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Dirichlet Branes on a Calabi-Yau Three-fold Orbifold

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Abstract

The D-brane spectrum of a $\mathbb{Z}_2 \times \mathbb{Z}_2$ Calabi-Yau three-fold orbifold of toroidally compactified Type IIA and Type IIB string theory is analysed systematically. The corresponding K-theory groups are determined and complete agreement is found. New kinds of stable non-BPS D-branes are found, whose stability regions are far more complicated than the previously discussed non-BPS D-branes. The decay channels of non-BPS D-branes beyond their stability regions are identified. Finally the T-dual orbifold is analysed and a suitable K-theory is found.

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1 Introduction

The study of non-BPS D-branes[1] has provided a better understanding of several aspects of string theory. The classification of D-branes in terms of K-theory [4, 5, 6] can be physically understood by thinking of D-branes as solitonic solutions of the tachyon field of higher dimensional brane-anti-brane systems [7, 8, 5, 9]. The study of the tachyon potential using string field theory [10, 11, 12, 13] has given strong evidence to the conjecture that a brane-anti-brane system annihilates into the vacuum. Due to the presence of a single decay mode unstable non-BPS D-branes have been interpreted as sphaleron solutions of string theory [14] and some understanding of bosonic D-branes [15] has now also been developed. The confinement process of the unbroken gauge group boson on the world-volume of annihilating brane-anti-brane systems [16, 5] has been better understood [17, 18, 19].

Another aspect of recent developments has involved the study of stable non-BPS D-branes through the boundary state formalism [20, 21]. This was first considered in a specific context in [7, 22]. Over the past year the boundary state formalism has been used to construct many more examples of non-BPS D-branes [23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33]. These constructions have tested the connection between D-branes and K-theory [34, 25, 29] as well as testing S-dualities beyond the BPS constraints. In particular the dualities between heterotic and Type II theories [24] and heterotic and Type I theories [35, 36, 37] have been investigated. Due to the fact that certain non-BPS D-branes are the lightest states carrying a particular charge it has been possible to identify their duals and compare regions of stability [36] as well as interaction properties [37] in the dual theories.

The integrally (rather than torsion) charged non-BPS D-branes have so far been constructed on $\mathbb{Z}_2$ orbifolds [2] and have been interpreted in the blow-up [23] as coming from certain volume minimising non-supersymmetric cycles of the manifold which in the orbifold limit shrink to one-cycles. Here we discuss a $\mathbb{Z}_2 \times \mathbb{Z}_2$ orbifold of Type IIA and IIB. In particular we study BPS and non-BPS D-branes on such an orbifold. The orbifold is a particular limit of a Calabi-Yau three-fold, and as such preserves N=2, D=4 supersymmetry. The action of the two generators g_1 and g_2 will be taken as

$$g_1(x^0,\ldots,x^9) = (x^0,x^1,x^2,x^3,x^4,-x^5,-x^6,-x^7,-x^8,x^9) \quad (1.1)$$
$$g_2(x^0,\ldots,x^9) = (x^0,x^1,x^2,-x^3,-x^4,x^5,x^6,-x^7,-x^8,x^9) \quad (1.2)$$

and we label $g_3 = g_1 g_2$. Thus both $\mathbb{Z}_2$'s have fixed points. This orbifold has been studied previously [38, 39, 40, 41, 42]. In particular in [39] fractional D-branes on it were discussed and it was noted that there are two kinds of fractional D-branes, those that live at the fixed hyperplanes of one of the g_i (we shall refer to these as *singly* fractional D-branes) and those that live at the fixed hyperplanes of all g_i (we shall refer to these as *totally* fractional D-branes).

[1]For some recent reviews see [1, 2, 3].

[2]In [23, 30] this had been generalised somewhat to a Calabi-Yau $\mathbb{Z}_2 \times \mathbb{Z}_2'$ orbifold, where $\mathbb{Z}_2'$ is freely acting.

In this paper we use the boundary state formalism to investigate the full D-brane spectrum, BPS and non-BPS on this orbifold and study the stability regions of non-BPS $\hat{\text{D}}$-branes on the orbifold. We give a boundary state description of both kinds of fractional D-branes as well as bulk D-branes wrapping the special Lagrangian three-cycle of the Calabi-Yau orbifold. We find non-BPS $\hat{\text{D}}$-branes similar to those of [29]. We also find that there are new kinds of non-BPS $\hat{\text{D}}$-branes on this orbifold, and we refer to these too as *truncated* $\hat{\text{D}}$-branes [29]. The truncated $\hat{\text{D}}$-branes break all supersymmetry and are charged under twisted R-R fields. The relevant K-theory groups are evaluated and complete agreement between the D-branes we have constructed and K-theory is found.

Some of the new $\hat{\text{D}}$-branes have very unusual domains of stability and besides coupling to twisted R-R sectors they couple to certain twisted NS-NS sectors as well. We investigate the decay channels of the various $\hat{\text{D}}$-branes. Besides the usual decays into brane-anti-brane pairs of BPS fractional D-branes there are other decay channels in which non-BPS $\hat{\text{D}}$-branes decay into one another. There are also certain decay channels whose decay products are unknown. When the decay products are known the masses and charges of the relevant objects are the same at critical radii, indicating that the process is a marginal deformation in the conformal field theory [8, 43, 44].

The construction of boundary states describing the D-branes is given in section 2, as well as the appendices. In section 3 we compute the relevant equivariant K-theory groups and find complete agreement with section 2. In section 4 we consider the compact orbifold and we discuss in detail the minimal charge configurations of the D-branes. The stability regions of the truncated non-BPS $\hat{\text{D}}$-branes are analysed in section 5. The new kind of truncated $\hat{\text{D}}$-brane exhibits some remarkable stability properties as given in equation (4.8). In section 6 we identify most of the decay channels for the various $\hat{\text{D}}$-branes. Finally section 7 discusses D-branes on the T-dual orbifold. A new K-theory group suitable for such an orbifold is defined.

2 Non-compact orbifold

In this section we discuss the non-compact orbifold, while later on the directions $x^3, \ldots, x^8$ will be compactified on circles. We first briefly discuss the massless spectrum, and then construct relevant D-brane boundary states.

The $\mathbb{Z}_2 \times \mathbb{Z}_2$ orbifold discussed here has been studied in the past [38, 39]. In particular its closed string spectrum has been worked out in some detail [38]. Here we point out some of the points of present importance. Each g_i, $i = 1, 2, 3$ gives rise to a twisted sector. Both the twisted NS and R sectors are massless and have zero modes, those of the R sector are in the directions unaffected by g_i, while the zero modes in the NS sector are in the directions inverted by g_i. The lowest lying states in the twisted R-R sector transform as a tensor product of $SO(4)$ (which contains the spacetime SO(2)), and are further required to be invariant under the remaining orbifold projections. The twisted NS-NS sector ground states transform as bispinors of an internal $SO(4)$, and have to be invariant under the other elements of the orbifold group. They give rise to four dimensional scalars. We denote the g_i-twisted sectors by NS-NS,Tg_i and R-R,Tg_i.

Boundary states corresponding to physical D-branes consist of linear combinations of boundary states from the various closed string sectors which are GSO and orbifold invariant. We refer to a D-brane as of type $(r;\mathbf{s}) = (r; s_1, s_2, s_3)$, if it is a $r+s_1+s_2+s_3$-brane, and extends along $r+1$, s_1, s_2, s_3 of the directions $(x^0,\ldots,x^2,x^9)$, (x^3,x^4), (x^5,x^6), (x^7,x^8), respectively.[3] Due to the presence of fermionic zero modes in the various sectors requiring GSO and orbifold invariance places restrictions on r and the s_i. These conditions are analysed in Appendix B.

A boundary state which corresponds to a physical D-brane has to satisfy certain consistency criteria. In particular the string with endpoints on the D-brane has to be a suitably projected open string. It turns out that in the $\mathbb{Z}_2 \times \mathbb{Z}_2$ orbifold there are four kinds of combinations of boundary states from the various closed string sectors which correspond to physical D-branes. These include a bulk D-brane

$$|B(r;\mathbf{s})\rangle_{\text{NS-NS}} + \varepsilon|B(r;\mathbf{s})\rangle_{\text{R-R}} \;, \tag{2.1}$$

where $\varepsilon = \pm 1$, and a fractional D-brane

$$|B(r\mathbf{s};)\rangle_{\text{NS-NS}} + \varepsilon|B(r;\mathbf{s})\rangle_{\text{R-R}} + \sum_{i=1}^{3}\varepsilon_i\left(|B(r;\mathbf{s})\rangle_{\text{NS-NS,T}g_i} + \varepsilon|B(r;\mathbf{s})\rangle_{\text{R-R,T}g_i}\right) , \tag{2.2}$$

for which $\varepsilon_i\varepsilon_j = \varepsilon_k$ for $i \neq j \neq k \neq i$; both these objects are BPS as can be seen for example from the vanishing of the cylinder amplitudes computed in appendix A. In the above $\varepsilon, \varepsilon_i$ indicate the sign of the various R-R charges. Further there are truncated non-BPS $\hat{\text{D}}$-branes charged under one type of twisted R-R field

$$|B(r;\mathbf{s})\rangle_{\text{NS-NS}} + \varepsilon|B(r;\mathbf{s})\rangle_{\text{R-R,T}g_i}, \quad i = 1,2,3\,. \tag{2.3}$$

Their boundary state is very similar to that of the $\hat{\text{D}}$-branes in [29]. Finally there is a second type of truncated non-BPS $\hat{\text{D}}$-brane whose boundary state is

$$\begin{aligned}|B(r;\mathbf{s})\rangle_{\text{NS-NS}} + \varepsilon_i|B(r;\mathbf{s})\rangle_{\text{R-R,T}g_i} + \varepsilon_i\varepsilon_j|B(r;\mathbf{s})\rangle_{\text{NS-NS,T}g_k} + \varepsilon_j|B(r;\mathbf{s})\rangle_{\text{R-R,T}g_j}\,, \\ i,j,k = 1,2,3, \quad \text{s.t. } \epsilon_{ijk} \neq 0\,.\end{aligned} \tag{2.4}$$

This is a new kind of $\hat{\text{D}}$-brane which couples to a twisted NS-NS sector as well as to twisted R-R sectors.

A second consistency condition is that a string beginning on any one of the above branes and ending on a different one must describe an open string. In [29] this condition ensured that for

[3] With this convention g_i acts trivially on the s_i directions.

any given (r, s) if a fractional and truncated (r, s)-branes existed then the fractional brane would be of smaller mass and charge rendering the truncated brane unstable. Applying the condition to the branes above we again conclude that if a fractional and truncated brane exist for some $(r; \mathbf{s})$ then it is the fractional object that is minimal and stable. Similarly one shows that if for some $(r; \mathbf{s})$ both truncated objects exist it is the one in equation (2.4) that is minimally charged and so is fundamental.

Given these consistency conditions the D-brane spectrum can then be determined. The detailed analysis can be found in appendix B. Bulk and fractional D-branes exist for $(r; \mathbf{s})$ of the form

$$(r;0,0,0)\,,(r;0,0,2)\,,(r;0,2,0)\,,(r;2,0,0)\,,(r;0,2,2)\,,(r;2,0,2)\,,(r;2,2,0)\,,(r;2,2,2)\,, \quad (2.5)$$

where r is even/odd for Type IIA/IIB. The elementary objects above are *totally* fractional branes which live on the fixed point of the whole orbifold group ($x^3 = \ldots = x^8 = 0$) and are charged under all three twisted R-R sectors. The corresponding K-theory group should be $\mathbb{Z}^{\oplus 4}$. The totally fractional branes correspond to a single brane in the covering space. Two such D-branes with the same bulk and g_i-twisted R-R charges but opposite remaining charges can come together to form a *singly* fractional brane, which is stuck to the fixed plane of g_i. Two singly fractional branes with opposite twisted R-R charge can come together and form a bulk brane. Further bulk D-branes exist for $(r; 1, 1, 1)$. Since these are charged under the untwisted R-R field, the corresponding K-group should be $\mathbb{Z}$.

Truncated $\hat{\text{D}}$-branes with a boundary state given by equation (2.3) exist for $(r; \mathbf{s})$ of the form[4]

$$(r;0,1,1),\ (r;1,0,1),\ (r;1,1,0),\ (r;2,1,1),\ (r;1,2,1),\ (r;1,1,2)\,, \quad (2.6)$$

with r even/odd for Type IIA/IIB. These $\hat{\text{D}}$-branes are stuck at the fixed points of the g_i under which they are charged, and one expects the corresponding K-group to be $\mathbb{Z}$. Truncated $\hat{\text{D}}$-branes with a boundary state given by equation (2.4) exist for $(r; \mathbf{s})$ of the form

$$\begin{aligned}&(r;0,0,1),\ (r;1,0,0),\ (r;0,1,0),\ (r;2,0,1),\ (r;1,2,0),\ (r;2,1,0),\\ &(r;0,2,1),\ (r;1,0,2),\ (r;0,1,2),\ (r;2,2,1),\ (r;1,2,2),\ (r;2,1,2)\,.\end{aligned} \quad (2.7)$$

Here r is even/odd for Type IIA/IIB. The basic such branes are stuck at the fixed points of the whole orbifold group and are charged under two twisted R-R fields suggesting that the K-group should be $\mathbb{Z} \oplus \mathbb{Z}$. However, as with the fractional branes there are also $\hat{\text{D}}$-branes with the above $(r; \mathbf{s})$ which are charged under only g_i-twisted R-R fields and as such are only stuck to the fixed points of g_i.

[4]We disregard here the tachyon that arises when considering decompactified $s_i > 0$ truncated branes [29], as on compactification the D-brane will stabilise for certain values of the compactification radii.

3 K-theory analysis in uncompactified theory

The K-groups relevant to this orbifold are the $\mathbb{Z}_2 \times \mathbb{Z}_2$-equivariant K-groups [45] with compact support

$$K^*_{\mathbb{Z}_2\times\mathbb{Z}_2}(\mathbb{R}^{a;b,c,d})\,, \tag{3.1}$$

where $a = 0, \ldots, 4$ and $b, c, d = 0, 1, 2$. In the above the directions a are left invariant by $G = \mathbb{Z}_2 \times \mathbb{Z}_2$, c and d are inverted by the first $\mathbb{Z}_2$ and b, c are inverted by the second $\mathbb{Z}_2$. These groups exhibit complex Bott periodicity in a, b, c and d thus the answer depends only on the parity of a, b, c and d. Further there is a symmetry between b, c, d. For example $K^*_G(\mathbb{R}^{a;b,c,d}) = K^*_G(\mathbb{R}^{a;c,b,d})$. As a result we need only compute very few terms. Since the representation ring of $\mathbb{Z}_2 \times \mathbb{Z}_2$ is $\mathbb{Z}^{\oplus 4}$ we have

$$K^*_G(\mathbb{R}^{a;b,c,d}) = \begin{cases} \mathbb{Z}^{\oplus 4} & a + * \text{ even} \\ 0 & a + * \text{ odd} \end{cases} \tag{3.2}$$

for b, c, d even. Fractional branes in the decompactified theory can be charged under three twisted R-R fields as well as the untwisted R-R field, the above K-groups confirm the presence of all fractional branes. Since g_1 acts trivially on $\mathbb{R}^{0;1,0,0}$ we have

$$K^*_G(\mathbb{R}^{0;1,0,0}) = K^*_{\mathbb{Z}_2}(\mathbb{R}^{0,1}) \otimes R[\mathbb{Z}_2]\,, \tag{3.3}$$

where on the right hand side $\mathbb{Z}_2$ inverts the line $\mathbb{R}^{0,1}$. The right-hand side groups have been obtained in [25, 29]. Thus we have for c, d even

$$K^*_G(\mathbb{R}^{a;1,c,d}) = \begin{cases} \mathbb{Z}^{\oplus 2} & a + * \text{ even} \\ 0 & a + * \text{ odd} \end{cases} \tag{3.4}$$

Similar results hold for permutations of $1, c, d$. This is in agreement with the presence of truncated $\hat{\mathrm{D}}$-branes in (2.7). In order to compute $K^*_G(\mathbb{R}^{0;1,1,2k})$ we re-write it as

$$K^*_G(\mathbb{R}^{0;1,1,0}) = K^*_G(\mathbb{R}^{0;1,0,0} \times D^1, \mathbb{R}^{a;1,0,0} \times S^0)\,, \tag{3.5}$$

where $S^0 \subset D^1 \subset \mathbb{R}^{0;0,1,0}$ are the zero-sphere (two points) and the one-disc (the interval). By homotopy equivalence we have

$$K^*_G(X \times D^1) = K^*_G(X)\,, \tag{3.6}$$

for any X and

$$K^*_G(\mathbb{R}^{0;1,0,0} \times S^0) = K^*_{\mathbb{Z}_2}(\mathbb{R}^{0,1})\,, \tag{3.7}$$

where on the right hand side $\mathbb{Z}_2$ inverts the real line $\mathbb{R}^{0,1}$. We may now use the long exact sequence

$$\begin{aligned} \cdots \;\to\; & K^{-1}_G(\mathbb{R}^{0;1,0,0} \times S^0) \to K_G(\mathbb{R}^{0;1,0,0} \times \mathbb{R}^{0;0,1,0}) \;\to\; K_G(\mathbb{R}^{0;1,0,0} \times D^1) \\ \to\; & K_G(\mathbb{R}^{0;1,0,0} \times S^0) \;\to\; K^1_G(\mathbb{R}^{0;1,0,0} \times \mathbb{R}^{0;0,1,0}) \to\; K^1_G(\mathbb{R}^{0;1,0,0} \times D^1) \;\to \cdots \end{aligned} \tag{3.8}$$

which in turn becomes

$$0 \to K_G(\mathbb{R}^{0;1,1,0}) \to \mathbb{Z} \oplus \mathbb{Z} \to \mathbb{Z} \to K^1_G(\mathbb{R}^{0;1,1,0}) \to 0\,. \tag{3.9}$$

It is not difficult to see that the map $\mathbb{Z} \oplus \mathbb{Z} \to \mathbb{Z}$ is surjective. This gives for d even

$$K^*_G(\mathbb{R}^{a;1,1,d}) = \begin{cases} \mathbb{Z} & a+* \text{ even} \\ 0 & a+* \text{ odd} \end{cases} \tag{3.10}$$

Together with the permutations of $1, 1, d$ this confirms the presence of truncated $\hat{\mathrm{D}}$-branes in equation (2.6). Finally we have

$$K^*_G(\mathbb{R}^{0;1,1,1}) = K^*_G(\mathbb{R}^{0;1,1,0} \times D^1, \mathbb{R}^{a;1,1,0} \times S^0)\,, \tag{3.11}$$

where $S^0 \subset D^1 \subset \mathbb{R}^{0;0,0,1}$. Using equation (3.6) the results above and

$$K^*_G(\mathbb{R}^{0;1,1,0} \times \mathcal{S}^0) = K^*_{\mathbb{Z}_2}(\mathbb{R}^{0,2})\,, \tag{3.12}$$

the following sequence is exact

$$0 \to K_G(\mathbb{R}^{0;1,1,1}) \to \mathbb{Z} \to \mathbb{Z} \oplus \mathbb{Z} \to K^1_G(\mathbb{R}^{0;1,1,1}) \to 0\,. \tag{3.13}$$

One can show using this exact sequence that [46]

$$K^*_G(\mathbb{R}^{a;1,1,1}) = \begin{cases} \mathbb{Z} & a+* \text{ odd} \\ 0 & a+* \text{ even} \end{cases} \tag{3.14}$$

This confirms the presence of bulk BPS branes with $s_1 = s_2 = s_3 = 1$. We have demonstrated complete agreement between K-theory and D-branes on this orbifold.

4 The compactified orbifold

From now on we take the directions $x^3, \ldots, x^8$ to be compactified on circles of radii $R_3, \ldots, R_8$. This introduces new fixed points at $x^i = \pi R_i$ and gives rise to new twisted sectors - in total 48 twisted R-R sectors and 48 twisted NS-NS sectors. D-branes with $s_i > 0$ will be charged under the twisted sectors over which they stretch. The structure of the boundary states and in particular the normalisation is described in appendix A. The allowed charges for various D-branes are restricted by factorisation. In particular a brane charged under many R-R charges cannot have an arbitrary choice of minimal positive and negative charges. This was already encountered in [29] where for example a $\hat{\mathrm{D}}(r, 2)$-brane in the $\mathcal{I}_n(-1)^{F_l}$ orbifolds was charged under four twisted R-R fields but the minimally charged branes had to have an even number of negative charges.

This behaviour is typical and we encounter it here as well. The restriction arises as a result of cylinder-annulus consistency. The process of checking what sign freedom one has in the closed string sector in order to factorise on a consistent open string amplitude is laborious, here we summarise the results.

In the following subsections we shall discuss the D-brane spectrum of the compactified orbifold, paying attention to the allowed charges, as well as stability regions and decay products of the non-BPS branes. For simplicity we concentrate on the fractional D$(r;0,0,0)$ and D$(r;0,0,2)$-branes and truncated $\hat{\mathrm{D}}(r;0,0,1)$ and $\hat{\mathrm{D}}(r;0,1,1)$-branes. The extension of these results to the other branes is obvious.

4.1 The D$(r;0,0,0)$-brane

The fully fractional D$(r;0,0,0)$-brane's consistent boundary state is given by

$$\begin{aligned} |D(r;0,0,0),\varepsilon,\varepsilon_i\rangle \;&=\; |D(r;0,0,0)\rangle_{\text{NS-NS}} + \varepsilon|D(r;0,0,0)\rangle_{\text{R-R}} \\ &\quad + \sum_{i=1}^{3}\varepsilon_i\left(|D(r;0,0,0)\rangle_{\text{NS-NS,T}g_i} + \varepsilon|D(r;0,0,0)\rangle_{\text{R-R,T}g_i}\right) , \end{aligned} \tag{4.1}$$

with $\varepsilon_3 = \varepsilon_1\varepsilon_2 = \pm 1$. If we denote by $[a;b,c,d]$ the four charges under which the D-brane is charged (with a corresponding to the bulk charge, and b,c,d to the twisted charges) the allowed configurations of minimal charge are

$$\begin{gathered} [1;1,1,1],\ [1;1,-1,-1],\ [1;-1,1,-1],\ [1;-1,-1,1], \\ [-1;-1,-1,-1],\ [-1;-1,1,1],\ [-1;1,-1,1],\ [-1;1,1,-1]\,. \end{gathered} \tag{4.2}$$

From this it is easy to see that a singly fractional brane is a sum of two totally fractional branes. For example $[2;0,2,0] = [1;1,1,1] + [1;-1,1,-1]$ or $[-2;0,2,0] = [-1;-1,1,1] + [-1;1,1,-1]$.

4.2 The D$(r;0,0,2)$-brane

We take the fully fractional D$(r;0,0,2)$-brane to extend along x^7 and x^8 and to be fixed at $x^3 = \ldots = x^6 = 0$. Its' boundary state is

$$\begin{aligned} |D(r;0,0,2),\varepsilon,\varepsilon_i,\theta_j\rangle \;&=\; \sum_{w_7,w_8} e^{i(\theta_7 w_7+\theta_8 w_8)}(|D(r;0,0,2),w_7,w_8\rangle_{\text{NS-NS}} + \varepsilon|D(r;0,0,2),w_7,w_8\rangle_{\text{R-R}}) \\ &\quad + \sum_{i\in\{1,2\}}\Big(\varepsilon_i(|D(r;0,0,2)\rangle_{\text{NS-NS,T}g_{i,1}} + \varepsilon|D(r;0,0,2)\rangle_{\text{R-R,T}g_{i,1}}) \\ &\quad + \varepsilon_i e^{i\theta_7}(|D(r;0,0,2)\rangle_{\text{NS-NS,T}g_{i,2}} + \varepsilon|D(r;0,0,2)\rangle_{\text{R-R,T}g_{i,2}}) \\ &\quad + \varepsilon_i e^{i\theta_8}(|D(r;0,0,2)\rangle_{\text{NS-NS,T}g_{i,3}} + \varepsilon|D(r;0,0,2)\rangle_{\text{R-R,T}g_{i,3}}) \end{aligned}$$

$$
\begin{aligned}
&+\varepsilon_i e^{i(\theta_7+\theta_8)}\big(|D(r;0,0,2)\rangle_{\text{NS-NS,T}g_{i,4}} + \varepsilon|D(r;0,0,2)\rangle_{\text{R-R,T}g_{i,4}}\big)\Big) \\
&+\varepsilon_1\varepsilon_2 \sum_{w_7,w_8} e^{i(\theta_7 w_7+\theta_8 w_8)}\big(|D(r;0,0,2),w_7,w_8\rangle_{\text{NS-NS,T}g_3} \\
&\qquad +\varepsilon|D(r;0,0,2),w_7,w_8\rangle_{\text{R-R,T}g_3}\big)
\end{aligned}
\tag{4.3}
$$

From this it is apparent that D($r;0,0,2$)-branes can only have an even number of negative R-R,Tg_1 and R-R,Tg_2 charges, and further that the sign of the R-R,Tg_3 charge is fixed by the other twisted charges. A D($r;0,0,2$)-brane with all positive R-R,Tg_1 and bulk R-R charges, but negative remaining charges (take $\varepsilon = \varepsilon_1 = +1$, $\theta_7 = \theta_8 = 0$, $\varepsilon_3 = -1$) can be added to a D($r;0,0,2$)-brane with all positive charges, to give a brane which is only charged under bulk R-R and R-R,Tg_1 fields, and so is a singly fractional brane (it can move off the $x^3 = x^4 = 0$ fixed plane).

4.3 The $\hat{\mathbf{D}}(r;0,0,1)$-brane

We consider the $\hat{\text{D}}(r;0,0,1)$-brane to stretch along the x^8 direction and to be fixed at $x^3 = \ldots = x^7 = 0$. In order to be consistent with the open string partition function the $\hat{\text{D}}(r;0,0,1)$-brane boundary state must be written in the form

$$
\begin{aligned}
|\hat{D}(r;0,0,1),\varepsilon_i,\theta\rangle \;=\; &\sum_w e^{i\theta w}|\hat{D}(r;0,0,1),w\rangle_{\text{NS-NS}} + \varepsilon_1|\hat{D}(r;0,0,1)\rangle_{\text{R-R,T}g_{1,1}} \\
&+\varepsilon_1 e^{i\theta}|\hat{D}(r;0,0,1)\rangle_{\text{R-R,T}g_{1,2}} + \varepsilon_2|\hat{D}(r;0,0,1)\rangle_{\text{R-R,T}g_{2,1}} \\
&+\varepsilon_2 e^{i\theta}|\hat{D}(r;0,0,1)\rangle_{\text{R-R,T}g_{2,2}} + \varepsilon_1\varepsilon_2\sum_w e^{i\theta w}|\hat{D}(r;0,0,1),w\rangle_{\text{NS-NS,T}g_3}\,.
\end{aligned}
\tag{4.4}
$$

Denoting by $[a,b;c,d]$ the four charges a $\hat{\text{D}}(r;0,0,1)$-brane carries (two from the g_1-twisted sector, say a,b and two from the g_2 twisted sector, say c,d), the allowed $\hat{\text{D}}(r;0,0,1)$-branes with minimal charge are

$$
\begin{gathered}
[1,1;1,1],\ [-1,-1;1,1],\ [1,1;-1,-1],\ [1,-1;1,-1], \\
[-1,1;1,-1],\ [1,-1;-1,1],\ [-1,1;-1,1],\ [-1,-1;-1,-1]\,.
\end{gathered}
\tag{4.5}
$$

A singly charged $\hat{\text{D}}(r;0,0,1)$-brane with minimal charge comes from $[1,1;1,1]+[1,1;-1,-1] = [2,2;0,0]$ or $[1,-1;1,-1]+[1,-1,-1,1] = [2,-2;0,0]$.

Since the stability region of this $\hat{\text{D}}$-brane is so different from the non-BPS branes previously encountered we discuss this case in detail. The partition function for a string with both end-points on this $\hat{\text{D}}$-brane is

$$
\int \frac{dt}{2t}\,\text{Tr}_{NS-R}\left(\frac{1+(-1)^F g_1}{2}\,\frac{1+(-1)^F g_3}{2}\,e^{-2tH_o}\right)
$$

$$
\begin{aligned}
&= \frac{V_{r+1}}{4(2\pi)^{r+1}} \int \frac{dt}{2t}(2t)^{-(r+1)/2} \left(\frac{f_3^8(\tilde{q}) - f_2^8(\tilde{q})}{f_1^8(\tilde{q})} \right) \sum_{n_8 \in \mathbb{Z}} e^{-2t\pi n_8^2/R_8^2} \prod_{m=3}^{7} \sum_{w_m \in \mathbb{Z}} e^{-2t\pi w_m^2 R_m^2} \\
&\quad - \frac{V_{r+1}}{(2\pi)^{r+1}} \int \frac{dt}{2t}(2t)^{-(r+1)/2} \frac{f_3^4(\tilde{q}) f_4^4(\tilde{q})}{f_1^4(\tilde{q}) f_2^4(\tilde{q})} \prod_{m=3,4} \sum_{w_m \in \mathbb{Z}} e^{-2t\pi w_m^2 R_m^2} \\
&\quad - \frac{V_{r+1}}{(2\pi)^{r+1}} \int \frac{dt}{2t}(2t)^{-(r+1)/2} \frac{f_3^4(\tilde{q}) f_4^4(\tilde{q})}{f_1^4(\tilde{q}) f_2^4(\tilde{q})} \prod_{m=5,6} \sum_{w_m \in \mathbb{Z}} e^{-2t\pi w_m^2 R_m^2} \\
&\quad + \frac{V_{r+1}}{(2\pi)^{r+1}} \int \frac{dt}{2t}(2t)^{-(r+1)/2} \frac{f_3^4(\tilde{q}) f_4^4(\tilde{q})}{f_1^4(\tilde{q}) f_2^4(\tilde{q})} \sum_{w_7 \in \mathbb{Z}} e^{-2t\pi w_7^2 R_7^2} \sum_{n_8 \in \mathbb{Z}} e^{-2t\pi n_8^2/R_8^2} \\
&\approx \frac{V_{r+1}}{4(2\pi)^{r+1}} \int \frac{dt}{2t}(2t)^{-(r+1)/2} \tilde{q}^{-1} \Big(\tilde{q}^{2(n_8/R_8)^2} + \tilde{q}^{2(w_7 R_7)^2} + \sum_{i=3}^{7} \tilde{q}^{2(w_i R_i)^2} (\tilde{q}^{2(n_8/R_8)^2} + \tilde{q}^{2(w_7 R_7)^2}) \\
&\quad + \tilde{q}^{2(w_3 R_3)^2 + 2(w_5 R_5)^2} + \tilde{q}^{2(w_3 R_3)^2 + 2(w_6 R_6)^2} + \tilde{q}^{2(w_4 R_4)^2 + 2(w_5 R_5)^2} + \tilde{q}^{2(w_4 R_4)^2 + 2(w_6 R_6)^2} \Big) \, . \qquad (4.6)
\end{aligned}
$$

We have expanded to relevant order in the last line of the amplitude. It is then clear that the tachyon cancels provided[5]

$$
R_7 \geq \frac{1}{\sqrt{2}} \, , \qquad R_8 \leq \sqrt{2} \, , \qquad (4.7)
$$

$$
R_3^2 + R_5^2 \geq \frac{1}{2} \, , \quad R_4^2 + R_5^2 \geq \frac{1}{2} \, , \quad R_3^2 + R_6^2 \geq \frac{1}{2} \, , \quad R_4^2 + R_6^2 \geq \frac{1}{2} \, . \qquad (4.8)
$$

Note that by fixing say R_3, R_4 to suitable values R_5, R_6 can take on any value and the truncated D-brane will still be stable.

In order to analyse the decay products of this $\hat{\mathrm{D}}$-brane we take it to carry a unit of positive twisted R-R charge in each of the four sectors under which it is charged (two R-R,Tg_1 and two R-R,Tg_2). As can be seen from the above equations there are three distinct decay channels: R_8 can increase beyond $\sqrt{2}$, R_7 can decrease below $1/\sqrt{2}$, and the bounds for R_3, R_4, R_5, R_6 in equation (4.8) can be violated.

For $R_8 > \sqrt{2}$ the $\hat{\mathrm{D}}(r;0,0,1)$-brane decays, conserving mass and charge, into a pair of totally fractional $\mathrm{D}(r;0,0,0)$-branes, with opposite bulk and R-R,Tg_3 charge. In the notation of equation (4.1) one of the fractional D-branes has $\varepsilon = \varepsilon_1 = \varepsilon_2 = \varepsilon_3 = 1$, and the other has

[5] It is straightforward to see how this generalises to other $\hat{\mathrm{D}}(0;2k,2k',1)$-branes. For example for a $\hat{\mathrm{D}}(0;2,0,1)$-brane stretching along the directions x^3, x^4, x^8 the stability region is

$$
R_7 \geq \frac{1}{\sqrt{2}} \, , \qquad R_8 \leq \sqrt{2} \, ,
$$

$$
\frac{1}{R_3^2} + R_5^2 \geq \frac{1}{2} \, , \quad \frac{1}{R_4^2} + R_5^2 \geq \frac{1}{2} \, , \frac{1}{R_3^2} + R_6^2 \geq \frac{1}{2} \, , \quad \frac{1}{R_4^2} + R_6^2 \geq \frac{1}{2} \, .
$$

$\varepsilon = \varepsilon_1 = -\varepsilon_2 = \varepsilon_3 = -1$. The easiest way to compare the mass and charges of the original truncated brane with those of the decay products is by perusal of the relevant normalisation constants computed in appendix A. The normalisation constants of the $\hat{\mathrm{D}}(r;0,0,1)$-brane are (cf. equations (A.36) and (A.37))

$$\mathcal{N}_{td,U}^2 = \frac{V_{r+1}}{(2\pi)^{r+1}}\frac{1}{64}\frac{R_8}{\prod_{m=3}^{7} R_m}\,, \tag{4.9}$$

$$\mathcal{N}_{td,Tg_1,i}^2 = \frac{V_{r+1}}{(2\pi)^{r+1}}\frac{2^3}{64}\frac{1}{R_3R_4}\,, \qquad i = 1,2\,, \tag{4.10}$$

$$\mathcal{N}_{td,Tg_3,i}^2 = \frac{V_{r+1}}{(2\pi)^{r+1}}\frac{2^3}{64}\frac{1}{R_5R_6}\,, \qquad i = 1,2\,, \tag{4.11}$$

while those of the two $\mathrm{D}(r;0,0,0)$-branes (cf. equations (A.32) and (A.33))

$$\mathcal{N}_{f,U}^2 = 4\frac{V_{r+1}}{(2\pi)^{r+1}}\frac{1}{128}\frac{1}{\prod_{m=3}^{8} R_m}\,, \tag{4.12}$$

$$\mathcal{N}_{f,Tg_1,i}^2 = \frac{V_{r+1}}{(2\pi)^{r+1}}\frac{2^4}{128}\frac{1}{R_3R_4}\,, \qquad i = 1,2\,, \tag{4.13}$$

$$\mathcal{N}_{f,Tg_3,i}^2 = \frac{V_{r+1}}{(2\pi)^{r+1}}\frac{2^4}{128}\frac{1}{R_5R_6}\,, \qquad i = 1,2\,. \tag{4.14}$$

The factor of 4 comes from the fact that there are two BPS branes. These agree at the critical radius.

The second decay channel occurs when we decrease R_7 below $1/\sqrt{2}$. The decay products are a pair of totally fractional $\mathrm{D}(r;0,0,2)$-branes. In the notation of section 4.3 one of the D-branes has $\epsilon, \epsilon_1, \epsilon_3 = +1$, and $\theta_7, \theta_8 = 0$, while the other brane has $\epsilon, \epsilon_1, \epsilon_3 = -1$ and $\theta_7, \theta_8 = 0$. Comparing the normalisation constants confirms that charge and mass are conserved in the decay.

The third decay channel seems much more complicated. Naively one might expect a decay into a brane-anti-brane combination of fractional branes stretching along x^8 and one of $x^3\,, x^4\,, x^5\,, x^6$. However, there are no such fractional branes. If the dacay is to be into branes with two internal directions, then by considering the Dirac quantisation condition these will have to be fractional branes, which presumably will be somehow bent away from the axes, and not have a boundary state description of the type analysed in this paper.

It is also possible to analyse the stability of the $\hat{\mathrm{D}}(r;0,0,1)$-brane charged only under,say, the g_1-twisted R-R sector. Its stability region is

$$R_5, R_6, R_7 \geq \frac{1}{\sqrt{2}}\,, \qquad\qquad R_8 \leq \sqrt{2}\,, \tag{4.15}$$

and the R_7 and R_8 decay channels follow easily from the decay channels of the $\hat{\mathrm{D}}(r;0,0,1)$-brane charged under the g_1 and g_2-twisted R-R sectors discussed above.

4.4 The $\hat{\mathbf{D}}(r;0,1,1)$-brane

Finally, we discuss a $\hat{\rm D}(r;0,1,1)$-brane which we take to extend along x^8 and x^6, and be fixed at $x^5=x^7=0$ and to consist of two identical copies at (x^3,x^4) and $(-x^3,-x^4)$, as explained in appendix A. The allowed boundary states are of the form

$$\begin{aligned}|\hat{D}(r;0,1,1),\varepsilon_i,\theta_j\rangle =&\sum_{w_6,w_8} e^{i(\theta_6 w_6+\theta_8 w_8)}|\hat{D}(r;0,1,1),w_6,w_8\rangle_{\text{NS-NS}}\\ &+\varepsilon_1|\hat{D}(r;0,1,1)\rangle_{\text{R-R,T}g_{1,1}}+\varepsilon_1 e^{i\theta_6}|\hat{D}(r;0,1,1)\rangle_{\text{R-R,T}g_{1,2}}\\ &+\varepsilon_1 e^{i\theta_8}|\hat{D}(r;0,1,1)\rangle_{\text{R-R,T}g_{1,3}}+\varepsilon_1 e^{i(\theta_6+\theta_8)}|\hat{D}(r;0,1,1)\rangle_{\text{R-R,T}g_{1,4}}\,.\end{aligned} \tag{4.16}$$

Hence, such truncated branes can only carry an even number of negative twisted R-R charges. The stability of this brane is very similar to those encountered in $\mathbb{Z}_2$ orbifolds, we simply state the results here. It is stable for

$$R_5,R_7\geq\frac{1}{\sqrt{2}}\,,\qquad\qquad R_6,R_8\leq\sqrt{2}\,. \tag{4.17}$$

The $\hat{\rm D}$-brane can decay in two different ways: by increasing R_6 or R_8 beyond $\sqrt{2}$, or by decreasing R_5 or R_7 below $1/\sqrt{2}$. In the first the decay is into a pair of singly charged $\hat{\rm D}(r;0,0,1)$-branes, which carry the same charges as the $\hat{\rm D}(r;0,1,1)$-brane. It is straightforward to check that at the critical radius ($R_6=\sqrt{2}$) mass and charge are conserved. The second decay channel corresponds to decreasing R_5 below $1/\sqrt{2}$. The $\hat{\rm D}$-brane decays into a pair of singly charged $\hat{\rm D}(r;0,2,1)$-branes, whose charges are the same at the four fixed points on which the $\hat{\rm D}(r;0,1,1)$-brane ends and opposite at the other four fixed points. Again both mass and charge are conserved at the transition. The decay products of the $\hat{\rm D}(r;0,1,1)$-branes are different from the decay products previously encountered. In particular the brane does not decay into a brane-anti-brane pair of separately BPS objects, rather the decay is into other non-BPS objects.

5 T-duality

In this section we discuss the T-dual of the $\mathbb{Z}_2\times\mathbb{Z}_2$ orbifold. We shall T-dualise along the x^5 direction. The orbifold we consider will then be a $\mathbb{Z}_2\times\mathbb{Z}_2$ orbifold where the generators are $g_1=\mathcal{I}_{5678}(-1)^{F_l}$ and $g_2=\mathcal{I}_{3478}$, where F_l is the left moving space-time fermion number. One imposes GSO and orbifold invariance of the boundary states in the different closed string sectors. As in the $\mathcal{I}_4(-1)^{F_l}$ orbifolds there is an ambiguity regarding which part of the twisted sector states to keep: the $\mathcal{I}_4(-1)^{F_l}$ odd states or the $\mathcal{I}_4(-1)^{F_l}$ even states? For the case of the invariance under g_i for the g_i-twisted sector we follow the prescription of [29]. In particular we require the g_1- and g_3- twisted sectors to be odd under g_1 and g_3 respectively, while the g_2-twisted sectors are to be even under g_2. We further shall require the g_1-twisted sector to be g_2 even and

g_3 odd, the g_2-twisted sector to be g_1 and g_3 even and the g_3-twisted sector to be g_2 even and g_1 odd. This ensures that we can construct fractional branes. With these clarifications one obtains table 1 for Type IIB

	GSO	g_1	g_2	g_1g_2
NS-NS	–	–	–	–
R-R	$r+s_1+s_2+s_3$ odd	s_2+s_3 odd	s_1+s_3 even	s_1+s_2 odd
NS-NS,Tg_1	s_2+s_3 odd	s_2+s_3 odd	s_3 even	s_2 odd
R-R,Tg_1	$r+s_1$ even	–	s_1 even	s_1 even
NS-NS,Tg_2	s_1+s_3 even	s_3 even	s_1+s_3 even	s_1 even
R-R,Tg_2	$r+s_2$ odd	s_2 odd	–	s_2 odd
NS-NS,Tg_3	s_1+s_2 odd	s_2 odd	s_1 even	s_1+s_2 odd
R-R,Tg_3	$r+s_3$ even	s_3 even	s_3 even	–

Table 1: Restrictions on r and s_i as a result of requiring GSO and orbifold invariance for boundary states in the various closed string sectors for Type IIB.

With these restrictions it is not difficult to identify bulk, fractional and both kinds of truncated branes. As before if two kinds of branes can exist for a given $(r;\mathbf{s})$it is the fractional rather than truncated (or the truncated with the NS-NS twisted sector rather than the other truncated brane) that will be fundamental and of minimal charge. For Type IIB we then have bulk wrapped branes for r, s_1, s_3 odd s_2 even (only charged under the untwisted R-R sector), and bulk and fractional branes (latter charged under four kinds of charges in the decompactified theory) for r, s_1, s_3 even s_2 odd. Truncated branes charged under R-R,Tg_1 and R-R,Tg_2 fields exist for r, s_1 even s_2, s_3 odd. Truncated branes charged under R-R,Tg_1 and R-R,Tg_3 fields exist for r, s_1, s_2, s_3 even. Truncated branes charged under R-R,Tg_2 and R-R,Tg_3 fields exist for r, s_3 even s_1, s_2 odd. Truncated branes charged only under R-R,Tg_1 exist for r, s_1, s_2 even s_3 odd. Truncated branes charged only under R-R,Tg_2 exist for r even s_1, s_2, s_3 odd. Truncated branes charged only under R-R,Tg_3 exist for r, s_2, s_3 even s_1 odd. For Type IIA in the above the parity of r changes.

One can develop a K-theory understanding of this. We define a new kind of K-theory which we call $K^*_{\mathbb{Z}_2\times\mathbb{Z}_2^\pm}$. Elements of this are pairs of isomorphism classes of bundles (E,F), which are equivariant under the action of the first $\mathbb{Z}_2$ and we are given an isomorphism between (E,F) and $(g^*(E), g^*(F))$ where in this case $g=\mathcal{I}_{5678}$ and $g^*(E)$ is the pullback of E by g. Further we are given an isomorphism which maps (E,F) to $(h^*(F), h^*(E))$, where $h^*(E)$ is the pullback of E by h, the generator of the geometric part of the second $\mathbb{Z}_2$ (in other words in this case $h=\mathcal{I}_{3478}$). A version of Hopkins' formula then is

$$K^*_{\mathbb{Z}_2\times\mathbb{Z}_2^\pm}(\mathbb{R}^{a;b,c,d}) \cong K^*_{\mathbb{Z}_2\times\mathbb{Z}_2}(\mathbb{R}^{a+1;b,c+1,d})\,, \qquad (5.18)$$

where the second group is just the usual equivariant K-group. We have computed the latter in section 3, and using the above relation exact agreement between K-theory and the string results follows. The stability and decay of the the truncated branes can easily be obtained from the results of the T-dual scenario described in detail in the previous sections.

A Construction and normalisation of boundary states

In this appendix we determine the normalisation constants of the boundary states for the orbifold theories under consideration.

A.1 The uncompactified case

In each (bosonic) sector of the theory we can construct the boundary state

$$|B(r;\mathbf{s}),k,\eta\rangle = \exp\left(\sum_{l>0}^{\infty}\left[\frac{1}{l}\alpha^{\mu}_{-l}S_{\mu\nu}\tilde{\alpha}^{\nu}_{-l}\right] + i\eta\sum_{m>0}^{\infty}\left[\psi^{\mu}_{-m}S_{\mu\nu}\tilde{\psi}^{\nu}_{-m}\right]\right)|B(r;\mathbf{s}),k,\eta\rangle^{(0)}\,, \tag{A.1}$$

where, depending on the sector, l and m are integer or half-integer, and k denotes the momentum of the ground state. We shall always work in light-cone gauge with light-cone directions x^0 and x^9; thus μ and ν take the values $1,\ldots,8$. We shall also drop the dependence on α'.

The parameter $\eta = \pm$ describes the two different spin structures [21, 20], and the matrix S encodes the boundary conditions of the Dp-brane which we shall always take to be diagonal

$$S = \mathrm{diag}(-1,\ldots,-1,1,\ldots,1)\,, \tag{A.2}$$

where $p+1$ entries are equal to -1, $7-p$ entries are equal to $+1$, and $p = r+s_1+s_2+s_3$. If there are fermionic zero modes, the ground state in (A.1) satisfies additional conditions discussed in the following appendix.

In order to obtain a localised D-brane, we have to take the Fourier transform of the above boundary state, where we integrate over the directions transverse to the brane,

$$|B(r,\mathbf{s}),y,\eta\rangle = \int\left(\prod_{\mu\ \mathrm{transverse}} dk^{\mu}e^{ik^{\mu}y_{\mu}}\right)dk^0e^{ik^0y_0}\,dk^9e^{ik^9y_9}\,|B(r;\mathbf{s}),k,\eta\rangle\,, \tag{A.3}$$

y denotes the location of the boundary state, and in the g_i-twisted sector the momentum integral only involves transverse directions that are not inverted by the action of g_i. In the following we shall typically consider (without loss of generality) the case $y=0$ in which case the boundary state is denoted by $|B(r;\mathbf{s}),\eta\rangle$.

The invariance of the boundary state under the GSO-projection always requires that the physical boundary state is a linear combination of the two states corresponding to $\eta = \pm$. Using the conventions of appendix B in [29], these linear combinations are of the form

$$
\begin{aligned}
|B(r;\mathbf{s})\rangle_{\text{NS-NS}} &= \frac{1}{2}\Big(|B(r;\mathbf{s}),+\rangle_{\text{NS-NS}} - |B(r;\mathbf{s}),-\rangle_{\text{NS-NS}}\Big)\,, && \text{(A.4)}\\
|B(r;\mathbf{s})\rangle_{\text{R-R}} &= 2i\Big(|B(r;\mathbf{s}),+\rangle_{\text{R-R}} + |B(r;\mathbf{s}),-\rangle_{\text{R-R}}\Big)\,, && \text{(A.5)}\\
|B(r;\mathbf{s})\rangle_{\text{NS-NS,T}g_i} &= \frac{1}{2}\Big(|B(r;\mathbf{s}),+\rangle_{\text{NS-NS,T}g_i} + |B(r;\mathbf{s}),-\rangle_{\text{NS-NS,T}g_i}\Big)\,, && \text{(A.6)}\\
|B(r;\mathbf{s})\rangle_{\text{R-R,T}g_i} &= i\Big(|B(r;\mathbf{s}),+\rangle_{\text{R-R,T}g_i} + |B(r;\mathbf{s}),-\rangle_{\text{R-R,T}g_i}\Big)\,, && \text{(A.7)}
\end{aligned}
$$

where, depending on the theory in question, these states are actually GSO-invariant provided that r and s_i satisfy suitable conditions discussed in section 2 ($i = 1, 2, 3$). The normalisation constants have been introduced for later convenience.

In order to solve the open-closed consistency condition the actual D-brane state is a linear combination of physical boundary states from different sectors. There are three elementary cases to consider, fully fractional, and the two truncated D-branes. (cf. equations 2.2)-(2.4) In the fully fractional case, the D-brane state can be written as

$$
\begin{aligned}
|D(r;\mathbf{s})\rangle &= \mathcal{N}_{f,U}\left(|B(r;\mathbf{s})\rangle_{\text{NS-NS}} + \epsilon\,|B(r;\mathbf{s})\rangle_{\text{R-R}}\right)\\
&\quad + \sum_{i=1}^{3} \epsilon_i\, \mathcal{N}_{f,Tg_i}\Big(|B(r;\mathbf{s})\rangle_{\text{NS-NS,T}g_i} + \epsilon\,|B(r;\mathbf{s})\rangle_{\text{R-R,T}g_i}\Big)\,, \qquad \text{(A.8)}
\end{aligned}
$$

where $\epsilon = \pm$ determines the sign of the charge with respect to the untwisted R-R sector charge, while $\epsilon_i = \pm$, $i = 1, 2, 3$ determines the sign of the charge with respect to the R-R,Tg_i charge. For consistency when $i \neq j \neq k \neq i$ we have $\epsilon_i = \epsilon_j \epsilon_k$. The closed string cylinder diagram is then of the form

$$
\begin{aligned}
\mathcal{A} &= \int dl\, \langle B(r;\mathbf{s})|\, e^{-lH_c}\, |B(r;\mathbf{s})\rangle\\
&= \frac{1}{2}\mathcal{N}_{f,U}^2 \int dl\, l^{(p-9)/2} \left(\frac{f_3^8(q) - f_4^8(q) - f_2^8(q)}{f_1^8(q)}\right)\\
&\quad + \sum_{i=1}^{3} \frac{1}{2}\mathcal{N}_{f,Tg_i}^2 \int dl\, l^{(r+s_i-5)/2} \left(\frac{f_3^4(q) f_2^4(q) - f_2^4(q) f_3^4(q)}{f_1^4(q) f_4^4(q)}\right)\,, \qquad \text{(A.9)}
\end{aligned}
$$

where the functions f_i are defined as in [28], $q = e^{-2\pi l}$, and the closed string Hamiltonian is given by

$$
H_c = \pi k^2 + 2\pi \sum_{\mu=1}^{8} \left[\sum_{l>0}^{\infty} (\alpha_{-l}^{\mu} \alpha_l^{\mu} + \tilde{\alpha}_{-l}^{\mu} \tilde{\alpha}_l^{\mu}) + \sum_{m>0}^{\infty} m(\psi_{-m}^{\mu} \psi_m^{\mu} + \tilde{\psi}_{-m}^{\mu} \tilde{\psi}_m^{\mu})\right] + 2\pi C_c\,. \qquad \text{(A.10)}
$$

Here the constant C_c is -1 in the NS-NS sector, and zero in all other sectors. The corresponding open string amplitude is obtained by the modular transformation $t = 1/2l$, $\tilde{q} = e^{-\pi t}$,

$$\begin{aligned}\mathcal{A} &= 2^{(7-p)/2}\mathcal{N}_{f,U}^2 \int \frac{dt}{2t} t^{-(p+1)/2} \left(\frac{f_3^8(\tilde{q}) - f_2^8(\tilde{q}) - f_4^8(\tilde{q})}{f_1^8(\tilde{q})} \right) \\ &+ \sum_{i=1}^{3} 2^{(3-r-s_i)/2}\mathcal{N}_{f,Tg_i}^2 \int \frac{dt}{2t} t^{-(r+s_i+1)/2} \left(\frac{f_3^4(\tilde{q})f_4^4(\tilde{q}) - f_4^4(\tilde{q})f_3^4(\tilde{q})}{f_1^4(\tilde{q})f_2^4(\tilde{q})} \right) . \end{aligned} \tag{A.11}$$

This is to be compared with the open string one-loop diagram,

$$\begin{aligned} &\int \frac{dt}{2t} \mathrm{Tr}_{NS-R} \left(\frac{1+(-1)^F}{2} \frac{1+g_1+g_2+g_3}{4} e^{-2tH_o} \right) \\ &= \frac{V_{p+1}}{(2\pi)^{p+1}} 2^{-(p+7)/2} \int \frac{dt}{2t} t^{-(p+1)/2} \left(\frac{f_3^8(\tilde{q}) - f_4^8(\tilde{q}) - f_2^8(\tilde{q})}{f_1^8(\tilde{q})} \right) \\ &+ \sum_{i=1}^{3} \frac{V_{r+s_i+1}}{(2\pi)^{r+s_i+1}} 2^{-(r+s_i+3)/2} \int \frac{dt}{2t} t^{-(r+s_i+1)/2} \left(\frac{f_3^4(\tilde{q})f_4^4(\tilde{q}) - f_4^4(\tilde{q})f_3^4(\tilde{q})}{f_1^4(\tilde{q})f_2^4(\tilde{q})} \right) , \end{aligned} \tag{A.12}$$

where V_{p+1} is the (infinite) $p+1$ dimensional volume of the brane, whilst V_{r+s_i+1} is the volume of the projection onto the directions unaffected by g_i. The open string Hamiltonian is given by

$$H_o = \pi p^2 + \pi \sum_{\mu=1}^{8} \left[\sum_{l>0}^{\infty} \alpha_{-l}^{\mu}\alpha_l^{\mu} + \sum_{m>0}^{\infty} m\psi_{-m}^{\mu}\psi_m^{\mu} \right] + \pi C_o , \tag{A.13}$$

where, in the R sector, l and m run over the positive integers for NN and DD directions, and over positive half integers for ND directions. In the NS sector, the moding of the fermions (and therefore the values for m) are opposite to those in the R sector. C_o is zero in the R sector and is $\frac{4-t}{8}$ in the NS sector, where t is the number of ND directions. Comparison of equations (A.12) and (A.11) then gives

$$\mathcal{N}_{f,U}^2 = \frac{V_{p+1}}{(2\pi)^{p+1}} \frac{1}{128} , \tag{A.14}$$

$$\mathcal{N}_{f,Tg_i}^2 = \frac{V_{r+s_i+1}}{(2\pi)^{r+s_i+1}} \frac{1}{8} . \tag{A.15}$$

Consider next the singly fractional D-branes charged under the R-R,Tg_i field. These can be thought of as superpositions of two totally fractional D-branes with opposite twisted charges in the other two twisted sectors. These pairs can move off the fixed points of g_j for $j \neq i$, such that they lie at positions x^μ and $-x^\mu$ for the directions transverse to the brane fixed by g_j and not by g_i. In particular their boundary states will now look like

$$\begin{aligned}|D(r;\mathbf{s})\rangle = \mathcal{N}_{f,U}\,(&|B(r;\mathbf{s}),x^\mu\rangle_{\text{NS-NS}} + |B(r;\mathbf{s}),-x^\mu\rangle_{\text{NS-NS}} \\ &+\epsilon(|B(r;\mathbf{s}),x^\mu\rangle_{\text{R-R}} + |B(r;\mathbf{s}),-x^\mu\rangle_{\text{R-R}}\,)) \\ &+\epsilon_i\,\mathcal{N}_{f,Tg_i}\Big(|B(r;\mathbf{s}),x^\mu\rangle_{\text{NS-NS,T}g_i} + |B(r;\mathbf{s}),-x^\mu\rangle_{\text{NS-NS,T}g_i} \\ &+\epsilon\,|B(r;\mathbf{s}),x^\mu\rangle_{\text{R-R,T}g_i} + \epsilon\,|B(r;\mathbf{s}),-x^\mu\rangle_{\text{R-R,T}g_i}\Big)\ . \end{aligned} \tag{A.16}$$

The above normalisation is consistent with the fact that the cylinder diagram for such D-branes factorises on

$$2\int\frac{dt}{2t}\text{Tr}_{NS-R}\left(\frac{1+(-1)^F}{2}\frac{1+g_i}{2}e^{-2tH_o}\right)\ . \tag{A.17}$$

Such singly fractional D-branes correspond to, in the covering space, two Type II D-branes stuck at the fixed point of g_i. There are four kinds of open strings stretching between two such D-branes so one would expect a factor of four in front of the trace above. However, the orbifold identifies certain strings leaving only two independent ones (namely the string that has endpoint on the same brane and the string that has ends on opposite branes). Similarly a bulk D-brane is a configuration of four totally fractional D-branes whose twisted charges cancel. It's boundary state is a sum over four NS-NS and four R-R boundary states at positions mapped to one another under the orbifold action. The cylinder diagram for such a D-brane factorises on

$$4\int\frac{dt}{2t}\text{Tr}_{NS-R}\left(\frac{1+(-1)^F}{2}e^{-2tH_o}\right)\ . \tag{A.18}$$

The analysis for the case of the truncated D̂-branes is similar. The truncated D-brane of the type given by equation (2.3) charged only under the R-R,Tg_1 sector, say, has a boundary state

$$\begin{aligned}|\hat{D}(r;\mathbf{s})\rangle \;=\; & \mathcal{N}_{ts,U}\,(|B(r;\mathbf{s}),x^\mu\rangle_{\text{NS-NS}} + |B(r;\mathbf{s}),-x^\mu\rangle_{\text{NS-NS}}\,) \\ &+\epsilon\mathcal{N}_{ts,Tg_1}\Big(|B(r;\mathbf{s}),x^\mu\rangle_{\text{R-R,T}g_1} + |B(r;\mathbf{s}),-x^\mu\rangle_{\text{R-R,T}g_1}\Big)\ ,\end{aligned} \tag{A.19}$$

where $\epsilon = \pm$ determines the sign of the R-R,Tg_1 sector charge and x^μ denotes some of the directions x^3, x^4 which are transverse to the D̂-brane. The closed string tree diagram now only produces some of the terms of (A.9), and the corresponding open string amplitude is

$$\begin{aligned}&4\int\frac{dt}{2t}\text{Tr}_{NS-R}\left(\frac{1+g_1(-1)^F}{2}e^{-2tH_o}\right) \\ &\quad= 4\frac{V_{p+1}}{(2\pi)^{p+1}}2^{-(p+3)/2}\int\frac{dt}{2t}t^{-(p+1)/2}\left(\frac{f_3^8(\tilde{q})-f_2^8(\tilde{q})}{f_1^8(\tilde{q})}\right) \\ &\qquad -4\frac{V_{r+s_1+1}}{(2\pi)^{r+s_1+1}}2^{(1-s_1-r)/2}\int\frac{dt}{2t}t^{-(r+s_1+1)/2}\left(\frac{f_4^4(\tilde{q})f_3^4(\tilde{q})}{f_1^4(\tilde{q})f_2^4(\tilde{q})}\right)\ .\end{aligned} \tag{A.20}$$

Comparison with the corresponding closed string calculation then gives

$$\mathcal{N}_{t1,U}^2 = \frac{V_{p+1}}{(2\pi)^{p+1}}\frac{1}{8}, \tag{A.21}$$

$$\mathcal{N}_{t1,Tg_1}^2 = \frac{V_{r+s_1+1}}{(2\pi)^{r+s_1+1}}2. \tag{A.22}$$

The truncated D-brane of the type given in equation (2.4) is charged under the R-R,Tg_i and R-R,Tg_j sectors ($i \neq j$) has a boundary state given by

$$\begin{aligned} |\hat{D}(r;\mathbf{s})\rangle &= \mathcal{N}_{td,U}\,|B(r;\mathbf{s})\rangle_{\text{NS-NS}} + \epsilon_i \mathcal{N}_{td,Tg_i}\,|B(r;\mathbf{s})\rangle_{\text{R-R,T}g_i} \\ &\quad + \epsilon_j \mathcal{N}_{td,Tg_j}\,|B(r;\mathbf{s})\rangle_{\text{R-R,T}g_j} + \epsilon_i\epsilon_j \mathcal{N}_{td,Tg_k}\,|B(r;\mathbf{s})\rangle_{\text{NS-NS,T}g_k} \end{aligned} \tag{A.23}$$

where $\epsilon_i = \pm$ determines the sign of the R-R,Tg_i sector charge and $j \neq k \neq i$ with $k = 1,2,3$. The closed string tree diagram produces only some of the terms of (A.9), and the corresponding open string amplitude is

$$\begin{aligned} &\int \frac{dt}{2t}\text{Tr}_{NS-R}\left(\frac{1+g_i(-1)^F}{2}\frac{1+g_j(-1)^F}{2}e^{-2tH_o}\right) \\ &\quad = \frac{V_{p+1}}{(2\pi)^{p+1}}2^{-(p+5)/2}\int\frac{dt}{2t}t^{-(p+1)/2}\left(\frac{f_3^8(\tilde{q})-f_2^8(\tilde{q})}{f_1^8(\tilde{q})}\right) \\ &\quad\quad - \sum_{\alpha\in\{i,j,k\}}\rho(\alpha)\frac{V_{r+s_\alpha+1}}{(2\pi)^{r+s_\alpha+1}}2^{-(r+s_\alpha+1)/2}\int\frac{dt}{2t}t^{-(r+s_\alpha+1)/2}\left(\frac{f_4^4(\tilde{q})f_3^4(\tilde{q})}{f_1^4(\tilde{q})f_2^4(\tilde{q})}\right), \end{aligned} \tag{A.24}$$

where $\rho(i) = \rho(j) = -\rho(k) = 1$. Comparison with the corresponding closed string calculation then gives

$$\mathcal{N}_{t2,U}^2 = \frac{V_{p+1}}{(2\pi)^{p+1}}\frac{1}{64}, \tag{A.25}$$

$$\mathcal{N}_{t2,Tg_\alpha}^2 = \frac{V_{r+s_\alpha+1}}{(2\pi)^{r+s_\alpha+1}}\frac{1}{4}. \tag{A.26}$$

A.2 The compactified case

The construction in the compactified case is essentially the same as in the above uncompactified case; however there are the following differences.

1. In the localised boundary state (A.3) the integral over compact transverse directions is replaced by a sum

$$\int dk^\nu e^{ik^\nu y_\nu} \longrightarrow \sum_{m^\nu\in\mathbb{Z}} e^{im^\nu y_\nu/R_\nu}, \tag{A.27}$$

where R_ν is the radius of the compact x^ν direction.
2. In the two untwisted sectors, the ground state is in addition characterised by a winding number w_ν for each compact direction that is tangential to the world-volume of the brane. In the g_i twisted sectors, the ground states will also be characterised by winding numbers in each of the s_i directions tangential to the world-volume of the brane. The localised bound state (A.3) then also contains a sum over these winding states

$$\sum_{w_\mu} e^{i\theta^\mu w_\mu}\,, \tag{A.28}$$

where θ^μ is a Wilson line; as required by orbifold invariance, $\theta^\mu \in \{0,\pi\}$.
3. For general s_i, the contribution in the twisted sectors consists of a sum of terms that are associated to 2^{s_i} of the 16 different twisted sectors that define the endpoints of the world-volume of the brane in the internal space. For convenience we may assume that one of the 2^{s_i} fixed points is always the origin.
4. The open and closed string Hamiltonians, H_o and H_c, each acquire an extra term $1/4\pi(\sum_\mu w_\mu^2)$.

Let us now construct in more detail the boundary state for a fully fractional D$(r;\mathbf{s})$ brane. This is of the form

$$\begin{aligned}|D(r;\mathbf{s})\rangle =& \mathcal{N}_{f,U}\left(|B(r;\mathbf{s})\rangle_{\text{NS-NS}} + \varepsilon\,|B(r;\mathbf{s})\rangle_{\text{R-R}}\right)\\ &+\sum_{i=1}^{3}\varepsilon_i\,\mathcal{N}_{f,Tg_i}\sum_{\alpha_i=1}^{2^{s_i}} e^{i\theta_{\alpha_i}}\Big(|B(r;\mathbf{s})\rangle_{\text{NS-NS,T}g_{\alpha_i}} + \epsilon\,|B(r;\mathbf{s})\rangle_{\text{R-R,T}g_{\alpha_i}}\Big)\,,\end{aligned} \tag{A.29}$$

where α_i labels the different fixed points between which the brane stretches (where we choose the convention that $\mathrm{T}g_{i_1}$ is the twisted sector at the origin), and θ_{α_i} is the Wilson line that is associated to the difference of the fixed point α_i and the origin. The closed string tree diagram is now

$$\begin{aligned}\mathcal{A}_c &= \int dl\,\langle B(r;\mathbf{s})|\,e^{-lH_c}\,|B(r;\mathbf{s})\rangle\\ &= \frac{1}{2}\mathcal{N}_{f,U}^2\int dl\; l^{(r-3)/2}\left(\frac{f_3^8(q)-f_2^8(q)-f_4^8(q)}{f_1^8(q)}\right)\\ &\quad\times\prod_{m=1}^{s_1+s_2+s_3}\sum_{w_{j_m}\in\mathbb{Z}} e^{-l\pi R_{j_m}^2 w_{j_m}^2}\prod_{m=1}^{6-s_1-s_2-s_3}\sum_{n_{k_m}\in\mathbb{Z}} e^{-l\pi(n_{k_m}/R_{k_m})^2}\\ &\quad+\sum_{i=1}^{3}\frac{2^{s_1+s_2+s_3-s_i}}{2}\mathcal{N}_{f,Tg_i}^2\int dl\; l^{(r-3)/2}\left(\frac{f_3^4(q)f_2^4(q)-f_2^4(q)f_3^4(q)}{f_1^4(q)f_4^4(q)}\right)\\ &\quad\times\prod_{m=1}^{s_i}\sum_{w_{j_m}\in\mathbb{Z}} e^{-l\pi R_{j_m}^2 w_{j_m}^2}\prod_{m=1}^{2-s_i}\sum_{n_{k_m}\in\mathbb{Z}} e^{-l\pi(n_{k_m}/R_{k_m})^2}\,,\end{aligned} \tag{A.30}$$

where R_{j_m}, $m=1,\dots,s_1+s_2+s_3$ are the radii of the circles that are tangential to the world-volume of the brane, and R_{k_m}, $i=1,\dots,6-s_1-s_2-s_3$ are the radii of the directions transverse to the brane. Upon the substitution $t=1/2l$, using the Poisson resummation formula (see for example [7, 28]), this amplitude becomes

$$\begin{aligned}\mathcal{A}_c &= \mathcal{N}_{f,U}^2 \frac{\prod_{m=1}^{6-s_1-s_2-s_3} R_{k_m}}{\prod_{m=1}^{s_1+s_2+s_3} R_{j_m}} 2^{(7-r)/2} \int \frac{dt}{2t} t^{-(r+1)/2} \left(\frac{f_3^8(\tilde{q}) - f_2^8(\tilde{q}) - f_4^8(\tilde{q})}{f_1^8(\tilde{q})} \right) \\ &\quad \times \prod_{m=1}^{s_1+s_2+s_3} \sum_{n_{j_m}\in\mathbb{Z}} e^{-2t\pi n_{j_m}^2/R_{j_m}^2} \prod_{m=1}^{6-s_1-s_2-s_3} \sum_{w_{k_m}\in\mathbb{Z}} e^{-2t\pi w_{k_m}^2 R_{k_m}^2} \\ &\quad + \sum_{i=1}^{3} \frac{\prod_{m=1}^{2-s_i} R_{k_m}}{\prod_{m=1}^{s_i} R_{j_m}} 2^{(1-r)/2} 2^{s_1+s_2+s_3-s_i} \mathcal{N}_{f,Tg_i}^2 \int \frac{dt}{2t} t^{-(r+1)/2} \left(\frac{f_3^4(\tilde{q}) f_4^4(\tilde{q}) - f_4^4(\tilde{q}) f_3^4(\tilde{q})}{f_1^4(\tilde{q}) f_2^4(\tilde{q})} \right) \\ &\quad \times \prod_{m=1}^{s_i} \sum_{n_{j_m}\in\mathbb{Z}} e^{-2t\pi n_{j_m}^2/R_{j_m}^2} \prod_{m=1}^{2-s_i} \sum_{w_{k_m}\in\mathbb{Z}} e^{-2t\pi w_{k_m}^2 R_{k_m}^2} .\end{aligned}$$

This is to be compared with the open string amplitude

$$\begin{aligned}\int \frac{dt}{2t} &\quad \mathrm{Tr}_{NS-R} \left(\frac{1+(-1)^F}{2} \frac{1+g_1+g_2+g_3}{4} e^{-2tH_o} \right) \\ &= \frac{V_{r+1}}{8(2\pi)^{r+1}} 2^{-(r+1)/2} \int \frac{dt}{2t} t^{-(r+1)/2} \left(\frac{f_3^8(\tilde{q}) - f_4^8(\tilde{q}) - f_2^8(\tilde{q})}{f_1^8(\tilde{q})} \right) \\ &\quad \times \prod_{m=1}^{s_1+s_2+s_3} \sum_{n_{j_m}\in\mathbb{Z}} e^{-2t\pi n_{j_m}^2/R_{j_m}^2} \prod_{m=1}^{6-s_1-s_2-s_3} \sum_{w_{k_m}\in\mathbb{Z}} e^{-2t\pi w_{k_m}^2 R_{k_m}^2} \\ &\quad \sum_{i=1}^{3} \frac{V_{r+1}}{2(2\pi)^{r+1}} 2^{-(r+1)/2} \int \frac{dt}{2t} t^{-(r+1)/2} \frac{f_3^4(\tilde{q}) f_4^4(\tilde{q}) - f_4^4(\tilde{q}) f_3^4(\tilde{q})}{f_1^4(\tilde{q}) f_2^4(\tilde{q})} \\ &\quad \times \prod_{m=1}^{s_i} \sum_{n_{j_m}\in\mathbb{Z}} e^{-2t\pi n_{j_m}^2/R_{j_m}^2} \prod_{m=1}^{2-s_i} \sum_{w_{k_m}\in\mathbb{Z}} e^{-2t\pi w_{k_m}^2 R_{k_m}^2} .\end{aligned} \tag{A.31}$$

By comparison this then fixes the normalisation constants as

$$\mathcal{N}_{f,U}^2 = \frac{V_{r+1}}{(2\pi)^{r+1}} \frac{1}{128} \frac{\prod_{m=1}^{s_1+s_2+s_3} R_{j_m}}{\prod_{m=1}^{6-s_1-s_2-s_3} R_{k_m}} , \tag{A.32}$$

$$\mathcal{N}_{f,Tg_i}^2 = \frac{V_{r+1}}{(2\pi)^{r+1}} \frac{2^{4-(s_1+s_2+s_3-s_i)}}{128} \frac{\prod_{m=1}^{s_i} R_{j_m}}{\prod_{m=1}^{2-s_i} R_{k_m}} . \tag{A.33}$$

The extension of this to the singly fractional and bulk D-branes is obvious.

The analysis for the truncated D-branes is almost identical. The boundary state of a truncated D-brane of the type given in equation (2.3) is the truncation of (A.29) to the untwisted NS-NS and the g_i-twisted R-R sectors with the addition of boundary states at mirror positions as in equation (A.19). The open string amplitude contains also only the corresponding terms. Furthermore, since the projection operator is now[6] $4 \times \frac{1}{2}(1 + g_i(-1)^F)$ each of the terms that appears is sixteen times as large as in the fractional case above. This implies that the relevant normalisation constants are given as

$$\mathcal{N}^2_{t1,U} = \frac{V_{r+1}}{(2\pi)^{r+1}} \frac{1}{8} \frac{\prod_{m=1}^{s_1+s_2+s_3} R_{j_m}}{\prod_{m=1}^{6-s_1-s_2-s_3} R_{k_m}}, \tag{A.34}$$

$$\mathcal{N}^2_{t1,Tg_i} = \frac{V_{r+1}}{(2\pi)^{r+1}} \frac{2^{4-(s_1+s_2+s_3-s_i)}}{8} \frac{\prod_{m=1}^{s_i} R_{j_m}}{\prod_{m=1}^{2-s_i} R_{k_m}}. \tag{A.35}$$

The second type of truncated $\hat{\text{D}}$-brane (equation (2.4)) contains the untwisted NS-NS , twisted R-R,Tg_i, R-R,Tg_j and NS-NS,Tg_k boundary states. The projection operator is $\frac{1}{4}(1+g_i(-1)^F)(1+g_j(-1)^F)$ thus fixing the normalisations to be

$$\mathcal{N}^2_{t2,U} = \frac{V_{r+1}}{(2\pi)^{r+1}} \frac{1}{64} \frac{\prod_{m=1}^{s_1+s_2+s_3} R_{j_m}}{\prod_{m=1}^{6-s_1-s_2-s_3} R_{k_m}}, \tag{A.36}$$

$$\mathcal{N}^2_{t2,Tg_\alpha} = \frac{V_{r+1}}{(2\pi)^{r+1}} \frac{2^{4-(s_1+s_2+s_3-s_\alpha)}}{64} \frac{\prod_{m=1}^{s_\alpha} R_{j_m}}{\prod_{m=1}^{2-s_\alpha} R_{k_m}}, \tag{A.37}$$

where $\alpha \in \{i, j, k\}$.

B Consistency conditions of boundary states

In this appendix we discuss the invariance of the boundary states under the GSO and orbifold actions. Since the action of the orbifold group is purely geometrical the GSO projection is the same in twisted and untwisted sectors namely we have

$$\text{NS-NS} \quad \frac{1}{4}(1 + (-1)^F)(1 + (-1)^{\tilde{F}}), \tag{B.1}$$

$$\text{R-R} \quad \frac{1}{4}(1 + (-1)^F)(1 \mp (-1)^{\tilde{F}}), \tag{B.2}$$

for Type IIA and IIB, respectively.

The GSO invariance of each of the sectors' boundary states was computed in detail in [29]. Here we simply restate those results in terms of r and s_i. We shall work in the light-cone gauge with the directions x^0, x^9 always Dirichlet [47]. In the untwisted NS-NS sector it is easy to see that

$$|B(r;\mathbf{s})\rangle_{\text{NS-NS}} = \frac{\mathcal{N}}{2}(|B(r;\mathbf{s}),+\rangle_{\text{NS-NS}} - |B(r;\mathbf{s}),-\rangle_{\text{NS-NS}}) \tag{B.3}$$

[6] cf equation (A.20)

is GSO invariant for all r, s_i. The exact values of the normalisation constants of the boundary states will depend on the kind of D-brane the boundary state is a part of, and are computed in the Appendix. In the untwisted R-R sector

$$|B(r;\mathbf{s})\rangle_{\text{R-R}} = \frac{4i\mathcal{N}}{2}(|B(r;\mathbf{s}),+\rangle_{\text{R-R}} + |B(r;\mathbf{s}),-\rangle_{\text{R-R}}) \tag{B.4}$$

is a GSO invariant boundary state if $r + s_1 + s_2 + s_3$ is even/odd for Type IIA/B, respectively. For the NS-NS,Tg_1 sector we find that

$$|B(r;\mathbf{s})\rangle_{\text{NS-NS,T}g_1} = \mathcal{N}_T(|B(r;\mathbf{s}),+\rangle_{\text{NS-NS,T}g_1} + |B(r;\mathbf{s}),-\rangle_{\text{NS-NS,T}g_1}) \tag{B.5}$$

is a GSO invariant boundary state provided $s_2 + s_3$ is even, while

$$|B(r;\mathbf{s})\rangle_{\text{R-R,T}g_1} = i\mathcal{N}_T(|B(r;\mathbf{s}),+\rangle_{\text{R-R,T}g_1} + |B(r;\mathbf{s}),-\rangle_{\text{R-R,T}g_1}) \tag{B.6}$$

is GSO invariant for $r + s_1$ even/odd for Type IIA/IIB. Similarly in the NS-NS,Tg_2 sector $s_1 + s_3$ is to be even and for the R-R,Tg_2 sector $r + s_2$ has to be even/odd for Type IIA/IIB. Finally for the NS-NS,Tg_3 $s_1 + s_2$ is to be even and for the R-R,Tg_2 sector $r + s_3$ has to be even/odd for Type IIA/IIB.

Next one requires the boundary states to be invariant under g_i. As usual [29] this places no restrictions on the untwisted NS-NS sector, and in the untwisted R-R sector it requires for s_1+s_2, s_2+s_3 and s_1+s_3 to be all even. The R-R,Tg_i boundary state has no restrictions placed on it by g_i since it has no zero modes in the directions which g_i inverts, while the NS-NS,Tg_i boundary state's invariance under g_i is equivalent to the GSO condition on that boundary state [29].

New non-trivial restrictions arise by requiring the g_i-twisted sector's boundary state be invariant under g_j, $j \neq i$. We consider the invariance of NS-NS,Tg_1 and R-R,Tg_1 under g_2 in some detail. The other conditions will follow in a similar way. Since the NS-NS,Tg_1 sector has zero modes in the directions x^7 and x^8 (as well as x^5 and x^6) g_2 will have a non-trivial representation on these zero-modes, given by[7]

$$g_2 = \prod_{\mu=7,8}(\sqrt{2}\psi_0^\mu)\prod_{\mu=7,8}(\sqrt{2}\tilde{\psi}_0^\mu)\,. \tag{B.7}$$

This operator squares to one. It is not difficult to see that

$$g_2|B(r;\mathbf{s}),\eta\rangle^0_{\text{NS-NS,T}g_1} = (-1)^{s_3}|B(r;\mathbf{s}),\eta\rangle^0_{\text{NS-NS,T}g_1} \tag{B.8}$$

and hence that s_3 has to be even.[8] Similarly since the R-R,Tg_2 sector has zero modes in directions x^3 and x^4, g_2 has a non-trivial representation on this sector as

$$g_2 = \prod_{\mu=3,4}(\sqrt{2}\psi_0^\mu)\prod_{\mu=3,4}(\sqrt{2}\tilde{\psi}_0^\mu)\,, \tag{B.9}$$

[7] We use the same convention here for g_2 as was used for $\mathcal{I}_2$ in [29]. Since this is a supersymmetric theory this can be viewed as a condition for supersymmetry preservation by the orbifold action.

[8] Note that in the notation of equations (B.4) and (B.5) Appendix B in [29] for the NS-NS,Tg_1 state $a = b = 1$.

and so

$$g_2|B(r;\mathbf{s}),\eta\rangle^0_{\text{R-R,T}g_1} = (-1)^{s_1}|B(r;\mathbf{s}),\eta\rangle^0_{\text{R-R,T}g_1}\,. \tag{B.10}$$

Thus s_1 has to be even.[9] Performing a similar analysis for the other twisted sectors one finds that in Type IIB the following boundary states are GSO and orbifold invariant

$$\begin{array}{rl} |B(r;\mathbf{s})\rangle_{\text{NS-NS}} & \text{for all } r \text{ and } s_i, \\ |B(r;\mathbf{s})\rangle_{\text{R-R}} & \text{for either } r \text{ odd and } s_i \text{ all even or } r \text{ even and } s_i \text{ all odd}, \\ |B(r;\mathbf{s})\rangle_{\text{NS-NS,T}g_i} & \text{for } s_j, s_k \text{ even, } j,k \neq i, \\ |B(r;\mathbf{s})\rangle_{\text{R-R,T}g_i} & \text{for } r \text{ odd and } s_i \text{ even.} \end{array}$$

For Type IIA the conditions are the same for s_i but r has to be even.

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