can be simplified into two lemmas, with the extensions represented by certain distinguished triangles in $\mathbb{F}H$ -$\underline{\text{mod}}$.

2.3 Triangles in the stable module categories of blocks

In this section, we fix n to be greater than 1. All tensor products are over $\mathbb{F}$ unless otherwise stated.

Definition 2.7. *Consider the i^{th} digit of the base-p presentation a_i with $0 \leq a_i \leq p-2$.*

1. *Define a_i' to be $p-2-a_i$;*

2. *For an integer a with $0 \leq a_i \leq p-2$ for some i,*

   - *define $a(i') = (a_0, ..., a_{i-1}, a_i', a_{i+1}, ..., a_{n-1})$ to be the number acquired by replacing the digit a_i by a_i'.*
   - *$a(\bar{i}) = (a_0, ..., a_{i-1}, p-1, a_{i+1}, ..., a_{n-1})$ be the number acquired by replacing the digit a_i by $p-1$.*

Remark. Note that $a_i'' = a_i$. Using this we will define a pairing (later) between integers $0 \leq a \leq q-2$ which we will often use later.

We are going to build up some lemmas, culminating to a general description of certain distinguished triangles of $\mathbb{F}H$ -$\underline{\text{mod}}$ for further calculation.

Lemma 2.8. *Let i be an integer with $0 \leq i \leq n-1$. Let l be any integer and t be an integer with $0 \leq t \leq p-2$. Let $j = l - p^i(p+1+t)$ and $\breve{j} = l - p^i(p-1-t)$. Then any non-zero $\mathbb{F}H$-homomorphism from $U_j V^{p-1}$ to $U_{\breve{j}} V^{p-1}$ has a cokernel isomorphic to $U_l V_i^t$.*

Proof See Lemma 4 of [Chuang01] (further referenced to Lemma 2.1 and 2.2 of [Carlson83]). Note the proof from [Chuang01] can be directly adapted for arbitrary i. Also note U_i in [Chuang01] becomes U_{-i} here. □

Lemma 2.9. *We have non-split short exact sequences of the following $\mathbb{F}H$-modules:*

$$0 \to U_{-p^{i+1}} V_i^{b_i'} \to U_{-p^i(p-1-b_i)} V_i^{p-1} \to V_i^{b_i} \to 0$$

$$0 \to U_{-p^{i+1}} M_{b(i')} \to U_{-p^i(p-1-b_i)} M_{b(\bar{i})} \to M_b \to 0. \tag{2.2}$$

Recall that M_a is the restriction of S_a from G-modules to H-modules.

Proof The two short exact sequences can be obtained similarly to the proof of Lemma 6 of [Chuang01]: Using the previous lemma we have an exact sequence

$$U_j V_i^{p-1} \to U_{\breve{j}} V_i^{p-1} \to U_{p^i(p-1-b_i)} V_i^{p-1} \to V_i^{b_i} \to 0$$

where $j = -p^i(p-1-b_i+2p)$ and $\breve{j} = -p^i(p+1+b_i)$. Using the previous lemma, the first homomorphism has a cokernel isomorphic to $U_{-p^{i+1}} V^{b_i'}$ which gives the first sequence. The second sequence is obtained by tensoring the first sequence at each term with $V_0^{b_0}, ..., V_{n-1}^{b_{n-1}}$ except $V_i^{b_i}$. Then using (2.1) and restriction to see this is the desired result. □

Lemma 2.10. *Let b_i be an integer with $0 \le b_i \le p-2$ with $0 \le i \le n-1$. Then we have the following triangle in $\mathbb{F}H$-$\underline{\text{mod}}$:*

$$U_{p^i(1+b_i')} \Omega M_{b(\bar{i})} \to U_{p^{i+1}} \Omega M_{b(i')} \to M_b \rightsquigarrow \tag{2.3}$$

Proof The short exact sequence (2.2) in $\mathbb{F}H$-mod induces a triangle

$$U_{-p^{i+1}} M_{b(i')} \to U_{-p^i(p-1-b_i)} M_{b(\bar{i})} \to M_b \rightsquigarrow$$

in $\mathbb{F}H$-$\underline{\text{mod}}$. To obtain the stated triangle from the short exact sequence, we take it as a triangle in the stable module category and perform the following steps:

1. Tensor throughout the triangle obtained by $U_{p^{i+1}}$,
2. Relabel b_i by b_i' (and vice versa).
3. Rotate the triangle (c.f. definition 1.63(c)) two places to the left. (Put the rightmost term to the leftmost and shift by Σ^{-1}. In stable $\mathbb{F}H$-module category Σ^{-1} is represented by applying Ω; perform this twice.).

□

Remark. This triangle does lie in a particular block of $\mathbb{F}H$, depending on the parity of b. When $M_{b(\bar{i})}$ is the Steinberg module (i.e. $b(\bar{i}) = q-1$), we regard that module as zero module. That is partly justified by the fact the Steinberg module restricts to a projective module.

Indecomposable $\mathbb{F}H$-modules M_a with $a = (a_0, ..., a_{n-1})$ and all but one a_i equal to $p-1$ have periodic Heller translates. Furthermore, every such Heller translates are isomorphic to $U_i M_b$ for some suitable i and b. More precisely:

Lemma 2.11. *Fix i to be an integer with $0 \leq i \leq n-1$. Let b_i be integers such that $0 \leq b_i \leq p-2$. m is an integer. Then*
$$\begin{cases} \Omega^m M_{(p-1,...,b_i,...,p-1)} \cong U_{-mp^{i+1}} M_{(p-1,...,b_i',...,p-1)} & \text{if } m \text{ is odd.} \\ \Omega^m M_{(p-1,...,b_i,...,p-1)} \cong U_{-mp^{i+1}} M_{(p-1,...,b_i,...,p-1)} & \text{if } m \text{ is even.} \end{cases}$$

Proof We only need to prove $\Omega M_{(p-1,...,b_i,...,p-1)} = U_{-p^{i+1}} M_{(p-1,...,b_i',...,p-1)}$, since both cases are only the m^{th} iteration of it. Using (2.2) with all digits as $p-1$ except b_i we have

$$0 \to U_{-p^{i+1}} M_{(p-1,...,b_i',...,p-1)} \to U_{-p^i(p-1-b_i)} M_{p^n-1} \to M_{(p-1,...,b_i,...,p-1)} \to 0.$$

Since the middle term is projective it is regarded as zero in stable module category. Hence the first term is isomorphic to the Heller translate of the last term by the axioms, which is exactly the desired equation. □

Remark. The highlight in the preceding lemma is the $-mp^{i+1}$ subscript of U regardless of whether subscript of M has b_i or b_i' as its i^{th} digit.

2.4 Extension lemmas

We have to determine the possible extension of some $\mathbb{F}G$-modules. This is for us to work through perverse equivalence later. To achieve this, we transfer $\mathbb{F}G$-modules to $\mathbb{F}H$-modules using the restriction functor. Since

$$\mathrm{Ext}^1_{\mathbb{F}G}(M, N) = \underline{\mathrm{Hom}}_{\mathbb{F}G}(\Omega M, N) = \underline{\mathrm{Hom}}_{\mathbb{F}H}(\Omega M{\downarrow_H}, N{\downarrow_H}),$$

to find out the necessary extension needed in our construction later we introduce two lemmas. First, we follow the route in [Chuang01] and utilise Carlson's calculations on Ext groups of simple $\mathbb{F}G$-modules [Carlson83], then we adapt the result to $\mathbb{F}H$-$\underline{\mathrm{mod}}$ using the restriction functor and generalises it. We end up with a refinement similar to the one in [Chuang01, lemma 6]. Second, we need another piece of information which turns out to be a direct calculation of stable homomorphism group ($\underline{\mathrm{Hom}}$) of some modules, which generalises a lemma from Holloway [Holloway01].

Lemma 2.12. *For $0 \leq i \leq n-1$, suppose b_i, c_i ranges from 0 to $p-1$ and $j, \breve{j}$ are*

integers. Then the dimension of

$$\mathrm{Ext}^1_{\mathbb{F}H}(U_j M_b, U_{\breve{j}} M_c) \cong \underline{\mathrm{Hom}}_{\mathbb{F}H}(\Omega U_j M_b, U_{\breve{j}} M_c)$$

is determined by the number of n-tuples $(l, k_0, ..., k_{n-1})$ *of integers satisfying:*

$$b_l, c_l \leq p-2,$$

$$j - \breve{j} + \sum_{\substack{i=0 \\ i \neq l}}^{n-1} p^i(b_i - c_i + 2k_i) + p^l(-b_l - c_l + 2k_l - 2) \equiv 0 \pmod{p^n - 1} \tag{2.4}$$

with also

$$\max\{0, c_i - b_i\} \leq k_i \leq c_i$$

for $i \neq l$ *and*

$$\max\{0, b_l + c_l + 2 - p\} \leq k_l \leq \min\{b_l, c_l\}.$$

Proof The proof is similar to the proof in [Chuang01]. Consider the summation

$$\sum_{\substack{i=0 \\ i \neq l}}^{n-1} p^i(b_i - c_i + 2k_i) + p^l(-b_l - c_l + 2k_l - 2) \equiv 0 \pmod{p^n - 1}. \tag{2.5}$$

Adapt Carlson's theorem [Carlson83, Theorem 4.1] in Ext groups of simple $\mathbb{F}G$-modules. Fixing $r = 1$, it splits into two cases.

1. When p is odd, condition (1) forces $e_i = 0$ and $f_i = 0$ except for one f_i, record this subscript as l. Condition (3) gives the first constraint, and the condition 2.4 is a simplified version after substitution.

2. When $p = 2$, fixing $r = 1$ forces all but one e_i to be 0, again we record that subscript l, this forces $b_l = c_l = 0$. The requirements on k_i in our version is a precise replacement of (3') in [Carlson83]. Then, in order to see the last two equations here agree with the original, note we factored the first term $-2(2^l)$ (since $e_l = 1$) into the summand to yield the -2 in p^l term. With the fact that $b_l = 0$, the terms inside bracket of p^l are indeed equal.

Finally, the $j - \breve{j}$ term in (2.4) is introduced using a spectral sequence argument as in [Chuang01], and our proof is complete. □

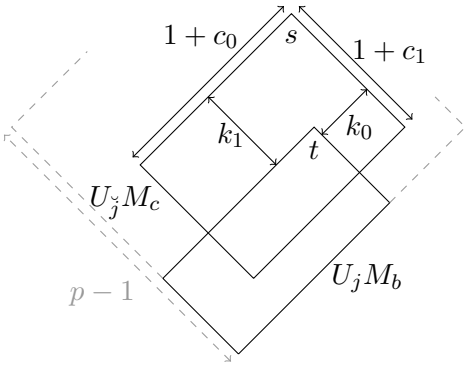
Lemma 2.13. *The dimension of the stable morphism group* $\underline{\mathrm{Hom}}_{\mathbb{F}H}(U_j M_b, U_{\breve{j}} M_c)$ *is equal to the number of n-tuples* $(k_0, ..., k_{n-1})$ *of integers satisfying:*

1. $\max\{0, c_i - b_i\} \leq k_i \leq c_i$ *for all* i.

2. *There exists an integer* l *such that* $k_l < b'_l$.

3.
$$j - \breve{j} + \sum_{i=0}^{n-1} p^i(b_i - c_i + 2k_i) \equiv 0 \pmod{p^n - 1} \tag{2.6}$$

Proof Considering the restriction of $\mathbb{F}G$-simple modules are of a special class of $\mathbb{F}H$-modules with the shape of hypercuboids. The $\mathbb{F}H$-modules M_b has irreducible top U_b and the length of its sides $(1 + b_0, 1 + b_1, ..., 1 + b_{n-1})$. All components within the cuboid is decided by its position (c.f. [Holloway01, pg.35]). Now consider $\underline{\mathrm{Hom}}_{\mathbb{B}}(U_j M_b, U_{\breve{j}} M_c)$, it has become a consideration of the head of $U_j M_b$'s position in $U_{\breve{j}} M_c$. A two-dimensional illustration (cuboid becomes rectangle) is shown here [Holloway01, figure 5.2]:



where $s = \breve{j} + c$ and $t = j + b$.

Condition (1) restricts the position of the modules such that $U_j M_b$ contains the socle of $U_{\breve{j}} M_c$. Condition (2) rules out the possibility of such a map factoring through the injective hull of $U_j M_b$. Shown by dashed line in the figure, if the injective hull, which is known to have size $(p - 1, ..., p - 1)$, covered $U_{\breve{j}} M_c$, the map factors through projectives (=injectives) hence quotiented out of $\underline{\mathrm{Hom}}(U_j M_b, U_{\breve{j}} M_c)$. Lastly condition (3) locate the head of $U_j M_b$ in the component of $U_{\breve{j}} M_c$. □

Remark. The proof is a generalisation of [Holloway01, Theorem 5.2.1 (2)].

These two lemmas build up arithmetic constraints for a certain type of extension. Now we would tailor the lemmas into two of particular situation. But first we have to decode the modulo equations (2.4) and (2.6) in both of the lemmas. Temporarily ignore the term $j - \breve{j}$ in (2.4) and (2.6) and regard p as an indeterminate. We define the p^i*-digit* to be the coefficient with the term p^i. The following two inequalities aim at looking at these p^i-digits.

Lemma 2.14. *With b_i, c_i, k_i, l, p as defined and restricted under lemma 2.12 and lemma 2.13, we have*

$$0 \leq |b_i - c_i| \leq b_i - c_i + 2k_i \leq b_i + c_i \leq 2p - 2 \tag{2.7}$$

$$-p \leq -b_l - c_l + 2k_l - 2 \leq -2. \tag{2.8}$$

Proof This statement is a technicality mentioned but not shown explicitly in [Carlson83, Theorem 4.1]. Without loss of generality assume $b_i \geq c_i$. For the inequality signs in (2.7), the first sign is obvious, second sign because $0 \leq k_i$, third sign because $k_i \leq c_i$ and fourth as $b_i, c_i \leq p - 1$. We turn to (2.8) and the maximum value is

$$-b_l - c_l + 2c_l - 2 = c_l - b_l - 2 \leq -2.$$

Considering the minimum value of (2.8), we split into two cases:

1. When $b_l + c_l \leq p - 2$ we have $-b_l - c_l - 2 \geq -p$
2. When $b_l + c_l > p - 2$ we have $-b_l - c_l + 2(b_l + c_l + 2 - p) - 2 = b_l + c_l + 2 - 2p \geq -p$.

Combining all the arguments gives the two inequalities. □

In the next part we will be defining some new symbols and terms that are needed to express clearly the upcoming results. These end up with two corollaries of Lemma 2.12 and Lemma 2.13, which show that the triangles (2.3) are exactly what are needed to verify our main theorem. The proof of the arguments are much like [Carlson83, Theorem 4.1] with extra consideration for subscripts of U expressed in the statement by the term $j - \breve{j}$.

Definition 2.15. *Recall $\mathbb{S}$ is the set of non-isomorphic simple $\mathbb{B}$-modules (Definition 2.15). Define sets*

$$I_i = \{S_a \mid a_{i+1} = ... = a_{n-1} = p - 1\}$$

for $0 \leq i \leq n - 1$ to be subsets of simple $\mathbb{B}$-modules.

Note that $I_{n-1} = \mathbb{S}$ since we have no restriction on S_a, and a filtration

$$I_\bullet = (\emptyset = I_{-1} \subset I_0 \subset I_1 \subset \ldots \subset I_{n-1} = \mathbb{S})$$

on the complete set of non-isomorphic simple $\mathbb{B}$-modules.

Definition 2.16. *Fix a prime p. We say a simple module S_a is* in layer *i if $S_a \in I_{i+1} - I_i$. We also define that an integer a and the $\mathbb{F}H$-module M_a are* in layer *i if S_a is.*

Remark. This is equivalent to saying a has base-p presentation $(a_0, ..., a_i, p-1, ..., p-1)$ with $a_i \neq p-1$, or by base-p arithmetic,

$$p^n - p^{i+1} \leq a \leq p^n - p^i - 1.$$

Definition 2.17. *Fix a prime p. Let a be an integer with $0 \leq a \leq q-2$ with base-p presentation $(a_0, ..., a_{n-1})$. Let a be an integer in the layer s. That is, we have*

$$a = (a_0, ..., a_s, p-1, ..., p-1) \qquad \textit{with} \qquad a_s \leq p-2.$$

The partner *of a, denoted a' (see notation below for clarification of use), is*

$$a(s') = (a_0, ..., a'_s, p-1, ..., p-1);$$

The completion *of a, denoted $\overline{a}$, is*

$$a(\overline{s}) = (a_0, ..., a_{s-1}, p-1, p-1, ..., p-1).$$

(c.f. Definition 2.16 and Definition 2.7)

Notation. *Recall that we defined $a'_i = p - s - a_i$. It will not contradict if we apply the following: If there is a subscript on the letter concerned, the prime treats it as a base-p digit i.e. $a'_i = p - 2 - a_i$. Otherwise it is treated as the partner of the integer defined just above, i.e. $a' = (a_0, ..., a'_i, ..., a_{n-1})$ for i the layer of a.*

Remark. The partner defined here turns out to be the correspondence of simple $\mathbb{B}$-modules used for our trick later. For odd primes, an even number is a partner of an odd number of the same layer, and vice versa. For $p = 2$, the partner of every integer is itself. We also point out that under this involution, the filtration in Definition 2.16 is fixed.

Definition 2.18. *For an integer m with $1 \leq m \leq p^{n-1}$, define r_m to be the integer such that p^{r_m} divides m with a p'-integer quotient. Let*

$$m = \sum_{i=0}^{n-1} p^i m_i = \sum_{i=r_m}^{n-1} p^i m_i$$

be its base-p expression. For an integer s, define

$$\lfloor m \rfloor_s := \sum_{i=s}^{n-1} p^i m_i,$$

the floor of m at s *and*

$$\lceil m \rceil^s = \sum_{i=s}^{n-1} p^i m_i + p^s,$$

the ceiling of m at s.

Remark. We hide the subscript m of r when it is obvious which m we are referring to. Note that for any m, the floor(resp. ceiling) of m at s is the nearest integer smaller(resp. greater) than or equal to m that is divisible by p^s.

Proposition 2.19. *Let m be an integer with $1 \leq m \leq p^{n-1}$. Let M_c be a module in layer i for some $i \leq r = r_m$, and M_b be a module in layer s with $s > r$.*

(a) If m and b satisfy $m_{s-1} + b_s < p - 1$, then

$$\underline{\mathrm{Hom}}_{\mathbb{F}H}(U_{\lfloor m \rfloor_s p} M_b, U_{mp} M_c) = 0;$$

(b) If m and b satisfy $m_{s-1} + b_s = p - 1$ and $m_{s-2} = ... = m_0 = 0$, then

$$\underline{\mathrm{Hom}}_{\mathbb{F}H}(U_{\lfloor m \rfloor_s p} M_b, U_{mp} M_c)$$

is of dimension 1 when $c = \overline{b}$. The corresponding unique non-split extension of $U_{mp} M_c$ by $U_{\lfloor m \rfloor_s p} M_b$ is represented by a distinguished triangle

$$U_{mp} \Omega M_c \to U_{\lceil m \rceil^s p} \Omega M_{b'} \to U_{\lfloor m \rfloor_s p} M_b \rightsquigarrow$$

in $\mathbb{F}H$ -$\underline{\mathrm{mod}}$.

Proof The condition (2.6) in Lemma 2.13 requires

$$\lfloor m \rfloor_s p - mp + \sum_{i=0}^{n-1} p^i (b_i - c_i + 2k_i) \equiv 0 \qquad (\mathrm{mod}\ p^n - 1)$$

for some k_i satisfying condition 1 in Lemma 2.13 and a particular k_l for condition 2 in Lemma 2.13 if a non-zero stable homomorphism exists. Note that

$$\lfloor m \rfloor_s p - mp = -\sum_{i=r}^{s-1} p^{i+1} m_i = -\sum_{i=r+1}^{s} p^i m_{i-1}.$$

Merging the term $mp - \lfloor m \rfloor_s p$ (the $j - \breve{j}$ term) into the last expression, we have that

$$\sum_{i=0}^{r} p^i(b_i - c_i + 2k_i) + \sum_{i=r+1}^{s} p^i(b_i - (p-1) + 2k_i - m_{i-1}) + \sum_{i=s+1}^{n-1} p^i(2k_i) \tag{2.9}$$

has to be divisible by $p^n - 1$. Now we consider the actual value of expression (2.9). From (2.7), the p^{r+1} to p^s digits lie between $-(p-1)$ and $2(p-1)$ and other digits lie between 0 and $2(p-1)$. Hence, for a possible extension to exist, (2.9) evaluates to one of the four following values: 0, $p^n - 1$, $2p^n - 2$ or $-p^n + 1$. Firstly, it cannot be $-(p^n - 1)$ because it requires (2.9) to have all terms at $-(p-1)$ (this is the smallest possible value of any p^i-digits in 2.9) but it must have a non-negative p^0 term. Secondly note that $c_l = p - 1$ will force $b'_l < p - 1 - b_l \leq k_l$. So, the range of l mentioned in condition 2 of theorem 2.13 is restricted to $0 \leq l \leq r$. For this particular l, we have

$$b_l - c_l + 2k_l \leq b_l - c_l + b'_l + c_l = p - 2.$$

This rules out the possibility for (2.9) to be $2p^n - 2$. Thirdly, the last inequality indicates the sum up to p^l-term:

$$\sum_{i=0}^{l} p^i(b_i - c_i + 2k_i)$$

(since l cannot be greater than r as $b_l, c_l \leq p - 2$) lies between 0 and $p^{l+1} - 2$, which can never be -1 modulo p^{l+1}. Adding up the remaining terms of (2.9) will not change this. However, if the expression (2.9) is equal to $p^n - 1$, we have the expression equal to -1 modulo p^{l+1}, creating a contradiction. Thus, the argument above boils down to the conclusion that (2.9) is zero. Now we split into the following two cases:

(a) If the condition $m_{s-1} + b_s < p - 1$ holds, it forces the value of the p^s-digit to be at least 1. Since the sum

$$\sum_{i=0}^{r} p^i(b_i - c_i + 2k_i) + \sum_{i=r+1}^{s-1} p^i(b_i - (p-1) + 2k_i - m_{i-1}) \tag{2.10}$$

must be greater than $-(p^s - p)$, adding the next term p^s will make the subtotal

$$\sum_{i=0}^{r} p^i(b_i - c_i + 2k_i) + \sum_{i=r+1}^{s} p^i(b_i - (p-1) + 2k_i - m_{i-1}) \tag{2.11}$$

strictly greater than zero. With other p^i-digits ($i > s$) non-negative, we conclude that it cannot be zero hence the dimension of $\underline{\mathrm{Hom}}_{\mathbb{F}H}(U_{\lfloor m \rfloor_s p} M_b, U_{mp} M_c)$ is zero.

(b) Note that the other condition requires $m_{i-1} = 0$ on every digit except the p^s-digit. Hence from (2.7) each digit has to be non-negative. Furthermore, p^s-digit is non-negative too, so every digit has to be zero. We can conclude that $b_i = c_i$ for every i by (2.7) (with $k_i = 0$) except when $i = s$, in which case we have $b_s = p-1-m_{s-1}$. Note that the condition 2 in lemma 2.13 is automatically satisfied by k_r in this case, which is shown by the fact $c_r \leq p-2$ in the definition. Thus, we have $b'_r = p-2-c_r \geq 0 = k_r$.

Now we consider the only possible non-trivial extension. The conditions require c such that $c_i = b_i$ for $0 \leq i \leq n-1$ except when $i = s$ and $c_s = p-1$ as given by (b), which means that $c = \overline{b}$. Now put $i = s$, tensor the sequence (2.3) by $U_{\lfloor m \rfloor_s p}$, it induces the triangle

$$U_{p^s(1+b'_s)}U_{\lfloor m \rfloor_s p}\Omega M_{\overline{b}} \to U_{p^s}U_{\lfloor m \rfloor_s p}\Omega M_{b'} \to U_{\lfloor m \rfloor_s p}M_b \rightsquigarrow .$$

The indices of the two U_i's in the middle term add up to the ceiling of m by s. Now by assumption, $b_s = p-1-m_{s-1}$, so

$$p^s(1+b'_s) + \lfloor m \rfloor_s p = p^s m_{s-1} + (m - m_{s-1}p^{s-1})p = mp.$$

The previous triangle becomes

$$U_{mp}\Omega M_{\overline{b}} \to U_{\lceil m \rceil^{r+1} p}\Omega M_{b'} \to U_{\lfloor m \rfloor_s p}M_b \rightsquigarrow .$$

□

Proposition 2.20. *Let m be an integer, $1 \leq m \leq p^{n-1}$. Let M_c be a module in layer i with $i \leq r = r_m$, and M_b in layer s with $s > r$ such that $m_{s-1} + b_s \geq p-1$. Then*

$$\underline{\mathrm{Hom}}_{\mathbb{F}H}(U_{\lceil m \rceil^s p}\Omega M_{b'}, U_{mp}M_c) = 0.$$

Proof Applying our assumption to Theorem 2.12, the condition requires

$$\lceil m \rceil^s p - mp + \sum_{\substack{i=0 \\ i \neq l}}^{n-1} p^i(b_i - c_i + 2k_i) + p^l(-b_l - c_l + 2k_l - 2) \tag{2.12}$$

to be divisible by $p^n - 1$ for an l with $0 \leq l \leq r < s$, by condition 1 of 2.12. We are going to show that there is no solution, by considering all the possible values. We have

$$\lceil m \rceil^s p - mp + p^s(b'_s - (p-1) + 2k_s) - p^l(-b_l - c_l + 2k_l - 2) \geq p^{r+1} + p^s - p^r(p) > 0,$$

so the expression (2.12) must be greater than zero.

Note that $b'_s - m_{s-1} \leq -1$. So the maximum value of the expression is

$$p^{s+1} + p^s(b'_s + p - 1 - m_{s-1}) + \sum_{\substack{i=0 \\ i \neq l,s}}^{n-1} 2(p-1)p^i - 2p^l < \sum_{i=0}^{n-1} 2(p-1)p^i = 2p^n - 2.$$

The only remaining possibility is that the expression (2.12) is equal to $p^n - 1$. Similar to proposition 2.19 we consider the partial sum of the expression up to the p^l-digit inclusive. In view of the inequality on p^l-digits in (2.8), it should lie between $-p^{l+1}$ and -2. However, if the whole expression is equal to $p^n - 1$ it should have remainder -1 modulo p^{l+1}, a contradiction. □

Chapter 3

Main construction and proof

The non-trivial perverse autoequivalence suggested at the start of the thesis will be described and proved in this chapter. Then we shall discuss some consequences of this autoequivalence. We shall approach this by constructing a string of algebras such that

- all of these algebras have equivalent derived categories (hence also their associated stable module categories),
- the derived equivalences between successive algebras are elementary perverse, and
- the last one is Morita equivalent to the first one

to give the aforementioned perverse autoequivalence. In this chapter we shall first show the autoequivalence is a composition of elementary perverse equivalences, while we shall prove it is itself a perverse equivalence in chapter 4.

We have defined the following notation: Let p be a prime number and n be a natural number, $q = p^n$. $\mathbb{F}_q$ is the (unique) finite field of q elements, $\mathbb{F} = \overline{\mathbb{F}_q}$ the algebraic closure of $\mathbb{F}_q$, which is a field of characteristic p. The group $G = SL_2(q)$ is the special linear group of degree two over $\mathbb{F}_q$. Denote the normaliser of a Sylow p-subgroup in G by H, which is also a Borel subgroup of G. The direct sum of all full defect blocks of $\mathbb{F}G$ is denoted by $\mathbb{B}$ and $\mathbb{F}H$ is the direct sum of local Brauer correspondents of the constituents of $\mathbb{B}$. Let S_a for $0 \leq a \leq q-1$ be the non-isomorphic simple $\mathbb{F}G$-modules and M_a their restrictions to H. Let U_i for $0 \leq i \leq q-2$ be representatives of non-isomorphic $\mathbb{F}H$ simples, all of which are one-dimensional. We also have endo-functors of $\mathbb{F}H$-mod: $U_i \otimes -$ for $0 \leq i \leq q-1$ and we abbreviated them as U_i for convenience. These functors induce non-trivial self-Morita equivalences. Recall base-p notation (Definition 2.1) is used to represent natural numbers. Simple $\mathbb{F}G$-modules have been indexed by base-p n-tuples using the Steinberg tensor product theorem (Theorem

2.1) and a filtration is defined on the set of simple modules (Definition 2.15). We have defined a partner $a \leftrightarrow a'$ between integers from 0 to $q-2$ (Definition 2.17) which is an involution mapping odd to even numbers (and vice versa) when $p \neq 2$ and is the identity map on natural numbers when $p = 2$. Also we have set up floor(resp. ceiling) at s for an integer as the nearest integer below(resp. above) it divisible by p^s (definition 2.18).

3.1 Construction and proof

First we define our successive elementary perverse tilts of algebras (see Example 1.80). We then explore their induced equivalences in their stable module categories to prove our main theorem.

Definition 3.1. *Define inductively a string of algebras $\mathbb{A}_m$, $1 \leq m \leq p^{n-1}$, and a bijection β_m of the complete set of non-isomorphic simple $\mathbb{A}_m$-modules, $\mathbb{S}_m$, to the complete set of non-isomorphic simple $\mathbb{B}$-modules, $\mathbb{S}$, by the following.*

First, define $\mathbb{A}_0 = \mathbb{B}$ with the identical bijection $\beta_0 := \mathbb{S} \to \mathbb{S}_0$. Suppose $\mathbb{A}_{m-1}$ and β_{m-1} is already defined, let $\mathbb{A}_m$ be a symmetric algebra such that $\mathbb{A}_m$ is a perverse tilt from $\mathbb{A}_{m-1}$ with derived equivalence

$$F_m : D^b(\mathbb{A}_{m-1}) \to D^b(\mathbb{A}_m)$$

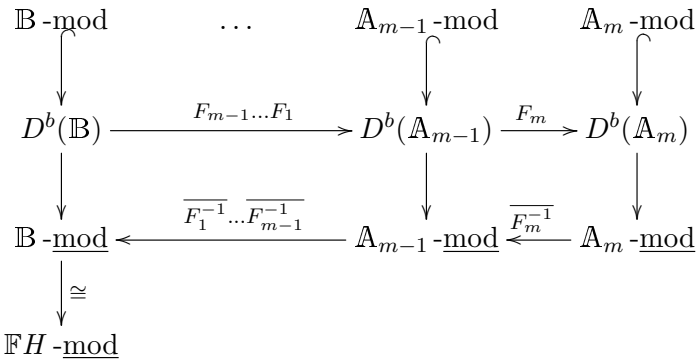
perverse relative to

$$(\beta_{m-1}(0 \subset I_{r_m} \subset \mathbb{S}), \varepsilon : p(0) = 1; p(1) = 0).$$

Such algebras $\mathbb{A}_m$ are symmetric [Rickard89] and defined up to Morita equivalence by Proposition 1.75. Now we also define a bijection $\beta_m : \mathbb{S} \to \mathbb{S}_m$ via $\beta_{F_m}\beta_{m-1}$, the composition of the earlier induced bijection and the bijection of simple modules required in the perverse equivalence (Lemma 1.77). We also transfer the numbering of simple modules from $\mathbb{S}$ to $\mathbb{S}_m$.

We also define specially $\mathbb{A}_{p^{n-1}} = \mathbb{A}$ and $F : D^b(\mathbb{B}) \to D^b(\mathbb{A})$ as the composition of F_m for $1 \leq m \leq p^{n-1}$.

We say *execute step m* when we apply functor F_m on the derived categories of $\mathbb{A}_{m-1}$ and $\mathbb{A}_m$. The above construction and our main proposition later can be illustrated by the following diagram.



Referring to the above plan, we study the simple modules of the new algebra $\mathbb{A}_m$ from an inductive approach from the previously defined algebras. The idea is to describe the image of simple $\mathbb{A}_m$-modules in $\mathbb{F}H$ -mod. More concretely let M_a^m be the image of simple $\mathbb{A}_m$-modules T_a in the stable module category expressed as $\mathbb{F}H$ -mod. Now we can describe M_a^m using induction from M_a^{m-1}, with the rules introduced in Example 1.80.

It turns out the terms and extensions are controllable and the result is being summarized into the proposition below. In the following proposition and lemma when we say a module we mean an $\mathbb{F}H$-module.

Proposition 3.2. *Fix a number m between 0 and p^{n-1}. The set of all $\mathbb{F}H$-modules M_b^m, the correspondents of simple $\mathbb{A}_m$-module T_b in $\mathbb{F}H$ -mod for $0 \leq b \leq q-2$, can be partitioned into three sets*

$$J_m \cup K_m \cup L_m$$

such that, depending on parity of $k_s = \lfloor m \rfloor_s / p^s$ (of m),

1. *J_m consists of M_b^m in layer $s \leq r_m$. The module $M_b^m \in J_m$ is isomorphic to*

$$\begin{cases} U_{mp} M_b & \text{if } k_s \text{ is even.} \\ U_{mp} M_{b'} & \text{if } k_s \text{ is odd.} \end{cases}$$

2. *K_m consists of M_b^m of layer $s > r_m$, with*

$$\begin{cases} m_{s-1} + b_s \geq p-1 & \text{if } k_s \text{ is even.} \\ m_{s-1} + b'_s \geq p-1 & \text{if } k_s \text{ is odd.} \end{cases}$$

The module $M_b^m \in K_m$ is isomorphic to

$$\begin{cases} U_{\lceil m \rceil^s p} \Omega M_{b'} & \text{if } k_s \text{ is even.} \\ U_{\lceil m \rceil^s p} \Omega M_b & \text{if } k_s \text{ is odd.} \end{cases}$$

3. L_m consists of the remaining modules, that is, those with b of layer $s > r_m$ with

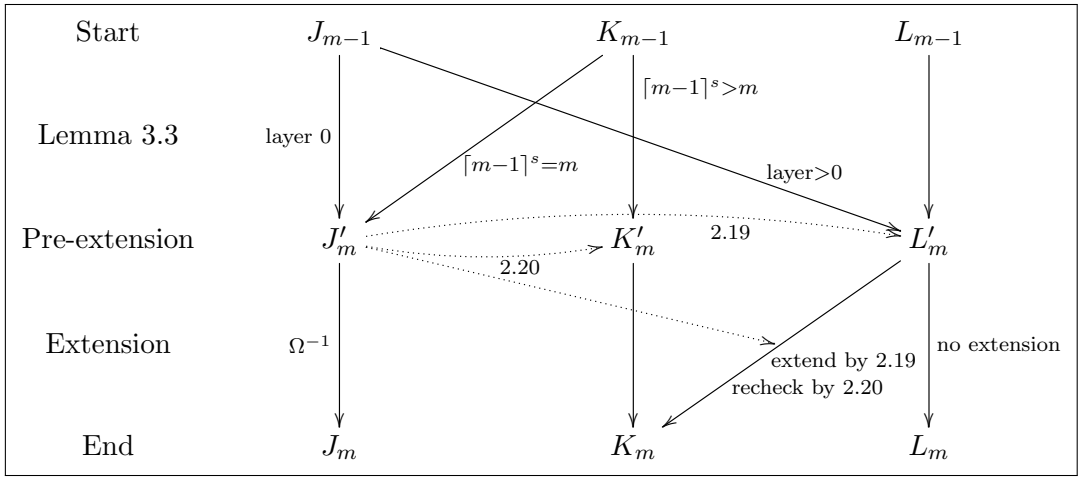
$$\begin{cases} m_{s-1} + b_s < p - 1 & \text{if } k_s \text{ is even.} \\ m_{s-1} + b'_s < p - 1 & \text{if } k_s \text{ is odd.} \end{cases}$$

The module $M_b^m \in L_m$ is isomorphic to

$$\begin{cases} U_{\lfloor m \rfloor_s p} M_b & \text{if } k_s \text{ is even.} \\ U_{\lfloor m \rfloor_s p} M_{b'} & \text{if } k_s \text{ is odd.} \end{cases}$$

Remark. We can check this is indeed a partition by considering modules in layers. J_m contains every module of layer $\leq r_m$, K_m and L_m splits modules in layers $> r_m$. Note that for $p = 2$ the statements are the same disregarding parity of k_s since $b = b'$ and $b_s = b'_s = 0$.

We will prove this by induction. It is a two-step approach for each inductive step on m. The scheme of the two steps approach is illustrated via the following diagram:



First, we have to rewrite the partition. Note that since the partition (based primarily on the parameter r) depends on m and subscripts such as $\lceil m \rceil^s$ and $\lfloor m \rfloor_s$ are involved, the induction assumption (from $m - 1$) is not in a very usable form for m.

Hence the first job is to rewrite it into a new partition (with respect to m)

$$\{M_b^{m-1} \mid 0 \leq b \leq q-2\} = J'_m \cup K'_m \cup L'_m$$

such that J'_m corresponds to simple $\mathbb{A}_m$-modules that makes up the foundation of the perverse equivalence F_m. The first step is concluded in the following lemma:

Lemma 3.3. *The set of M_a^{m-1}, rewriting in the perspective of m and k_s of m (instead of $m-1$), reorganised from the partition in Proposition 3.2, is partitioned into*

1. *J'_m consists of M_b^{m-1} with the layer of $b \leq r_m$. M_b^{m-1} is isomorphic to*

$$\begin{cases} U_{mp}\Omega M_b & \text{if } k_s(m) \text{ is even.} \\ U_{mp}\Omega M_{b'} & \text{if } k_s(m) \text{ is odd.} \end{cases}$$

2. *K'_m consists of M_b^{m-1} of layer $s > r_m$, with*

$$\begin{cases} m_{s-1} + b_s \geq p-1 & \text{if } k_s \text{ is even.} \\ m_{s-1} + b'_s \geq p-1 & \text{if } k_s \text{ is odd.} \end{cases}$$

   M_b^{m-1} is isomorphic to

$$\begin{cases} U_{\lceil m \rceil^s p}\Omega M_{b'} & \text{if } k_s \text{ is even.} \\ U_{\lceil m \rceil^s p}\Omega M_b & \text{if } k_s \text{ is odd.} \end{cases}$$

3. *L'_m consists of the remaining modules, that is, those with b of layer $s > r_m$ with*

$$\begin{cases} m_{s-1} + b_s < p-1 \text{ or } (m_{s-1} + b_s = p-1 \text{ and } m_i = 0 \text{ for } 0 \leq i \leq s-2) & \text{if } k_s \text{ is even.} \\ m_{s-1} + b'_s < p-1 \text{ or } (m_{s-1} + b'_s = p-1 \text{ and } m_i = 0 \text{ for } 0 \leq i \leq s-2) & \text{if } k_s \text{ is odd.} \end{cases}$$

   M_b^{m-1} is isomorphic to

$$\begin{cases} U_{\lfloor m \rfloor_s p} M_b \text{ if } k_s \text{ is even.} \\ U_{\lfloor m \rfloor_s p} M_{b'} \text{ if } k_s \text{ is odd.} \end{cases}$$

Before we prove these, we need a proposition concerning counting in base-p numbers.

Proposition 3.4. *Let m be an integer, $1 \leq m \leq p^{n-1} - 1$, m_i be the i^{th} digit of its base-p presentation. The following must hold for a certain natural number r with $0 \leq r \leq n-2$. (It will be the r_m in Definition 2.18.)*

1. *m/p^r is a p'-integer.*

2. *$0 < m_r = (m-1)_r + 1 < p$.*

3. *m is divisible by p^s for all integers s with $0 \leq s \leq r$.*

4. *$0 = m_s < (m-1)_s = p - 1$ for all s with $0 \leq s < r$.*

Proof This is the nature of counting (adding 1) by base-p numbers, especially when carrying happens on some digits. In particular, r indicates the first place without a carry when $m - 1$ is added by 1. □

Any perspective viewing the parameters in proving Lemma 3.3 and Proposition 3.2 are included in this proposition. It is worth noting that we only introduce the parameter k_s for precision of statements. One might refer to Table A.4 and A.5 to reference how these work in real terms. It is easier to understand the argument with the tables. In the following proofs, we shall only argue for the statement starting with k_s even. It is easy to see similarly for odd k_s by exchanging $b \leftrightarrow b'$.

Proof of Lemma 3.3 The idea of this proof is to construct J'_m from the (new) layer constraint and K'_m and L'_m according further to the inherited format as $\mathbb{F}H$-modules.

- Consider J'_m: since $r_m \geq 0$, M_b^{m-1} with b of layer 0 always belongs to J'_m, thus their $\mathbb{F}H$-mod correspondents, using 2.11, can be written as

$$U_{(m-1)p} M_b \cong U_{mp} \Omega M_{b'}.$$

  Note that k_0 changes parity from $m-1$ to m. If $r_m = 0$, then we have already found the whole set J'_m. If $r_m > 0$, we have $r_{m-1} = 0$ and $(m-1)_{s-1} = p - 1$ for $0 < s \leq r_m$. Hence for b of layer s, all of M_b^{m-1} is in K_{m-1}. From the condition we have also

$$U_{\lceil m-1 \rceil^s p} \Omega M_{b'} \cong U_{mp} \Omega M_{b'}$$

  since $\lceil m-1 \rceil^s = m$ for $0 < s \leq r_m$. Note that k_s is again of different parity since the floor function is differed by p^s.

Combining both, now we have grouped into J'_m modules of form M_b^{m-1} of layer $s < r_m$ and isomorphic to $U_{mp}\Omega M_{b'}$ with k_s odd.

- Now we consider the set $K'_m \cup L'_m$, consisting of M_b^{m-1} with layer of b greater than r_m. We have to consider all 3 sources from the $(m-1)^{th}$ statement. First we translate the expressions and conditions to direct terms of m (so to avoid digits of $m-1$ in the subscript).

  - First we form the set K'_m, which we consider to include all modules of M_b^{m-1} maintaining a ceiling U-subscript. They must come only from K_{m-1} since by increasing m neither J_{m-1} nor L_{m-1} can contribute a ceiling U-subscript. Note that the formation of the set J'_m takes away all modules of layer $s \leq r_m$ from K_{m-1}. They are precisely those with $\lceil m-1 \rceil^s = m$. Note further that $\lceil m-1 \rceil^s$ cannot be $m-1$ since this will force a contradiction with its own condition[1]. Thus every ceiling U-subscript from the set $K_{m-1} \backslash J'_m$ has to stay and we take $K'_{m-1} = K_{m-1} \backslash J'_m$. Thus modules in K'_m have the form

$$U_{\lceil m \rceil^s p}\Omega M_{b'} \qquad \text{if } m_{s-1} + b_s \geq p-1. \tag{3.1}$$

    The last condition can be switched from $m-1$ to m directly because of the proposition 3.4: The only case where they differ is $m_{s-1} < (m-1)_s - 1$, which is equivalent to p^s divides m and those modules are moved away from K_{m-1} to J'_m.

  - Now we consider the remaining set L'_m, which consists of modules with a floor U-subscript. Since m is only increased by 1, modules in K_{m-1} will not be rewritten[2] into L'_m. First we consider modules coming from L_{m-1}. To rewrite the U-subscript from $m-1$ into m we need to consider the case $\lfloor m-1 \rfloor_s \neq \lfloor m \rfloor_s$. However this only happens when m is divisible by p^s hence $(m-1)_{s-1} = p-1$ thus this case is not included in L_{m-1}. We can safely change the subscript $\lfloor m-1 \rfloor_s$ from modules in L_{m-1} to $\lfloor m \rfloor_s$. Secondly we consider modules coming from J_{m-1}. If there are such modules then $r_{m-1} > s > 0$ and hence $r_m = 0$. Their expression $U_{(m-1)p}M_b$, translate to m is equal to

$$U_{\lfloor m \rfloor_s p}M_b, \text{ with } (m-1)_{s-1} + b_s = b_s < p-1.$$

[1] $(m-1)_{s-1} = 0$; $b_s \leq p-2$ forces $(m-1)_{s-1} + b_s < p-1$

[2] $\lceil m-1 \rceil^s$ cannot be $m-1$ and if $\lceil m-1 \rceil^s = m$ it has been assigned to J'_m.

This expression coincides with those coming from L_{m-1}. In conclusion we have the modules in L'_m isomorphic to

$$U_{\lfloor m \rfloor_s p} M_b \text{ with } (m-1)_{s-1} + b_s < p-1. \tag{3.2}$$

Translating $(m-1)_{s-1} + b_s < p-1$ in terms of m (instead of $m-1$) cause it to split into two:

$$m_{s-1} + b_s < p-1 \qquad \text{or} \qquad m_{s-1} + b_s = p-1 \text{ and } m_{s-2} = ... = m_0 = 0. \tag{3.3}$$

It is easy to check that k_s does not change when $r_m < s$. By these arguments we have successfully re-partitioned M_b^{m-1} as indicated in the lemma. □

Of course, the re-partition in Lemma 3.3 is tailored such that we can apply the correspondence in perverse equivalence in a fairly convenient manner. The set J'_m corresponds to the foundation of the perverse equivalence F_m. K'_m and L'_m is grouped by expression and needed to check for extensions.

Proof of Proposition 3.2 Now we start the main proof by considering the algebra $\mathbb{A}_0 = \mathbb{B}$. The module M_b^0 is simply M_b and the set of such modules is partitioned as $J_0 \cup K_0 \cup L_0 = \mathbb{S} \cup \emptyset \cup \emptyset$. Thus the statement is true at $m = 0$ (assuming large enough r_0), which allows us to start the induction for $m \geq 1$. Now assume the statement is true for an $m-1$. Let $S_b \in \mathbb{S}_{m-1}$ be a simple $\mathbb{A}_{m-1}$-module corresponding to a simple $\mathbb{A}_m$-module $T_b \in \mathbb{S}_m$. The induction step requires us to find the image of T_b in $\mathbb{F}H$ -$\underline{\text{mod}}$ via M_b^{m-1}. Using the stable category equivalent of Example 1.79, we have:

$$\overline{F_m^{-1}}(T) = \begin{cases} \Omega^{-1} S & \text{if } \beta_m(T) \in I_{r_m} \\ S_{\mathbb{S}'} & \text{otherwise.} \end{cases}$$

- For $\beta_m(T_b) \in I_{r_m}$ is equivalent to say b is of layer between 0 and r_m inclusive. The corresponding $\mathbb{F}H$-module of S_b is in the set J'_m. Then T_b corresponds to

$$\Omega^{-1}(U_{mp} \Omega M_b) \cong U_{mp} M_b$$

  in $\mathbb{F}H$ -$\underline{\text{mod}}$ which proves the statement for the set J_m.

- Before we argue on the modules that require us to check for extensions, note that J'_m includes all modules of layer $\leq r_m$, hence we disregard b or b' when considering

extensions. Further see that we can replace b by b' in Proposition 2.19 and 2.20 thus they always apply to L'_m and K'_m regardless of the parity of k_s.

- For the remaining T_b's such that $\beta_m(T_b) \notin I_{r_m}$, we need to find the universal extension of S_b by the set $\mathbb{S}'$, where $\beta_{m-1}(\mathbb{S}') = I_{r_m}$, which is equivalent to consider the universal extension of any element in $K'_m \cup L'_m$ by J'_m in $\mathbb{F}H$ -$\underline{\text{mod}}$. For any module in L'_m we first use Proposition 2.19 to check the required extension. Only modules corresponding to (3.2) satisfying the last condition in (3.3) have one-dimensional extensions. The module after extension is isomorphic in $\mathbb{F}H$ -$\underline{\text{mod}}$ to

$$U_{\lceil m \rceil^s p} \Omega M_{b'} \tag{3.4}$$

according to triangle (2.3). The rest of the modules in L'_m satisfying the first condition in (3.3) are not extendible, so have their corresponding expression remains the same. With this, we show Proposition 3.2 is true for L_m. Now note that (3.4) has exactly the same expression as those in K'_m, see (3.1). Their respective condition can be joined up perfectly as $m_{s-1} + b_s \geq p - 1$, exactly what the m^{th} proposition statement and Proposition 2.20 required. Now Proposition 2.20 has shown that all modules have no more available extensions. This has formed the required K_m part of the partition of the induction statement.

Thus we have successfully show that the statement for m is true for all three parts of the partition. Hence it is true by induction up to $\mathbb{A}_{p^{n-1}}$. □

Corollary 3.5. *The image of simples S_a of $\mathbb{A} = \mathbb{A}_{p^{n-1}}$ in $\mathbb{F}H$ -$\underline{\text{mod}}$ is $U_q M_{a'} \cong U_1 M_{a'}$.*

Proof By the induction statement all modules correspond to $J_{p^{n-1}}$ for $\mathbb{A}$. So all the simple modules correspond to $\mathbb{F}H$-modules $U_q M'_a$, since $k_s = p^{n-1-s}$ is an odd number for odd primes. When $p = 2$, $a = a'$ so it makes no discrimination. □

Now our desired result is imminent.

Theorem 3.6. *There is an derived autoequivalence of the direct sum of all full defect blocks $\mathbb{B}$ of $\mathbb{F}G$ exchanges the principal block with the non-principal block for odd primes. The derived autoequivalence can be realised by*

$$F : D^b(\mathbb{B}) \to D^b(\mathbb{B})$$

which is the composition of elementary perverse tiltings F_m with filtration

$$(\emptyset \subset \mathcal{I}_{r_m} \subset \mathbb{S})$$

and perversity function $\varepsilon(0) = -1, \varepsilon(1) = 0$ *for* $1 \leq m \leq p^{n-1}$.

Remark. Since we are going to prove $\mathbb{A}$ and $\mathbb{B}$ are Morita equivalent, $F : D^b(\mathbb{B}) \to D^b(\mathbb{A})$ will be consider as an equivalence

$$F : D^b(\mathbb{B}) \to D^b(\mathbb{B}).$$

Proof This theorem is immediate after we show that $\mathbb{A}$ is Morita equivalent with $\mathbb{B}$. Now consider the image of simple $\mathbb{A}$-module in its stable category,

$$\mathbb{A}\text{-}\underline{\text{mod}} = = = \mathbb{F}H\text{-}\underline{\text{mod}} \xrightarrow[=]{U_{-1}\otimes -} \mathbb{F}H\text{-}\underline{\text{mod}} \xrightarrow[=]{\text{Ind}} \mathbb{B}\text{-}\underline{\text{mod}}$$

$$T_a = = = = = = = U_1 M_{a'} \longmapsto \longrightarrow M_{a'} \longmapsto \longrightarrow S_{a'}$$

Note that both the functor $U_{-1} \otimes -$ and induction (the functor is ${}_{\mathbb{B}}\mathbb{B}_{\mathbb{F}H} \otimes_{\mathbb{F}H} -$) are stable equivalences of Morita type, and their composition maps simple $\mathbb{A}$-modules to simple $\mathbb{B}$-modules. Thus using Theorem 1.42 we conclude that $\mathbb{A}$ is Morita equivalent to $\mathbb{B}$. This has proved the first half of the theorem. For the second half consider the principal block $\mathbb{B}_0$ through the perverse tilts yields one of the blocks in $\mathbb{B}$. Similar to the proof of theorem above, note that corollary 3.5 indicate that the even-numbered simple $\mathbb{B}$-modules have stable images that are odd-numbered simple $\mathbb{B}$-modules. Thus the block obtained by tilting $\mathbb{B}_0$ using 3.1 should be $\mathbb{B}_1$. □

The following corollary is also obvious.

Corollary 3.7. *For an odd prime p, the principal block and non-principal block of $\mathbb{F}\,SL_2(q)$ are derived equivalent.*

This construction can also be seen as a generalisation of Morita equivalence for the two full defect blocks in $\mathbb{F}\,SL_2(p)$. For odd prime p, the two blocks are known to be Morita equivalent as Brauer Tree algebras.

Example 3.8. *When $n = 1$, there are no layered structure of simples (all belongs to the same layer). Our construction suggests a global shift of* [1]*, which by properties of perverse equivalence is a Morita equivalence. Or to see it directly, apply Linckelmann's*

theorem (1.42) to the functor Ω^{-1}. The pairing 2.17 agrees with the result given by Brauer tree algebras.

Remark. The last step of our construction ($m = p^{n-1}$) is combinatorically viable but actually redundant in the sense that it is always a global shift. So $\mathbb{A}_{p^{n-1}-1}$, $\mathbb{A}_{p^{n-1}}$ and $\mathbb{B}$ are Morita equivalent. Sometimes we might want to use the reduced version (without the last step). We define $F' = F_{p^{n-1}-1} \circ \cdots \circ F_1$. Unless otherwise stated, our discussion is on F (the full version).

The smallest non-trivial example is $SL_2(2^2)$. See the dedicated chapter later. There is also an appendix of further examples using (and explaining) a more intuitive approach of the construction.

3.2 Remarks

This construction has some intriguing properties that can be explored. The most extensive one would be considering this composition of elementary perverse equivalences. This we shall leave until the next chapter to better introduce tools and techniques in perverse equivalences to handle these problems. In this section, we explore how the construction fits into current knowledge.

3.2.1 Relations with other known constructions

The first thing worth discussing is whether this construction is the lift of U_1 in derived category. This is one of the motivation we start our construction. From the proof, F is being constructed, or precisely we obtain $\mathbb{A}$ from $\mathbb{B}$ via the functor $U_1 \otimes -$ in stable category. Hence it is natural to ask whether we can find an equivalence $Y : D^b(\mathbb{B}) \to D^b(\mathbb{b})$ such that

$$\begin{array}{ccc} D^b(\mathbb{B}) & \xrightarrow{F} & D^b(\mathbb{B}) \\ \updownarrow{\scriptstyle \sim Y} & & \updownarrow{\scriptstyle \sim Y} \\ D^b(\mathbb{b}) & \xrightarrow{U_1} & D^b(\mathbb{b}) \end{array}$$

commutes. For that, we have a negative answer.

Proof If the picture does commute, by iterating the construction $q-1$ times,

$$\begin{array}{ccccccc} D^b(\mathbb{B}) & \xrightarrow{F} & D^b(\mathbb{B}) & \xrightarrow{F} & \dots & \xrightarrow{F} & D^b(\mathbb{B}) \\ \updownarrow{\scriptstyle \sim Y} & & \updownarrow{\scriptstyle \sim Y} & & & & \updownarrow{\scriptstyle \sim Y} \\ D^b(\mathbb{b}) & \xrightarrow{U_1} & D^b(\mathbb{b}) & \xrightarrow{U_1} & \dots & \xrightarrow{U_1} & D^b(\mathbb{b}) \end{array}$$

and this amounts to

$$\begin{array}{ccc} D^b(\mathbb{B}) & \xrightarrow{F^{q-1}} & D^b(\mathbb{B}) \\ \updownarrow{\scriptstyle \sim Y} & & \updownarrow{\scriptstyle \sim Y} \\ D^b(\mathbb{b}) & \xrightarrow{U_{q-1}=U_0} & D^b(\mathbb{b}) \end{array}$$

which $U_0 \otimes -$ is the identity functor. Consider F^q cannot be identity as the perversity function is non-zero (we shall prove F^q is a perverse equivalence in chapter 4), we obtain a contradiction. □

This has shown that there is no canonical perverse equivalence effectively lifting the functor U_1 from local derived category, because such perverse equivalence cannot have finite order. We will show later for $SL_2(4)$ it is a lift of U_1 with addition of two spherical twists.

3.2.2 Role of the Steinberg block

Consider from the perspective that we use extensively the Steinberg module (although as zero module) in our proof, one might wonder what is the role the Steinberg module would play in all these constructions. It is the very fact that it is being used as zero module to obtain an unified expression that makes its role interesting yet hard to identify in the construction. Furthermore, the restriction of the Steinberg module in $\mathbb{F}H$ is projective, and it is not considered in Green or Brauer correspondence, but it comes up naturally in the triangles in our exposition as zeroes.

Example 3.9. *Consider the general construction on $\mathbb{B}$, there is exactly one module extended by the Steinberg module in each step, except the last general shift. To a certain extent, one can even regard the last step has a zero extended by zero. In particular for $SL_2(4)$, we have use in step 1, the triangle $(St = 0 \to cone(V \to 0)[-1] \to V \rightsquigarrow)$ on V.*

We have, in the process, always regarded it as zero due to its projective nature. However on the other hand, it is not hard to seek a generalisation of our construction by guessing its perversity. One might take the Steinberg module having perversity p^n.

Or one might regard Steinberg module having infinite perversity - from the fact that we lost the correspondence in local categories, or by the fact that its base-p subscript is an infinite cycle of $p-1$'s - a view better observed in next chapter. It is tempting to include it into the construction due to the fact that we might almost complete the filtrations and posets (also explained in next chapter) by introducing it as the minimal element. However we leave it out at this moment since the real consequence is not known and there is ambiguity in defining such autoequivalences including simple projective modules.

Chapter 4

Composition of the construction

To further explore the construction, we need to consider the composition of elementary perverse tilts for F_m defined in 3.1.

While it is not easy to decide in general whether the composition of two (or more) perverse equivalences is a perverse equivalence, for the composition of the equivalences F_m for $1 \leq m \leq p^{n-1}$ it is quite easy. We notice that the filtrations of these elementary perverse equivalences form a chain of nested sets. That will show our construction F is, in essence, one perverse equivalence altogether.

Theorem. *There exists an equivalence $F : D^b(\mathbb{B}) \to D^b(\mathbb{B})$ perverse with respect to $(I_\bullet, I_\bullet, \pi)$ where*

$$I_\bullet = (\emptyset = I_{-1} \subset I_0 \subset \cdots \subset I_{n-1} = \mathbb{S})$$

and I_r is defined as 2.15. The perversity function is

$$\pi : i \mapsto -p^{n-1-i}.$$

Remark. The correspondence of simple is $b \leftrightarrow b'$ but the filtrations remains the same on both sides.

Proof Using refineability of Proposition 1.76, we can always refine the filtration defined by F_m to the one in Definition 2.15. By composability from the same proposition we just need to add up the perversity function according to the new filtration, which amounts to counting for i the number of integers between 1 and p^{n-1} divisible by p^i (c.f. the definition of F_m). □

In order to handle further compositions, we will introduce the notion of poset perverse equivalence. Mentioned by Chuang-Rouquier in [Chuang, Rouquier13] and

explored by Dreyfus-Schmidt[Dreyfus-Schmidt13], poset perverse equivalence allows a better perspective to consider compositions of perverse equivalences. Considering the dual construction and Frobenius automorphism, we can also obtain an interesting Frobenius invariant perverse equivalence $D^b(\mathbb{B}) \to D^b(\mathbb{B})$.

4.1 Poset Perverse Equivalence

Considering whether the composition of perverse equivalences continues to be perverse in a natural way, the most important aspect is whether the filtrations of Serre subcategories of the equivalences concerned are compatible (c.f. Definition 1.75). In other words, if we can find filtrations of Serre subcategories in each step, such that the perverse equivalences being composed can work on this common filtration, then this composition concerned is perverse. In abelian categories with finite composition series, we can study these using the concept of poset perverse equivalence. The forthcoming definition (Definition 4.1) focuses on the fact that the composition factors of $H^r(F(S))$ are the only guidance and restriction on how to stack up quotients of Serre subcategories, and these naturally form a poset structure on it. As it turns out, poset perverse equivalences give a more natural setting to discuss whether composing two perverse equivalences results in a perverse equivalence naturally. Or even further it might be a more precise way to describe a perverse equivalence (but not without setbacks). In this chapter we assume all categories have finite composition series upon objects and a finite number of non-isomorphic simple objects. The following definition is a generalisation of Definition 1.77.

Definition 4.1. *Let $\mathscr{C}$ be an abelian category with $\mathbb{S}$ the set of non-isomorphic simple objects. Let $\mathscr{D}$ be another abelian category with $\mathbb{T}$ the set of non-isomorphic simple objects. A derived equivalence $F : D^b(\mathscr{C}) \to D^b(\mathscr{D})$ is perverse relative to $(\mathbb{S}, \prec, \omega)$, where $\prec$ is a poset structure on $\mathbb{S}$ and $\omega : \mathbb{S} \to \mathbb{Z}$, if and only if*

1. *There is a one-to-one correspondence $\beta_F : \mathbb{S} \to \mathbb{T}$.*
2. *Define $S_\prec = \{T \in \mathbb{S} \mid T \prec S\}$. The composition factors of $H^r(F(S))$ are in $\beta(S_\prec)$ for $r \neq -\omega(S)$ and there is a filtration $L_1 \subset L_2 \subset H^{-\omega(S)}(F(S))$ such that the composition factors of L_1 and of $H^{-\omega(S)}(F(S))/L_2$ are in $\beta(S_\prec)$ and L_2/L_1 is isomorphic to $\beta(S)$.*

When considering a perverse equivalence $F : D^b(\mathscr{C}) \to D^b(\mathscr{D})$ we can transfer the partial order structure on $\mathbb{S}$ to $\mathbb{T}$ via β_F.

One obvious problem of introducing a new definition is the compatibility of two notions. It is not hard to notice that they are interchangeable, provided that the poset perverse equivalence exists.

Lemma 4.2. *A derived equivalence $E : D^b(\mathscr{C}) \to D^b(\mathscr{D})$ that is perverse relative to $(\mathbb{S}, \prec, \omega)$ is also perverse relative to $(I_\bullet, \phi)$ for a certain filtration on simple objects $I_\bullet$ and perversity function ϕ. On the other hand, given a derived equivalence E perverse relative to $(I_\bullet, \phi)$, E is also perverse relative to $(\mathbb{S}, \prec, \omega)$ for certain partial order $<$ and perversity function ω.*

Proof If E is a poset perverse equivalence with respect to a partial order $\prec$, refine to a total order and let $I_\bullet$ be the corresponding maximal filtration on $\mathbb{S}$. That is, for each i, $I_i - I_{i-1} = \{S_i\}$ I_{i-1}. Now define perversity function $\phi(i) = \omega(S_i)$. Conversely, if E is perverse relative to $(I_\bullet, \phi)$, we define a partial order on $\mathbb{S}$: Define $S_i \prec S_j$ if and only if there exist a layer I_k such that $S_i \in I_k$ and $S_j \notin I_k$. Now define $\omega(S_i) = \phi(k)$ for the only k satisfying $S_i \in I_k - I_{k-1}$. One can easily check each definition makes each fulfil the other description of perverse equivalence. □

Definition 4.3. *We say that the datum $(\mathbb{S}, \prec, \omega)$ is compatible with the datum $(I_\bullet, \phi)$ if the following conditions hold*

1. *If $S_a \prec S_b$ then there exists an i such that $S_a \notin I_i$, $S_b \in I_i$.*
2. *If $S_a, S_b \in I_i - I_{i-1}$ then $\omega(S_a) = \omega(S_b) = p(i)$.*

Let $E : D^b(\mathscr{C}) \to D^b(\mathscr{D})$ be a perverse equivalence (filtered or poset). Consider the image of $E(S)$ with S running through all simple objects in $\mathscr{C}$. We can define a partial order $\prec$ by

$$S \prec S' \text{ if } H^*(E(S')) \text{ contains a copy of } S \text{ in its composition factors.}$$

This is defined as the coarsest partial order on E. The following is an example written as a proposition.

Proposition 4.4. *The coarsest relation on the perverse equivalence F is*

$$a < b \text{ if } a = \overline{b}.$$

Proof The coarsest relation for F_m is the relation that express the extension happened in the proof of Proposition 3.2, which is a module S_a extended by $S_{\overline{a}}$(c.f. definition 2.17). Consider all modules have been extended by this way we are done. □

We call a proposition similar to 1.76 for the properties of poset perverse equivalence.

Proposition 4.5. *Let $\mathscr{C}$ be an abelian category with finite composition series and a complete set of non-isomorphic simple objects $\mathbb{S}_{\mathscr{C}}$. Let $E : D^b(\mathscr{C}) \to D^b(\mathscr{C}')$ be an equivalence perverse relative to $(\mathbb{S}_{\mathscr{C}}, \prec, \omega)$.*

1. *E^{-1} is perverse relative to $(\beta_E(\mathbb{S}_{\mathscr{C}}), \beta_E(\prec), -\omega \circ \beta_E^{-1})$.*

2. *Let $E' : D^b(\mathscr{C}') \to D^b(\mathscr{C}'')$ be perverse relative to $(\beta_E(\mathbb{S}_{\mathscr{C}}), \beta_E(\prec), \omega')$ then $E' \circ E$ is perverse relative to $(\mathbb{S}_{\mathscr{C}}, \prec, \omega + \omega')$*

3. *If $\omega = 0$ then E restricts to an equivalence $\mathscr{C} \to \mathscr{C}'$.*

Proof The proof is the same as Proposition 1.76 except for item 2. Consider the homology $H^*(E(S))$ of the image of a simple object S under E. The composition factors of $H^*(E(S))$ contain a copy of $\beta_E(S)$ in the $-\phi_I(S)^{th}$ degree and the rest of the factors R satisfy $\beta_E^{-1}R \prec S$. Similarly, the homology $H^*(E'E(S))$ has a copy of $\beta_{E'}\beta_E(S)$ at degree $-\omega(S) - \omega'(S)$, and all the remaining composition factors R' of the homology satisfy $\beta_E^{-1}\beta_{E'}^{-1}R' \prec S$. Then by Definition 4.1 the derived equivalence $E : D^b(\mathscr{C}) \to D^b(\mathscr{C}'')$ we have constructed is perverse relative to $(\mathbb{S}, \prec, \omega + \omega')$. □

In the above proposition and proof, one might see the clumsiness of the notations. Provided that a perverse equivalence exists between two categories, they should share the information defined such as the bijection or the partial order. In the next section we shall suggest a way to improve this.

4.2 Composing Perverse Equivalences

Let A be a finite dimensional symmetric algebra and $\mathbb{S}_A$ the complete set of simple A-modules. Any filtration on $\mathbb{S}_A$ and perversity function ϕ always generate another finite symmetric algebra[Rickard89]. This property makes it ideal to consider continually applying two or more filtrations and perversity functions to generate new algebras. To facilitate this, we abstract a perverse equivalence into data.

Definition 4.6. *Let A be a symmetric algebra. We say A is* n-indexed, *if A has n non-isomorphic simple A-modules indexed by the set $\{0, \ldots, n-1\}$. We denote by β_A the bijective function $\{0, \ldots, n-1\} \to \mathbb{S}_A$ induced by the index, where $\mathbb{S}_A$ is the set of non-isomorphic simple A-modules.*

We define a perverse n-datum I *to be a pair* $(I_\bullet, \phi)$ *which consists of a filtration*

$$I_\bullet = (\emptyset = I_0 \subset I_1 \subset ... \subset I_r = \{0, \dots, n-1\})$$

on the set $\{0, \dots, n-1\}$ *and a perversity function*

$$\phi : \{0, \dots, r\} \to \mathbb{Z}\,.$$

Let I *be a perverse* n*-datum. For an* n*-indexed algebra* A *we can define* ${}^IA$ *to be another finite dimensional algebra, such that there exists an equivalence* $E : D^b(A) \to D^b({}^IA)$ *that is perverse relative to* $(\beta_A(I_\bullet), \phi)$*. By Rickard [Rickard89]* ${}^IA$ *is a symmetric algebra, and by [Chuang, Rouquier13]* ${}^IA$ *is defined up to Morita equivalence.*

Note that we directly regard ${}^IA$ *as* n*-indexed by taking* $\beta_{{}^IA} = \beta_E\beta_A$ *(c.f. Lemma 1.78).*

A similar definition may be done using poset instead of filtration. But since in that case ${}^IA$ does not always exist we are not going to define it at all.

Remark. Fixing an index on simple A-modules is very important since different indexing may lead to different algebras when applying the same perverse n-datum.

We can consider continually applying another perverse n-datum to the new algebra. This means that combinatorial data can be effectively transferred from one algebra to another perverse equivalent algebra. The essence of perverse equivalence has not changed, and all the properties in 1.76 still hold. It is now easier to discuss a string of perverse tilts since we automated the transfer process of filtrations by pullback to natural numbers. As an example using the new definition, we can rewrite composability of 1.76 starting directly from an algebra:

Proposition 4.7. *Let* $I = (I_\bullet, \phi_I)$ *and* $J = (J_\bullet, \phi_J)$ *be two perverse* n*-data. Suppose* $I_\bullet = J_\bullet$ *and let* A *be any* n*-indexed algebra. Then the composition of equivalence* $D^b(A) \to D^b({}^IA) \to D^b({}^{IJ}A)$ *is perverse relative to* $(I_\bullet, \phi_I + \phi_J)$*.*

To decide whether two (not necessary elementary) perverse equivalences commute naturally, using filtered perverse equivalences would require a complex expression on maps of projective modules. However, if using poset equivalence, the condition for it becomes tautology. As an example we consider Proposition 2.80 in [Chuang, Rouquier13]. Now using poset perverse equivalence it is actually easier to state the condition, namely

if both equivalence are compatible to a certain poset. It actually makes the description in 2.80 a bit cumbersome in hindsight.

Definition 4.8. *We say I is an elementary perverse n-datum, if $I_\bullet$ is a two-layer filtration $\emptyset \subset L \subset \{0 \dots n-1\}$ and for its perversity function ε, we have $\varepsilon(0) = 0$ and $\varepsilon(1) = 1$.*

Proposition 4.9. *Let A be an n-indexed algebra and I, J be elementary perverse n-data. Their corresponding sets and functions are marked by the subscript I or J. Then the following are equivalent:*

1. *The derived equivalence $D^b({}^{JI}A) \to D^b({}^{IJ}A)$ obtained by applying $IJI^{-1}J^{-1}$ to ${}^{JI}A$ restricts to an equivalence between $({}^{JI}A)$-mod and $({}^{IJ}A)$-mod.*

2. *For P any indecomposable projective module with the simple head indexed by an element in $L_J - L_I$ and Q similarly indexed by an element in $L_I - L_J$, every homomorphism $P \to Q$ and homomorphism $Q \to P$ factors through an indecomposable projective module R numbered by $L_I \cap L_J$.*

3. *For $\mathbb{S}_A$ the set of simple A-modules, there exists a partial order $\prec$ on $\mathbb{S}_A$ such that the perverse equivalence $E : D^b(A) \xrightarrow{\sim} D^b({}^{I}A)$ is perverse relative to $(\mathbb{S}, \prec, \omega_I)$ and $E' : D^b(A) \xrightarrow{\sim} D^b({}^{J}A)$ is perverse relative to $(\mathbb{S}, \prec, \omega_J)$, where*

$$\omega_I(S) = \begin{cases} 0 \text{ if } \beta_A^{-1}(S) \in L_I \\ 1 \text{ otherwise,} \end{cases}$$

and similarly for ω_J.

Proof We will show (1) $\Leftrightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (1). In fact, (1) $\Leftrightarrow$ (2) is done in 2.80 (c.f. [Chuang, Rouquier13, p.25]) and actually (2) $\Rightarrow$ (3) is implied in the same proof:

(2) $\Rightarrow$ (3): Since (2) is satisfied, by 2.80 the canonical equivalence $D^b(A) \xrightarrow{\sim} D^b({}^{JI}A)$ exists, and is induced by perverse n-datum $(\emptyset \subset (L_I \cap L_J) \subset (L_I \cup L_J) \subset \{0, \dots, n-1\}, \phi')$ where $\phi'(i) = i$ for $i = 0, 1, 2$. This datum is compatible with the partial order $\prec$ on $\mathbb{S}_A$ defined by $S \prec S' \prec S''$ whenever $\beta_A^{-1}(S) \in L_I \cap L_J$, $\beta_A^{-1}(S') \in L_I \triangle L_J$, $\beta_A^{-1}(S'') \in L_I \cup L_J$. Thus (3) holds.

(3) $\Rightarrow$ (1): Since E and E' are perverse relative to the same partial order $\prec$, they are equivalences by shifting on the same poset structure of Serre subcategories. Thus we can apply the perverse n-datum J onto ${}^{I}A$ and the composition $D^b(A) \to D^b({}^{I}A) \to$

$D^b({}^{JI}A)$ is perverse relative to $(\mathbb{S}, \prec, \omega')$ where

$$\omega'(S) = \begin{cases} 0 \text{ if } \beta_A^{-1}(S) \in (L_I \cap L_J) \\ 1 \text{ if } \beta_A^{-1}(S) \in (L_I \triangle L_J) \\ 2 \text{ otherwise.} \end{cases}$$

Similarly we have $D^b(A) \to D^b({}^{J}A) \to D^b({}^{IJ}A)$ is perverse relative to $(\mathbb{S}, \prec, \omega')$. Since they have the same perversity we can deduce (1) by composing the inverse of one with the other to form a perversity zero equivalence. Thus the condition (1) hold because of 4 in 1.76. □

Remark. The set $I \triangle J = (I \cup J) - (I \cap J)$ is the symmetric difference of I and J.

4.3 Conjugation with other functors

Consider the fact that a perverse equivalence is defined up to Morita equivalence, given a perverse equivalence, one can compose/conjugate it with some known functor inducing Morita equivalence to obtain further results. The first one we consider is the Frobenius automorphism on the module category of $\mathbb{B}$-modules. Then we consider duality, which give some insights to the nature of F via Grothendieck group.

4.3.1 Frobenius Twist

The equivalence F we have demonstrated so far is not Frobenius invariant. This can be easily observed, since the Frobenius automorphism on H-modules maps the $\mathbb{F}H$-module U_1 to U_p, thus hinting the construction is twisted to another self-equivalence. However, we can find a self-equivalence which is Frobenius invariant by introducing a composition of functors generated using the Frobenius automorphism on $\mathbb{F}G$-modules. To start, we define:

Definition 4.10. *Let F be the autoequivalence on $D^b(\mathbb{B})$ defined in Theorem 4. The Frobenius automorphism σ of $\mathbb{B}$ induces an automorphism on $D^b(\mathbb{B})$, which we also call σ. Define F^σ,* the Frobenius conjugate *of F, to be the functor $\sigma F \sigma^{-1}$. Define $F^{\sigma^i} = \sigma^i F \sigma^{-i}$ similarly for $0 \leq i \leq n-1$.*

Let E be the composition of functors F^{σ^i} for i from 0 to $n-1$ in that order, i.e.

$$E = F^{\sigma^{n-1}} \circ F^{\sigma^{n-2}} \circ ... \circ F^\sigma \circ F.$$

Since σ restricts to an equivalence on $\mathbb{B}$-mod, the new functors F^{σ^i} introduced are all perverse equivalences. However, they are associated with different filtrations and perversity functions. More precisely, $F^{\sigma^i} : D^b(\mathbb{B}) \to D^b(\mathbb{B})$ is perverse relative to

$$(I_\bullet^{\sigma^i}, I_\bullet^{\sigma^i}, \pi : j \mapsto -p^{n-1-j}).$$

Now we want to transcribe F and its Frobenius twists into poset perverse equivalence.

Proposition 4.11. *The coarsest partial order $\prec^i$ compatible with F^{σ^i} is*

$$S_a \prec^i S_b \text{ if } p^i a \equiv \overline{p^i b} \mod p^n - 1.$$

Proof This is obvious from Proposition 4.3. □

We define another partial order on $\mathbb{B}$ below. It is a refinement of the earlier partial order. We can compose F and its Frobenius twists under this partial order.

Definition 4.12. *Define a partial order $<_{\mathbb{B}}$ on the set $\mathbb{S}$: $S_a <_{\mathbb{B}} S_b$ if, for their n-digit base-p presentations, a has more digits of $p-1$ and $a_i = p-1$ whenever $b_i = p-1$.*

Remark. This partial order is Frobenius invariant because multiplication by p (see chapter 2) only rotates the digits modulo $q-1$. It is also preserved under partner involution (i.e. $\beta_F(<_{\mathbb{B}}) = <_{\mathbb{B}}$) since the partners have same digits of $p-1$ at the same place.

With the partial order being set up we focus on the perversity.

Definition 4.13. *Define a map*

$$\omega : \mathbb{S} \to \mathbb{Z}$$

on simple $\mathbb{B}$-modules such that ω is the composition of the layer map and π. We further define

$$\omega^{\sigma^i} : \mathbb{S} \to \mathbb{Z}$$

to be ω pre-composed by σ^i on the set $\mathbb{S}$.

Proposition 4.14. *The equivalence F^{σ^i} is perverse relative to $(\mathbb{S}_a, <_{\mathbb{B}}, \omega^{\sigma^i})$.*

Proof What we need is $<_{\mathbb{B}}$ compatible with F^{σ^i}. Since $<_{\mathbb{B}}$ is Frobenius invariant, it equates to be compatible with F, but this is obvious. □

Thus we have the following:

Theorem 4.15. *The functor E defined in 4.10 is perverse relative to*

$$(\mathbb{S}_a, <_{\mathbb{B}}, \sum_{i=0}^{n-1} \omega^{\sigma^i}).$$

Furthermore, E is Frobenius invariant.

Proof The first assertion comes from the fact that each F^{σ^i} is compatible with $(\mathbb{S}, <_{\mathbb{B}})$. Now Frobenius conjugation on E yields a functor

$$F^{\sigma^n} \circ F^{\sigma^{n-1}} \circ ... \circ F^{\sigma}$$

which is a cyclic permutation of E (note $F^{\sigma^n} = F$). Since by the first assertion they are all compatible with the same partial order, they commute by Proposition 4.5. Thus after rearrangement we get back E. The new perversity function is just the sum of all perversity functions from F and its Frobenius twists. To see that the sum is Frobenius invariant, observe that applying σ rotates the sum. □

We are going to complete this subsection by showing that we can define a filtration on $\mathbb{B}$ for E which is Frobenius invariant. By doing this we further see we can group simple $\mathbb{B}$-modules using partitions of n and in our case, the perversity function on S_a only depends on the partition.

Definition 4.16. *Let $S_a \in \mathbb{S}$ for an integer a, $0 \leq a \leq p^n - 2$. Assign a partition $\lambda_a \dashv n$ to S_a using the following steps.*

1. *Denote by $(Z_a)_i$ the layer of $p^i a$ modulo $p^n - 1$*
2. *Let $\lambda = (\lambda_0, \ldots, \lambda_{n-1})$ be a n-tuple, where λ_j is the number of times $n - 1 - j$ occurred in $(Z_a)_i$ for $0 \leq i \leq n - 1$.*

We can show that λ is indeed a partition, i.e. $\lambda_0 \geq \lambda_1 \geq \cdots \geq \lambda_{n-1}$, by the following lemma.

Lemma 4.17. *Let a be in layer $i < n - 1$, then p^a modulo $p^n - 1$ is in layer $j + 1$.*

Proof If a is in layer $j < n - 1$ then we have

$$p^n - p^{j+1} \leq a \leq p^n - p^j - 1.$$

Multiplying by p we have

$$p^{n+1} - p^{j+2} \leq pa \leq p^{n+1} - p^{j+1} - p.$$

Since $i+2<n$, $p^{n+1}=p^n+p-1$ modulo p, we have

$$p^n-p^{j+2}+(p-1)\leq pa\leq p^n-p^{j+1}-1$$

showing pa is of layer $j+1$. □

This shows for any $(Z_a)_i=j<n-1$ we have $(Z_a)_{i+1}=j+1$, hence λ is a partition.

By the above we have defined a map

$$l:\mathbb{S}\to\{\text{Partitions of } n\}$$

and we defined another function

$$\begin{aligned}\phi':\{\text{Partitions of } n\}&\to\mathbb{Z}\\ \lambda=(\lambda_0,\ldots,\lambda_{n-1})&\mapsto\sum_{i=0}^{n-1}-\lambda_i p^{n-1-i}\end{aligned}$$

Proposition 4.18. *For the definitions above we have the following:*

1. *The partial order $<_{\mathbb{B}}$ we defined in Definition 4.12 is collapsed by l into the reverse dominance order of partitions.*
2. *The map ϕ' is injective.*
3. *$\phi'(a)<\phi'(b)$ when $\lambda_a<\lambda_b$ in lexicographical order*[1]*.*

Proof Let $a<_{\mathbb{B}}b$ be two integers and $a_i=p-1\neq b_i$ for some i. Now notice that $(Z_a)_{i+1}<(Z_b)_{i+1}=n-1$. Then by previous lemma we have 1. For two partitions λ and λ', let j be the greatest integer such that $\lambda_j\neq\lambda'_j$. Then $\phi'(a)-\phi'(b)\neq 0 \mod p^j$ thus we have 2. The last one is just a check on the sum. □

Example 4.19. *Consider* 77 *in* $p^n=3^6$*, its* 6*-digit base-*3 *presentation is* $(2,1,2,2,0,0)$*, then we have*

$$(Z_a)=(5,5,3,4,5,4)$$

hence 77 *correspond to a* 6*-partition* $(3,2,1)$*. The function $q'\circ l$ maps S_{77} to* $3(-1)+2(-3)+1(-9)=-18$*.*

Compare 62 *under the same setting, it has presentation* $(2,2,0,2,0,0)$*. We have*

$$(Z_a)=(5,5,4,5,3,4)$$

[1] i.e. The partition (n) is the last (greatest) term and (1^n) is the first (smallest) term.

hence 62 *corresponds to the same* 6*-partition.*

Now we apply the new filtration on simple $\mathbb{B}$-modules and the perversity function on the filtration.

Definition 4.20. *Define I to be a perverse $(q-1)$-data by the following: First, assign any integer $a \in \{0, \ldots, q-2\}$ to the set J_λ, where $\lambda \dashv n$ is the partition representing a. Then we order partitions using lexicographical order, $\prec$ and let*

$$I_\lambda = \bigcup_{\kappa \prec \lambda} J_\kappa.$$

and set

$$I_\bullet = (\emptyset \subset I_{(1^n)} \subset I_{(1^{n-2}2)} \subset \cdots \subset I_{(n)} = \mathbb{S}).$$

The perversity function is taken as ϕ' defined in Definition 4.16.

Define $E' : D^b(\mathbb{B}) \to D^b({}^I\mathbb{B})$ to be a derived equivalence generated by the perverse $q-1$-data on (naturally $q-1$-indexed) $\mathbb{B}$.

Remark. We have indexed the filtration by partitions, but this does not affect anything for using the idea of perverse data.

Lastly, we use a theorem to conclude this Frobenius invariant result.

Theorem 4.21. *Let $E' : D^b(\mathbb{B}) \to D^b({}^I\mathbb{B})$ be the derived equivalence defined using the above data, perverse relative to*

$$(\emptyset \subset I_{(n)} \subset \ldots I_{(1^n)} = \mathbb{S}, \phi')$$

We have the equivalence $E = F^{\sigma^{n-1}} \ldots F : D^b(\mathbb{B}) \to D^b(\mathbb{B})$ is compatible with E'. Therefore

1. *${}^I\mathbb{B}$ is Morita equivalent to $\mathbb{B}$.*
2. *E' is Frobenius invariant.*
3. *E' is of increasing perversity (i.e. The perversity function is strictly increasing, see[Craven10]).*

Proof All these can be achieved as long as we show E' is actually E, or showing E' is compatible with E. First we show the filtration of E' is a refinement of the poset order in E. Recall that every extension in E (which comes from various Frobenius

twists of F) has more $p-1$'s in its base-p expression (since it is a composition of F and its Frobenius twists) and thus maps to a lower filtrate in E'. So the partial order defined in E is compatible with the filtration of E'. Now it remains to check the perversity functions of the two equivalences are the same on all simple modules S_a. This is not difficult to see since

$$\sum_{i=0}^{n-1} \omega^{\sigma^i} = \sum_{i=0}^{n-1} -p^{n-1-(Z_a)_i} = \sum_{i=0}^{n-1} -\lambda_i p^{n-1-i}$$

by rearranging according to p-powers.

Thus we have checked their perversity is equal on all simple modules and thus E and E' is compatible. The fact that it is of increasing perversity comes from E''s perversity function ϕ' is obtained as sum of p-powers from an n-partition. □

4.3.2 Duality

Given a perverse equivalence, one can construct a dual perverse equivalence, see [Chuang, Rouquier13]. Applying to our construction F, it amounts to another autoequivalence $F^* : D^b(\mathbb{B}^{op}) \to D^b(\mathbb{B}^{op})$ perverse relative to the same filtration (or partial order) but negative perversity function.

Moreover, we can compose a perverse equivalence with functors that induce Morita equivalence to produce new equivalence. First, recall that for a finite-dimensional algebra A one has the duality functor $\mathrm{Hom}_k(-,k)$ (c.f. Definition 1.12) that maps an A-module to a module-A by $(a.f)(m) = f(am)$. For group algebras we also have an anti-automorphism $g \to g^{-1}$. These allows us to define contravariant duality:

Definition 4.22. *Let A be an finite dimensional algebra with an anti-automorphism $\alpha : A \to A$. We have an equivalence*

$$\Phi : \qquad A\text{-mod} \longrightarrow A^{op}\text{-mod} \longrightarrow (A\text{-mod})^{op}$$

$$X \longmapsto {}^{\alpha}X \longmapsto {}^{\alpha}X^*$$

We denote the object ${}^{\alpha}X^*$ by X°.

Let $F : D^b(\mathbb{B}) \to D^b(\mathbb{B})$ denote the equivalence in Definition 3.1 which sends a simple $F(S)$ to a chain complex X_S. Now conjugating F with the contravariant

duality we just defined we have

$$D^b(\mathbb{B}\text{-mod})^{op} \cong D^b(\mathbb{B}\text{-mod}^{op}) \xrightarrow{\Phi} D^b(\mathbb{B}) \xrightarrow{F} D^b(\mathbb{B}) \xrightarrow{\Phi^{-1}} D^b(\mathbb{B}\text{-mod}^{op}) \cong D^b(\mathbb{B}\text{-mod})^{op}$$

which sends S° to X_S°.

Now taking off the opposite, we have established an equivalence $G : D^b(\mathbb{B}\text{-mod}) \to D^b(\mathbb{B}\text{-mod})$ such that it is perverse relative to $(S_\bullet^\circ, -\pi)$.

Observing that the filtration $S_\bullet^\circ$ is the same as $S_\bullet$, we deduce that G is the inverse of F. Furthermore note that the perverse equivalence $G^{-1}F$, equivalent to F^2, sends X_S° to X_S. Thus we are able to obtain that F^2 is the identity on the Grothendieck group.

Lemma 4.23. *The functor F^2 induces the identity on the Grothendieck group.*

Proof We have seen that F^2 sends X_S° to X_S. Consider the generating property of the simple modules in the derived category, the only remaining task is to show X_S° corresponds to the same element in Grothendieck group as X_S. But this is obvious since X_s° is just negating the numbering on X_S. □

Chapter 5

The case of $G = SL_2(4)$

The first non-trivial equivalence introduced by our construction involves $G = SL_2(2^2)$. It is a somewhat special case, since $p = 2$ and thus we have only one non-semisimple block instead of two. However it is good enough for us to demonstrate the construction while decoding some intriguing features. Thus we dedicate a chapter to this example.

The group $SL_2(4)$ is isomorphic to the alternating group A_5 of five elements. A Sylow 2-subgroup of A_5 is the Klein 4-group, and can be chosen so that its normaliser in A_5 is A_4. This example has been studied by many, since its representation type is tame.

There are 4 non-isomorphic simple $\mathbb{F}A_5$-modules: The trivial module $k = S_0$, two modules $V = S_1$, $W = S_2$ which are two-dimensional and the (4-dimensional) Steinberg module S_3. Their corresponding indecomposable projective covers have Loewy series as follows:

$$P_k = \begin{array}{c} k \\ V \quad W \\ k \quad k \\ W \quad V \\ k \end{array} \qquad P_V = \begin{array}{c} V \\ k \\ W \\ k \\ V \end{array} \qquad P_W = \begin{array}{c} W \\ k \\ V \\ k \\ W \end{array} \qquad St.$$

We shall ignore S_3 as it is lies in the Steinberg block St, a simple block of $\mathbb{F}G$. The remaining simple modules form a full defect block, the principal block $\mathbb{B}_0$. There are 3 non-isomorphic simple $\mathbb{F}A_4$-modules. The trivial module $k = U_0$, U_1 and U_2, all one-dimensional. Their corresponding indecomposable projective covers have Loewy

series (with U suppressed) as follows:

$$Q_0 = \begin{matrix} & 0 & \\ 1 & & 2 \\ & 0 & \end{matrix} \qquad Q_1 = \begin{matrix} & 1 & \\ 2 & & 0 \\ & 1 & \end{matrix} \qquad Q_2 = \begin{matrix} & 2 & \\ 0 & & 1. \\ & 2 & \end{matrix}$$

The restrictions of simple A_5-modules to A_4 are given by

$$k{\downarrow_{A_4}} = 0 \qquad V{\downarrow_{A_4}} = M_1 = \begin{matrix} 1 \\ 2 \end{matrix} \qquad W{\downarrow_{A_4}} = M_2 = \begin{matrix} 2 \\ 1 \end{matrix} \qquad St{\downarrow_{A_4}} = M_3 = \begin{matrix} 0 \\ 1 \\ 0 \end{matrix} \quad 2 = Q_0.$$

Adopting the terminology of 'layers' from Definition 2.16, W is in layer 0 and the remaining simple $\mathbb{F}A_5$-modules are in layer 1. The only non-trivial step of the procedure carried out in Definition 3.1 is at $m = 1$, for which the one-sided tilting complex of $\mathbb{B}_0$-modules is

$$P_k \oplus P_V \oplus P_k \xrightarrow{\alpha} P_W$$

where $\alpha : P_k \to P_W$ is a presentation of the simple module W. The following is a table of the images of simple $\mathbb{A}_i$-modules in $D^b(\mathbb{B}_0)$, and in $\mathbb{F}A_4$-$\underline{\text{mod}}$.

simples numbered / in algebra	0	1	2	0	1	2
$\mathbb{B} = \mathbb{A}_0$	k	V	W	0	$\begin{matrix} & 1 & \\ 2 & & \end{matrix}$	$\begin{matrix} 2 & \\ & 1 \end{matrix}$
$\mathbb{A}_1$	$\begin{matrix} k \\ W \end{matrix}$	V	$W[1]$	$\begin{matrix} 2 & & 0 \\ & 1 & \end{matrix}$	$\begin{matrix} & 1 & \\ 2 & & \end{matrix}$	$\begin{matrix} 1 & \\ & 0 \end{matrix}$
$\mathbb{A}_2 = \mathbb{A}$	$\begin{matrix} k \\ W \end{matrix}[1]$	$V[1]$	$W[2]$	1	$\begin{matrix} & 2 & \\ 0 & & \end{matrix}$	$\begin{matrix} 0 & \\ & 2 \end{matrix}$
in category:	$D^b(\mathbb{B}_0)$			$\mathbb{F}A_4$-$\underline{\text{mod}}$		

The benefit of writing in $\mathbb{F}A_4$-$\underline{\text{mod}}$ may not be obvious in this example, but will be immense when generalised in either p or n. See the appendix for more examples.

5.1 Spherical twist

In section 3.2 we have showed that our automorphism F of $D^b(\mathbb{B})$ does not translate to the autoequivalence $U_1 \otimes -$ of $D^b(\mathbb{b})$ via any equivalence $D^b(\mathbb{B}) \to D^b(\mathbb{b})$. In this

section we show that it does correspond to a composition of $U_1 \otimes -$ with two spherical twists of $D^b(\mathbb{b})$.

The notion of spherical twist comes from algebraic geometry. It mimics the geometric construction of twisting a sphere - hence it is always an autoequivalence. For a brief introduction related to our needs, see [Grant10, section 1.3]. For a more systematic approach to spherical twists see [Seidel-Thomas01].

Definition 5.1. *A projective A-module P is called* spherical *if* $\mathrm{End}(P) \cong k[x]/(x^2)$. *The spherical twist of P is the equivalence $D^b(A) \to D^b(A)$ given by*

$$\Phi_P(X) = \mathrm{cone}(\mathrm{Hom}_{D^b(A)}(P, X) \otimes_k X \to X),$$

where the map is the evaluation map.

Here we are only concerned with spherical twists in $D^b(\mathbb{F}A_4)$, in which all the indecomposable projective modules are spherical. We describe the effect of such twists on $D^b(\mathbb{F}A_4)$ directly.

Definition 5.2. *Define Φ_S to be the spherical twist obtained by twisting the spherical projective module P_S, the projective cover of S, which is a simple A_4-module.*

These spherical twists take the following values on simple $\mathbb{F}A_4$-modules.

$$\Phi_S(T) \cong \begin{cases} T & \text{if } T \neq S \\ P_T \to T, \text{ where } T \text{ is in degree zero,} & \text{if } T = S. \end{cases}$$

To travel between the global and local derived categories $D^b(\mathbb{B}_0)$ and $D^b(A_4)$, we fix an equivalence $\mathbf{Ric} : D^b(\mathbb{B}_0) \to D^b(A_4)$ given by the two sided tilting complex of A_5-bimodule-A_5

$$(0 \to P \to \mathbb{F}A_5 \to 0)^*.$$

where $\mathbb{F}A_5$ is in degree zero, P is a projective A_5-bimodule-A_5 cover of the augmentation ideal of $\mathbb{F}A_5$ and the star denotes its dual as A_5-bimodule-A_5. The symbols used below are defined as in the chapter earlier.

Note: Non-projective term is in degree zero.

Complex expressed by A_5-modules	Complex expressed by A_4-modules
k	0 (This is the trivial A_4-module)
V	$\begin{matrix}1\\2\end{matrix} \to Q_2 \simeq \begin{matrix}2\\0\end{matrix}[-1]$
W	$\begin{matrix}2\\1\end{matrix} \to Q_1 \simeq \begin{matrix}1\\0\end{matrix}[-1]$
$P_W \to \begin{matrix}W\\k\\V\end{matrix} \simeq \begin{matrix}k\\W\end{matrix}[1]$	1
$P_V \to \begin{matrix}V\\k\\W\end{matrix} \simeq \begin{matrix}k\\V\end{matrix}[1]$	2

N.B. Equal sign indicates quasi-isomorphic chain complexes on both sides. Conjugating our construction with **Ric** yields the following:

A_4-simples	$\mathbf{Ric}^{-1}$	$F\,\mathbf{Ric}^{-1}$	$\mathbf{Ric}\,F\,\mathbf{Ric}^{-1}$
0	k	$\begin{matrix}k\\W\end{matrix}[1]$	1
1	$\begin{matrix}k\\W\end{matrix}[1]$	$\mathrm{cone}\left\{\begin{matrix}k\\W\end{matrix}[1] \to W[3]\right\}$	$\mathrm{cone}\left\{1 \to \begin{matrix}1\\0\end{matrix}[2]\right\}$
2	$\begin{matrix}k\\V\end{matrix}[1]$	$\begin{matrix}k\\V\quad W\end{matrix}[2]$	$\begin{matrix}2\quad 1\\0\end{matrix}[1]$

This derived equivalence is the same as $U_1\Phi_2\Phi_1$, as demonstrated below:

A_4-simples	Φ_1	$\Phi_2\Phi_1$	$U_1\Phi_2\Phi_1$
0	0	0	1
1	$\begin{matrix}0\quad 2\\1\end{matrix}[1]$	$\mathrm{cone}\left\{\begin{matrix}1\quad 0\\2\quad 1\end{matrix}[1] \to \begin{matrix}0\\1\end{matrix}[1]\right\}$	$\mathrm{cone}\left\{\begin{matrix}2\quad 1\\0\quad 2\end{matrix}[1] \to \begin{matrix}1\\2\end{matrix}[1]\right\}$
2	2	$\begin{matrix}1\quad 0\\2\end{matrix}[1]$	$\begin{matrix}2\quad 1\\0\end{matrix}[1]$

We need to show

$$\mathrm{cone}\left\{\begin{matrix}2\quad 1\\0\end{matrix}[1] \to \begin{matrix}1\\2\end{matrix}[1]\right\} \qquad \text{and} \qquad \mathrm{cone}\left\{1 \to \begin{matrix}1\\0\end{matrix}[2]\right\}$$

quasi-isomorphic. We rewrite the latter using projective resolution and it is quasi isomorphic to

$$\begin{matrix} 1 & & 2 & & 0 \\ & 0 & & 1 & \end{matrix} \to Q_1$$

with Q_1 in degree -1 and the map takes the composition module 1 in the socle to the socle of Q_1. Now, this complex have a quasi-isomorphic map into

$$\text{cone}\left\{\begin{matrix} 2 & & 1 & & 1 \\ & 0 & & [1] \to & & [1] \\ & & & & 2 & \end{matrix}\right\} \quad \cong \quad \begin{matrix} 1 & & 2 & & 1 \\ & & & \to & & \\ & & 0 & & 2 & \end{matrix}$$

with the rightmost term in degree -1. Hence proving the two entries in earlier tables to be quasi-isomorphic.

The Grothendieck group of $D^b(\mathbb{B})$ is a free abelian group on 3 generators (the classes of its simple modules). The group automorphism induced by F' is given by

$$[k] \mapsto [k] + [W]; \qquad [V] \mapsto [V]; \qquad [W] \mapsto -[W].$$

Thus the automorphism induced by F is

$$[k] \mapsto -[k] - [W]; \qquad [V] \mapsto -[V]; \qquad [W] \mapsto [W].$$

This map has determinant 1 when viewed as an automorphism of $\mathcal{K}(\mathbb{B}\text{-}\underline{\text{mod}})$. The Grothendieck group of $\mathbb{B}\text{-}\underline{\text{mod}}$ is a cyclic group of order 4 generated by $[k]$ with $[V] = [W] = 2[k]$ thus the automorphism induced by F is the identity on $\mathcal{K}(\mathbb{B}\text{-}\underline{\text{mod}})$. Note that U_1 conveniently induces the identity on $\mathbb{b}\text{-}\underline{\text{mod}}$. This information agrees with that for a composition of two spherical twists.

Appendix A

Further examples

This appendix hopes to illustrate the constructions we have demonstrated in more detail but with an unofficial flavour. In particular, we will use Loewy layers to exhibit the structures of various modules used in the paper. We build our Loewy structures from the socle (bottom) and socle series are arranged horizontally throughout the appendix. Our aim is to give the reader some more intuition about the content of the construction, in particular Proposition 3.2 and Lemma 3.3. Although there is also intuition coming from the quivers we do not go into any further detail here. More information on quivers and relations of special linear groups of finite fields can be found in [Koshita98][Koshita94] .

A.1 Restriction of modules

Recall our notation: p is a positive prime, n is an integer and $q = p^n$. $\mathbb{F}$ is an algebraically closed field of characteristic p, $G = SL_2(q)$ and V is the natural two-dimensional representation of G. H is the normaliser of a Sylow p-subgroup of G. For $0 \leq a \leq q-1$, S_a is a simple $\mathbb{F}G$-module and they form a complete set of simple $\mathbb{F}G$-modules. Let M_a be the restriction of S_a as an $\mathbb{F}H$-module. For $0 \leq a \leq q-2$, U_a is a simple one-dimensional $\mathbb{F}H$-module.

Example A.1. *For $0 \leq a \leq p-1$, the restriction M_a of $S_a = V^a = Sym^a(V)$ is an indecomposable uniserial $\mathbb{F}H$-module with top U_a and socle U_{-a}. Each Loewy layer has the one dimensional module indexed by two less than the module immediately above (for*

the proof of this see [Holloway01, Chapter 5]). That is,

$$M_a = \begin{matrix} U_a \\ U_{a-2} \\ \vdots \\ U_{-a} \end{matrix}$$

with $a+1$ Loewy layers.

Taking the Frobenius twist of such a module results in multiplying the subscript of all composition factors by p (see chapter 2). Taking tensor products of these uniserial modules is simply taking tensor products of each component and crisscrossing them, since all simple H-modules are 1-dimensional. These facts determine the structure of M_a, for $0 \leq a \leq q-1$, via the Steinberg tensor product theorem.

Example A.2. *When $p = 5$, $n = 2$, the $\mathbb{F}H$-module $M_{16} = M_{(1,3)}$ is a tensor product of M_1 and M_{15} ($V^1 \otimes V_1^3$). (Recall $V^n = Sym^n V$.) So we have the following Loewy structure*

$$M_1 \cong \begin{matrix} U_1 \\ U_{-1} \end{matrix}, \qquad M_3 \cong \begin{matrix} U_3 \\ U_1 \\ U_{-1} \\ U_{-3} \end{matrix}$$

and hence

$$M_{15} \cong \begin{matrix} U_{15} \\ U_5 \\ U_{-5} \\ U_{-15} \end{matrix} \qquad \text{and} \qquad M_{16} \cong \begin{matrix} & & U_{16} & & \\ & U_{14} & & U_6 & \\ U_4 & & U_{20} & & \\ & U_{18} & & U_{10} & \\ & & U_8 & & \end{matrix}.$$

Notation. *For clarity and convenience, we suppress the U's in Loewy structures yet to appear. We have also taken modulo $p^n - 1$ on the subscript when possible because they are isomorphic (see chapter 2). Note that, from now on, a 0 is representing the trivial module U_0.*

As one can see illustrated in the example above, kH-modules restricted from simple $\mathbb{F}G$-modules always have 'hypercuboid' Loewy structures. The Ext group between simple $\mathbb{F}G$-modules is well-studied, see [Carlson83]. Chuang ([Chuang01]) has extended this knowledge of modules of the form $U_i M_a$ using a spectral sequence argument.

Now we consider the structure of projective kH-modules, using the restriction of the Steinberg $\mathbb{F}G$-module (which is known to be projective in the defining characteristic case).

Example A.3. *For $p = 5$, $n = 2$ the module $M_{24} = M_{(4,4)}$ is the restriction from G to H of the Steinberg module (hence projective), which has Loewy structure*

$$\begin{array}{ccccccccccccccc}
 & & & & & & 0 & & & & & & & & \\
 & & & & & 22 & & & 14 & & & & & & \\
 & & & & 20 & & & 12 & & & 4 & & & & \\
 & & & 18 & & & 10 & & & 2 & & & 18 & & \\
 & & 16 & & & 8 & & & 0 & & & 16 & & & 8 \\
 & & & 6 & & & 22 & & & 14 & & & 6 & & \\
 & & & & 20 & & & 12 & & & 4 & & & & \\
 & & & & & 10 & & & 2 & & & & & & \\
 & & & & & & 0 & & & & & & & &
\end{array}.$$

Example A.4. *Take $M_1 \cong \begin{matrix} 1 \\ 23 \end{matrix}$ for $p = 5$, $n = 2$. The projective cover of M_1, denoted by P_1, is $U_1 \otimes M_{24}$ (see A.3, add 1 to every number). Thus the Heller translate is given by the exact sequence*

$$0 \to \Omega M_1 \to P_1 \to M_1 \to 0,$$

hence

$$\Omega M_1 = \begin{array}{ccccccccccccccc}
 & & & & & & 15 & & & & & & & & \\
 & & & & & 21 & & & 13 & & & 5 & & & \\
 & & & & 19 & & & 11 & & & 3 & & & 19 & \\
 & & 17 & & & 9 & & & 1 & & & 17 & & & 9 \\
 & & & 7 & & & 23 & & & 15 & & & 7 & & \\
 & & & & 21 & & & 13 & & & 5 & & & & \\
 & & & & & 11 & & & 3 & & & & & & \\
 & & & & & & 1 & & & & & & & &
\end{array}.$$

Example A.5. *Continuing from Example A.4, by tensoring U_1 with the Heller translate*

of M_1 we have the following Loewy structure

$$U_1\Omega M_1 = \begin{array}{ccccccccccc} & & & & & 16 & & & & & \\ & & & & 22 & & 14 & & 6 & & \\ & & & 20 & & 12 & & 4 & & 20 & \\ & & 18 & & 10 & & 2 & & 18 & & 10 \\ & & & 8 & & 0 & & 16 & & 8 & \\ & & & & 22 & & 14 & & 6 & & \\ & & & & & 12 & & 4 & & & \\ & & & & & & 2 & & & & \end{array} .$$

Recall M_{16} for $p = 5$, $n = 2$ from Example A.2. Note that M_{16} is isomorphic to the upper-right rectangular part given in the picture (separated by dotted line). The remainder is isomorphic to M_{22}. That is, using this rather intuitive way we have obtained a distinguished triangle in the stable $\mathbb{F}H$-module category:

$$M_{22} \to U_1\Omega M_1 \to M_{16} \rightsquigarrow .$$

One can check using Proposition 2.12 that $Ext^1_G(S_{16}, S_{22})$ is one dimensional. What the above triangle indicated is that the non-split extension in the stable $\mathbb{F}G$-module category is isomorphic to the non-projective summand of the induction of $U_1\Omega M_1$ from H to G.

An intuitive view would be that the restriction of modules with partner subscripts (as defined in chapter 2) gives two modules which are complements of each other. The two partner modules join (not in the form of extension, only shape-wise) together to form a side (as in rectangles, or hypercuboids for higher dimension) of length p when presented in the $\mathbb{F}H$-module category. (e.g. M_1 and M_{16} in Example A.2 to A.5.) This new 'module' has a strictly 'lower' deficit than a projective $\mathbb{F}H$-module which has all sides of length p as hypercuboids. We will give some further examples for the case when $n = 3$ in section A.3. Before that, in the next section we first give a full example of the modules involved in Proposition 3.3 and Lemma 3.4.

A.2 Tables: $SL_2(3^2)$, $SL_2(5^2)$ and $SL_2(3^3)$

Here we list the local stable correspondence and the extensions which occur in each elementary perverse step in our proof when $G = SL_2(3^2)$, $SL_2(5^2)$ and $SL_2(3^3)$. In the tables for $SL_2(3^2)$ we give the corresponding simple collection and tilting complexes.

Local stable equivalent tables are enough to remake the process in derived category as shown in section 1.5 earlier, though the expression of the result in the derived category is messy, as one would expect from observing table A.1b.

Table A.1: Case for $SL_2(3^2)$

(a) The local stable category equivalent for $B_0(SL_2(3^2))$

$B_0(3^2)$	Layer 1			Layer 0
$\mathbb{A}_0$	M_0	M_2	M_4	$M_6 = U_3\Omega M_7$
step 1			$M_6 = U_3\Omega M_7$	Ω^{-1}
$\mathbb{A}_1$	M_0	M_2	$U_1\Omega M_1$	$U_3M_7 = U_6\Omega M_6$
step 2	$U_3M_7 = U_6\Omega M_6$	$M_8 = 0$		Ω^{-1}
$\mathbb{A}_2$	$U_1\Omega M_3$	$U_1\Omega M_5$	$U_1\Omega M_1$	$U_6M_6 = U_1\Omega M_7$
step 3	Ω^{-1}	Ω^{-1}	Ω^{-1}	Ω^{-1}
$\mathbb{A}_3$	U_1M_3	U_1M_5	U_1M_1	U_1M_7

(b) The construction as in $D^b(SL_2(3^2))$ - the corresponding complex for simples in $B_0(\mathbb{A}_m)$

$B_0(3^2)$	Layer 1			Layer 0
	S_0	S_2	S_4	S_6
$\mathbb{A}_0$	S_0	S_2	S_4	S_6
step 1			map to $S_6[1]$ and take co-cone	$[1]$
$\mathbb{A}_1$	S_0	S_2	S_4 S_6	$S_6[1]$
step 2	map to $S_6[2]$ and take co-cone	*		$[1]$
$\mathbb{A}_2$	co-cone$(S_0 \to S_6[2])$	S_2	S_4 S_6	$S_6[2]$
$\mathbb{A}_3$	cone$(S_0 \to S_6[2])$	$S_2[1]$	$S_4[1]$ S_6	$S_6[3]$

*This step take its map to 0 and obtain its cocone - equivalent to doing nothing.

(c) Summands for one sided tilting complex for both blocks in $SL_2(9)$

The underlined term (if it exists) is in the zeroth degree. All terms not shown are zero.

Tilting from $B_0(SL_2(9))$ to	Layer 1			Layer 0
$B_0(\mathbb{A}_0)$	P_0	P_2	P_4	P_6
$B_0(\mathbb{A}_1)$	P_0	P_2	P_4	$\to \underline{P_4} \to P_6 \to$
$B_0(\mathbb{A}_2)$	P_0	P_2	P_4	$\to \underline{P_0} \to P_4 \to P_6 \to$
$B_0(\mathbb{A}_3) = B_1(SL_2(9))$	$P_0[-1]$	$P_2[-1]$	$P_4[-1]$	$\to \underline{0} \to P_0 \to P_4 \to P_6 \to$
Tilting from $B_1(SL_2(9))$ to	Layer 1			Layer 0
$B_1(\mathbb{A}_0)$	P_1	P_3	P_5	P_7
$B_1(\mathbb{A}_1)$	P_1	P_3	P_5	$\to \underline{P_3} \to P_7 \to$
$B_1(\mathbb{A}_2)$	P_1	P_3	P_5	$\to \underline{P_1} \to P_3 \to P_7 \to$
$B_1(\mathbb{A}_3) = B_0(SL_2(9))$	$P_1[-1]$	$P_3[-1]$	$P_5[-1]$	$\to \underline{0} \to P_1 \to P_3 \to P_7 \to$

The full one-sided tilting complex from $\mathbb{B}$ to $\mathbb{A}_m$ is the direct sum (of a certain number of copies) of the tilting complex above.

Table A.2: Exchanging blocks of $SL_2(5^2)$ under the construction, local stable category equivalent

Shorthand: $U_i\Omega^j M_a$ written as ${}^j_i M_a$ U subscripts modulo $5^2-1=24$

$B_0(5^2)$	Layer 1										Layer 0	
	S_0	S_2	S_4	S_6	S_8	S_{10}	S_{12}	S_{14}	S_{16}	S_{18}	S_{20}	S_{22}
$\mathbb{A}_0$	$M_{(0,0)}$	$M_{(2,0)}$	$M_{(4,0)}$	$M_{(1,1)}$	$M_{(3,1)}$	$M_{(0,2)}$	$M_{(2,2)}$	$M_{(4,2)}$	$M_{(1,3)}$	$M_{(3,3)}$	${}^1_5M_{(3,4)}$	${}^1_5M_{(1,4)}$
step 1									${}^1_5M_{(1,4)}$	${}^1_5M_{(3,4)}$	Ω^{-1}	Ω^{-1}
$\mathbb{A}_1$	$M_{(0,0)}$	$M_{(2,0)}$	$M_{(4,0)}$	$M_{(1,1)}$	$M_{(3,1)}$	$M_{(0,2)}$	$M_{(2,2)}$	$M_{(4,2)}$	${}^1_1M_{(1,0)}$	${}^1_1M_{(3,0)}$	${}^1_{10}M_{(0,4)}$	${}^1_{10}M_{(2,4)}$
step 2						${}^1_{10}M_{(0,4)}$	${}^1_{10}M_{(2,4)}$	$M_{(4,4)}=0$			Ω^{-1}	Ω^{-1}
A_2	$M_{(0,0)}$	$M_{(2,0)}$	$M_{(4,0)}$	$M_{(1,1)}$	$M_{(3,1)}$	${}^1_1M_{(0,1)}$	${}^1_1M_{(2,1)}$	${}^1_1M_{(4,1)}$	${}^1_1M_{(1,0)}$	${}^1_1M_{(3,0)}$	${}^1_{15}M_{(3,4)}$	${}^1_{15}M_{(1,4)}$
step 3				${}^1_{15}M_{(1,4)}$	${}^1_{15}M_{(3,4)}$						Ω^{-1}	Ω^{-1}
$\mathbb{A}_3$	$M_{(0,0)}$	$M_{(2,0)}$	$M_{(4,0)}$	${}^1_1M_{(1,2)}$	${}^1_1M_{(3,2)}$	${}^1_1M_{(0,1)}$	${}^1_1M_{(2,1)}$	${}^1_1M_{(4,1)}$	${}^1_1M_{(1,0)}$	${}^1_1M_{(3,0)}$	${}^1_{20}M_{(0,4)}$	${}^1_{20}M_{(2,4)}$
step 4	${}^1_{20}M_{(0,4)}$	${}^1_{20}M_{(2,4)}$	$M_{(4,4)}=0$								Ω^{-1}	Ω^{-1}
$\mathbb{A}_4$	${}^1_1M_{(0,3)}$	${}^1_1M_{(2,3)}$	${}^1_1M_{(4,3)}$	${}^1_1M_{(1,2)}$	${}^1_1M_{(3,2)}$	${}^1_1M_{(0,1)}$	${}^1_1M_{(2,1)}$	${}^1_1M_{(4,1)}$	${}^1_1M_{(1,0)}$	${}^1_1M_{(3,0)}$	${}^1_1M_{(3,4)}$	${}^1_1M_{(1,4)}$
$\mathbb{A}_5$	${}_1M_{15}$	${}_1M_{17}$	${}_1M_{19}$	${}_1M_{11}$	${}_1M_{13}$	${}_1M_5$	${}_1M_7$	${}_1M_9$	${}_1M_1$	${}_1M_3$	${}_1M_{23}$	${}_1M_{21}$
$B_1(5^2)$	S_1	S_3	S_5	S_7	S_9	S_{11}	S_{13}	S_{15}	S_{17}	S_{19}	S_{21}	S_{23}
$\mathbb{A}_0$	$M_{(1,0)}$	$M_{(3,0)}$	$M_{(0,1)}$	$M_{(2,1)}$	$M_{(4,1)}$	$M_{(1,2)}$	$M_{(3,2)}$	$M_{(0,3)}$	$M_{(2,3)}$	$M_{(4,3)}$	${}^1_5M_{(2,4)}$	${}^1_5M_{(0,4)}$
step 1								${}^1_5M_{(0,4)}$	${}^1_5M_{(2,4)}$	$M_{(4,4)}=0$	Ω^{-1}	Ω^{-1}
$\mathbb{A}_1$	$M_{(1,0)}$	$M_{(3,0)}$	$M_{(0,1)}$	$M_{(2,1)}$	$M_{(4,1)}$	$M_{(1,2)}$	$M_{(3,2)}$	${}^1_1M_{(0,0)}$	${}^1_1M_{(2,0)}$	${}^1_1M_{(4,0)}$	${}^1_{10}M_{(1,4)}$	${}^1_{10}M_{(3,4)}$
step 2						${}^1_{10}M_{(1,4)}$	${}^1_{10}M_{(3,4)}$				Ω^{-1}	Ω^{-1}
$\mathbb{A}_2$	$M_{(1,0)}$	$M_{(3,0)}$	$M_{(0,1)}$	$M_{(2,1)}$	$M_{(4,1)}$	${}^1_1M_{(1,1)}$	${}^1_1M_{(3,1)}$	${}^1_1M_{(0,0)}$	${}^1_1M_{(2,0)}$	${}^1_1M_{(4,0)}$	${}^1_{15}M_{(2,4)}$	${}^1_{15}M_{(0,4)}$
step 3			${}^1_{15}M_{(0,4)}$	${}^1_{15}M_{(2,4)}$	$M_{(4,4)}=0$						Ω^{-1}	Ω^{-1}
$\mathbb{A}_3$	$M_{(1,0)}$	$M_{(3,0)}$	$M_{(0,2)}$	${}^1_1M_{(2,2)}$	${}^1_1M_{(4,2)}$	${}^1_1M_{(1,1)}$	${}^1_1M_{(3,1)}$	${}^1_1M_{(0,0)}$	${}^1_1M_{(2,0)}$	${}^1_1M_{(4,0)}$	${}^1_{20}M_{(1,4)}$	${}^1_{20}M_{(3,4)}$
step 4	${}^1_{20}M_{(1,4)}$	${}^1_{20}M_{(3,4)}$									Ω^{-1}	Ω^{-1}
$\mathbb{A}_4$	${}^1_1M_{(1,3)}$	${}^1_1M_{(3,3)}$	${}^1_1M_{(0,2)}$	${}^1_1M_{(2,2)}$	${}^1_1M_{(4,2)}$	${}^1_1M_{(1,1)}$	${}^1_1M_{(3,1)}$	${}^1_1M_{(0,0)}$	${}^1_1M_{(2,0)}$	${}^1_1M_{(4,0)}$	${}^1_1M_{(2,4)}$	${}^1_1M_{(0,4)}$
$\mathbb{A}_5$	${}_1M_{16}$	${}_1M_{18}$	${}_1M_{10}$	${}_1M_{12}$	${}_1M_{14}$	${}_1M_6$	${}_1M_8$	${}_1M_0$	${}_1M_2$	${}_1M_4$	${}_1M_{22}$	${}_1M_{20}$

Table A.3: Mapping from principal block to non-principal block of $SL_2(3^3)$, local stable category equivalent.

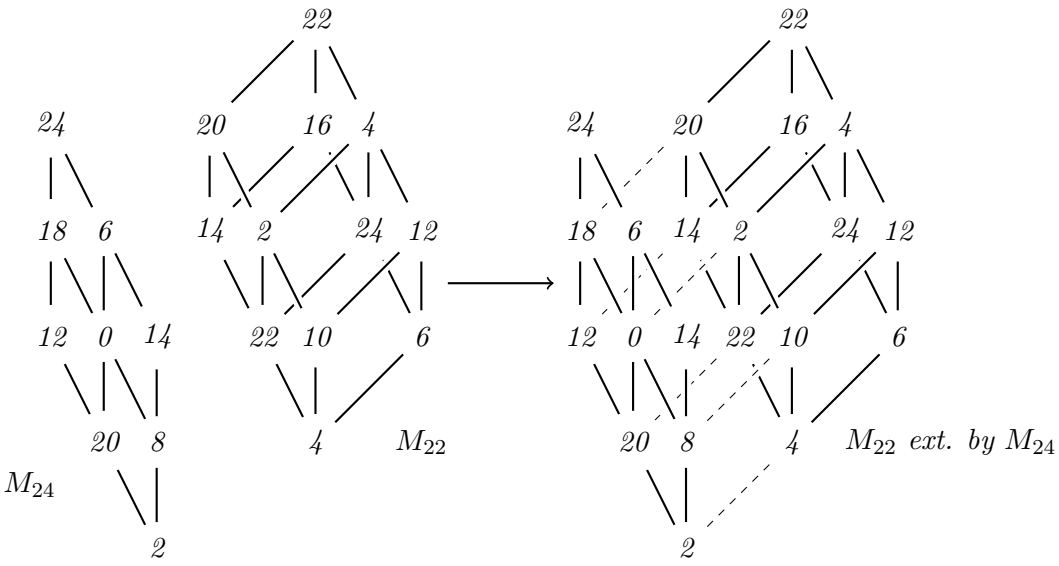
Shorthand: $U_i\Omega^j M_a$ written as ${}^j_i M_a$ $\quad$ U subscripts modulo $3^3 - 1 = 26$

$B_0(3^2)$	Layer 2									Layer 1			Layer 0
	S_0	S_2	S_4	S_6	S_8	S_{10}	S_{12}	S_{14}	S_{16}	S_{18}	S_{20}	S_{22}	S_{24}
b_2	$b_2 = 0$					$b_2 = 1$				$b_2 = 2$			
b_1										$b_1 = 0$		$b_1 = 1$	$b_1 = 2$
$\mathbb{A}_0 = \mathbb{B}_0$	M_0	M_2	M_4	M_6	M_8	M_{10}	M_{12}	M_{14}	M_{16}	M_{18}	M_{20}	M_{22}	${}^1_3 M_{25}$
step 1; $r = 0$												${}^1_3 M_{25}$	Ω^{-1}
$\mathbb{A}_1$	M_0	M_2	M_4	M_6	M_8	M_{10}	M_{12}	M_{14}	M_{16}	M_{18}	M_{20}	${}^1_9 M_{19}$	${}^1_6 M_{24}$
step 2; $r = 0$										${}^1_6 M_{24}$	$M_{26} = 0$		Ω^{-1}
$\mathbb{A}_2$	M_0	M_2	M_4	M_6	M_8	M_{10}	M_{12}	M_{14}	M_{16}	${}^1_9 M_{21}$	${}^1_9 M_{23}$	${}^1_9 M_{19}$	${}^1_9 M_{25}$
step 3; $r = 1$						${}^1_9 M_{19}$	${}^1_9 M_{21}$	${}^1_9 M_{23}$	${}^1_9 M_{25}$	Ω^{-1}	Ω^{-1}	Ω^{-1}	Ω^{-1}
$\mathbb{A}_3$	M_0	M_2	M_4	M_6	M_8	${}^1_1 M_1$	${}^1_1 M_3$	${}^1_1 M_5$	${}^1_1 M_7$	${}_9 M_{21}$	${}_9 M_{23}$	${}_9 M_{19}$	${}^1_{12} M_{24}$
step 4; $r = 0$										${}^1_{12} M_{24}$	$M_{26} = 0$		Ω^{-1}
$\mathbb{A}_4$	M_0	M_2	M_4	M_6	M_8	${}^1_1 M_1$	${}^1_1 M_3$	${}^1_1 M_5$	${}^1_1 M_7$	${}^1_{18} M_{18}$	${}^1_{18} M_{20}$	${}_9 M_{19}$	${}^1_{15} M_{25}$
step 5; $r = 0$												${}^1_{15} M_{25}$	Ω^{-1}
$\mathbb{A}_5$	M_0	M_2	M_4	M_6	M_8	${}^1_1 M_1$	${}^1_1 M_3$	${}^1_1 M_5$	${}^1_1 M_7$	${}^1_{18} M_{18}$	${}^1_{18} M_{20}$	${}^1_{18} M_{22}$	${}^1_{18} M_{24}$
step 6; $r = 1$	${}^1_{18} M_{18}$	${}^1_{18} M_{20}$	${}^1_{18} M_{22}$	${}^1_{18} M_{24}$	$M_{26} = 0$					Ω^{-1}	Ω^{-1}	Ω^{-1}	Ω^{-1}
$\mathbb{A}_6$	${}^1_1 M_9$	${}^1_1 M_{11}$	${}^1_1 M_{13}$	${}^1_1 M_{15}$	${}^1_1 M_{17}$	${}^1_1 M_1$	${}^1_1 M_3$	${}^1_1 M_5$	${}^1_1 M_7$	${}_{18} M_{18}$	${}_{18} M_{20}$	${}_{18} M_{22}$	${}^1_{21} M_{25}$
step 7; $r = 0$												${}^1_{21} M_{25}$	Ω^{-1}
$\mathbb{A}_7$	${}^1_1 M_9$	${}^1_1 M_{11}$	${}^1_1 M_{13}$	${}^1_1 M_{15}$	${}^1_1 M_{17}$	${}^1_1 M_1$	${}^1_1 M_3$	${}^1_1 M_5$	${}^1_1 M_7$	${}_{18} M_{18}$	${}_{18} M_{20}$	${}^1_1 M_{19}$	${}^1_{24} M_{24}$
step 8; $r = 0$										${}^1_{24} M_{24}$	$M_{26} = 0$		Ω^{-1}
$\mathbb{A}_8$	${}^1_1 M_9$	${}^1_1 M_{11}$	${}^1_1 M_{13}$	${}^1_1 M_{15}$	${}^1_1 M_{17}$	${}^1_1 M_1$	${}^1_1 M_3$	${}^1_1 M_5$	${}^1_1 M_7$	${}^1_1 M_{21}$	${}^1_1 M_{23}$	${}^1_1 M_{19}$	${}^1_1 M_{25}$
$\mathbb{A}_9$	${}_1 M_9$	${}_1 M_{11}$	${}_1 M_{13}$	${}_1 M_{15}$	${}_1 M_{17}$	${}_1 M_1$	${}_1 M_3$	${}_1 M_5$	${}_1 M_7$	${}_1 M_{21}$	${}_1 M_{23}$	${}_1 M_{19}$	${}_1 M_{25}$

A.3 Brief illustration for $SL_2(3^3)$

Here we demonstrate how the construction is generalised to larger n using an example with $n = 3$. In the following, the horizontal layers are the Loewy layers of the module, while the skeleton gives the extension relations between simple components. The first example is a generalisation of the case where $n = 2$.

Example A.6. *When $p = 3$, $n = 3$. The following illustrate the module M_{24}, M_{22} as well as M_{22} extended by M_{24} on step $m = 1$.*

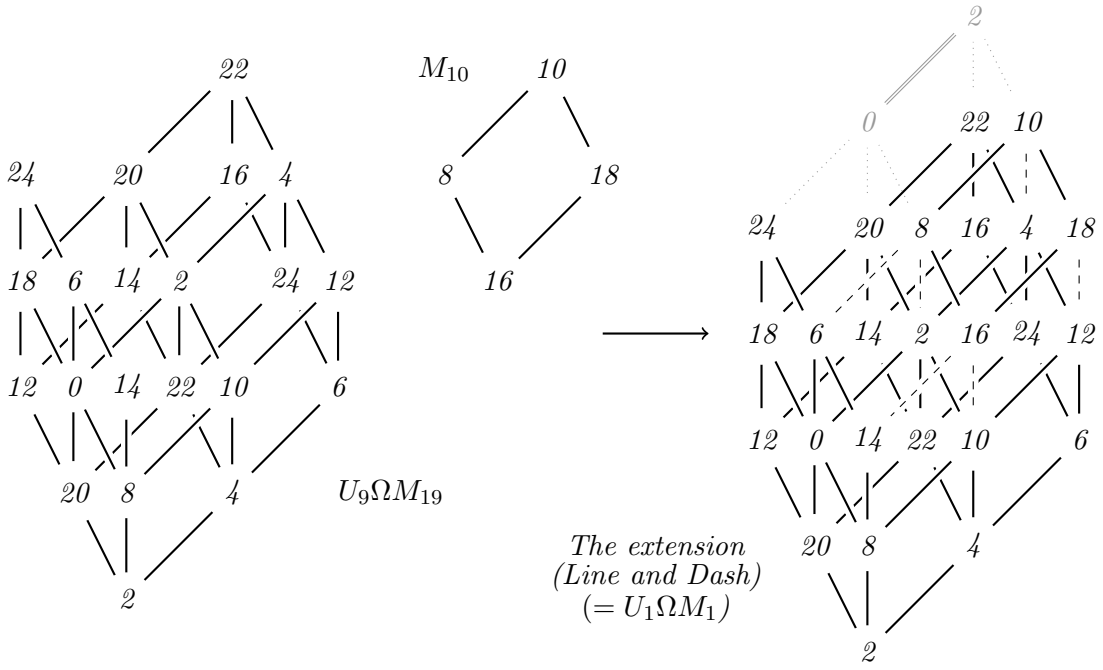


Consider every projective as a $3 \times 3 \times 3$ cube. The resulting module (illustrated by the picture on the right) is isomorphic to the Heller translate of a module with shape $(1, 0, 2)$, which corresponds to M_{19}. Comparing the socles of M_{19} and the module which is isomorphic to M_{22} extended by M_{24}, one can get a distinguished triangle

$$M_{24} \to U_9\Omega M_{19} \to M_{22} \rightsquigarrow .$$

When m is a multiple of $p(= 3)$ (with $r_m = 1$) we put the pre-fabricated modules from previous step to modules 1 layer lower.

Example A.7. *Continuing the previous example, M_{10} is extended by the module formed above ($U_9\Omega M_{19}$) to get a module isomorphic to the Heller translate of U_1M_1.*



This process can be expressed by the distinguished triangle $U_9\Omega M_{19} \to U_1\Omega M_1 \to M_{10} \rightsquigarrow$*, which occurs in step 3.*

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