Optimal Continuous-Time Hedging with Leptokurtic Returns*

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Abstract

We examine the behaviour of optimal mean–variance hedging strategies at high rebalancing frequencies in a model where stock prices follow a discretely sampled exponential Lévy process and one hedges a European call option to maturity. Using elementary methods we show that all the attributes of a discretely rebalanced optimal hedge, i.e. the mean value, the hedge ratio and the expected squared hedging error, converge pointwise in the state space as the rebalancing interval goes to zero. The limiting formulae represent 1-D and 2-D generalized Fourier transforms which can be evaluated much faster than backward recursion schemes, with the same degree of accuracy.

In the special case of a compound Poisson process we demonstrate that the convergence results hold true if instead of using an infinitely divisible distribution from the outset one models log returns by multinomial approximations thereof. This result represents an important extension of Cox, Ross, and Rubinstein (1979) to markets with leptokurtic returns.

Keywords: hedging error, Fourier transform, mean–variance hedging, locally optimal strategy, exponential Lévy process, incomplete market, option pricing

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1 Introduction

It is an empirical fact that equity returns are non-normal on high sampling frequencies, contradicting the assumption of the Black–Scholes model. The departure from normality grows as the rebalancing interval becomes shorter; as a rule of thumb kurtosis is inversely proportional to the length of the rebalancing interval. We examine a model where log returns are generated by a Lévy process, that is a process with stationary, independent increments, which provides a simple but flexible framework to analyze impacts of excess kurtosis on option hedging. It is in fact the only possible representation of a continuous-time model with IID returns.

In this model we examine the performance of two hedging strategies: the dynamically optimal strategy that minimizes the unconditional expected squared hedging error among all trading strategies, and the so-called locally optimal strategy which resembles Black–Scholes delta hedging in that it only depends on the stock price and time to maturity.

The paper makes a contribution in three directions: i) we give closed-form expressions for the representative agent price, the optimal hedging coefficient and the unconditional expected squared hedging error of the optimal hedging strategy in the limit as the rebalancing interval goes to zero; ii) we compare, in closed form, the performance of locally optimal and dynamically optimal hedging strategies; iii) we show that the continuous-time results for compound Poisson processes can be obtained as a limit of multinomial lattice models. Elsewhere, Černý (2004c) demonstrates that the gain in computational speed afforded by the closed-form results, as compared to traditional backward recursion schemes, is highly significant and can be likened to the difference between the Black–Scholes formula and its binomial implementation.

Modelling of excess kurtosis in equity return data has attracted considerable attention since mid 1960s. Building on the geometric Brownian motion of Osborne (1959) and Samuelson (1965) researchers have proposed different parametric distributions for log returns, often generated as normal mixtures, see Mandelbrot (1963), Press (1967), Praetz (1972), Clark (1973), Madan and Seneta (1990), and Eberlein et al. (1998). All of the above are special cases of the exponential Lévy process.

In exponential Lévy models it is no longer possible to construct a dynamic self-financing portfolio that replicates the option. One can visualize this situation in discrete time by imagining a multinomial stock price lattice (cf. Maller et al. 2004) instead of the standard binomial tree (cf. Cox et al. 1979 and Madan et al. 1989). Since the hedged position is risky (synonymously, market is incomplete), it is necessary to formulate a reward-for-risk measure telling us which option prices are sensible, and which, in contrast, lead to near-arbitrage opportunities (good deals). Based on the seminal work of Von Neumann and Morgenstern (1944), it has become customary in economic literature to use utility functions for this purpose.

Markowitz (1952) pioneered the use of quadratic utility in static portfolio selection, and his results were extended to dynamic setting and other utility functions by Merton (1969). Hodges and Neuberger (1989) were the first to apply dynamic optimal portfolio selection with a random endowment (short/long position in the option) to option valuation, computing so-called *reservation prices,* which make buyer/seller indifferent between undertaking a given option trade or not trading at all. If, instead, one allows the option trade to shift the efficient frontier one obtains so-called good-deal price bounds, see Cochrane and Saá-Requejo (2000), Černý and Hodges (2002), and Černý (2003) who shows that the good–deal bounds are *robust across different utility functions*.

Except in special cases both the reservation prices and good-deal prices are difficult to evaluate in closed form. Duffie and Richardson (1991) and Henderson (2002) give closed form solutions of the expected utility maximization for non-trivial futures and option positions in a market with basis risk in the case of a quadratic utility and exponential utility, respectively. Young (2004) derives reservation catastrophe bond prices in a term structure model with exponential utility. In all three cases mentioned above asset prices are continuous.

When the option trade is infinitesimally small one obtains so-called *representative agent price*, which in some circumstances can be computed even in the presence of jumps. Representative agent prices in the exponential Lévy model with power or exponential utility give rise to so-called Esscher risk-neutral measures which permit fast pricing via Fourier transform, see Madan and Milne (1991), Gerber and Shiu (1994), Carr and Madan (1999) and Fujiwara and Miyahara (2003). For recent results on representative agent prices in Heston's stochastic volatility model see Hobson (2004).

Among the different utility functions the quadratic utility is the most tractable and gives the best hope of recovering closed-form reservation and/or good-deal prices in the presence of jumps. Historically, option hedging under mean–variance criteria has focused on minimization of expected squared hedging error (which is the case in this paper) rather than on maximization of expected utility. It turns out, however, that the two problems are equivalent. Černý (2004b), Chapter 12 shows how to reinterpret the results of mean-variance hedging to obtain option price bounds parameterized by unconditional Sharpe ratio.

Dynamic mean–variance hedging has been examined in great generality since the beginning of 1990s, but the theoretical characterization of optimal solutions is still in the process of being completed, see Pham (2000), Schweizer (2001), Arai (2005) and Černý and Kallsen (2005). Some explicit formulae are available for continuous price processes and in particular diffusions, see Heath et al. (2001), Laurent and Pham (1999), Lim (2006), Biagini et al. (2000). For discontinuous price processes the characterization of optimal solutions is more patchy. When the opportunity set is deterministic, which is the case in the present paper, Schweizer (1994) shows the solution can be obtained from the Föllmer-Schweizer decomposition of the contingent claim. For jump-diffusions this decomposition is computed explicitly in Colwell and Elliott (1993) and in a geometric Lévy model by Hubalek et al. (2005).

A general solution is known in the case when the state space is finite, see Černý (2004a), and also Bertsimas et al. (2001) and Schweizer (1995). The optimal hedging strategy is determined by *exogenous* state variables including stock price, volatility and other factors as the case may be, and one *endogenous* state variable represented by the value of the self-financing hedging portfolio.

One of the remarkable features of dynamic mean-variance hedging is that the optimal hedge is an (affine) function of the *endogenous* state variable, which introduces path dependency absent in the Black–Scholes hedge. It is therefore interesting to examine suboptimal but purely *exogenous* hedging strategies, which has lead to the concept of local risk minimization, see Föllmer and Schweizer (1991), Schweizer (1991), Colwell and Elliott (1993), Hofmann et al. (1992). In their original definition locally risk-minimizing strategies are *not* self-financing. To emphasize that in the present paper we only use self-financing strategies we call the suboptimal path-independent strategy generated by local risk minimization *locally optimal*. Heath et al. (2001) compare the performance of locally optimal and dynamically optimal hedging strategies in stochastic volatility models. This paper performs the same task in a model with IID returns.

From the practical point of view one wishes to compute the exogenous components of the solution as functions of the exogenous state variables. In the discrete setup this is done by backward recursion, in continuous time one obtains non-linear partial difference-differential equations which are then solved by numerical methods not dissimilar to the backward recursion, see Bertsimas et al. (2001), Heath et al. (2001). The present paper and Hubalek et al. (2005) sidestep the need to perform the backward recursion numerically by expressing the option pay-off as a linear combination of exponential affine terms in log stock price, computing the variance-optimal characteristic function of log returns, and thereby obtaining all exogenous coefficients of the solution in closed form.

Černý (2004c) uses the results presented here to evaluate option price bounds in a calibrated model of FTSE 100 returns and concludes that while it may be rational to observe option prices with implied volatility above historical volatility due to the presence of hedging risk, the optimal deltas are linked to historical volatility of the stock rather than to the implied volatility of the option. This calls into question the practice of option hedging based on risk-neutral models fitted to implied volatility surfaces, because the fitted volatility is typically much higher than the historical volatility of the underlying stock returns.

The paper is organized as follows: Section 2 summarizes the main results, Section 3 writes down the optimal hedging strategy in a discrete-time model with IID stock returns, Section 4 describes the solution in a model where stock prices are obtained by discrete sampling from a geometric Lévy process, Section 5 gives convergence proofs for the setup of Section 4, Section 6 proves convergence for multinomial lattices, and Section 7 concludes.

2 Overview of the problem and its solution

We fix a time horizon T and consider a sequence of discrete-time models with number of trading dates $n \in \mathbb{N}$ and rebalancing interval $\triangle = T/n$. We assume that the log returns in each model are IID and that the unconditional distribution of log return at time T coincides across all models. It turns out that in the limit the unconditional distribution must be infinitely divisible and its characteristic function has the so-called Lévy–Khintchin representation:

$$\begin{aligned}\phi_T(u) &:= \mathrm{E}\left[\mathrm{e}^{\mathrm{i}u\ln(S_T/S_0)}\right] = \mathrm{e}^{\kappa(\mathrm{i}u)T},\\ \kappa(v) &:= \mu v + \frac{\sigma^2}{2}v^2 + \int_{\mathbb{R}} \left(\mathrm{e}^{vx} - 1 - vh(x)\right) M(\mathrm{d}x),\end{aligned}$$

where

$$\mu \in \mathbb{R}, \sigma^2 \geq 0, h(x) = x1_{|x|\leq 1},\ M(0) = 0,\ \text{and}$$
$$\mu(A) := \int_A \min(x^2, 1)M(\mathrm{d}x) \text{ is a finite measure on } \mathbb{R}.$$

The aim of the paper is to give closed-form solution of the mean–variance hedging problem

$$\inf_{\vartheta} \mathrm{E}\left[(V_T^{x,\vartheta} - H_T)^2\right],$$

where H_T is the pay-off of a European call option, as the rebalancing interval $\triangle$ approaches zero. Here $V_T^{x,\vartheta}$ is the terminal value of a self-financing portfolio containing ϑ_t shares at time t and starting with initial wealth $V_0 = x$. The trading strategy ϑ is assumed to be adapted to the filtration generated by stock prices. At each rebalancing frequency the optimal hedging strategy φ is characterized by the mean value process H, in practice very similar to Black–Scholes value, the locally optimal hedge ξ (similar to Black–Scholes delta), and the expected squared hedging error to maturity $\varepsilon^2(\varphi)$.

The limiting values as $\triangle$ approaches 0 are obtained in closed form by Fourier transform. The convergence is to be understood on the 3-dimensional state space given by calendar time t, current stock price S and the current value of the hedging portfolio V. For the mean value process we have

$$H(t, \ln S) = \int_{(\alpha+1)+\mathrm{i}\mathbb{R}} \psi(u)\mathrm{e}^{(\hat{\kappa}(u)-r)(T-t)}\mathrm{e}^{u\ln S}\mathrm{d}u, \tag{2.1}$$

$$\hat{\kappa}(u) := \kappa(u) - \bar{a}\left(\kappa(u+1) - \kappa(u) - \kappa(1)\right), \tag{2.2}$$

$$\bar{a} := \frac{\kappa(1) - r}{\kappa(2) - 2\kappa(1)}, \tag{2.3}$$

$$\psi(u) := \frac{\mathrm{e}^{-(u-1)k}}{2\pi u(u-1)}, \tag{2.4}$$

where $\psi(u)$ are the (generalized) Fourier coefficients of the option pay-off:

$$H_T = \int_{(\alpha+1)+\mathrm{i}\mathbb{R}} \psi(u)\mathrm{e}^{u\ln S_T}\mathrm{d}u,$$

with $\alpha > 0$ such that $\kappa(2 + 2\alpha)$ is finite. The dynamically optimal strategy φ is given by

$$\varphi(t, \ln S, V) = \xi(t, \ln S) + \bar{a}\frac{H(t, \ln S) - V}{S}, \tag{2.5}$$

$$\xi(t, \ln S) = \int_{(\alpha+1)+\mathrm{i}\mathbb{R}} \mathrm{e}^{(u-1)\ln S}\psi(u)\mathrm{e}^{(\hat{\kappa}(u)-r)(T-t)}\frac{\bar{A}(u)}{\bar{A}(1)}\,\mathrm{d}u, \tag{2.6}$$

$$\bar{A}(u) := \kappa(u+1) - \kappa(u) - \kappa(1). \tag{2.7}$$

Note that the notion of locally optimal hedge ξ used in this paper is similar but not identical to that of *locally risk minimizing strategy* introduced in Schweizer (1991). The main difference stems from the fact that we think of ξ as being self-financing, whereas local risk minimization allows for funds to be added or withdrawn along the way, in such a way that the amount of extra funds is zero *on average*.

The expected squared hedging error of the two strategies with $x = H_0$ is given as follows:

$$\begin{aligned}\varepsilon^2(\varphi) &= \bar{\varepsilon}^2(0, \ln S_0, 1),\\ \varepsilon^2(\xi) &= \bar{\varepsilon}^2(0, \ln S_0, 0),\end{aligned}$$

where

$$\begin{aligned}\bar{\varepsilon}^2(t, \ln S, \delta) &= \int_{G^2} \left(\prod_{j=1,2} \mathrm{e}^{\hat{\kappa}(u_j)T + u_j \ln S} \psi(u_j) \right) \frac{\mathrm{e}^{\bar{C}(u_1,u_2)(T-t)} - \mathrm{e}^{-\delta \bar{b}(T-t)}}{\bar{C}(u_1, u_2) + \delta \bar{b}} \\ &\quad \times \left(\bar{B}(u_1, u_2) - \frac{\bar{A}(u_1)\bar{A}(u_2)}{\bar{A}(1)} \right) \mathrm{d}u_1 \mathrm{d}u_2, \qquad (2.8)\\ \bar{B}(u_1, u_2) &:= \kappa(u_1 + u_2) - \kappa(u_1) - \kappa(u_2), \qquad (2.9)\\ \bar{C}(u_1, u_2) &:= \bar{B}(u_1, u_2) + \bar{a}\left(\bar{A}(u_1) + \bar{A}(u_2)\right), \qquad (2.10)\\ \bar{b} &:= \bar{a}(\kappa(1) - r). \qquad (2.11)\end{aligned}$$

The interpretation of the parameters in the solution goes as follows: $\bar{a}$ is the optimal proportion of wealth invested in the stock by an agent with unit risk tolerance and short investment horizon, $\sqrt{\mathrm{e}^{\bar{b}T} - 1}$ is the unconditional Sharpe ratio of the optimal dynamic strategy investing only in the stock over the interval $[0, T]$, $\sqrt{\kappa(2) - 2\kappa(1)}$ is the instantaneous volatility of stock returns, and $\kappa(1) - r$ is the expected rate of excess return.

The convergence proofs for the mean value process and for the hedging strategy are completely general. The convergence of unconditional hedging errors requires a technical condition that we are unable to verify in general, but which holds in a large class of models where either i) $\sigma^2 = 0$ and $\int_{|x|<1} |x|^\gamma M(\mathrm{d}x) < \infty$ for some $\gamma < 2$, or ii) $\sigma^2 > 0$, see Theorem 5.13. Thus the convergence of the hedging errors $\varepsilon(\varphi)$ and $\varepsilon(\xi)$ is verified in all models with non-zero Brownian motion component and in all frequently encountered pure jump models including compound Poisson, variance gamma and generalized hyperbolic models.

3 Discrete-time model

3.1 Geometry of least squares

Theorem 3.1 *Let $\mathcal{H}$ be a complex pre-Hilbert space with inner product $(.,.)$ and let $X_1, X_2 \in \mathcal{H}$ be linearly independent. Then for any $Y \in \mathcal{H}$ and for any $\beta_1, \beta_2 \in \mathbb{C}$ we have*

$$(3.1)\quad \|Y - \beta_1 X_1 - \beta_2 X_2\|^2 = \|Y - \hat\beta_1 X_1 - \hat\beta_2 X_2\|^2 + \|(\hat\beta_1 - \beta_1)X_1 + (\hat\beta_2 - \beta_2)X_2\|^2,$$

where $\hat\beta_1, \hat\beta_2$ is the unique solution of the normal equations

$$(3.2)\qquad \Big(Y - \hat\beta_1 X_1 - \hat\beta_2 X_2, X_1\Big) = 0,$$

$$(3.3)\qquad \Big(Y - \hat\beta_1 X_1 - \hat\beta_2 X_2, X_2\Big) = 0.$$

Furthermore we have

$$(3.4)\qquad \hat\beta_1 = \frac{\Big(Y - \hat\beta_2 X_2, X_1\Big)}{\|X_1\|^2} = \frac{(M_{X_2}Y, M_{X_2}X_1)}{\|M_{X_2}X_1\|^2} = \frac{(Y, M_{X_2}X_1)}{\|M_{X_2}X_1\|^2},$$

$$(3.5)\qquad \hat\beta_2 = \frac{\Big(Y - \hat\beta_1 X_1, X_2\Big)}{\|X_2\|^2} = \frac{(M_{X_1}Y, M_{X_1}X_2)}{\|M_{X_1}X_2\|^2} = \frac{(Y, M_{X_1}X_2)}{\|M_{X_1}X_2\|^2},$$

$$(3.6)\qquad \|Y - \hat\beta_1 X_1 - \hat\beta_2 X_2\|^2 = \|M_{X_2}Y\|^2 - |\hat\beta_1|^2 \|M_{X_2}X_1\|^2$$

$$(3.7)\qquad = \|Y\|^2 - \frac{|(Y, X_2)|^2}{\|X_2\|^2} - |\hat\beta_1|^2 \left(\|X_1\|^2 - \frac{|(X_1, X_2)|^2}{\|X_2\|^2}\right),$$

where $M_X Y = Y - X(Y,X)/\|X\|^2$ is the orthogonal projection of Y away from X.

PROOF. We will need the following standard properties of inner product:

i) $(Y - X, X) = 0 \Rightarrow \|Y - X\|^2 = \|Y\|^2 - \|X\|^2$

ii) $(X_2, M_{X_2}X_1) = 0$

iii) $(Y, M_{X_2}X_1) = (M_{X_2}Y, M_{X_2}X_1) = (M_{X_2}Y, X_1)$.

Suppose that the matrix

$$A = \begin{bmatrix} \|X_1\|^2 & (X_2, X_1) \\ (X_1, X_2) & \|X_2\|^2 \end{bmatrix}$$

is singular, then there is a complex vector $z \in \mathbb{C}^2$, $z \neq 0$, such that $Az = 0$, implying

$$0 = z^* A z = \|z_1 X_1 + z_2 X_2\|^2,$$

which contradicts the assumption of linear independence of X_1 and X_2. Therefore A is an invertible square matrix and hence the solution $\hat\beta_1, \hat\beta_2$ of the system (3.2), (3.3) exists and it is unique. Equation (3.1) then follows from property i).

The first equalities in (3.4) and (3.5) follow immediately from the normal equations. Writing $X_1 = X_2(X_1, X_2)/\|X_2\|^2 + M_{X_2}X_1$ equations (3.2) and (3.3) imply

$$\left(Y - \hat{\beta}_1 X_1 - \hat{\beta}_2 X_2, M_{X_2}X_1\right) = 0,$$

whereby properties ii) and iii) yield

$$0 = \left(M_{X_2}Y - \hat{\beta}_1 M_{X_2}X_1, M_{X_2}X_1\right).$$

Solving for $\hat{\beta}_1$ proves the second and third equality in (3.4), where $\|M_{X_2}X_1\| > 0$ by linear independence. Furthermore, property i) implies

$$\|M_{X_2}Y - \hat{\beta}_1 M_{X_2}X_1\|^2 = \|M_{X_2}Y\|^2 - |\hat{\beta}_1|^2\|M_{X_2}X_1\|^2,$$

while the normal equation (3.3) gives

$$\begin{aligned} Y - \hat{\beta}_1 X_1 - \hat{\beta}_2 X_2 &= M_{X_2}\left(Y - \hat{\beta}_1 X_1 - \hat{\beta}_2 X_2\right) \\ &= M_{X_2}Y - \hat{\beta}_1 M_{X_2}X_1, \end{aligned}$$

proving (3.6). Equation (3.5) follows from (3.4) by symmetry. $\square$

Remark 3.2 *The least squares theorem plays a dual role in the derivations that follow. A real-valued version of the theorem is used to determine the hedging error of locally optimal and dynamically optimal hedging strategies, while the complex-valued version is instrumental in obtaining bounds for the variance-optimal characteristic function, the hedging coefficients and the hedging error as one passes to continuous-time limit.*

Theorem 3.3 *Fix $n \in \mathbb{N}$, and let $\{Z_i\}_{i=1,\ldots,n}$ be a collection of IID random variables on the probability space $\{\Omega, \mathscr{F}, P\}$ such that $0 < \mathrm{Var}(\exp(Z_i)) < \infty$. Let $\{\mathscr{F}_i\}_{i=1,\ldots,n}$ be the information filtration generated by the random variables $\{Z_i\}$, with $\mathscr{F}_0$ trivial. Fix $S_0 > 0$ and define the price process of a risky asset $\{S_i\}$:*

$$S_i = S_0 \exp\left(\sum_{k=1}^{j} Z_k\right),$$

and a risk-free bank account with total return R (implying the rate of return $R - 1$ per period). Let $E_i[.]$ denote the expectation conditional on $\mathscr{F}_i$.

Suppose that there is a contingent claim $H_n = f(\ln S_n)$ such that $\mathrm{E}[H_n^2] < \infty$. Then the dynamically optimal hedging strategy φ solving

$$\inf_{\vartheta} \mathrm{E}[\left(V_n^{x,\vartheta} - H_n\right)^2],$$

subject to ϑ_i being $\mathscr{F}_i$-measurable with

$$
\begin{aligned}
V_i^{x,\vartheta} &= RV_{i-1}^{x,\vartheta} + \vartheta_{i-1}(S_i - RS_{i-1}) \\
V_0^{x,\vartheta} &= x
\end{aligned}
$$

is given by

$$
\begin{aligned}
\varphi_i &= \xi_i + aR\frac{H_i - V_i^{x,\varphi}}{S_i}, \\
H_i &= \mathrm{E}_i[(1 - aX_{i+1})H_{i+1}]/(bR), \qquad (3.8) \\
\xi_i &= \mathrm{Cov}_i\left(H_{i+1}, S_{i+1}\right)/\mathrm{Var}_i\left(S_{i+1}\right) \\
&= \mathrm{E}_i\left[\left(H_{i+1} - RH_i\right)X_{i+1}\right]/\left(S_i \mathrm{E}_i\left[X_{i+1}^2\right]\right), \\
X_i &= \exp(Z_i) - R, \\
a &= \mathrm{E}_i\left[X_{i+1}\right]/\mathrm{E}_i\left[X_{i+1}^2\right], \\
b &= 1 - \left(\mathrm{E}_i\left[X_{i+1}\right]\right)^2/\mathrm{E}_i\left[X_{i+1}^2\right].
\end{aligned}
$$

The hedging performance of the dynamically optimal and the locally optimal strategies is given by

$$
\begin{aligned}
\mathrm{E}_j\left[(V_n^{x,\varphi} - H_n)^2\right] &= \left(R^2 b\right)^{n-j}\left(V_j^{x,\varphi} - H_i\right)^2 + \varepsilon_j^2(\varphi), \\
\mathrm{E}_j\left[\left(V_n^{x,\xi} - H_n\right)^2\right] &= \left(R^2\right)^{n-j}\left(V_j^{x,\xi} - H_i\right)^2 + \varepsilon_j^2(\xi), \\
\varepsilon_j^2(\varphi) &= \mathrm{E}_j\left[\varepsilon_{j+1}^2(\varphi)\right] + \left(R^2 b\right)^{n-j-1}\mathrm{ESRE}_j\left(H_{j+1}\right), \qquad (3.9) \\
\varepsilon_j^2(\xi) &= \mathrm{E}_i\left[\varepsilon_{j+1}^2(\xi)\right] + \left(R^2\right)^{n-j-1}\mathrm{ESRE}_j\left(H_{j+1}\right), \qquad (3.10) \\
\varepsilon_n^2\left(\varphi\right) &= \varepsilon_n^2\left(\xi\right) = 0, \\
\mathrm{ESRE}_j\left(H_{j+1}\right) &= \mathrm{E}_j\left[\left(RH_j + \xi_j S_j X_{j+1} - H_{j+1}\right)^2\right] \\
&= \mathrm{E}_j\left[H_{j+1}^2\right] - bR^2 H_j^2 - \frac{\left(\mathrm{E}_j\left[X_{j+1}H_{j+1}\right]\right)^2}{\mathrm{E}_i\left[X_{j+1}^2\right]} \qquad (3.11) \\
&= \mathrm{Var}_j\left(H_{j+1}^2\right) - \frac{\left(\mathrm{Cov}_j\left(S_{j+1}, H_{j+1}\right)\right)^2}{\mathrm{Var}_j\left(S_{j+1}\right)}.
\end{aligned}
$$

PROOF. The proof for a model with finite number of states is given in Černý (2004b), Chapter 12. The general discrete-time IID case is handled identically, using Theorem 3.1 with $Y = H_{i+1}, X_1 = 1, X_2 = X_{i+1}$ and with the inner product given by

$$(X, Y) = \mathrm{E}_i\left[X\bar{Y}\right].$$

Integrability is shown as follows. Equation (3.11) implies $H_i^2 \leq \mathrm{E}_i\left[H_{i+1}^2\right]/b$ and consequently by the law of iterated expectations

$$\mathrm{E}\left[H_i^2\right] \leq \mathrm{E}\left[H_{i+1}^2\right]/b < \infty.$$

Since $\mathrm{E}\left[H_n^2\right] < \infty$ and $b > 0$ by assumption, it follows that $\mathrm{E}\left[H_i^2\right] < \infty$ for all i. $\square$

3.2 Distribution of log returns under the variance-optimal measure

Equation (3.8) motivates the introduction of a signed martingale measure Q

$$\frac{dQ}{dP} := \prod_{j=1}^{n} \frac{1 - a\left(\mathrm{e}^{Z_j} - R\right)}{b}.$$

For any $\mathscr{F}_t$-measurable random variable X with $E\left[X^2\right] < \infty$ we define the conditional expectation under Q following Černý (2004a)

$$\hat{\mathrm{E}}_j\left[X\right] := \mathrm{E}_j\left[X \prod_{k=j+1}^{n} \frac{1 - a\left(\mathrm{e}^{Z_k} - R\right)}{b}\right], \tag{3.12}$$

The expression (3.12) is well defined even if the density process of Q is zero at time j. The definition (3.12) is consistent with the theory of $\mathscr{E}$-martingales proposed by Choulli, Krawczyk, and Stricker (1998). Q is the variance-optimal (signed) measure in the sense of Schweizer (1995) and it coincides with the minimal martingale measure, cf. Schäl (1994), Schweizer (1995, Section 4.3).

By virtue of (3.8) and (3.12) we can write

$$H_j = R^{j-n}\hat{\mathrm{E}}_j\left[H_n\right]. \tag{3.13}$$

To obtain a more explicit expression for the mean value process $\{H_j\}$ it is important to know the characteristic function of log returns under measure Q. To this end, we define the characteristic function of one-period log returns,

$$\phi(v) := \mathrm{E}\left[\exp(\mathrm{i}vZ_1)\right],$$

and note that

$$\phi(v) = \mathrm{E}_{j-1}\left[\exp(\mathrm{i}vZ_j)\right] \text{ for } j = 1, \ldots, n.$$

The mean–variance analysis only makes sense if the variance of stock returns is finite, $\mathrm{E}\left[S_n^2\right] < \infty$. For technical reasons that will become clear in Section 3.3 we will require slightly more:

Standing assumption 1 There is $\alpha > 0$ such that the characteristic function of log returns is well defined on the strip $\operatorname{Re} v \in \mathbb{R}$, $\operatorname{Im} v \in [-2 - 2\alpha, 0]$.

We can now define the (pseudo)-characteristic function of log returns under Q

$$\hat{\phi}(v) := \hat{\mathrm{E}}\left[\exp(\mathrm{i}vZ_1)\right]. \tag{3.14}$$

Lemma 3.4 *The function* $\hat{\phi}(v)$ *is well defined on the strip* $\operatorname{Re} v \in \mathbb{R}$, $\operatorname{Im} v \in [-1-2\alpha, 0]$ *and we have*

$$\begin{aligned}
\hat{\phi}(v) &= \frac{1+aR}{b}\phi(v) - \frac{a}{b}\phi(v-\mathrm{i}),\\
a &:= \frac{\phi(-\mathrm{i}) - R}{\phi(-2\mathrm{i}) - 2R\phi(-\mathrm{i}) + R^2},\\
b &:= 1 - \frac{\left(\phi(-\mathrm{i}) - R\right)^2}{\phi(-2\mathrm{i}) - 2R\phi(-\mathrm{i}) + R^2}.
\end{aligned}$$

Furthermore

$$\hat{\phi}(v) = \hat{\mathrm{E}}_{j-1}\left[\exp(\mathrm{i}vZ_j)\right] \text{ for } j = 1, \dots, n.$$

PROOF. From the law of iterated expectations we obtain

$$\begin{aligned}
\hat{\mathrm{E}}_{j-1}\left[\exp(\mathrm{i}vZ_j)\right] &= \mathrm{E}_{j-1}\left[\mathrm{e}^{\mathrm{i}vZ_j}\prod_{k=j}^{n}\frac{1-a\left(\mathrm{e}^{Z_k}-R\right)}{b}\right]\\
&= \mathrm{E}_{j-1}\left[\mathrm{e}^{\mathrm{i}vZ_j}\frac{1-a\left(\mathrm{e}^{Z_j}-R\right)}{b}\prod_{k=j}^{n}\mathrm{E}_{k-1}\left[\frac{1-a\left(\mathrm{e}^{Z_k}-R\right)}{b}\right]\right]\\
&= \mathrm{E}_{j-1}\left[\mathrm{e}^{\mathrm{i}vZ_j}\frac{1-a\left(\mathrm{e}^{Z_j}-R\right)}{b}\right]\\
&= \frac{1+aR}{b}\phi(v) - \frac{a}{b}\phi(v-\mathrm{i}).
\end{aligned}$$

Since $\phi(v)$ is well defined on $\mathbb{R}\times\mathrm{i}[-2-2\alpha,0]$ $\phi(v-\mathrm{i})$ is well defined on $\mathbb{R}\times\mathrm{i}[-1-2\alpha,1]$ and therefore $\hat{\phi}(v)$ is well defined on $\mathbb{R}\times\mathrm{i}[-1-2\alpha,0]$. □

We are now ready to compute the mean value process in terms of $\hat{\phi}$.

3.3 Mean value process as a Fourier transform

The main idea of the paper is to express the option pay-off as a sum of terms which are exponentially affine in the log stock price and then make use of the risk-neutral characteristic function (3.14) when evaluating the expectation $H_j = R^{j-n}\hat{\mathrm{E}}_j[H_n]$.

Lemma 3.5 *Let $f(\ln S_T) := (S_T - \mathrm{e}^k)^+$ be the pay-off of a call option with strike e^k. Then for any $\alpha > 0$ we have*

$$f(x) = \int_G \psi(u)\mathrm{e}^{ux}\mathrm{d}u, \tag{3.15}$$

$$\psi(u) := \frac{\mathrm{e}^{-(u-1)k}}{2\pi u(u-1)}, \tag{3.16}$$

$$G := \alpha + 1 + \mathrm{i}\mathbb{R}. \tag{3.17}$$

PROOF. To obtain $\psi(u)$ one computes the inverse Fourier transform of the modified option pay-off $g(\ln S_T) := \left(\mathrm{e}^{\ln S_T} - \mathrm{e}^k\right)^+ S_T^{-1-\alpha}$. The multiplication by $S_T^{-1-\alpha}$ is performed to achieve absolute integrability of the pay-off as a function of $\ln S_T$. Function $g(x)$ is continuous and therefore by Theorem 11' in Chandrasekharan (1989) we have

$$g(x) = \int_{\mathbb{R}} \chi(v)\mathrm{e}^{-\mathrm{i}vx}\mathrm{d}v,$$

$$\chi(v) := \frac{1}{2\pi}\int_{\mathbb{R}} g(x)\mathrm{e}^{-\mathrm{i}vx}\mathrm{d}x.$$

A short computation yields $\chi(v) = \mathrm{e}^{-(\alpha+\mathrm{i}v)k} / \left(2\pi(\alpha+\mathrm{i}v)(1+\alpha+\mathrm{i}v)\right)$ and consequently also (3.15)-(3.17). □

Remark 3.6 *Carr and Madan (1999) evaluate the inverse Fourier transform with respect to the log strike k, which happens to yield the harmonic decomposition (3.15)-(3.17). In general, and particularly for derivative securities that do not feature a striking price such as the log contract or powers thereof, one has to perform the Fourier transform with respect to $\ln S_T$.*

Remark 3.7 *A formula analogous to (3.15)-(3.17) is valid for all claims that grow at most linearly in S_T as $S_T \to \infty$. The Fourier coefficients $\psi(u)$ are easily computed from the inverse Fourier transform; this can be done for a put option and, a fortiori, for any portfolio of put and call options including spreads, strangles, straddles etc. In the unlikely case that it is impractical to evaluate the Fourier coefficients in closed form one can compute them efficiently using the fast Fourier transform.*

Remark 3.8 *Another way to achieve integrability is to employ the put–call parity and examine a put option instead of a call,*

$$\left(S_T - \mathrm{e}^k\right)^+ - S_T + \mathrm{e}^k = \left(\mathrm{e}^k - S_T\right)^+,$$

the advantage being that the put pay-off only needs to be multiplied by $S_T^{-\alpha}$ to achieve integrability with respect to $\ln S_T$. The hedging errors from being short one call option and from being short one call and long one stock are the same, since the addition of the stock just shifts the number of shares in the hedging portfolio by 1.

Lemma 3.9 *The discrete-time mean value process $H_j := R^{j-n}\hat{\mathrm{E}}_j[H_n]$ is given as follows:*

$$H_j = \int_G \tilde{\psi}_j(u)\mathrm{e}^{u\ln S_j}\mathrm{d}u, \tag{3.18}$$

$$\tilde{\psi}_j(u) := R^{j-n}\psi(u)\left(\hat{\phi}(-\mathrm{i}u)\right)^{n-j}. \tag{3.19}$$

PROOF. Proceed by direct calculation:

$$R^{n-j}H_j = \hat{\mathrm{E}}_j[H_n] = \mathrm{E}_j\left[H_n \prod_{k=j+1}^{n} \frac{1-a\left(\mathrm{e}^{Z_k}-R\right)}{b}\right] \tag{3.20}$$

$$= \mathrm{E}_j\left[\int_G \left(\prod_{k=j+1}^{n} \frac{1-a\left(\mathrm{e}^{Z_k}-R\right)}{b}\right)\psi(u)\mathrm{e}^{u\ln S_n}\mathrm{d}u\right] \tag{3.21}$$

$$= \int_G \psi(u)\mathrm{e}^{u\ln S_j}\mathrm{E}_j\left[\prod_{k=j+1}^{n} \frac{1-a\left(\mathrm{e}^{Z_k}-R\right)}{b}\mathrm{e}^{u\ln(S_k/S_{k-1})}\right]\mathrm{d}u \tag{3.22}$$

$$= \int_G \psi(u)\mathrm{e}^{u\ln S_j}\left(\mathrm{E}_j\left[\prod_{k=j+1}^{n}\mathrm{E}_{k-1}\left[\frac{1-a\left(\mathrm{e}^{Z_k}-R\right)}{b}\mathrm{e}^{uZ_k}\right]\right]\right)\mathrm{d}u \tag{3.23}$$

$$= \int_G \psi(u)\mathrm{e}^{u\ln S_j}\left(\hat{\phi}(-\mathrm{i}u)\right)^{n-j}\mathrm{d}u. \tag{3.24}$$

Here (3.20) follows from (3.12) and (3.13), equation (3.21) follows from (3.15), equation (3.22) follows from Fubini's theorem, equation (3.23) follows from the law of iterated expectations and (3.24) follows from Lemma 3.4. □

With (3.18) in hand it is now easy to evaluate the locally optimal hedge ξ_j and the expected squared hedging errors to maturity $\varepsilon^2(\varphi)$ and $\varepsilon^2(\xi)$.

Lemma 3.10 *The discrete-time locally optimal hedging strategy is given by*

$$S_j\xi_j = \int_G \mathrm{e}^{u\ln S_j}\tilde{\psi}_{j+1}(u)A(u)/A(1)\mathrm{d}u, \tag{3.25}$$

$$A(u) := \phi(-\mathrm{i}(u+1)) - \phi(-\mathrm{i}u)\phi(-\mathrm{i})$$

PROOF. By direct calculation using Theorem 3.3 we have

$$\xi_j = \mathrm{Cov}_j\left(H_{j+1}, S_{j+1}\right)/\mathrm{Var}_j(S_{j+1}),$$

$$\frac{\mathrm{Cov}_j\left(H_{j+1}, S_{j+1}\right)}{S_j} = \mathrm{E}_j\left[H_{j+1}\mathrm{e}^{Z_{j+1}}\right] - \phi\left(-\mathrm{i}\right)\mathrm{E}_j\left[H_{j+1}\right].$$

Substitute for H_{j+1} from (3.18) and use Fubini's theorem to obtain:

$$\frac{\mathrm{Cov}_j\left(H_{j+1}, S_{j+1}\right)}{S_j} = \int_G \mathrm{e}^{u \ln S_j} \tilde{\psi}_{j+1}(u) A(u) \mathrm{d}u.$$

By definition of ϕ we have

$$\frac{\mathrm{Var}_j(S_{j+1})}{S_j^2} = \phi\left(-2\mathrm{i}\right) - \left(\phi\left(-\mathrm{i}\right)\right)^2,$$

and the claim follows.

□

Lemma 3.11 *The dynamically optimal and the locally optimal expected squared hedging errors are given by the following formulae:*

$$\begin{aligned}
\varepsilon_j^2(\varphi) &= h(j,1), \\
\varepsilon_j^2(\xi) &= h(j,0),
\end{aligned}$$

where

$$\begin{aligned}
h(j,\delta) &:= \int_{G^2} \mathrm{e}^{(u_1+u_2)\ln S_j} \psi(u_1)\psi(u_2) \\
&\quad \times \left(\left(\phi(-\mathrm{i}(u_1+u_2)\right)^{n-j} - b^{1-\delta}\left(b^{\delta}\hat{\phi}(-\mathrm{i}u_1)\hat{\phi}(-\mathrm{i}u_2)\right)^{n-j} \right) \mathrm{d}u_1 \mathrm{d}u_2, \\
&+ \quad (1-b^{1-\delta}) \int_{G^2} \sum_{k=j+1}^{n-1} \left(\phi(-\mathrm{i}(u_1+u_2)\right)^{k-j} \\
&\quad \times \prod_{l=1,2} \left(\mathrm{e}^{u_l \ln S_j} \psi(u_l) \left(b^{\delta/2}\hat{\phi}(-\mathrm{i}u_l)\right)^{n-k} \mathrm{d}u_l \right) \\
&- \quad \left(\phi(-2\mathrm{i}) - 2R\phi(-\mathrm{i}) + R^2\right) \int_{G^2} \sum_{k=j}^{n-1} \left(\phi(-\mathrm{i}(u_1+u_2)\right)^{k-j} \\
&\quad \times \prod_{l=1,2} \left(\mathrm{e}^{u_l \ln S_j} \psi(u_l) \left(b^{\delta/2}\hat{\phi}(-\mathrm{i}u_l)\right)^{n-1-k} \left(\frac{A(u_l)}{A(1)} + a\hat{\phi}(-\mathrm{i}u_l)\right) \mathrm{d}u_l \right)
\end{aligned}$$

PROOF. Proceed from Theorem 3.3, equations (3.9), (3.10) and (3.11), by direct calculation,

$$\begin{aligned}
h(j,\delta) &= \sum_{k=j}^{n-1} \left(b^{\delta}R^2\right)^{n-k-1} \mathrm{E}_j\left[\mathrm{ESRE}_k(H_{k+1})\right] \\
&= \sum_{k=j}^{n-1} \left(b^{\delta}R^2\right)^{n-1-k} \mathrm{E}_j\left[\mathrm{E}_k\left[H_{k+1}^2\right] - bR^2 H_k^2 - \left(S_k\xi_k + aRH_k\right)^2 \mathrm{E}_k\left[X_{k+1}^2\right]\right]
\end{aligned}$$

$$
\begin{aligned}
&= \mathrm{E}_j\left[H_n^2\right] - b^{1-\delta}\left(b^\delta R^2\right)^{n-j} H_j^2 + \sum_{k=j+1}^{n-1} \mathrm{E}_j\left[H_k^2\right]\left(b^\delta R^2\right)^{n-k}\left(1-b^{1-\delta}\right) \\
&\quad - \sum_{k=j}^{n-1}\left(b^\delta R^2\right)^{n-1-k} \mathrm{E}_j\left[\left(S_k\xi_k + aRH_k\right)^2\left(\phi(-2\mathrm{i}) - 2R\phi(-\mathrm{i}) + R^2\right)\right],
\end{aligned}
$$

Use equation (3.18) and Fubini's theorem to write

$$
\begin{aligned}
\mathrm{E}_j\left[H_k^2\right] &= R^{2(k-n)}\mathrm{E}_j\left[\left(\int_G \tilde{\psi}_k(u_1)\mathrm{e}^{u_1 \ln S_k}\mathrm{d}u_1\right)\left(\int_G \tilde{\psi}_k(u_2)\mathrm{e}^{u_2 \ln S_k}\mathrm{d}u_2\right)\right] \\
&= R^{2(k-n)}\mathrm{E}_j\left[\int_{G^2} \tilde{\psi}_k(u_1)\tilde{\psi}_k(u_2)\mathrm{e}^{(u_1+u_2)\ln S_k}\mathrm{d}u_1\mathrm{d}u_2\right] \\
&= R^{2(k-n)}\int_{G^2} \tilde{\psi}_k(u_1)\tilde{\psi}_k(u_2)\left(\phi\left(-\mathrm{i}\left(u_1+u_2\right)\right)\right)^{k-j}\mathrm{e}^{(u_1+u_2)\ln S_j}\mathrm{d}u_1\mathrm{d}u_2,
\end{aligned}
$$

with $\tilde{\psi}_k(u)$ defined in equation (3.19). Similarly, utilize formulae (3.18) and (3.25) to write

$$
\begin{aligned}
\mathrm{E}_j\left[\left(S_k\xi_k + aRH_k\right)^2\right] &= \int_{G^2} \mathrm{e}^{(u_1+u_2)\ln S_j}\left(\phi(-\mathrm{i}(u_1+u_2))\right)^{k-j} \\
&\quad \times \prod_{l=1,2}\left(\psi(u_l)\left(\hat{\phi}(-\mathrm{i}u_l)\right)^{n-1-k}\left(\frac{A(u_l)}{A(1)} + a\hat{\phi}(-\mathrm{i}u_l)\right)\mathrm{d}u_l\right).
\end{aligned}
$$

□

4 Towards continuous time

Consider a fixed time horizon $T \in \mathbb{R}_{++}$ divided into $n \in \mathbb{N}$ time intervals of length $\triangle$,

$$
\triangle := T/n.
$$

We wish to consider a general model where stock returns are IID at any sampling frequency $\triangle$. To achieve this *and* to avoid technicalities associated with optimization of dynamic portfolios generated by geometric Lévy processes we will work with a family of models, one for each sampling frequency, and we will characterize the hedging strategy and hedging errors in the limit as $\triangle \to 0$.

For a fixed $n \in \mathbb{N}$ let $\{Z_{jn} : j = 1, \ldots, n\}$ be independent and identically distributed real-valued random variables on the probability space $(\Omega_n, \mathscr{F}_n, P_n)$ and let $\{\mathscr{F}_{jn} : j = 0, \ldots, n\}$ be the information filtration generated by $\{Z_{jn}\}$

$$
\begin{aligned}
\mathscr{F}_{0n} &:= \{\Omega_n, \emptyset\}, \\
\mathscr{F}_{jn} &:= \sigma(\{Z_{kn} : 1 \le k \le j \le n\}).
\end{aligned}
$$

The expectation operator on $(\Omega_n, \mathscr{F}_n, P_n)$ will be denoted $\mathrm{E}_n[.]$. The expectation conditional on information at time j in model n is denoted by $\mathrm{E}_{jn}[.]$,

$$
\mathrm{E}_{jn}[.] := \mathrm{E}_n[.|\mathscr{F}_{jn}].
$$

For $S_0 \in \mathbb{R}_{++}$ define a triangular array of random variables $\{S_{jn}\}$ by setting

$$S_{jn} := S_0 \exp\left(\sum_{k=1}^{j} Z_{kn}\right) \text{ for } j = 0, \ldots, n.$$

Lemma 4.1 ***(Characterization of return distributions)*** *Suppose* $\ln S_{nn} \xrightarrow{\mathrm{d}} \bar{Z}$ *as* $n \to \infty$. *Then* $\bar{Z}$ *has an infinitely divisible distribution.*

PROOF. Infinite divisibility follows by Theorem 7.5.2 in Ash and Doléans-Dade (1999). □

Theorem 4.2 *Suppose that* $\bar{Z}$ *is infinitely divisible. Then*

1. *Its characteristic function has the Lévy–Khintchin representation*

   $$\phi(v) := \mathrm{E}\left[\mathrm{e}^{\mathrm{i}v\bar{Z}}\right] = \mathrm{e}^{\kappa(\mathrm{i}v)}, \tag{4.1}$$

   $$\kappa(u) := \mu u + \frac{\sigma^2}{2}u^2 + \tilde{\kappa}(u), \tag{4.2}$$

   $$\tilde{\kappa}(u) := \int_{\mathbb{R}} \left(\mathrm{e}^{ux} - 1 - uh(x)\right) M(\mathrm{d}x), \tag{4.3}$$

   where

   $$\mu \in \mathbb{R}, \sigma \in \mathbb{R}_+, h(x) = x 1_{|x|\leq 1},\ M(0) = 0,\ \textit{and} \tag{4.4}$$

   $$\mu(A) := \int_A h^2(x) M(\mathrm{d}x) \textit{ is a finite measure on } \mathbb{R}. \tag{4.5}$$

   Conversely, every representation (4.1)-(4.5) corresponds to an infinitely divisible distribution.

2. *The representation (4.1)-(4.5) is valid in the strip* $a \leq -\mathrm{Im}\,(v) = \mathrm{Re}\,(u) \leq b$, *such that*

   $$\mathrm{E}\left[S_T^y\right] < \infty \textit{ for } a \leq y \leq b.$$

PROOF. See Theorem 25.17 in Sato (1999). □

Lemma 4.1 justifies the following standing assumption.

Standing assumption 2 The conditional P_n-distribution of log returns is such that the resulting unconditional distribution of log returns coincides across all models with different values of $\triangle$, that is

$$\phi_n(v) := \mathrm{E}_{jn}\left[\exp(\mathrm{i}vZ_{(j+1)n})\right] = \mathrm{e}^{\kappa(\mathrm{i}v)\triangle}, \tag{4.6}$$

for all $n \in \mathbb{N}$ and $j = 1, \ldots, n$, with κ given in (4.2).

Assumption (4.6) guarantees that the unconditional distribution of log returns $\ln S_{nn}$ coincides for all n and it is infinitely divisible. Effectively, the stock prices in the discrete-time model are obtained by sampling from a geometric Lévy model at the frequency $1/\triangle$. We will relax the infinite divisibility assumption for finite n in Section 6.

The next assumption together with Theorem 4.3 is equivalent to the Standing assumption 1 in the context of this section:

Standing assumption 3 $\mathrm{E}_n\left[S_{nn}^{2+2\alpha}\right] < \infty$ for some $\alpha > 0$.

One can dispose of the requirement $\alpha > 0$ by using a transformation akin to put–call parity, see Remark 3.8.

Standing assumption 4 We wish to consider the non-trivial case when the stock return is risky: $\mathrm{Var}_n\left(S_{nn}\right) > 0$.

Standing assumption 5 The risk-free rate of return coincides across all models, i.e. there is $r \in \mathbb{R}$ such that the one-period risk-free total return in model n equals

$$R_n := \mathrm{e}^{r\triangle}.$$

Theorem 4.3 *Suppose* $\ln S_{nn}$ *is infinitely divisible with its Lévy–Khintchin representation given by (4.1)-(4.5). Under the standing assumptions 3 and 4 the integral representation (4.1)-(4.5) is valid at least in the strip* $0 \leq -\mathrm{Im}\,(v) = \mathrm{Re}\,(u) \leq 2 + 2\alpha$, *and we have*

$$\kappa(2) - 2\kappa(1) > 0.$$

PROOF. If $\mathrm{E}\left[S_{nn}^{2+2\alpha}\right] < \infty$ then for $0 \leq y \leq 2 + 2\alpha$ we have $\mathrm{E}\left[S_{nn}^{y}\right] < \infty$. The rest follows from Theorem 4.2. Regarding the variance of S_{nn} we can write explicitly

$$\mathrm{Var}_n\left(S_{nn}\right) = \mathrm{E}_n\left[S_{nn}^2\right] - \left(\mathrm{E}_n\left[S_{nn}\right]\right)^2 = \mathrm{e}^{\kappa(2)T} - \mathrm{e}^{2\kappa(1)T} > 0,$$

which implies $\kappa(2) > 2\kappa(1)$. □

Remark 4.4 *Under the standing assumption 3 the variance of stock returns is always finite, but the variance of* log *returns may be infinite if negative jumps arrive with sufficiently high intensity. It may even happen that the mean log return is infinitely negative,* $\mathrm{E}_n(\ln S_{nn}) = -\infty$.

Proposition 4.5 *Define the following functions*

$$\text{(4.7)} \qquad \bar{H}_n(t,x) := \int_G \mathrm{e}^{ux}\mathrm{e}^{r(t-T)}\psi(u)\left(\hat{\phi}_n(-\mathrm{i}u)\right)^{n(1-t/T)}\mathrm{d}u$$

$$\text{(4.8)} \qquad \bar{\xi}_n(t,x) := \mathrm{e}^{r(t+T/n-T)}\int_G \mathrm{e}^{(u-1)x}\psi(u)\left(\hat{\phi}_n\left(-\mathrm{i}u\right)\right)^{n(1-t/T-1/n)}\frac{A_n(u)}{A_n(1)}\mathrm{d}u$$

$$\bar{\varphi}_n(t,x,V) := \bar{\xi}_n(t,x) + a_n R_n e^{-x}\left(\bar{H}_n(t,x) - V\right)$$

$$
\begin{aligned}
\bar{\varepsilon}_n^2(t,x,\delta) \;:=\; & \int_{G^2} \mathrm{e}^{(u_1+u_2)x}\psi(u_1)\psi(u_2) \\
& \times \left((\phi_n(-\mathrm{i}(u_1+u_2))^{n-j} - b_n^{1-\delta}\left(b_n^\delta \hat{\phi}_n(-\mathrm{i}u_1)\hat{\phi}_n(-\mathrm{i}u_2)\right)^{n-j} \right) \mathrm{d}u_1\mathrm{d}u_2, \\
+ \; & (1-b_n^{1-\delta}) \int_{G^2} \sum_{k=\lfloor t/\triangle \rfloor+1}^{n-1} (\phi_n(-\mathrm{i}(u_1+u_2))^{k-j} \\
& \times \prod_{l=1,2} \left(\mathrm{e}^{u_l x}\psi(u_l)\left(b_n^{\delta/2}\hat{\phi}_n(-\mathrm{i}u_l)\right)^{n-k} \mathrm{d}u_l \right) \\
- \; & c_n \int_{G^2} \sum_{k=\lfloor t/\triangle \rfloor}^{n-1} (\phi_n(-\mathrm{i}(u_1+u_2))^{k-j} \mathrm{e}^{(u_1+u_2)x} \\
& \times \prod_{l=1,2} \left(\psi(u_l)\left(b_n^{\delta/2}\hat{\phi}_n(-\mathrm{i}u_l)\right)^{n-1-k} \left(\frac{A_n(u_l)}{A_n(1)} + a_n\hat{\phi}_n(-\mathrm{i}u_l) \right) \mathrm{d}u_l \right),
\end{aligned} \tag{4.9}
$$

with

$$
\begin{aligned}
A_n(u) &:= \phi_n(-\mathrm{i}(u+1)) - \phi_n(-\mathrm{i}u)\phi_n(-\mathrm{i}), \\
a_n &:= (\phi_n(-\mathrm{i}) - R_n)/c_n, \\
b_n &:= A_n(1)/c_n, \\
c_n &:= \phi_n(-2\mathrm{i}) - 2R_n\phi_n(-\mathrm{i}) + R_n^2
\end{aligned} \tag{4.10}
$$

$$
\hat{\phi}_n(u) := \frac{1+a_nR}{b_n}\phi_n(v) - \frac{a_n}{b_n}\phi_n(v-\mathrm{i}). \tag{4.11}
$$

Then in the n-period model described by standing assumptions 2-5 the mean value process H_{jn}, the optimal strategy φ_{jn}, the locally optimal strategy ξ_{jn} and the expected squared hedging error to maturity $\varepsilon^2_{jn}(\varphi)$, $\varepsilon^2_{jn}(\xi)$ are well defined and equal

$$
\begin{aligned}
H_{jn} &= \bar{H}_n(j\triangle, \ln S_{jn}), \\
\xi_{jn} &= \bar{\xi}_n(j\triangle, \ln S_{jn}), \\
\varphi_{jn} &= \bar{\varphi}_n(j\triangle, \ln S_{jn}, V_{jn}^{x,\varphi}), \\
\varepsilon^2_{jn}(\varphi) &= \bar{\varepsilon}^2_n(j\triangle, \ln S_{jn}, 1), \\
\varepsilon^2_{jn}(\xi) &= \bar{\varepsilon}^2_n(j\triangle, \ln S_{jn}, 0).
\end{aligned}
$$

PROOF. The statement follows directly from Lemmata 3.5, 3.9, 3.10, 3.11 in combination with the standing assumptions 2-5. □

The preceding proposition motivates the study of sufficient conditions for convergence

$$
\begin{aligned}
\bar{H}_n(t,x) &\rightarrow \bar{H}(t,x), \\
\bar{\xi}_n(t,x) &\rightarrow \bar{\xi}(t,x), \\
\bar{\varphi}_n(t,x,V) &\rightarrow \bar{\varphi}(t,x,V), \\
\bar{\varepsilon}_n^2(t,x,\delta) &\rightarrow \bar{\varepsilon}^2(t,x,\delta),
\end{aligned}
$$

for $t \in [0,T]$, $x \in \mathbb{R}$, $V \in \mathbb{R}$, $\delta \in \{0,1\}$ and $n \to \infty$.

5 Proofs of convergence

5.1 Variance-optimal measure

We now examine the behaviour of the variance-optimal characteristic function in the limit as $\triangle \to 0$.

Lemma 5.1 *The coefficients a_n and b_n satisfy*

(5.1) $$a_n = \bar{a} + \tilde{a}\triangle + o(\triangle),$$

(5.2) $$\bar{a} = \frac{\kappa(1) - r}{\kappa(2) - 2\kappa(1)}, \ \tilde{a} \text{ finite,}$$

(5.3) $$b_n = 1 - \bar{b}\triangle + o(\triangle),$$

(5.4) $$\bar{b} := \bar{a}\left(\kappa(1) - r\right) \geq 0.$$

Furthermore there is $\delta > 0$ such that for all $0 < \triangle < \delta$ we have

(5.5) $$0 < \left(\kappa(2) - 2\kappa(1)\right)/2 < \mathrm{E}_n\left[\left(\mathrm{e}^{Z_{1n}} - \mathrm{e}^{r\triangle}\right)^2\right]/\triangle < 3\left(\kappa(2) - 2\kappa(1)\right)/2,$$

(5.6) $$0 < \left(\kappa(2) - 2\kappa(1)\right)/2 < \mathrm{Var}_n\left(\mathrm{e}^{Z_{1n}}\right)/\triangle < 3\left(\kappa(2) - 2\kappa(1)\right)/2,$$

$$\mathrm{e}^{-(\bar{b}+1)\triangle} < b_n < \mathrm{e}^{(-\bar{b}+1)\triangle}.$$

PROOF. By direct calculation

$$
\begin{aligned}
\mathrm{E}_n\left[\mathrm{e}^{Z_{1n}} - \mathrm{e}^{r\triangle}\right] &= \mathrm{e}^{\kappa(1)\triangle} - \mathrm{e}^{r\triangle} = \left(\kappa(1) - r\right)\triangle + o(\triangle), \\
\mathrm{E}_n\left[\left(\mathrm{e}^{Z_{1n}} - \mathrm{e}^{r\triangle}\right)^2\right] &= \mathrm{e}^{\kappa(2)\triangle} - 2\mathrm{e}^{\kappa(1)\triangle}\mathrm{e}^{r\triangle} + \mathrm{e}^{2r\triangle} \\
&= \left(\kappa(2) - 2\kappa(1)\right)\triangle + o(\triangle), \\
\mathrm{Var}_n\left(\mathrm{e}^{Z_{1n}}\right) &= \mathrm{e}^{\kappa(2)\triangle} - \mathrm{e}^{2\kappa(1)\triangle} = \left(\kappa(2) - 2\kappa(1)\right)\triangle + o(\triangle),
\end{aligned}
$$

the rest follows from the Taylor expansion of a_n and b_n. The existence of $\delta > 0$ such that inequalities (5.5) and (5.6) hold follows from the fact that

$$\lim_{\triangle \to 0} \mathrm{E}_n\left[\left(\mathrm{e}^{Z_{1n}} - \mathrm{e}^{r\triangle}\right)^2\right]/\triangle = \lim_{\triangle \to 0} \mathrm{Var}_n\left(\mathrm{e}^{Z_{1n}}\right)/\triangle = \kappa(2) - 2\kappa(1) > 0.$$

Since $\lim_{\triangle\to 0}(b_n - 1)/\triangle = -\lim_{\triangle\to 0}(b_n^{-1} - 1)/\triangle = -\bar{b} \le 0$ we have $b_n < 1 + (-\bar{b}+1)\triangle$ and $b_n^{-1} < 1 + (\bar{b}+1)\triangle$ for all $\triangle$ sufficiently small which implies $b_n < \mathrm{e}^{(-\bar{b}+1)\triangle}$ and $b_n^{-1} < \mathrm{e}^{(\bar{b}+1)n}$. □

Lemma 5.2

$$\begin{aligned} \hat{\phi}_n(v) &= 1 + \hat{\kappa}(\mathrm{i}v)\triangle + o(\triangle) \\ \hat{\kappa}(u) &:= \kappa(u) - \bar{a}\left(\kappa(u+1) - \kappa(u) - \kappa(1)\right), \end{aligned} \tag{5.7}$$

where $\hat{\kappa}(u)$ is well defined on $[0, 1+2\alpha] \times \mathrm{i}\mathbb{R}$.

PROOF. Lemma 3.4 yields:

$$\begin{aligned} \hat{\phi}_n(v) &= \hat{\mathrm{E}}_n\left[\mathrm{e}^{\mathrm{i}vZ_{1n}}\right] \\ &= \frac{1 + a_n\mathrm{e}^{r\triangle}}{b_n}\phi_n(v) - \frac{a_n}{b_n}\phi_n(v-\mathrm{i}) \\ &= \frac{1 + a_n\mathrm{e}^{r\triangle}}{b_n}\mathrm{e}^{\kappa(\mathrm{i}v)\triangle} - \frac{a_n}{b_n}\mathrm{e}^{\kappa(1+\mathrm{i}v)\triangle}. \end{aligned}$$

Now expand the above around $\triangle = 0$ using (5.1) and (5.3)

$$\begin{aligned} \hat{\kappa}(u) &= \frac{\mathrm{d}}{\mathrm{d}\triangle}\left(\frac{1 + a_n\mathrm{e}^{r\triangle}}{b_n}\mathrm{e}^{\kappa(u)\triangle} - \frac{a_n}{b_n}\mathrm{e}^{\kappa(1+u)\triangle}\right)\Bigg|_{\triangle=0} \\ &= \bar{a}r + \tilde{a} + \bar{b}\left(1+\bar{a}\right) + \left(1+\bar{a}\right)\kappa(u) - (\tilde{a} + \bar{b}\bar{a}) - \bar{a}\kappa(u+1) \\ &= \bar{a}r + \bar{b} + \left(1+\bar{a}\right)\kappa(u) - \bar{a}\kappa(u+1), \end{aligned}$$

and on using the definition of $\bar{b}$ (5.4) we obtain the desired result (5.7). By Theorem 4.3 κ is well defined on $[0, 2+2\alpha]\times\mathrm{i}\mathbb{R}$ hence $\hat{\kappa}$ is well defined on $[0, 1+2\alpha]\times\mathrm{i}\mathbb{R}$. □

We are now ready to compute the continuous-time limit of the mean value process.

5.2 Mean value process

Lemma 5.3 *There are constants $K > 0$ and $\delta > 0$ such that for $0 < \triangle < \delta$ and $\mathrm{Re}\, u = 1 + \alpha$ we have $|\hat{\phi}_n(-\mathrm{i}u)| \le \mathrm{e}^{K\triangle}$.*

PROOF. Apply Theorem 3.1 with $Y = \mathrm{e}^{uZ_{1n}}$, $X_1 = 1$, $X_2 = \mathrm{e}^{Z_{1n}} - \mathrm{e}^{r\triangle}$ and inner product defined by $(X_1, X_2) = \mathrm{E}_n\left[X_1\bar{X}_2\right]$, which yields $\hat{\phi}_n(-\mathrm{i}u) = \hat{\beta}_1$. Equation (3.7) implies

$$b_n|\hat{\beta}_1|^2 \le \|Y\|^2,$$

which is equivalent to

$$|\hat{\phi}_n(-\mathrm{i}u)|^2 \le \phi_n(-2\mathrm{i}\mathrm{Re}\, u)/b_n = \mathrm{e}^{\kappa(2\mathrm{Re}\, u)\triangle}/b_n.$$

By Lemma 5.1 there is $\delta > 0$ such that $b_n^{-1} < \mathrm{e}^{(\bar{b}+1)\triangle}$ for $0 < \triangle < \delta$, therefore we have $|\hat{\phi}_n(-\mathrm{i}u)| \leq \mathrm{e}^{K\triangle}$ with $K = \left(\kappa\,(2+2\alpha) + \bar{b} + 1\right)/2$. $\square$

It is now possible to find the pointwise limit of the integrand in (4.7) and to apply the dominated convergence theorem to make the passage to the limit legitimate.

Theorem 5.4 *Under the standing assumptions the continuous-time limit of the mean value process is obtained by taking pointwise limit of the integrand in (4.7):*

$$\lim_{n\longrightarrow\infty} \bar{H}_n(t,x) = \bar{H}(t,x) := \int_G \psi(u)\mathrm{e}^{(\hat{\kappa}(u)-r)(T-t)}\mathrm{e}^{ux}\mathrm{d}u$$

PROOF. *Set $\tau := T - t$. Pointwise limit of $\left(\hat{\phi}_n(-\mathrm{i}u)\right)^{n(1-t/T)}$ is easily established from Lemma 5.2:*

$$\begin{aligned}\lim_{\triangle\to 0}\left(\hat{\phi}_n(-\mathrm{i}u)\right)^{\frac{\tau}{\triangle}} &= \lim_{\triangle\to 0}\left(1 + \triangle\hat{\kappa}(u) + o(\triangle)\right)^{\frac{\tau}{\triangle}} \\ &= \lim_{\triangle\to 0}\mathrm{e}^{\ln(1+\triangle\hat{\kappa}(u)+o(\triangle))\tau/\triangle} = \exp\left(\hat{\kappa}(u)\tau\right).\end{aligned}$$

Here $\ln$ *denotes the principal value of the logarithm. In addition, by Lemma 5.3 there is $K > 0$ such that*

$$\left|\left(\hat{\phi}_n(-\mathrm{i}u)\right)^{\frac{\tau}{\triangle}}\right| = \left|\left(\hat{\phi}_n(-\mathrm{i}u)\right)\right|^{\frac{\tau}{\triangle}} \leq \mathrm{e}^{K\tau}.$$

Since $|\mathrm{e}^{ux}| = \mathrm{e}^{\mathrm{Re}\,(u)x} = \mathrm{e}^{(1+\alpha)x}$, and $\int_G |\psi(u)|\,\mathrm{d}u$ is finite the statement of the theorem follows from (4.7) by dominated convergence. $\square$

5.3 Hedging strategy

To prove convergence of the hedging strategy $\bar{\xi}_n$ in (4.8) we will require an estimate of the characteristic function $\hat{\phi}_n$ that is somewhat stronger than the one provided by Lemma 5.3. This is because $|A_n(u)/A_1(u)|$ for A_n defined in (4.10) can be of the order $|\mathrm{Im}\,u|$ asymptotically as $|\mathrm{Im}\,u| \to \infty$, and $|\psi(u)\mathrm{Im}\,u|$ is no longer integrable. The necessary estimates are computed in Lemmata 5.5-5.9.

Lemma 5.5 *We have*

$$|e^z - 1| \leq 2\,|z| \;\; for \;\; |z| < 1/2$$

and

$$1 + x \leq \mathrm{e}^x \textit{ for any } x \in \mathbb{R}.$$

PROOF. See Abramowitz and Stegun (1992), 4.2.38, p. 70 and Hardy et al. (1952), 142, p. 103. $\square$

Lemma 5.6 *There are constants* $K_1, K_2 > 0$ *such that for* $0 \leq \mathrm{Re}\,(u) \leq 1+\alpha$

$$\begin{aligned} |\kappa(u+1)-\kappa(u)| &\leq K_1\left(1+\sqrt{K_2-\mathrm{Re}\,\kappa(u)}\right), \\ \mathrm{Re}\,\kappa(u) &\leq K_2. \end{aligned}$$

PROOF. Apply Theorem 3.1 with $Y = \mathrm{e}^{uZ_{1n}}$, $X_2 = 1$, $X_1 = \mathrm{e}^{Z_{1n}}$ and inner product defined by $(X_1, X_2) = \mathrm{E}_n\left[X_1\bar{X}_2\right]$. This in particular implies

$$\hat{\beta}_1 = \frac{\phi_n\left(-\mathrm{i}(u+1)\right)-\phi_n\left(-\mathrm{i}u\right)\phi_n\left(-\mathrm{i}\right)}{\phi_n\left(-2\mathrm{i}\right)-\left(\phi_n\left(-\mathrm{i}\right)\right)^2}.$$

Equation (3.7) yields

$$\begin{aligned} 0 &\leq \|Y\|^2-\|M_{X_2}Y\|^2-|\hat{\beta}_1|^2\left(\|X_1\|^2-\|M_2X_1\|^2\right), \\ 0 &\leq \phi_n\left(-2\mathrm{i}\mathrm{Re}\,u\right)-\left|\phi_n\left(-\mathrm{i}u\right)\right|^2-\frac{\left|\phi_n\left(-\mathrm{i}(u+1)\right)-\phi_n\left(-\mathrm{i}u\right)\phi_n\left(-\mathrm{i}\right)\right|^2}{\phi_n\left(-2\mathrm{i}\right)-\left(\phi_n\left(-\mathrm{i}\right)\right)^2} \\ &= \mathrm{e}^{\kappa(2\mathrm{Re}\,u)\triangle}-\mathrm{e}^{2\mathrm{Re}\,\kappa(u)\triangle}-\frac{\left|\mathrm{e}^{\kappa(u+1)\triangle}-\mathrm{e}^{(\kappa(u)+\kappa(1))\triangle}\right|^2}{\mathrm{e}^{\kappa(2)\triangle}-\mathrm{e}^{2\kappa(1)\triangle}}. \end{aligned}$$

Divide both sides by $\triangle$ and take a limit as $\triangle$ approaches 0 using Lemma 5.5 and the fact that the absolute value is continuous in $\mathbb{C}$, obtaining

$$0 \leq \kappa(2\mathrm{Re}\,u)-2\mathrm{Re}\,\kappa(u)-\frac{\left|\kappa(u+1)-\kappa(u)-\kappa(1)\right|^2}{\kappa(2)-2\kappa(1)}. \tag{5.8}$$

The statement of the lemma is obtained by setting

$$\begin{aligned} K_1 &= \max(\left|\kappa(1)\right|, \sqrt{2\left(\kappa(2)-2\kappa(1)\right)}), \\ K_2 &= \max_{0\leq\beta\leq 2+2\alpha}\kappa(\beta)/2. \end{aligned}$$

K_2 exists and is finite because the cumulant generating function is continuous in its strip of regularity. Equation (5.8) also implies

$$\begin{aligned} 2\mathrm{Re}\,\kappa(u) &\leq \kappa(2\mathrm{Re}\,u), \\ \mathrm{Re}\,\kappa(u) &\leq K_2. \end{aligned}$$

□

Lemma 5.7 *There is* $K_3 > 0$ *and* $\delta > 0$ *such that*

$$\left|\left(\hat{\phi}_n(-\mathrm{i}u)\right)^n\right| < \exp\left(\left(\mathrm{Re}\,\kappa(u)+K_3+K_3\left|\kappa(u+1)-\kappa(u)\right|\right)T\right),$$

for all $u \in \mathbb{C}$ *such that* $0 \leq \mathrm{Re}\,u \leq 1+\alpha$ *and for* $\delta > \triangle > 0$.

PROOF. Take δ sufficiently small, $0 < \triangle < \delta$, such that $|r|\,\delta < 1/4$. The limits $\lim_{\triangle\to 0} \frac{a_n}{b_n}$ and $\lim_{\triangle\to 0} \frac{1}{b_n}$ are finite, therefore there is $K > 0$ such that

$$\max\left(|a_n/b_n|\,,\left|\left(1+a_n e^{r\triangle}\right)/b_n\right|\right) < K \text{ for } 0 < \triangle < \delta.$$

For $|\kappa(u+1)-\kappa(u)|\,\triangle < 1/2$ we can write

$$\left|\left(\hat{\phi}_n(-\mathrm{i}u)\right)^n\right| = \mathrm{e}^{\mathrm{Re}\,\kappa(u)T}\left|1+\frac{a_n}{b_n}\left(\mathrm{e}^{r\triangle}-\mathrm{e}^{(\kappa(u+1)-\kappa(u))\triangle}\right)\right|^{\frac{T}{\triangle}},$$

and by Lemma 5.5 we have

$$\begin{aligned}\left|\left(\hat{\phi}_n(-\mathrm{i}u)\right)^n\right| &\leq \mathrm{e}^{\mathrm{Re}\,\kappa(u)T}\left(1+2K\left(|r|+|\kappa(u+1)-\kappa(u)|\right)\triangle\right)^{\frac{T}{\triangle}}\\ &\leq \mathrm{e}^{(\mathrm{Re}\,\kappa(u)+2K(|r|+|\kappa(u+1)-\kappa(u)|))T}.\end{aligned}$$

For $0 < \triangle < \delta$ and

$$|\kappa(u+1)-\kappa(u)|\,\triangle \geq 1/2 \tag{5.9}$$

we can estimate $|\hat{\phi}_n(-\mathrm{i}u)|$ very coarsely:

$$\begin{aligned}|\hat{\phi}_n(-\mathrm{i}u)| &\leq \left|\frac{1+a_n e^{r\triangle}}{b_n}\right|\mathrm{e}^{\mathrm{Re}\,\kappa(u)\triangle}+\left|\frac{a_n}{b_n}\right|\mathrm{e}^{\mathrm{Re}\,\kappa(u)\triangle}\mathrm{e}^{\mathrm{Re}\,(\kappa(u+1)-\kappa(u))\triangle}\\ &\leq K\mathrm{e}^{\mathrm{Re}\,\kappa(u)\triangle}(1+\mathrm{e}^{|\kappa(u+1)-\kappa(u)|\triangle})\\ &\leq 2K\mathrm{e}^{\mathrm{Re}\,\kappa(u)\triangle}\mathrm{e}^{|\kappa(u+1)-\kappa(u)|\triangle}.\end{aligned}$$

This yields

$$\begin{aligned}\left|\left(\hat{\phi}_n(-\mathrm{i}u)\right)^n\right| &\leq \mathrm{e}^{(\mathrm{Re}\,\kappa(u)\triangle+|\kappa(u+1)-\kappa(u)|\triangle+\ln(2K))T/\triangle}\\ &= \mathrm{e}^{(\mathrm{Re}\,\kappa(u)+|\kappa(u+1)-\kappa(u)|+\ln(2K)/\triangle)T}\\ &\leq \mathrm{e}^{(\mathrm{Re}\,\kappa(u)+|\kappa(u+1)-\kappa(u)|(1+2\ln(2K)))T},\end{aligned}$$

where the last inequality follows from (5.9). Statement of the lemma therefore follows by taking $K_3 = \max((1+2\ln(2K)), 2K, 2K\,|r|)$. □

Lemma 5.8 *For* $\mathrm{Re}\,u \leq 2+2\alpha$ *and for* $K_2 > 0$ *defined in Lemma 5.6 we have*

$$\mathrm{Re}\,\kappa(u) \leq 2K_2.$$

PROOF. From the Lévy–Khintchin representation (4.1)-(4.3) we obtain

$$\begin{aligned}\mathrm{Re}\,\kappa(u) &= \mu\mathrm{Re}\,u+\frac{\sigma^2}{2}\left((\mathrm{Re}\,u)^2-(\mathrm{Im}\,u)^2\right)\\ &\quad+\int_{\mathbb{R}}\left(\cos\left(x\mathrm{Im}\,u\right)\mathrm{e}^{x\mathrm{Re}\,u}-1-h(x)\mathrm{Re}\,u\right)M(\mathrm{d}x)\\ &\leq \mu\left(\mathrm{Re}\,u\right)+\frac{\sigma^2}{2}\left(\mathrm{Re}\,u\right)^2\\ &\quad+\int_{\mathbb{R}}\left(\mathrm{e}^{x\mathrm{Re}\,u}-1-h(x)\mathrm{Re}\,u\right)M(\mathrm{d}x)\\ &= \kappa(\mathrm{Re}\,u) \leq 2K_2.\end{aligned}$$

□

Lemma 5.9 *There is $K_4 > 0$ and $\delta > 0$ such that for all $\operatorname{Re} u = 1+\alpha$ and for all $0 < \triangle < \delta$ we have*

$$\left|\frac{A_n(u)}{A_n(1)}\right| = \left|\frac{\mathrm{e}^{\kappa(u+1)\triangle} - \mathrm{e}^{(\kappa(u)+\kappa(1))\triangle}}{\mathrm{e}^{\kappa(2)\triangle} - \mathrm{e}^{2\kappa(1)\triangle}}\right| < K_4\left(1 + |\kappa(u+1) - \kappa(u)|\right).$$

PROOF. By virtue of Lemma 5.1 $\left(\mathrm{e}^{\kappa(2)\triangle} - \mathrm{e}^{2\kappa(1)\triangle}\right)/\triangle > (\kappa(2) - 2\kappa(1))/2 > 0$ for all $\triangle$ sufficienly small and therefore it is enough to examine

$$\left|\mathrm{e}^{\kappa(u+1)\triangle} - \mathrm{e}^{(\kappa(u)+\kappa(1))\triangle}\right|/\triangle.$$

Take $\triangle < \delta := \min(1, 1/(2|\kappa(1)|))$. For $|\kappa(u+1) - \kappa(u)|\triangle < 1/2$, using Lemmata 5.5, 5.6 and 5.8, we find

$$\begin{aligned}\left|\mathrm{e}^{\kappa(u+1)\triangle} - \mathrm{e}^{(\kappa(u)+\kappa(1))\triangle}\right|/\triangle &= \mathrm{e}^{\operatorname{Re}\kappa(u)\triangle}\left|\mathrm{e}^{(\kappa(u+1)-\kappa(u))\triangle} - \mathrm{e}^{\kappa(1)\triangle}\right|/\triangle \\ &\leq \mathrm{e}^{2K_2\triangle}\left(|\kappa(u+1) - \kappa(u)| + |\kappa(1)|\right),\end{aligned}$$

whereas for $|\kappa(u+1) - \kappa(u)|\triangle \geq 1/2$ we can write

$$\begin{aligned}\left|\mathrm{e}^{\kappa(u+1)\triangle} - \mathrm{e}^{(\kappa(u)+\kappa(1))\triangle}\right|/\triangle &\leq \left(\mathrm{e}^{2K_2\triangle} + \mathrm{e}^{4K_2\triangle}\right)/\triangle \\ &\leq 2\left(\mathrm{e}^{2K_2\triangle} + \mathrm{e}^{4K_2\triangle}\right)|\kappa(u+1) - \kappa(u)|.\end{aligned}$$

It is therefore enough to take

$$K_4 = \frac{2\left(\mathrm{e}^{2K_2\delta} + \mathrm{e}^{4K_2\delta}\right)\max(|\kappa(1)|, 1)}{(\kappa(2) - 2\kappa(1))/2}.$$

□

Theorem 5.10 *The continuous-time limit of the locally optimal hedging strategy is obtained by taking pointwise limit of the integrand in (4.8):*

$$\lim_{n\to\infty}\bar{\xi}_n(t,x) = \bar{\xi}(t,x) := \int_G \mathrm{e}^{(u-1)x}\psi(u)\mathrm{e}^{(\hat{\kappa}(u)-r)(T-t)}\frac{\kappa(u+1) - \kappa(u) - \kappa(1)}{\kappa(2) - 2\kappa(1)}\,\mathrm{d}u.$$

PROOF. Set $\tau := T - t$ and take $0 < \triangle < \frac{\tau}{2}$. By Lemmata 5.6, 5.7 and 5.9 there are positive constants δ, K_1, K_2, K_3 and K_4 such that

$$\begin{aligned}\left|\left(\hat{\phi}_n(-\mathrm{i}u)\right)^{n(1-t/T-1/n)}\frac{A_n(u)}{A_n(1)}\right| &\leq \mathrm{e}^{\left(\operatorname{Re}\kappa(u)+K_3+K_3K_1\left(1+\sqrt{K_2-\operatorname{Re}(\kappa(u))}\right)\right)(\tau-\triangle)} \\ &\times \left(K_4(1+K_1) + K_1K_4\sqrt{K_2 - \operatorname{Re}\kappa(u)}\right),\end{aligned}$$

for $0 < \triangle < \delta$. We wish to show that the right hand side is bounded uniformly in $\triangle$. The expression

$$f(\operatorname{Re}\kappa(u)) := \operatorname{Re}\kappa(u) + K_3 + K_3K_1\left(1 + \sqrt{K_2 - \operatorname{Re}(\kappa(u))}\right)$$

is continuous in $\operatorname{Re}\kappa(u)$ for $\operatorname{Re}\kappa(u) < K_2$ and left-continuous at $\operatorname{Re}\kappa(u) = K_2$. Its limit as $\operatorname{Re}\kappa(u)$ approaches $-\infty$ is 0, therefore $f(.)$ is bounded from above by some constant $K > 0$. Hence we have

$$f(\operatorname{Re}\kappa(u))\tau \leq K\tau/2 + f(\operatorname{Re}\kappa(u))\tau/2 \text{ for all } 0 < \triangle < \min(\delta, \tau/2)\,.$$

The expression

$$\mathrm{e}^{K\tau/2+f(\operatorname{Re}\kappa(u))\tau/2}\left(K_4(1+K_1) + K_1K_4\sqrt{K_2 - \operatorname{Re}\kappa(u)}\right)$$

is continuous in $\operatorname{Re}\kappa(u)$ and its limit as $\operatorname{Re}\kappa(u) \to -\infty$ is 0, it is therefore bounded. We have thus shown an existence of $\tilde{K} > 0$ such that

$$\left|\left(\hat{\phi}_n\,(-\mathrm{i}u)\right)^{n(1-t/T-1/n)}\frac{A_n(u)}{A_n(1)}\right| \leq \tilde{K} \text{ for } 0 < \triangle < \min(\delta, \tau/2).$$

The statement of the theorem follows by dominated convergence. □

5.4 Hedging error

Theorem 5.11 *If*

(5.10) $$\int_{G^2} \mathrm{e}^{\operatorname{Re}\kappa(u_1+u_2)\frac{T-t}{4}} \prod_{l=1,2} |\psi(u_l)|\,|\kappa(u_l+1) - \kappa(u_l)|\,\mathrm{d}u_1\mathrm{d}u_2 < \infty,$$

then the continuous-time limit of the unconditional hedging error is obtained by taking pointwise limit under the integral sign in (4.9)

$$\begin{aligned}\lim_{n\to\infty} \bar{\varepsilon}_n^2(t,x,\delta) &= \bar{\varepsilon}^2(t,x,\delta) \\ &:= \int_{G^2} \frac{\mathrm{e}^{\left(\bar{C}(u_1,u_2)\right)(T-t)} - \mathrm{e}^{-\delta\bar{b}(T-t)}}{\bar{C}(u_1,u_2)+\delta\bar{b}}\left(\bar{B}(u_1,u_2) - \frac{\bar{A}(u_1)\bar{A}(u_2)}{\bar{A}(1)}\right) \\ &\qquad \times\left(\prod_{j=1,2}\mathrm{e}^{\hat{\kappa}(u_j)(T-t)+u_jx}\psi(u_j)\mathrm{d}u_l\right),\end{aligned} \tag{5.11}$$

with

(5.12) $$\bar{B}(u_1,u_2) := \kappa(u_1+u_2) - \kappa(u_1) - \kappa(u_2),$$

(5.13) $$\bar{A}(u) := \bar{B}(u,1),$$

(5.14) $$\bar{C}(u_1,u_2) := \bar{B}(u_1,u_2) + \bar{a}\left(\bar{A}(u_1)+\bar{A}(u_2)\right)$$

PROOF. The first two terms in (4.9) do not pose any problems since $|b_n^n|$, $|b_n - 1|/\triangle$, $|\hat{\phi}_n\,(-\mathrm{i}u_l)\,|^n$ and $|\phi_n\,(-\mathrm{i}(u_1+u_2))|^n$ are uniformly bounded by Lemmata 5.1, 5.3 and 5.8, respectively. It is more difficult to find integrable majorant for the last term in (4.9), particularly for k close to n. To this end Lemmata 5.3 and 5.9 yield

$$(5.15)\qquad \left|A_n(u)/A_n(1) + a_n\hat{\phi}_n(-\mathrm{i}u_l)\right| \leq (K_4 + 2(|\bar{a}|+1))\left(1 + |\kappa(u+1) - \kappa(u)|\right).$$

By Lemmata 5.3 and 5.8 there is $K > 0$ such that $|\hat{\phi}_n(-\mathrm{i}u_l)| < \mathrm{e}^{K\triangle}$ and $\operatorname{Re}\kappa(u_1+u_2) < K$ therefore

$$(5.16)\qquad |\hat{\phi}_n(-\mathrm{i}u_l)|^{n-j} \leq \mathrm{e}^{K(T-t)/2}|\hat{\phi}_n(-\mathrm{i}u_l)|^{(T-t)/4} \text{ for } j \leq n/2,$$

$$(5.17)\qquad |\phi_n(-\mathrm{i}(u_1+u_2))|^{j} \leq \mathrm{e}^{K(T-t)/2 + \operatorname{Re}\kappa(u_1+u_2)(T-t)/4} \text{ for } j \geq n/2.$$

Reasoning identical to that of Lemma 5.1 shows that there is $K > 0$ such that for all $\triangle$ sufficiently small

$$(5.18)\qquad 0 \leq c_n := \mathrm{E}_{jn}\left[X^2_{(j+1)n}\right] = \mathrm{e}^{\kappa(2)\triangle} - 2\mathrm{e}^{(\kappa(1)+r)\triangle} + \mathrm{e}^{2r\triangle} \leq K\triangle.$$

On combining the estimates (5.15)-(5.18) we obtain

$$\begin{aligned}
&\int_{G^2}\sum_{k=\lfloor t/\triangle\rfloor}^{n-1}\Bigg|c_n\left(\phi_n(-\mathrm{i}(u_1+u_2))\right)^{k-j}\mathrm{e}^{(u_1+u_2)x}\\
&\qquad\times\prod_{l=1,2}\left(\psi(u_l)\left(b_n^{\delta/2}\hat{\phi}_n(-\mathrm{i}u_l)\right)^{n-1-k}\left(A_n(u_l)/A_n(1) + a_n\hat{\phi}_n(-\mathrm{i}u_l)\right)\mathrm{d}u_l\right)\Bigg|\\
&\leq \tilde{K}\int_t^{\frac{T+t}{2}}\mathrm{d}\tau\int_G\prod_{l=1,2}\left(\left|\hat{\phi}_n(-\mathrm{i}u_l)\right|^{\frac{T-t}{4}}|\psi(u_l)|\left(1+|\kappa(u_l+1)-\kappa(u_l)|\right)\mathrm{d}u_l\right)\\
&\quad+\tilde{K}\int_{\frac{T+t}{2}}^{T}\mathrm{d}\tau\int_{G^2}\mathrm{e}^{\operatorname{Re}\kappa(u_1+u_2)\frac{T-t}{4}}\prod_{l=1,2}\left(|\psi(u_l)|\left(1+|\kappa(u_l+1)-\kappa(u_l)|\right)\mathrm{d}u_l\right).
\end{aligned}$$

The proof of Theorem 5.10 shows that the first integral on the right hand side is finite whereas the assumption (5.10) of this theorem guarantees convergence of the second integral. By dominated convergence we therefore have

$$\begin{aligned}
\lim_{n\to\infty}\bar{\varepsilon}_n^2(t,x,\delta) &= \int_{G^2}\prod_{l=1,2}\left(\psi(u_l)\mathrm{e}^{u_l x}\mathrm{d}u_l\right)\left(\mathrm{e}^{\kappa(u_1+u_2)(T-t)} - \mathrm{e}^{\left(\hat{\kappa}(u_1)+\hat{\kappa}(u_2)-\delta\bar{b}\right)(T-t)}\right)\\
&\quad+(1-\delta)\bar{b}\int_{G^2}\prod_{l=1,2}\left(\psi(u_l)\mathrm{e}^{u_l x}\mathrm{d}u_l\right)\int_t^T\mathrm{d}\tau\mathrm{e}^{\kappa(u_1+u_2)(\tau-t)}\mathrm{e}^{(\hat{\kappa}(u_1)+\hat{\kappa}(u_2)-\delta\bar{b})(T-\tau)}\\
&\quad-\int_{G^2}\prod_{l=1,2}\left(\psi(u_l)\mathrm{e}^{u_l x}\mathrm{d}u_l\right)\int_t^T\mathrm{d}\tau\mathrm{e}^{\kappa(u_1+u_2)(\tau-t)}\mathrm{e}^{(\hat{\kappa}(u_1)+\hat{\kappa}(u_2)-\delta\bar{b})(T-\tau)}\\
&\qquad\times\bar{A}(1)\prod_{l=1,2}\left(\bar{A}(u_l)/\bar{A}(1)+\bar{a}\right).
\end{aligned}$$

After a simple algebraic manipulation utilizing (5.2), (5.4), (5.7) and (5.12)-(5.14) we obtain

$$\begin{aligned}
\lim_{n\to\infty}\bar{\varepsilon}_n^2(t,x,\delta) &= \int_{G^2}\mathrm{d}u_1\mathrm{d}u_2\psi(u_1)\psi(u_2)\mathrm{e}^{(u_1+u_2)\ln S_0}\left(\bar{B}(u_1,u_2) - \frac{\bar{A}(u_1)\bar{A}(u_2)}{\bar{A}(1)}\right)\\
&\quad\times\int_t^T\mathrm{d}\tau\mathrm{e}^{\kappa(u_1+u_2)(T-\tau)}\mathrm{e}^{\left(\hat{\kappa}(u_1)+\hat{\kappa}(u_2)-\delta\bar{b}\right)(\tau-t)},
\end{aligned}$$

which on integration yields the formula (5.11) $\square$

To conclude this section, we wish to list sufficient conditions under which the assumption (5.10) is satisfied. Here we need a result that estimates the growth of $|\kappa(u+1)-\kappa(u)|$ for large values of $|\mathrm{Im}\, u|$.

Lemma 5.12 *Take $0 \leq \gamma \leq 2$ and suppose $\int_{|x|<1} |x|^\gamma M(\mathrm{d}x) < \infty$. Then there is $K > 0$ such that*

$$|\kappa(u+1)-\kappa(u)| < K\left(1 + |\mathrm{Im}\, u|^{\max(\gamma-1,0)} + \sigma^2 |\mathrm{Im}\, u|\right)$$

for all $\mathrm{Re}\, u = 1+\alpha$.

PROOF. To simplify notation let K represent a generic positive constant, not necessarily the same in each line. For any $\delta > 0$ write explicitly

$$\begin{aligned}
\tilde{\kappa}(u) &= \int_{\mathbb{R}} (\mathrm{e}^{ux} - 1 - uh(x))\, M(\mathrm{d}x), \\
\tilde{\kappa}(u+1) - \tilde{\kappa}(u) &= \int_{\mathbb{R}} (\mathrm{e}^{ux}(e^x - 1) - h(x))\, M(\mathrm{d}x) \\
&= Z_1(u,\delta) + \int_{|x|<\delta} (\mathrm{e}^{ux}\,(e^x-1) - x)\, M(\mathrm{d}x) \\
&= Z_2(u,\delta) + \int_{|x|<\delta} (\mathrm{e}^{ux} - 1)\,(e^x-1)\, M(\mathrm{d}x),
\end{aligned}$$

with $Z_1, Z_2 \in \mathbb{C}$, such that for given $\delta > 0$ $|Z_i(u,\delta)|$ is uniformly bounded for all $\mathrm{Re}\, u = 1+\alpha$. We continue with the estimation of the last integral, choosing ε and δ sufficiently small so that Lemma 5.5 applies

$$\begin{aligned}
I &= \left| \int_{|x|<\delta} (\mathrm{e}^{ux} - 1)\,(e^x - 1)\, M(\mathrm{d}x) \right| \\
&\leq \int_{\substack{|x|<\delta \\ |xu|<\varepsilon}} |\mathrm{e}^{ux} - 1|\, |e^x - 1|\, M(\mathrm{d}x) + \int_{\substack{|x|<\delta \\ |ux|\geq\varepsilon}} |\mathrm{e}^{ux} - 1|\, |e^x - 1|\, M(\mathrm{d}x), \\
&\leq K \int_{\substack{|x|<\delta \\ |ux|<\varepsilon}} |u|\, |x|^2\, M(\mathrm{d}x) + K \int_{\varepsilon/|u| \leq |x| < \delta} |x|\, M(\mathrm{d}x).
\end{aligned}$$

If $\gamma \leq 1$ it follows immediately that $I \leq K$. Suppose then that $\gamma > 1$; we carry on with

$$\begin{aligned}
I &\leq K \int_{|xu|<\varepsilon} |u|\, |x|^{2-\gamma}\, |x|^\gamma\, M(\mathrm{d}x) + \int_{\varepsilon/|u| \leq |x| < \delta} |x|\, (|xu|/\varepsilon)^{\gamma-1}\, M(\mathrm{d}x) \\
&\leq K \int_{|ux|<\varepsilon} |u|\, (\varepsilon/|u|)^{2-\gamma}\, |x|^\gamma\, M(\mathrm{d}x) + K\varepsilon^{1-\gamma}\, |u|^{\gamma-1} \int_{|x|<\delta} |x|^\gamma\, M(\mathrm{d}x) \\
&\leq K(1 + |\mathrm{Im}\, u|^{\gamma-1}),
\end{aligned}$$

where the last inequality utilizes

$$|u| = \sqrt{(1+\alpha)^2 + (\mathrm{Im}\, u)^2} \leq \sqrt{2}\,(|\mathrm{Im}\, u| + |1+\alpha|)\,.$$

This proves the case of $\sigma^2 = 0$. For $\sigma^2 > 0$ we have

$$\kappa(u+1) - \kappa(u) = \mu + \sigma^2(\mathrm{Re}\, u + 1/2) + \mathrm{i}\sigma^2 \mathrm{Im}\, u + \tilde{\kappa}(u+1) - \tilde{\kappa}(u),$$

and the statement of the lemma follows. □

Theorem 5.13 *The assumption of Theorem 5.11 is satisfied either if*

i) there is $\gamma < 2$ such that $\int_{|x|<1} |x|^\gamma \, M(\mathrm{d}x) < \infty$ and $\sigma = 0$, or

ii) $\sigma^2 > 0$.

PROOF. Without loss of generality we can take $t = 0$. We know from Lemma 5.8 that $\mathrm{e}^{\mathrm{Re}\,\kappa(u_1+u_2)\frac{T}{4}} < K$ for $u_1, u_2 \in G$. If i) holds then by Lemma 5.9 $|\kappa(u_l+1) - \kappa(u_l)| \leq K\left(1 + |\mathrm{Im}\, u|^{\gamma-1}\right)$ and therefore $\int_G |\psi(u_l)|\,|\kappa(u_l+1) - \kappa(u_l)| < \infty$, which implies

$$\int_{G^2} \mathrm{e}^{\mathrm{Re}\,\kappa(u_1+u_2)T/4} \prod_{l=1,2} |\psi(u_l)|\,|\kappa(u_l+1) - \kappa(u_l)|\, \mathrm{d}u_1 \mathrm{d}u_2 < \infty.$$

If $\sigma > 0$ then there are positive constants $\tilde{K}$ and $\hat{K}$ such that

$$\begin{aligned} \int_{G^2} \mathrm{e}^{\mathrm{Re}\,\kappa(u_1+u_2)T/2} & \prod_{l=1,2} |\psi(u_l)|\,|\kappa(u_l+1) - \kappa(u_l)|\, \mathrm{d}u_1 \mathrm{d}u_2 \\ & \leq \tilde{K}\left(1 + \int_{G^2} \mathrm{e}^{-\sigma^2 T(\mathrm{Im}\,(u_1+u_2))^2/4} \prod_{l=1,2} \frac{1}{1 + |\mathrm{Im}\, u_l|} \mathrm{d}u_1 \mathrm{d}u_2\right) \\ & \leq \hat{K}\left(1 + \int_{\substack{z_1>1 \\ z_2>1}} \frac{\mathrm{e}^{-\sigma^2 T(z_1-z_2)^2/4}}{z_1 z_2} \mathrm{d}z_1 \mathrm{d}z_2\right) \\ & = \hat{K}\left(1 + \int_{\mathbb{R}} \mathrm{e}^{-\sigma^2 T x^2/4} \frac{\ln(1+|x|)}{|x|} \mathrm{d}x\right) < \infty, \end{aligned}$$

where in the last integral we have performed the transformation $x = z_1 - z_2$, $y = z_1$ and integrated over y. □

6 Convergence of multinomial lattices

This section extends the results of Cox et al. (1979) and Madan et al. (1989) to incomplete markets. Consider a model where log returns follow a compound Poisson process with jump sizes and arrival intensities $\{x_j, \lambda_j\}_{j=1}^m$ and with deterministic drift μ. Let us define

$$\lambda = \sum_{j=1}^m \lambda_j,$$

and assume without loss of generality that $x_j \neq x_k$ for $j \neq k$, $x_j \neq 0$ and $\lambda_j > 0$. Now consider a multinomial approximation of this process at the rebalancing frequency $\triangle < 1/\lambda$, assigning to the log return values $\{\tilde{x}_j\}_{j=0}^m$ with probability $\{p_j\}_{j=0}^m$

$$\tilde{x}_0 = \mu\triangle, \tag{6.1}$$

$$\tilde{x}_j = x_j + \mu\triangle, \tag{6.2}$$

$$p_0 = 1 - \lambda\triangle, \tag{6.3}$$

$$p_j = \lambda_j\triangle. \tag{6.4}$$

The characteristic function of one-period log returns then reads:

Standing assumption 2'

$$\phi_n(-\mathrm{i}u) := \mathrm{E}_n\left[\mathrm{e}^{u\ln(S_{1n}/S_0)}\right] = \mathrm{e}^{u\mu\triangle}\left(1 + \sum_{j=1}^{m}\left(\mathrm{e}^{ux_j} - 1\right)\lambda_j\triangle\right). \tag{6.5}$$

Lemma 6.1 *The statement of Lemma 5.1 holds also for the conditional distribution of log returns given in (6.5), provided that we take*

$$\kappa(u) = u\mu + \sum_{j=1}^{m}\left(\mathrm{e}^{ux_j} - 1\right)\lambda_j. \tag{6.6}$$

PROOF. The statement of Lemma 5.1 depends only on the expression

$$\lim_{\triangle\to 0}\frac{\phi_n(-\mathrm{i}u) - 1}{\triangle} = u\mu + \sum_{j=1}^{n}\left(\mathrm{e}^{ux_j} - 1\right)\lambda_j \quad \text{for } u = 1, 2.$$

□

Lemma 6.2 *There are $K, \delta > 0$ such that for ϕ_n defined in (6.5) and for $\hat{\phi}_n$ defined in (4.11) we have*

1. $|\phi_n(-\mathrm{i}u)| \leq \mathrm{e}^{K\triangle}$,
2. $|\hat{\phi}_n(-\mathrm{i}u)| \leq \mathrm{e}^{K\triangle}$,
3. $|\phi_n(-\mathrm{i}(u+1)) - \phi_n(-\mathrm{i}u)\phi_n(-\mathrm{i})|/\triangle < K$,

for $\mathrm{Re}\, u \leq 2 + 2\alpha$ *and* $0 < \triangle < \delta$.

PROOF.

1. By virtue of Lemma 5.5

$$|\phi_\triangle(-\mathrm{i}u)| \leq \mathrm{e}^{\mathrm{Re}\, u\mu\triangle}\left(1+\sum_{j=1}^{m}\left(\mathrm{e}^{\mathrm{Re}\, ux_j}+1\right)\lambda_j\triangle\right) \leq \mathrm{e}^{\left(\mathrm{Re}\, u\mu+\sum_{j=1}^{m}\left(\mathrm{e}^{\mathrm{Re}\, ux_j}+1\right)\lambda_j\right)\triangle}.$$

2. The statement follows directly from Lemma 6.1 and the proof of Lemma 5.3.

3. We can write

$$\begin{aligned}|\phi_n(-\mathrm{i}(u+1))-\phi_n(-\mathrm{i}u)\phi_n(-\mathrm{i})|/\triangle &\leq |\phi_n(-\mathrm{i}(u+1))-\phi_n(-\mathrm{i}u)|/\triangle \\ &\quad+|\phi_n(-\mathrm{i}u)|\,|\phi_n(-\mathrm{i})-1|/\triangle.\end{aligned}$$

By virtue of 1. $|\phi_n(-\mathrm{i}u)|$ is uniformly bounded, it is therefore enough to examine

$$|\phi_n(-\mathrm{i}(u+1))-\phi_n(-\mathrm{i}u)|/\triangle \leq \mathrm{e}^{\mathrm{Re}\, u\mu\triangle}\left|(\mathrm{e}^{\mu\triangle}-1)/\triangle\right|$$
$$+\mathrm{e}^{\mathrm{Re}\, u\mu\triangle}\sum_{j=1}^{m}\left(\mathrm{e}^{\mathrm{Re}\, ux_j}+1\right)\lambda_j+\mathrm{e}^{(\mathrm{Re}\, u+1)\mu\triangle}\sum_{j=1}^{m}\left(\mathrm{e}^{(\mathrm{Re}\, u+1)x_j}+1\right)\lambda_j$$
$$|\phi_n(-\mathrm{i})-1|/\triangle \leq \left|(\mathrm{e}^{\mu\triangle}-1)/\triangle\right|+\mathrm{e}^{\mathrm{Re}\, u\mu\triangle}\sum_{j=1}^{m}\left(\mathrm{e}^{\mathrm{Re}\, ux_j}+1\right)\lambda_j.$$

Fixing $\delta > 0$, the expression $\left|(\mathrm{e}^{\mu\triangle}-1)/\triangle\right|$ is bounded for all $0 < \triangle < \delta$, which completes the proof.

□

Theorem 6.3 *The mean value process, the optimal hedging strategy and the expected squared error in the multinomial model (6.1)-(6.4) converge, as $\triangle$ approaches 0, to the continuous-time limits (2.1)-(2.11), with κ given in equation (6.6).*

PROOF. Lemma 6.2 together with Lemma 6.1 show that all estimated quantities appearing in the proof of Theorems 5.4, 5.10, and 5.11 are uniformly bounded by a constant for $\triangle$ small; the statement of the theorem therefore follows by dominated convergence. □

7 Conclusions

We have examined the behaviour of (locally) optimal mean–variance hedging strategies at high rebalancing frequencies in a model where stock returns follow a discretely sampled exponential Lévy process and one hedges a European call option to maturity. Using elementary methods we have shown that all the attributes of a discretely rebalanced (locally) optimal hedge, i.e. the mean value, the hedging coefficient and the expected squared hedging error, converge pointwise in the state space as the rebalancing interval goes to zero.

The limiting formulae represent 1-D and 2-D generalized Fourier transforms which can be evaluated much faster than backward recursion schemes, with the same degree of accuracy.

In the special case of a compound Poisson process we have demonstrated that the convergence results hold true if instead of using an infinitely divisible distribution from the outset one models log returns by multinomial approximations thereof. This result represents an important extension of Cox et al. (1979), Madan et al. (1989) and He (1990) to incomplete markets with leptokurtic returns.

The results related to dynamically optimal hedging have been obtained independently in Hubalek et al. (2005), who show that formulae (2.1)-(2.11) represent the optimal solution in the continuous-time exponential Lévy model and are applicable also in cases when the Fourier coefficients ψ are not absolutely integrable and the pay-off H must be obtained by taking principal value in the Fourier integral.

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