Generalised Sharpe Ratios and Asset Pricing in Incomplete Markets *

Aleš Černý (a.cerny@ic.ac.uk)
Imperial College Management School

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Abstract.
The paper presents an incomplete market pricing methodology generating asset price bounds conditional on the absence of attractive investment opportunities in equilibrium. The paper extends and generalises the seminal article of Cochrane and Saá-Requejo who pioneered option pricing based on the absence of arbitrage and high Sharpe Ratios. Our contribution is threefold:

We base the equilibrium restrictions on an arbitrary utility function, obtaining the Cochrane and Saá-Requejo analysis as a special case with truncated quadratic utility. We extend the definition of Sharpe Ratio from quadratic utility to the entire family of CRRA utility functions and restate the equilibrium restrictions in terms of Generalised Sharpe Ratios which, unlike the standard Sharpe Ratio, provide a consistent ranking of investment opportunities even when asset returns are highly non-normal. Last but not least, we demonstrate that for Itô processes the Cochrane and Saá-Requejo price bounds are invariant to the choice of the utility function, and that in the limit they tend to a unique price determined by the minimal martingale measure.

Keywords: Generalised Sharpe Ratio, price bounds, arbitrage, good deal, incomplete market, certainty equivalent, reward for risk measure, optimal portfolio, duality and martingale methods, minimal martingale measure

JEL classification code: G12, D40, C61

1. Introduction

Asset pricing in incomplete markets is an intriguing problem because of the price ambiguity one has to deal with. Traditionally this ambiguity is either removed completely by assuming a representative agent equilibrium or it is acknowledged in its fullest by looking at the no-arbitrage bounds. Arguably the former assumption is too strong and the latter assumption is too weak. Good-deal pricing introduces moderately

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strong equilibrium restrictions somewhere between the two extremes, postulating the absence of attractive investment opportunities – good deals – in equilibrium. Under the influence of CAPM and APT attractive investments became associated with high Sharpe Ratios, both in theoretical and empirical work (Ross (1976), Shanken (1992), Cochrane and Saá-Requejo (2000)), but Černý and Hodges (2001) show that one can impose the good-deal restrictions with considerable generality. The generic term 'good deals' was introduced by Cochrane and Saá-Requejo (2000) who were the first to successfully apply the good-deal restrictions to option pricing. The idea of Cochrane and Saá-Requejo was to restrict the availability of high Sharpe Ratios at every point in time. Using the dual discount factor restrictions and backward recursion they calculated option price bounds that are based on very believable equilibrium restrictions, yet are much narrower than the corresponding super-replication bounds.

The association of good deals with high Sharpe Ratios has its pitfalls. High Sharpe Ratios do not include all arbitrage opportunities, therefore to make the equilibrium restrictions meaningful one must eliminate not just high Sharpe Ratios (grey circle in Figure 1) but also arbitrage opportunities (dark triangle) and all the convex combinations between the two types of investments (black contour). As a result the

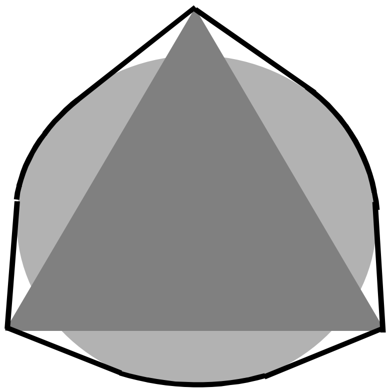

Figure 1. Black contour contains the set of good deals generated as a combination of high Sharpe Ratios (grey circle) and arbitrage opportunities (dark grey triangle).

equilibrium restriction of Cochrane and Saá-Requejo *cannot* be described by imposing restrictions on the standard Sharpe Ratio alone, but as we show here it is associated to a level of the *Arbitrage-Adjusted Sharpe Ratio* discussed in section 3.1.

To understand why pricing requires the use of Generalised Sharpe Ratios, it is useful to step back and examine the standard Sharpe Ratio. Sharpe Ratio is closely related to quadratic utility; there is a one-to-one relationship between the maximum quadratic utility attainable in a market and the market Sharpe Ratio. Crucially, Sharpe Ratio re-labels the levels of quadratic utility in such a fashion that the labels do not depend on the initial wealth.

The relationship with quadratic utility explains why Sharpe Ratio is *not* a good reward-for-risk measure. Quadratic utility has a bliss point, one is penalised for achieving wealth beyond this point. Consider two assets A and B with excess returns given in Table I. The optimal wealth

Table I. Asset A stochastically dominated by asset B

Probability	$\frac{1}{6}$	$\frac{1}{2}$	$\frac{1}{3}$	**Sharpe Ratio**
Return of Asset A	-1%	1%	2%	**1.0**
Return of Asset B	-1%	1%	11%	**0.8**

in market A does not extend beyond the bliss point, whereas in market B it does. This is why asset A achieves a higher Sharpe ratio of 1.0 than the unambiguously more attractive asset B (SR of 0.8). To obtain meaningful price bounds based on Sharpe ratio one must prevent such anomalous behaviour. The remedy is to make the utility non-decreasing after the bliss point – hence the need for truncated quadratic utility. The resulting wealth-independent labelling of the levels of truncated quadratic utility leads to the Arbitrage-Adjusted Sharpe Ratio. The Cochrane and Saá-Requejo set of good deals is simply the set of excess returns with high Arbitrage-Adjusted Sharpe Ratio.

Since truncated quadratic utility has none of the tractability of its non-truncated counterpart, it is natural to ask whether other utility functions are a viable alternative. For a given candidate utility function this means firstly defining the corresponding Generalised Sharpe Ratio, and secondly computing so called 'discount factor restrictions' corresponding to that GSR. For example, the Cochrane and Saá-Requejo set of equilibrium pricing kernels must satisfy $\mathrm{Var}(m) \leq h_A^2$ where m is the pricing kernel and h_A is the upper bound on Arbitrage-Adjusted Sharpe Ratio. Our first contribution is in showing how to derive this duality restriction for an arbitrary utility function. The second contribution is in extending the definition of the Sharpe ratio from quadratic utility to the entire CRRA family of utility functions and showing how such extension can, in principle, be performed for any utility function.

Our general approach permits to prove an interesting property of the Cochrane and Saá-Requejo good-deal bounds: for Itô price processes these bounds are invariant to the choice of the reward-for-risk measure (utility function). The representative agent equilibrium in this case always corresponds to pricing with the *minimal martingale measure* of

Föllmer and Schweizer (1989), closely related to the *numeraire portfolio* of Long (1990), see also Černý (1999) and Kallsen (2002).

As an answer to 'What are the discount factor restrictions implied by standard utility functions?' we can offer the following:

1. Truncated quadratic utility
$$1 + h_A^2(basis) \leq \mathrm{E}\left[m^2\right] \leq 1 + h_A^2 \tag{1}$$
2. Negative exponential (CARA) utility
$$\frac{1}{2}h_E^2(basis) \leq \mathrm{E}\left[m \ln m\right] \leq \frac{1}{2}h_E^2 \tag{2}$$
3. CRRA utility $0 < \gamma \neq 1$, and truncated CRRA utility $\gamma < 0$
$$\left(1 + h_\gamma^2(basis)\right)^{\frac{1-\gamma}{2\gamma^2}} \leq \mathrm{E}\left[m^{1-\frac{1}{\gamma}}\right] \leq \left(1 + h_\gamma^2\right)^{\frac{1-\gamma}{2\gamma^2}} \tag{3}$$
Truncated quadratic is a special case with $\gamma = -1$, $h_{-1} = h_A$.
4. Logarithmic utility $\gamma = 1$
$$\ln\left(1 + h_1^2(basis)\right) \leq -2\mathrm{E}\left[\ln m\right] \leq \ln\left(1 + h_1^2\right) \tag{4}$$
where $m > 0$ is the change of measure, h_A is the Sharpe Ratio adjusted for arbitrage, h_E, and h_γ are the Generalised Sharpe Ratios generated by the CARA and CRRA utility, respectively. All variables with attribute '*basis*' refer to the market containing only basis assets (that is without focus assets to be priced).
5. For Itô price processes the instantaneous restrictions coincide for all utility functions. Denoting ν the market price of risk vector, the no-good-deal restriction becomes
$$h^2(basis) \leq ||\nu||^2 \leq h^2.$$

For each of the utility functions in (1)-(4) the two inequalities are a direct consequence of the Extension Theorem, familiar from no-arbitrage pricing[1]. The left hand side inequalities are known in financial literature, although the authors do not seem to be aware of the common principle underlying all of them. These restrictions have been used to diagnose asset pricing models, and correspond to the above utility functions as follows

[1] For the derivation of the Extension Theorem in good-deal setting and for proofs of general properties of good-deal price bounds see Černý and Hodges (2001).

1. Hansen and Jagannathan (1991), Hansen et al. (1995),
2. Stutzer (1995),
3. with $0 < \gamma \neq 1$ Snow (1991), and
4. Bansal and Lehmann (1997).

The economic interpretation of the left hand side inequalities in (1)-(4) is simple: the best deal in a market containing only basis assets cannot be better than the best deal in a market including also the focus asset. The genuine no-good-deal restrictions are the right hand side inequalities, which quantify by how much the best deal can improve after the introduction of a focus asset. Here the only representative was the restriction (1) of Cochrane and Saá-Requejo (2000).

The Generalised Sharpe Ratios in (1)-(4) provide a scale-free measure of risk which behaves like the standard Sharpe Ratio for excess returns with small dispersion. We derive simple formulae that permit calculation of Generalised Sharpe Ratios for an arbitrary vector of excess return X. In the case of CRRA family of utility functions we have

$$h_\gamma^2(X) = \left(\max_\lambda \mathrm{E}\left[(1+\lambda X)^{1-\gamma}\right]\right)^{\frac{2\gamma}{1-\gamma}} - 1 \text{ for } 0 < \gamma < 1 \tag{5}$$

$$h_\gamma^2(X) = \left(\min_\lambda \mathrm{E}\left[(1+\lambda X)^{1-\gamma}\right]\right)^{\frac{2\gamma}{1-\gamma}} - 1 \text{ for } 1 < \gamma \tag{6}$$

$$h_\gamma^2(X) = \left(\min_\lambda \mathrm{E}\left[\max\left(1+\lambda X, 0\right)^{1-\gamma}\right]\right)^{\frac{2\gamma}{1-\gamma}} - 1 \text{ for } \gamma < 0 \tag{7}$$

$$h_\gamma^2(X) = e^{2\max_\lambda \mathrm{E}[\ln(1+\lambda X)]} - 1 \text{ for } \gamma = 1. \tag{8}$$

To obtain the Arbitrage-Adjusted Sharpe Ratio one computes h_{-1} in equation (7), to obtain the *standard Sharpe Ratio*[2] one simply removes the truncation at zero in the definition of h_{-1},

$$h^2(X) = \left(\min_\lambda \mathrm{E}\left[(1+\lambda X)^2\right]\right)^{-1} - 1. \tag{9}$$

[2] Since its first appearance in Sharpe (1966) there has been a number of generalisations of Sharpe Ratio within the portfolio management literature. These generalisations, important as they are, are captured in our definition of the *standard Sharpe Ratio* (9). In particular, the 'generalised Sharpe Ratio' of Dowd (1999) is obtained from (9) when X is a vector of risky excess returns, one of which represents the current portfolio.

The Generalised Sharpe Ratios proposed in this paper can be used with great advantage in portfolio management, because unlike the standard Sharpe Ratio they provide a consistent ranking of investment opportunities when asset returns are highly skewed.

Bernardo and Ledoit (2000) propose to base the definition of good deals on the gain-loss ratio. This reward-for-risk measure cannot be captured in our framework, for the following reason. In the present paper we fix the utility function and we measure good deals by the (appropriately rescaled) levels of expected utility. Bernardo and Ledoit, on the other hand, fix the level of expected utility that defines a good deal and they rank the good deals by changing the *shape* of the utility function. Namely, the gain-loss ratio is based on the Domar-Musgrave utility. With a piecewise linear utility in a frictionless market the maximum expected utility of a risky investment is either zero or plus infinity, and one can affect the outcome by changing the ratio of the slopes of the two linear parts of Domar-Musgrave utility function. The slope ratio at which expected utility switches from 0 to $+\infty$ is the market gain-loss ratio. The discount factor restrictions are similar in nature to those mentioned above

$$L_{basis} \leq \frac{\operatorname{ess\,sup} m}{\operatorname{ess\,inf} m} \leq L,$$

where L denotes the maximum gain-loss ratio in the market. The gain-loss does not work well in Itô process environment with continuous trading where typically $L_{basis} = +\infty$, as in, for example, the standard Black-Scholes model.

1.1. Organisation of the paper

The second section discusses one-period no-good-deal equilibria and the corresponding discount factor restrictions. The third section describes the link between the certainty equivalent gains and (Generalised) Sharpe Ratios; in particular it extends the definition of Sharpe Ratio to the entire family of CRRA utility functions. Section four gives two numerical examples which illustrate the computation of option price bounds in multiperiod model based on a number of Generalised Sharpe Ratios.

Section five translates the discrete time results into the Itô process framework and derives the instantaneous restrictions on the market price of risk. Section six quantifies the extent to which the instantaneous good-deal restrictions limit investment opportunities in the long run. Section seven explores the limiting cases of the instantaneous good-deal price bounds, and section eight concludes.

2. No-good-deal restrictions in one-period model

Consider a market with a finite number of states. Let r be the risk-free rate of return and let X be the vector of excess returns of basis risky assets. By θ denote the portfolio of basis assets. For a fixed utility function and fixed initial endowment V_0 it is natural to measure the attractiveness of a self-financing investment by the *certainty equivalent* of the resulting wealth V relative to the wealth of a riskless investment into the bank account. Specifically, the value of the best deal in a market characterised by excess return X will be denoted $a(X)$,with $a(X)$ defined implicitly as follow

$$U[(1+r)V_0 + a(X)] \triangleq \sup_{\theta} \mathrm{E}\left[U\left((1+r)V_0 + \theta X\right)\right], \tag{10}$$

having substituted for V from equation (40). The fact that the certainty equivalent $a(X)$ depends on V_0 is a nuisance, but it allows us to formulate and solve the pricing problem for *any utility function,* therefore formulation (10) is the most convenient at this point. Section 3 discusses the link between the certainty equivalent gain $a(X)$ and Generalised Sharpe Ratios.

Consider a focus asset Y. By $P_{\infty}(Y)$ we will denote the no-arbitrage price range of Y. Taking a fixed upper bound $\bar{a}$, we define the set of no-good-deal equilibrium prices of Y as

$$P_{\bar{a}}(Y) \triangleq \{p_y | a(X, Y-(1+r)p_y) \leq \bar{a}\} \cap P_{\infty}(Y). \tag{11}$$

Before we give a full characterisation of no-good-deal price bounds in finite dimension (Theorem 2), it is useful to provide the following classification of utility functions

DEFINITION 1. *Let $U(x)$ be a non-decreasing concave function defined on an interval $\mathcal{D} = (c, +\infty)$; $-\infty \leq c < +\infty$. We will distinguish the following three cases*

U1) *$U(x)$ is unbounded from above (and necessarily strictly increasing on $\mathcal{D}$).*

U2) *$U(x)$ is bounded from above and strictly increasing on $\mathcal{D}$.*

U3) *$U(x)$ is bounded from above and there is a threshold $\bar{x} \in \mathcal{D}$ such that $U(x)$ is constant for $x \geq \bar{x}$ and $U(x)$ is strictly increasing for $x \in \mathcal{D}$ such that $x < \bar{x}$. In this case we assume $(1+r)V_0 < \bar{x}$; utility can be improved by trading in risky assets.*

THEOREM 2. *Assume that utility function $U(.)$ is non-decreasing, differentiable and that it satisfies*

$$\lim_{x\to-\infty}\frac{x}{U(x)}=0.$$

Assume further that there is no arbitrage among the basis assets. Then

1. *The $a(X)$ The supremum in (10) is finite and it is attained by at least one portfolio θ. Moreover, the corresponding certainty equivalent $a(X)$ is finite. Let us denote its value by a_{basis}.*

2. *For any focus asset Y the set $P_{\bar{a}}(Y)$ is empty for $\bar{a} < a_{basis}$ and it is a non-empty interval for $\bar{a} > a_{basis}$.*

3. *In cases U1), U2) $P_{a_{basis}}(Y)$ is non-empty, in case U3) $P_{a_{basis}}(Y)$ may be empty.*

4. *For $\bar{a} > a_{basis}$ $P_{\bar{a}}(Y)$ contains a single point if and only if Y is a redundant asset (there is θ such that $Y = const + \theta X$).*

5. *If $a_{basis} \leq a_1 < a_2$ and Y is non-redundant then $P_{a_1}(Y)$ is inside $P_{a_2}(Y)$ which in turn is inside $P_\infty(Y)$ (the no-arbitrage price region for Y). If $P_{a_2}(Y)$ is strictly inside $P_\infty(Y)$ then $P_{a_1}(Y)$ is strictly inside $P_{a_2}(Y)$. In the case U1) $P_{a_2}(Y)$ is always strictly inside $P_\infty(Y)$.*

6. *As a_2 tends to infinity $P_{a_2}(Y)$ tends to the no-arbitrage price range, mathematically*

$$\bigcup_{a_2} P_{a_2}(Y) = P_\infty(Y). \tag{12}$$

7. *The no-arbitrage restriction in (11) is cosmetic in the following sense. If $p \in \{p_y | a(X, Y-(1+r)p_y) \leq \bar{a}\}$ then $p \in \mathrm{cl}P_\infty(Y)$, that is only the end points of $\{p_y | a(X, Y-(1+r)p_y) \leq \bar{a}\}$ may lie outside the no-arbitrage bounds and this may only happen in the cases U2), U3).*

Proof: *See Appendix A.*

Theorem 2 guarantees that good-deal prices are well defined. As one varies the upper bound $\bar{a}$ between a_{basis} and $+\infty$ the range of no-good-deal prices changes monotonically from representative agent prices to no-arbitrage bounds. With an unbounded utility the no-arbitrage bounds are never reached for finite $\bar{a}$, but they are approached as $\bar{a} \to +\infty$.

With a bounded utility function it may happen that the no-good-deal price region $P_{\bar{a}}(Y)$ hits one or both no-arbitrage bounds for finite $\bar{a}$, in such case $P_{\bar{a}}(Y)$ does not grow further beyond the no-arbitrage bounds as $\bar{a}$ increases.

In the rest of this section we will proceed in 2 steps. First we will explain how to find the highest a attainable in a complete market. In the second step we will show how, with the help of an extension theorem, this information can be used to find the no-good-deal price of an arbitrary focus asset. The second step will in a natural way lead to the dual discount factor restrictions.

Suppose the market X is complete and the state prices are given by a unique change of measure m; our aim is to find the maximum certainty equivalent gain $a(m)$ in this market. Instead of looking for the optimal investment strategy θ we will use an elegant trick, due to Pliska (1986), of searching for the optimal distribution of wealth, subject to the budget constraint dictated by the state prices m

$$\max_{\theta} \mathrm{E}\left[U\left((1+r)V_0+\theta X\right)\right] = \max_{\substack{V \\ \text{s.t.}\mathrm{E}[mV]=(1+r)V_0}} \mathrm{E}\left[U(V)\right],$$

whereby for $a(m)$ we simply have

$$U[(1+r)V_0+a(m)] \triangleq \max_{\substack{V \\ \text{s.t.}\mathrm{E}[mV]=(1+r)V_0}} \mathrm{E}\left[U(V)\right]. \tag{13}$$

In a finite state model the maximisation problem (13) is standard. Since there is just one linear constraint one solves (13) using unconstrained maximisation separately in each state with a Lagrange multiplier

$$\max_{\substack{V(\omega) \\ \omega\in\Omega}} U\left(V(\omega)\right) - \lambda m(\omega)V(\omega).$$

The first order conditions give

$$U'(V(\omega)) = \lambda m(\omega)$$

Denoting $I(.)$ the inverse function to the marginal utility $U'(.)$ we obtain

$$V = I(\lambda m) \tag{14}$$

and from the restriction $\mathrm{E}\left[mV\right] = (1+r)V_0$ we can recover the value of λ.

As an example let us apply the above procedure to the negative exponential utility. First we find the inverse of the marginal utility

$$\begin{aligned} U(V) &= -e^{-AV} \\ U'(V) &= Ae^{-AV} \\ I(x) &= -\frac{1}{A}\ln\frac{x}{A}. \end{aligned}$$

The optimal terminal wealth is then

$$V = I(\lambda m) = -\frac{1}{A}\ln\frac{\lambda m}{A}.$$

We recover the Lagrange multiplier from the budget constraint and plug this value back into the expression for optimal wealth

$$\mathrm{E}[mV] = (1+r)V_0$$
$$\lambda = Ae^{-A(1+r)V_0 - \mathrm{E}[m\ln m]}$$
$$V = (1+r)V_0 + \mathrm{E}\left[\frac{m}{A}\ln\frac{m}{A}\right] - \frac{1}{A}\ln\frac{m}{A}.$$

Finally, we recover the certainty equivalent of the optimal risky investment

$$U(V) = -me^{-A(1+r)V_0 - \mathrm{E}[m\ln m]}$$
$$\mathrm{E}[U(V)] = -e^{-A(1+r)V_0 - \mathrm{E}[m\ln m]}$$
$$a(m) = U^{-1}(\mathrm{E}[U(V)]) - (1+r)V_0 = \frac{1}{A}\mathrm{E}[m\ln m]. \tag{15}$$

2.1. Discount factor restrictions in good-deal pricing

We have just seen how one calculates the maximum attainable certainty equivalent $a(m)$ in a complete market. The crucial link between the complete and incomplete market is provided by the *extension theorem*[3] which asserts that any incomplete market without good deals can be embedded in a complete market that has no good deals. Let us denote by a_{basis} the certainty equivalent of the best deal attainable in the market containing only the basis assets. Two observations follow from the extension theorem. The best deal in the completed market cannot

[3] Interestingly, both the idea of Sharpe Ratio restrictions and the use of the extension theorem can be traced back to Ross, see Ross (1976) pg. 354 and the appendix of Ross (1978). For application of the extension theorem in good-deal pricing see Černý and Hodges (2001).

be worse than the best deal in the original market containing only basis assets. On the other hand, for any $\varepsilon > 0$ there is no good deal of size $a_{basis} + \varepsilon$ in the market containing just basis assets. Consequently, by extension theorem there must be a completion with a pricing kernel m for which $a(m) < a_{basis} + \varepsilon$. By letting $\varepsilon \to 0$ we obtain

$$a_{basis} = \inf_m a(m)$$

where m must price correctly all basis assets. This argument is inspired by 'fictitious completions' of Karatzas et al. (1991). In a finite state model the infimum is always attained by at least one pricing kernel.

THEOREM 3. *Assume that utility function $U(.)$ satisfies*

$$\lim_{x \to -\infty} \frac{x}{U(x)} = 0.$$

If there is no arbitrage among the basis assets then we have the following dual characterisation of good-deal price bounds:

1. *In the cases U1, U2 for $\bar{a} \geq a_{basis}$*

$$P_{\bar{a}}(Y) = \left\{ \frac{\mathrm{E}\,[mY]}{1+r} \middle| a_{basis} \leq a(m) \leq \bar{a}, \mathrm{E}\,[mX] = 0, m > 0 \right\}. \quad (16)$$

   Furthermore, for unbounded utility (U1) $m > 0$ in (16) can be omitted.

2. *In the case U3 for $\bar{a} \geq a_{basis}$ and Y non-redundant define*

$$\begin{aligned} \tilde{P}_{\bar{a}}(Y) &\triangleq \left\{ \frac{\mathrm{E}\,[mY]}{1+r} \middle| a_{basis} \leq a(m) \leq \bar{a}, \mathrm{E}\,[mX] = 0, m \geq 0 \right\}, \\ P_{\infty}(Y) &= (p_{-\infty}, p_{\infty}); p_{-\infty} < p_{\infty} \end{aligned}$$

   Then

$$P_{\bar{a}}(Y) \subseteq \tilde{P}_{\bar{a}}(Y) \subseteq P_{\bar{a}}(Y) \cup \{p_{-\infty}\} \cup \{p_{\infty}\}. \quad (17)$$

Proof: *See Appendix A.*

The restrictions of the type $a_{basis} \leq a(m)$ are well known in financial economics, where they have been employed to test different asset pricing models[4]. We are, however, primarily interested in the pricing

[4] See Stutzer for CARA utility, Bansal and Lehmann for log utility. Snow discusses the CRRA utility. In all these cases the discount factor restrictions are derived ad hoc from the Jensen's inequality.

implications of the extension theorem. Suppose that we want to find all prices of a focus asset that do not provide good deals of size $\bar{a}$ in the enlarged market. From the extension theorem all such prices must be supported by pricing kernels for which $a(m) \leq \bar{a}$. This is the dual no-good-deal discount factor restriction.

For example, for the CARA utility we have from (15) $Aa(m) = \mathrm{E}[m \ln m]$ and therefore the discount factor restrictions read

$$\begin{gathered} Aa_{basis} \leq \mathrm{E}[m \ln m] \leq A\bar{a} \\ \mathrm{E}[mX] = 0, \;\; m > 0. \end{gathered} \tag{18}$$

In conclusion, the market including both basis and focus assets does not provide deals better than $\bar{a}$, as measured by CARA utility, if (and only if) the focus assets are priced with no-arbitrage pricing kernels consistent with basis assets and satisfying the restriction (18).

Below we summarise the no-good-deal restrictions on the change of measure m for standard utility functions. The derivation proceeds as explained above between equations (13) and (15).

1. Truncated quadratic utility $U(V) = -(\bar{V} - V)^2$; $V < \bar{V}$ and $U(V) = 0$; $V \geq \bar{V}$

$$\left(\frac{1}{1 - Aa_{basis}}\right)^2 \leq \mathrm{E}\left[m^2\right] \leq \left(\frac{1}{1 - Aa}\right)^2 \tag{19}$$

2. Negative exponential utility $U(V) = -e^{-AV}$

$$Aa_{basis} \leq \mathrm{E}[m \ln m] \leq Aa \tag{20}$$

3. Power (isoelastic) utility $U(V) = \frac{V^{1-\gamma}}{1-\gamma}$; $0 < \gamma \neq 1, V > 0$

$$\left(1 + \frac{Aa_{basis}}{\gamma}\right)^{\frac{1}{\gamma}-1} \leq \mathrm{E}\left[m^{1-\frac{1}{\gamma}}\right] \leq \left(1 + \frac{Aa}{\gamma}\right)^{\frac{1}{\gamma}-1} \tag{21}$$

4. Logarithmic utility $U(V) = \ln V, V > 0$

$$\ln(1 + Aa_{basis}) \leq -\mathrm{E}[\ln m] \leq \ln(1 + Aa)\,. \tag{22}$$

In equations (19)- (22) A stands for the coefficient of absolute risk aversion evaluated at point $(1 + r)V_0$.

3. Generalised Sharpe Ratios

Having derived the state price restrictions (19)-(22) the task changes into interpreting the state price bounds as reward for risk measures, preferably ones that are close in nature to Sharpe Ratio. Note that if one uses a as the measure of attractiveness then one has to specify the coeffcient of absolute risk-aversion in restrictions (19)-(22). It turns out that for small Sharpe Ratios there is an unambiguous link between Sharpe Ratios and certainty equivalent gains, which we describe next.

To keep technicalities at minimum we assume that the excess return X has bounded support and that the utility function is sufficiently differentiable. From the Taylor expansion we obtain

$$\begin{aligned}\mathrm{E}\left[U\left(V_0+\theta X\right)\right] &= U(V_0)+U'(V_0)\theta\mathrm{E}\left[X\right]+\\ &+\frac{1}{2}U''(V_0)\theta^2\mathrm{E}\left[X^2\right]+o(\theta^2\mathrm{E}\left[X^2\right])\end{aligned}$$

and after maximisation with respect to θ we will have

$$\begin{aligned}\hat{\theta} &= \frac{U'(V_0)\mathrm{E}\left[X\right]}{U''(V_0)\mathrm{E}\left[X^2\right]}+o\left(\frac{\mathrm{E}\left[X\right]}{\mathrm{E}\left[X^2\right]}\right)\\ \max_\theta \mathrm{E}\left[U(V_0+\theta X)\right] &= U(V_0)-\frac{1}{2}\frac{\left(U'(V_0)\mathrm{E}\left[X\right]\right)^2}{U''(V_0)\mathrm{E}\left[X^2\right]}+o\left(\frac{\left(\mathrm{E}\left[X\right]\right)^2}{\mathrm{E}\left[X^2\right]}\right)\end{aligned} \quad (23)$$

Without loss of generality we can assume that $X\frac{\mathrm{E}[X]}{\mathrm{E}[X^2]}$ is small for all realisations of X so that the Taylor series approximation of $U\left(V_0+\hat{\theta}X\right)$can be made arbitrarily precise. At the same time, for a small certainty equivalent gain we can write

$$U(V_0+a)=U(V_0)+U'(V_0)a+o(a), \quad (24)$$

and the comparison of (23) and (24) gives

$$a=\frac{h^2(X)}{2A(V_0)}+o(h^2) \quad (25)$$

where $A(V)=-\frac{U''(V)}{U'(V)}$ is the coefficient of absolute risk-aversion, $h(X)$ is the Sharpe ratio of X

$$h(X)\triangleq\frac{\mathrm{E}\left[X\right]}{\sqrt{\mathrm{E}\left[X^2\right]-\left(\mathrm{E}\left[X\right]\right)^2}},$$

and $\lim_{h^2\to 0}\frac{o(h^2)}{h^2}=0$.

In conclusion, one could replace Aa in expressions (19)-(22) with $\frac{h^2}{2}$. Naturally, this is not the only transformation that satisfies the asymptotic property (25). For example, for small values of h^2 we have

$$h^2 = \frac{h^2}{1+h^2} + o(h^2) = e^{h^2} - 1 + o(h^2),$$

and indeed we might equally well replace Aa with any other function $f(h^2)$ as long as f is continuously differentiable around 0 with $f(0) = 0$ and $f'(0) = \frac{1}{2}$. The rest of this section describes how the ambiguity in choosing the function $f(h^2)$ is resolved for negative exponential, truncated quadratic and CRRA utility.

3.1. Truncated quadratic utility

To begin with, consider maximisation of *non-truncated* quadratic utility for a single asset with excess return X,

$$\max_{\theta} -\mathrm{E}\left[(1-\theta X)^2\right].$$

The optimal investment is

$$\hat{\theta} = \frac{\mathrm{E}[X]}{\mathrm{E}[X^2]} \tag{26}$$

and the maximum utility is an increasing function of the Sharpe Ratio

$$\max_{\theta} -\mathrm{E}\left[(1-\theta X)^2\right] = \frac{(\mathrm{E}[X])^2}{\mathrm{E}[X^2]} - 1 = -\frac{1}{1+h^2(X)}, \tag{27}$$

or conversely

$$h^2(X) = \frac{1}{\min_{\theta} \mathrm{E}\left[(1+\theta X)^2\right]} - 1. \tag{28}$$

The quadratic utility function has a bliss point at $\theta X = 1$; having more wealth than 1 actually *lowers* the expected utility. The optimal wealth *will not* extend beyond the bliss point if and only if $\hat{\theta} X \leq 1$, that is if

$$x_{\max}\mathrm{E}[X] \leq \mathrm{E}\left[X^2\right], \tag{29}$$

where $x_{\max} \triangleq \operatorname{ess\,sup} X$ is the highest excess return. If (29) is violated then the one-to-one relationship (27) between expected utility and Sharpe ratio tells us that the Sharpe Ratio of X cannot be a good measure of investment opportunities because by throwing some money

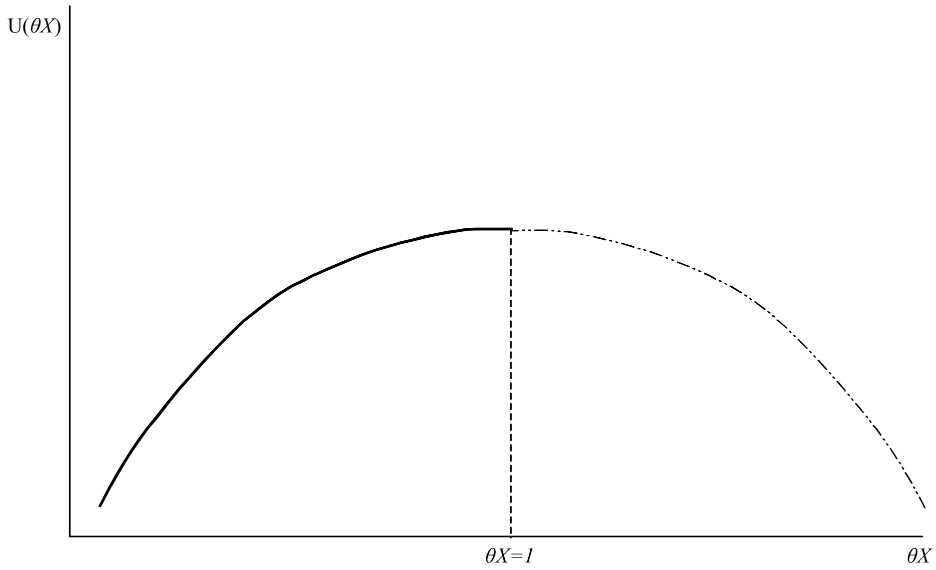

Figure 2. Quadratic utility has a bliss point

away in the states where $\hat{\theta}X > 1$ the Sharpe Ratio of X will actually increase. More specifically, we can replace the original excess return distribution X with a distribution X_{cap} *capped* at a fixed value x_{cap}. Initially x_{cap} is set at $x_{\max}$ and condition

$$x_{\text{cap}} \mathrm{E}\left[X_{\text{cap}}\right] \leq \mathrm{E}\left[X_{\text{cap}}^2\right] \tag{30}$$

is not satisfied. By lowering x_{cap} we increase the Sharpe Ratio of the capped distribution and make the difference between the left hand side and the right hand side in condition (30) smaller. The Sharpe Ratio of the capped distribution reaches its maximum just when

$$x_{\text{cap}} \mathrm{E}\left[X_{\text{cap}}\right] = \mathrm{E}\left[X_{\text{cap}}^2\right]. \tag{31}$$

At this point we have decomposed X into a pure Sharpe Ratio part X_{cap} and the pure arbitrage part $X - X_{\text{cap}}$.

EXAMPLE 4. *Consider a security with the following distribution of excess return X:*

Table II. Distribution of excess return

x	*-1%*	*1%*	*11%*
$\Pr(X = x)$	$\frac{1}{6}$	$\frac{1}{2}$	$\frac{1}{3}$

For the Sharpe Ratio we have

$$\begin{aligned} \mathrm{E}[X] &= 4 \\ \mathrm{Var}[X] &= \frac{25}{6} + \frac{9}{2} + \frac{49}{3} = 25 \\ h(X) &= \frac{4}{\sqrt{25}} = 0.8. \end{aligned}$$

Let us now see whether the bliss point condition (29) is violated. To this end

$$x_{\max}\mathrm{E}[X] = 11 \times 4 = 44,$$

whereas

$$\mathrm{E}\left[X^2\right] = \mathrm{Var}[X] - (\mathrm{E}[X])^2 = 25 - 16 = 9,$$

which means that the condition is indeed violated and one can increase the Sharpe Ratio by putting some money aside. Guessing that the truncation point will occur at $x_{cap} > 1\%$ *we can write the bliss point condition (31) as*

$$x_{cap}\left(\frac{1}{6} \times (-1) + \frac{1}{2} \times 1 + \frac{1}{3} \times x_{cap}\right) = \left(\frac{1}{6} \times (-1)^2 + \frac{1}{2} \times 1^2 + \frac{1}{3} \times x_{cap}^2\right).$$

Solving for x_{cap} *we find* $x_{cap} = 2$ *and*

$$\begin{aligned} h_A(X) &= h(X_{cap}) = \frac{\mathrm{E}[X_{cap}]}{\sqrt{\mathrm{E}\left[X_{cap}^2\right] - (\mathrm{E}[X_{cap}])^2}} = \\ &= \frac{1}{\sqrt{\frac{x_{cap}}{\mathrm{E}[X_{cap}]} - 1}} = \frac{1}{\sqrt{\frac{2}{1} - 1}} = 1 \end{aligned}$$

The arbitrage-adjusted Sharpe Ratio is 1 *compared to the standard Sharpe Ratio of* 0.8. *The pure Sharpe Ratio part of excess return* X *is*

$$X_{\mathrm{cap}} = [-1\% \; 1\% \; 2\%]$$

and the pure arbitrage part is

$$X_A = [0\%\ 0\%\ 9\%].$$

Figure 3 shows a slice of the 3D space of excess returns in the plane $x_1 + x_2 + x_3 = 1$; *it is, so to speak, bird's-eye view of the market from direction (1,1,1). The set of arbitrage opportunities (positive octant) appears as a dark grey triangle, the set of Sharpe Ratios greater than 1.0 appears as the medium grey circle and the Sharpe Ratios greater than 0.8 are inside the outer light grey circle. The decomposition into pure Sharpe Ratio and a pure arbitrage opportunity is captured as a movement from the original excess return* X *with low Sharpe Ratio to the truncated excess return* X_{cap} *with high Sharpe Ratio, away from the arbitrage opportunity* X_A*. The Arbitrage-Adjusted Sharpe Ratio of* X *is defined as the Sharpe Ratio of* X_{cap}.

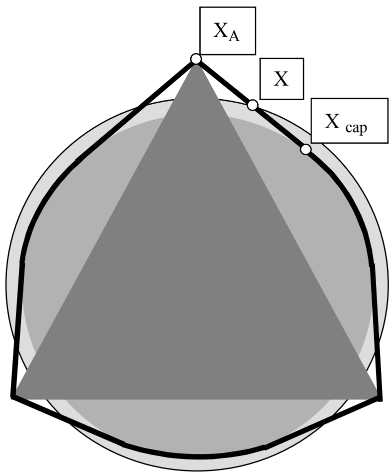

Figure 3. Illustration to Arbitrage-adjusted Sharpe Ratio: Movement from X towards X_{cap}, away from the arbitrage opportunity X_A, leads to higher Sharpe Ratio (circle with smaller radius).

Truncated quadratic utility formalises the 'throwing money away' procedure. With truncated utility one is neither rewarded nor penalised for achieving wealth levels above the bliss point, thus the excess return is effectively capped at a level where $\theta X = 1$.

Namely

$$\max_{\theta} -\mathrm{E}\left[(\max(1-\theta X, 0))^2\right] = \frac{(\mathrm{E}\left[X_{\mathrm{cap}}\right])^2}{\mathrm{E}\left[X_{\mathrm{cap}}^2\right]} - 1 = -\frac{1}{1+h^2(X_{\mathrm{cap}})}, \tag{32}$$

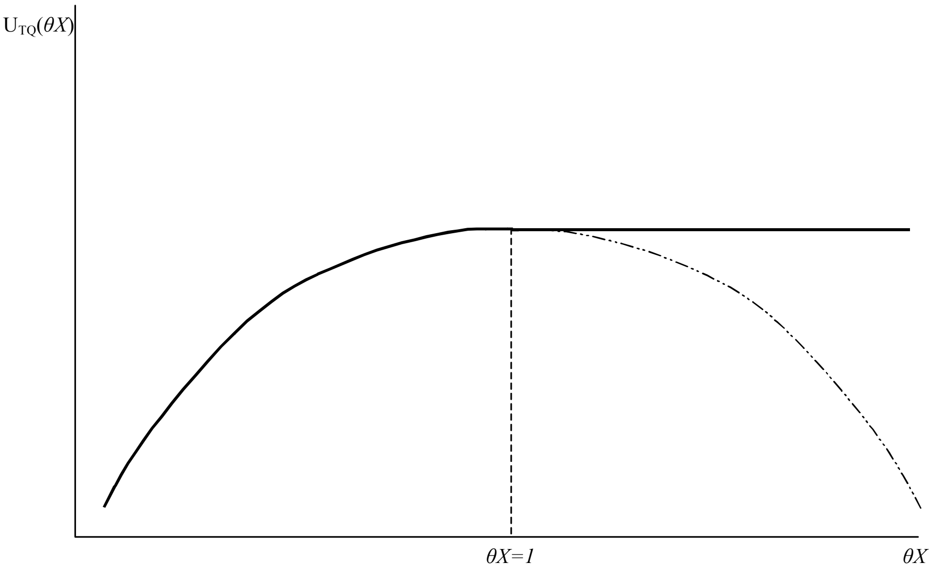

Figure 4.

where

$$X_{\text{cap}} = \min\left(X, \frac{1}{\hat{\theta}}\right) \tag{33}$$

with $\hat{\theta}$ being the optimal portfolio weight in (32). The first order conditions in (32) correspond exactly to

$$x_{\text{cap}} \mathrm{E}\left[X_{\text{cap}}\right] = \mathrm{E}\left[X_{\text{cap}}^2\right].$$

Appendix B shows that the above argument works with $\mathrm{E}[X] > 0$. For $\mathrm{E}[X] < 0$ the value $x_{\max}$ in condition (30) has to be replaced with $x_{\min} \equiv \operatorname{ess\,inf} X$ and the truncation proceeds from below.

We have argued intuitively that the Sharpe Ratio of X_{cap} is the arbitrage-adjusted Sharpe Ratio of X. Equation (32) then suggests how to define $h_A(X)$ using the truncated quadratic utility

$$h_A^2(X) \triangleq \frac{1}{\min_\theta \mathrm{E}\left[\max(1+\theta X, 0)^2\right]} - 1. \tag{34}$$

The obvious advantage of (34) is that it can be used with multiple assets, whereas the intuition of 'throwing money away' only works with one asset.

The proposition below shows that the Cochrane and Saá-Requejo set of good deals can be described by an upper bound on the arbitrage-adjusted Sharpe Ratio.

PROPOSITION 5. *The convex hull of* $\{X|h(X) \geq \bar{h}\}$ *and* $\{X|X \geq 0\}$ *coincides with* $\{X|h_A(X) \geq \bar{h}\} \cup \{X|X \geq 0\}$. *Graphically, if the medium grey circle in Figure 3 is described by* $h(X) \geq \bar{h}$ *then the area delineated by the black contour in the same Figure is described by* $h_A(X) \geq \bar{h}$.

Proof. *Denote by* B *the convex hull of* $\{X|h(X) \geq \bar{h}\}$ *and* $\{X|X \geq 0\}$, *and let* $C = \{X|h_A(X) \geq \bar{h}\} \cup \partial\{X|X \geq 0\}$. *If* $X \in B$ *then there is* $X_A \geq 0$ *and* X_h *with* $h(X_h) \geq \bar{h}$ *such that* $X = X_A + X_h$. *By virtue of (26) and (28)* $h(X_h) \geq \bar{h}$ *implies*

$$\begin{aligned}\bar{h}^2 &\leq \frac{1}{\mathrm{E}\left[(1+\hat{\theta}X_h)^2\right]} - 1\\ \textit{for } \hat{\theta} &= -\frac{\mathrm{E}\left[X_h\right]}{\mathrm{E}\left[X_h^2\right]}.\end{aligned}$$

Because the truncated utility is non-decreasing we have $\mathrm{E}\left[(1+\hat{\theta}X_h)^2\right] \geq \mathrm{E}\left[\max(1+\hat{\theta}\left(X_h + X_a\right), 0)^2\right]$, *therefore*

$$\bar{h} \leq \frac{1}{\mathrm{E}\left[\max(1+\hat{\theta}X, 0)^2\right]} - 1 \leq \frac{1}{\min_\theta \mathrm{E}\left[\max(1+\theta X, 0)^2\right]} - 1 = h_A(X)$$

and we have shown $X \in B \Rightarrow X \in C$.

Conversely, assume $X \in C$. *Then either* $X \in \{X|X \geq 0\}$ *(*X *lies in the positive orthant) and then trivially* $X \in B$, *or* $X \notin \{X|X \geq 0\}$ *and then by virtue of Theorem 2, part 1. there is finite* $\hat{\theta}$ *such that* $h_A^2(X) = \frac{1}{\mathrm{E}\left[\max(1+\hat{\theta}X,0)^2\right]} - 1$. *By virtue of (32), (33) we obtain a decomposition* $X = X_{\text{cap}} + X_A$ *such that* $h(X_{\text{cap}}) = h_A(X) \geq \bar{h}$ *and* $X_A \geq 0$, *proving that* $X \in B$.

Table III shows that standard Sharpe Ratio of 2.0 may seriously underestimate the true investment potential if the excess returns have high dispersion, whereas at the value of 0.5 this difference is negligible. The table shows arbitrage-adjusted Sharpe Ratios h_A against standard Sharpe Ratios for log-normally distributed returns. Because the returns are unbounded from above, the standard Sharpe Ratio is not an appropriate measure of risk. The difference between the AASR and SR is reported in the last column. The necessary calculations are given in the Appendix B.

Table III. Difference between h_A and h for lognormal risky return. h_A arbitrage-adjusted Sharpe Ratio, h standard Sharpe Ratio, $\bar{R}$ expected risky return, R risk-free return, σ return volatility

$\bar{R}$	R	σ	h	h_A	**%error** $\frac{h_A-h}{h}$
1.04	1.02	0.04	0.5	0.502	**0.5%**
1.06	1.02	0.08	0.5	0.503	**0.6%**
1.18	1.02	0.16	0.5	0.512	**2.4%**
1.04	1.02	0.02	1.0	1.085	**8.5%**
1.06	1.02	0.04	1.0	1.093	**9.3%**
1.18	1.02	0.08	1.0	1.140	**14.0%**
1.06	1.02	0.02	2.0	3.675	**83.7%**
1.10	1.02	0.04	2.0	3.824	**91.2%**
1.34	1.02	0.08	2.0	4.845	**142.3%**

3.2. Family of CRRA utility functions

Recall from (21) that the duality between pricing kernels and certainty equivalent gains in this case reads

$$\mathrm{E}_{t-1}\left[m_{t|t-1}^{1-\frac{1}{\gamma}}\right] = \left(1+\frac{Aa}{\gamma}\right)^{\frac{1}{\gamma}-1}. \tag{35}$$

The asymptotic relationship between certainty equivalent and Sharpe Ratio is $Aa = \frac{h^2}{2}$ which yields

$$\mathrm{E}_{t-1}\left[m_{t|t-1}^{1-\frac{1}{\gamma}}\right] = \left(1+\frac{h_\gamma^2}{2\gamma}\right)^{\frac{1}{\gamma}-1}. \tag{36}$$

By virtue of (25) all the generalised Sharpe Ratios h_γ defined by (36) have the same asymptotic behaviour for small values. It remains to check the consistency of this definition with the definition of the Arbitrage-Adjusted Sharpe Ratio, for which the duality is

$$\mathrm{E}_{t-1}\left[m_{t|t-1}^2\right] = 1 + h_A^2.$$

Recall that quadratic utility has $\gamma = -1$, substituting this value into equation (36) we obtain

$$\mathrm{E}_{t-1}\left[m_{t|t-1}^2\right] = \left(1-\frac{h_{-1}^2}{2}\right)^{-2},$$

and it is clear that h_{-1} from (36) is not equal to h_A even though asymptotically they are the same. Fortunately, there is an easy way out to achieve $h_{-1} = h_A$. It is enough to realise that asymptotically

$$\left(1 + \kappa\frac{h_\gamma^2}{2\gamma}\right)^{\frac{1}{\kappa}\left(\frac{1}{\gamma}-1\right)} = \left(1 + \frac{h_\gamma^2}{2\gamma}\right)^{\left(\frac{1}{\gamma}-1\right)} + o(h_\gamma^2)$$

for all κ. There are many choices of $\kappa(\gamma)$, for example $\kappa = -2$ or $\kappa = 2\gamma$, such that $h_{-1} = h_A$. A good way to pinpoint the 'right' value of κ is to look at the time scaling properties of the standard Sharpe Ratio and to compare them with the time scaling properties of the Generalised Sharpe Ratio h_γ, see Section 6. It turns out that one needs $\kappa = 2\gamma$. The discount factor restrictions then become

$$\left(1 + h_{\gamma basis}^2\right)^{\frac{1-\gamma}{2\gamma^2}} \leq \mathrm{E}\left[m^{\frac{\gamma-1}{\gamma}}\right] \leq \left(1 + h_\gamma^2\right)^{\frac{1-\gamma}{2\gamma^2}} \tag{37}$$

$$\frac{1}{2}\ln\left(1 + h_{1basis}^2\right) \leq -\mathrm{E}\left[\ln m_{t|t-1}\right] \leq \frac{1}{2}\ln\left(1 + h_1^2\right). \tag{38}$$

Comparing (37) with (21) and using the definition of certainty equivalent gain we obtain the computational definition of CRRA Sharpe Ratio for a given excess return X

$$\begin{aligned}
1 + h_\gamma^2(X) &= \left(\max_\lambda \mathrm{E}\left[(1+\lambda X)^{1-\gamma}\right]\right)^{\frac{2\gamma}{1-\gamma}} \text{ for } 0 < \gamma < 1 \\
1 + h_\gamma^2(X) &= e^{2\max_\lambda \mathrm{E}\ln(1+\lambda X)} \text{ for } \gamma = 1 \\
1 + h_\gamma^2(X) &= \left(\min_\lambda \mathrm{E}\left[(1+\lambda X)^{1-\gamma}\right]\right)^{\frac{2\gamma}{1-\gamma}} \text{ for } 1 < \gamma.
\end{aligned}$$

These definitions naturally extend to $\gamma < 0$ if the CRRA utility is truncated at the value of zero

$$\begin{aligned}
U(V) &= -\frac{(\bar{V} - V)^{1-\gamma}}{1-\gamma} \text{ for } V < \bar{V} \\
U(V) &= 0 \text{ for } V \geq \bar{V} \\
1 + h_\gamma^2(X) &= \left(\min_\lambda \mathrm{E}\left[\max(1+\lambda X, 0)^{1-\gamma}\right]\right)^{\frac{2\gamma}{1-\gamma}}.
\end{aligned}$$

3.3. Negative exponential utility

Interestingly, there is a special case where the relationship $h^2 = 2Aa$ holds for large certainty equivalent gains. By inverting the no-good-deal restriction (10) for negative exponential utility with an arbitrary random excess return X one obtains

$$a(X) = -\frac{1}{A}\ln\left(-\max_{\theta} -\mathrm{E}\left[e^{-A\theta X}\right]\right) = -\frac{1}{A}\ln\left(\min_{\theta}\mathrm{E}\left[e^{\theta X}\right]\right).$$

Hodges (1998) points out that for a *normally distributed* excess return X we have identically

$$\frac{1}{2}h^2(X) = -\ln\left(\min_{\theta}\mathrm{E}\left[e^{\theta X}\right]\right), \tag{39}$$

where $h(X)$ is a standard Sharpe Ratio, and consequently Hodges uses equation (39) to define the *Generalised Sharpe Ratio* h_E for an *arbitrarily distributed* excess return. The maximum Exponential Sharpe Ratio is hence related to the maximum certainty equivalent gain through (25)

$$\frac{1}{2}h_E^2 = Aa,$$

and one can write the state price restriction (20) in a scale-free form

$$\frac{1}{2}h_E^2(basis) \leq \mathrm{E}\left[m\ln m\right] \leq \frac{1}{2}h_E^2.$$

4. Two numerical examples

4.1. The relationship between one-period and multi-period model

Let us have a filtered probability space $(\Omega, \mathcal{F}, \mathbb{P}, \{\mathcal{F}_t\}_{t=0,1,\dots,T})$ with $\mathrm{E}_t[.]$ denoting the expectation conditional on the information at time t. We assume Ω finite. There are n risky securities with $\mathbb{R}^n$-valued processes S and D denoting their price and dividends respectively in money terms. Suppose that there is *short-term riskless borrowing* at a bounded rate r_t, that is an agent can borrow one unit of the numeraire in period t at the known rate r_t and repay $(1 + r_t) > 0$ units of the numeraire in the next period. It is natural to assume that the filtration $\{\mathcal{F}_t\}_{t=0,1,\dots,T}$ is generated by the processes S, D, r.

Let θ be an $\mathbb{R}^n$-valued portfolio process for 'risky securities'. If an agent uses self-financing strategies her wealth V_t evolves over time as follows

$$V_t = (1+r_{t-1})V_{t-1} + \theta_{t-1}X_t \quad (40)$$
$$X_t = S_t + D_t - (1+r_{t-1})S_{t-1}, \quad (41)$$

where $\frac{X_t}{S_{t-1}}$ can be interpreted as excess return when $S_{t-1} \neq 0$. No arbitrage means that there is a strictly positive $\mathcal{F}_t$-measurable variable[5] $m_{t|t-1}$ with $\mathrm{E}_{t-1}\left[m_{t|t-1}\right] = 1$ such that

$$\mathrm{E}_{t-1}\left[m_{t|t-1}X_t\right] = 0 \ ,$$

that is with artificial probabilities defined by $m_{t|t-1}$ the discounted wealth process is a martingale between $t-1$ and t

$$\mathrm{E}_{t-1}\left[m_{t|t-1}V_t\right] = (1+r_{t-1})V_{t-1} \ .$$

Now if we define unconditional change of measure m_T as

$$m_T = m_{1|0} \times m_{2|1} \times \ldots \times m_{T|T-1}$$

then from the law of iterated expectations $\mathrm{E}_0\left[m_T\right] = 1$ and we can define a new probability measure Q

$$\frac{dQ}{dP} = m_T \ .$$

It is useful to note that the density process m_t

$$m_t \equiv \mathrm{E}_t\left[m_T\right] = m_{1|0} \times m_{2|1} \times \ldots \times m_{t|t-1}$$

is related to the conditional change of measure as follows

$$m_{t|t-1} = \frac{m_t}{m_{t-1}}. \quad (42)$$

In a dynamic model the good-deal equilibria can be imposed in two ways, either as instantaneous restrictions of the Cochrane and Saá-Requejo type where the price bounds are evaluated in every period, or

[5] The variable $m_{t|t-1}$ can be visualised as the ratio between one-step risk-neutral probabilities and one-step objective probabilities at every node of a multinomial tree at time $t-1$. The ratio $\frac{m_{t|t-1}}{1+r_{t-1}}$ is known under a score of names: Intertemporal Marginal Rate of Substitution, stochastic discount factor, pricing kernel, or state price density.

Since there are finitely many securities the marketed subspace is finite dimensional and then by Theorem 6 in (Clark, 1993) a strictly positive valuation operator exists which is nothing else than the conditional change of measure $m_{t|t-1}$.

as unconditional bounds whereby one assumes a fixed position in the focus asset at the beginning and thereafter only dynamically trades in the basis assets, as in Hodges (1998). In this paper we will discuss the former approach.

By C_T let us denote an $\mathcal{F}_T$-measurable random variable representing the payoff of a derivative security. We say that the $\mathcal{F}$-adapted processes $\{C_t^H(\bar{a})\}$, $\{C_t^L(\bar{a})\}$ defined by

$$\begin{aligned}
C_t^H(\bar{a}) &\triangleq \sup\{p | a_t \left(X_{t+1}, C_{t+1}^H - (1+r)p\right) \leq \bar{a}\} \\
C_t^L(\bar{a}) &\triangleq \inf\{p | a_t \left(X_{t+1}, C_{t+1}^L - (1+r)p\right) \leq \bar{a}\} \\
C_T^L(\bar{a}) &\triangleq C_T^H(\bar{a}) \triangleq C_T
\end{aligned}$$

are the *instantaneous good-deal bounds*. From the Theorem 3 we have

$$\begin{aligned}
C_t^H &= \sup\left\{ \frac{\mathrm{E}_t\left[m_{t+1|t} C_{t+1}^H\right]}{1+r_t} \middle| \mathrm{E}_t\left[m_{t+1|t} X_{t+1}\right] = 0, a_t(m_{t+1|t}) \leq \bar{a} \right\} \\
C_t^L &= \inf\left\{ \frac{\mathrm{E}_t\left[m_{t+1|t} C_{t+1}^L\right]}{1+r_t} \middle| \mathrm{E}_t\left[m_{t+1|t} X_{t+1}\right] = 0, a_t(m_{t+1|t}) \leq \bar{a} \right\}.
\end{aligned}$$

Here the one-step conditional change of measure $m_{t+1|t}$ assumes the role of m from the one-period model and $a_t(X_{t+1})$ is defined in the natural way from (10)

$$U[(1+r_t)V_t + a_t(X_{t+1})] \triangleq \sup_{\theta_t} \mathrm{E}_t\left[U\left((1+r_t)V_t + \theta_t X_{t+1}\right)\right].$$

4.2. Pricing with Logarithmic Sharpe Ratio

This is a simple example set up in such a way that the price bounds can be computed in Excel[6] without using Visual Basic. Consider a model with a constant risk-free rate $r = 5\%$ p.a. where the expected rate of return on the stock is 10% p.a. and annual volatility is 20%. The stock price moves in a recombining trinomial lattice calibrated to the stated volatility and expected return with logarithmic upstep $u = 0.035$. Each time period represents one week and stock returns are by assumption independent. Our aim is to price an at-the-money European call option with strike price $K = 100$ and 6 weeks to maturity. The calibrated objective probabilities of movement in the lattice are $p_1 = 0.348, p_2 = 0.350, p_3 = 0.302$ for the upstep, middle and downstep respectively.

[6] Spreadsheet available from author's web site.

We assume that the above model is a true representation of stock price movements rather than an approximation to a diffusion model. Then, in the absence of other securities, the market is incomplete and the no-arbitrage price of the option is not unique. More specifically, the risk-neutral probabilities $q = (q_1, q_2, q_3)$ have one free parameter, and satisfy

$$\begin{pmatrix} q_1(\alpha) \\ q_2(\alpha) \\ q_3(\alpha) \end{pmatrix} = \begin{pmatrix} 0.341 \\ 0.333 \\ 0.325 \end{pmatrix} + \alpha \begin{pmatrix} 0.378 \\ -0.770 \\ 0.392 \end{pmatrix}$$

with $-0.830 < \alpha < 0.433$ parametrizing the range of no-arbitrage pricing kernels.

The maximum logarithmic Sharpe Ratio in the absence of the option can be found by *minimizing*[7] the central expression in equation (22)

$$\min_{\substack{\alpha \\ -0.830<\alpha<0.433}} \quad -\sum_{i=1}^{3} p_i \ln \frac{q_i(\alpha)}{p_i}$$

which gives $\hat{\alpha} = -0.0224$, $\hat{q} = (0.3329, 0.3505, 0.3166)$ and $-\sum_{i=1}^{3} p_i \ln \frac{q_i(\hat{\alpha})}{p_i} = 0.00065$. From expression (38) the basis logarithmic Sharpe Ratio is

$$h_{1\,basis} = \sqrt{\exp\left(-2\sum_{i=1}^{3} p_i \ln \frac{q_i(\hat{\alpha})}{p_i}\right) - 1} = 0.0361$$

weekly, equivalent to 0.573 per annum.

To decide which discount factors are admissible in equilibrium after the option is introduced, we must decide what level of Sharpe Ratio constitutes a good deal. One can either target an absolute level of Sharpe Ratio, say 2.0 p.a., or use a relative measure of c times the basis Sharpe Ratio, that is only those risk-neutral probabilities are admissible which satisfy

$$-\sum_{i=1}^{3} p_i \ln \frac{q_i(\alpha)}{p_i} \le \ln(1 + (c\, h_{1\,basis})^2). \tag{43}$$

We take $c = 2$ and find numerically

$$-0.0615 \le \alpha \le 0.0157. \tag{44}$$

[7] Alternatively, one can solve the primal portfolio problem

$$\max_{\beta} \mathrm{E} \ln \left[\beta R + (1-\beta) R^f\right]$$

where R is the risky return and R^f is the risk-free return.

The admissible risk-neutral probabilities are a convex combination of vectors q_L and q_U corresponding to the lower and upper bound on α in (44)

$$q_L = \begin{pmatrix} 0.3181 \\ 0.3806 \\ 0.3013 \end{pmatrix} \quad q_U = \begin{pmatrix} 0.3473 \\ 0.3211 \\ 0.3315 \end{pmatrix}. \tag{45}$$

With this range of discount factors we can price our option, bearing in mind that at every node of the lattice we have to keep track of the highest and lowest no-good-deal price C_t^H and C_t^L

$$\begin{aligned} C_t^H &= \frac{\max(\mathrm{E}_t^{q_L}\left[C_{t+1}^H\right], \mathrm{E}_t^{q_U}\left[C_{t+1}^H\right])}{1.00407} \\ C_t^L &= \frac{\min(\mathrm{E}_t^{q_L}\left[C_{t+1}^L\right], \mathrm{E}_t^{q_U}\left[C_{t+1}^L\right])}{1.00407} \\ C_5^H &= C_5^L = (S_5 - K)^+ \end{aligned}$$

The results are reported in a spreadsheet (Figure 5) with the middle price being the unique price which would result from taking $c = 1$. This price *coincides with representative equilibrium price* of the option for a representative agent with logarithmic utility of terminal wealth.

It is interesting to note that at $t = 5$ the option is a redundant asset in all states but one. The effect of this state, however, spreads quickly and at $t = 2$ the option is not redundant in any state. The option price bounds for different values of c are summarised in Table IV. The value $c = +\infty$ corresponds to the no-arbitrage (super-replication) bounds.

Table IV. No-good-deal option price bounds

multiple of basis GSR	$c = 1$	$c = 2$	$c = 4$	$c = 10$	$c = +\infty$
implied GSR p.a. (h_1)	0.57	1.15	2.29	5.73	$+\infty$
C_0^L	3.02	2.95	2.86	2.60	0.58
C_0^H	3.02	3.08	3.16	3.33	3.57

4.2.1. *Graphical representation of good-deal state prices*

The good-deal discount factors corresponding to different values of c are displayed in Figure 6. The triangle contains all no-arbitrage risk-neutral probabilities for the three states, with the objective probability corresponding to the point P. The risk-neutral probability measures

t=0	t=1	t=2	t=3	t=4	t=5	t=6
						23.37
						23.37
						23.37
					19.22	19.12
					19.22	19.12
					19.22	19.12
				15.22	15.12	15.03
				15.22	15.12	15.03
				15.22	15.12	15.03
			11.36	11.26	11.17	11.07
			11.36	11.26	11.17	11.07
			11.36	11.26	11.17	11.07
		7.88	7.67	7.44	7.35	7.25
		7.86	7.65	7.44	7.35	7.25
		7.84	7.63	7.44	7.35	7.25
	5.09	4.79	4.46	4.13	3.66	3.56
	5.04	4.75	4.43	4.10	3.66	3.56
	4.99	4.70	4.39	4.07	3.66	3.56
3.08	2.79	2.46	2.11	1.67	1.24	0.00
3.02	2.73	2.41	2.06	1.63	1.18	0.00
2.95	2.66	2.35	2.01	1.59	1.13	0.00
	1.28	1.01	0.72	0.43	0.00	0.00
	1.23	0.96	0.68	0.39	0.00	0.00
	1.17	0.92	0.64	0.36	0.00	0.00
		0.30	0.149	0.00	0.00	0.00
		0.27	0.13	0.00	0.00	0.00
		0.25	0.11	0.00	0.00	0.00
			0.00	0.00	0.00	0.00
			0.00	0.00	0.00	0.00
			0.00	0.00	0.00	0.00
				0.00	0.00	0.00
				0.00	0.00	0.00
				0.00	0.00	0.00
					0.00	0.00
					0.00	0.00
					0.00	0.00
						0.00
						0.00
						0.00

Figure 5. Option price bounds with $c = 2$.

which give less than 4 times the basis logarithmic Sharpe Ratio, that is those which satisfy equation (43) with $c = 4$, are contained in the oval area[8] σ_2 and those which only give double of the basis Sharpe Ratio

[8] Note that, unlike in the case of bounded utility functions, the no-good-deal state prices derived from the log utility are *strictly inside* the no-arbitrage triangle for all $c < +\infty$. Consequently, the no-good-deal price bounds are strictly sharper than the no-arbitrage price bounds for all $c < +\infty$.

are within the smaller oval area σ_1. The segment A_1A_2 contains all the no-arbitrage risk-neutral measures that are consistent with the stock returns, and among those measures segments B_1B_2 and C_1C_2 represent the good-deal risk-neutral probabilities consistent with $c = 4$ and $c = 2$ respectively.

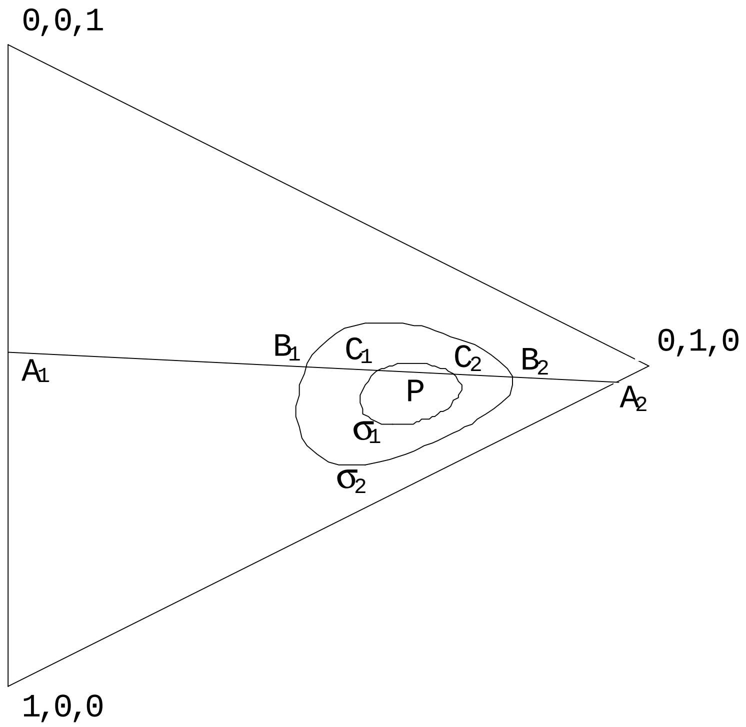

Figure 6. Admissible good-deal risk-neutral probabilities. Points C_1 and C_2 correspond to q_L and q_H from equation (45).

4.3. FTSE 100 Equity Index Option pricing

This is a heavy duty version of the trinomial tree model above. Here we use a 50-nomial tree calibrated to historical weekly returns of FTSE 100 index in the period 2/1/84 to 1/11/2001. We use a range of CRRA utility functions with $\gamma = \pm 0.25, \pm 0.5, \pm 1, \pm 2, \pm 5, \pm 50$. Starting value of the index is 5100 and the call option is 5% out of the money with six weeks to maturity and hedging once a week. Unlike the trinomial implementation, this model has a high degree of incompleteness, consequently the bounds are computed directly from the primal util-

gamma 0.25, basis GSR 0.228					
multiple of basis GSR	1	2	4	10	infty
lower price bound	32.44	28.78	25.30	17.56	0.00
upper price bound	32.44	39.61	58.83	156.46	474.71

gamma -0.25, basis GSR 0.229					
multiple of basis GSR	1	2	4	10	infty
lower price bound	31.84	25.38	11.07	0.00	0.00
upper price bound	31.84	36.09	40.52	53.25	474.71

gamma 0.5, basis GSR 0.230					
multiple of basis GSR	1	2	4	10	infty
lower price bound	32.29	28.42	24.56	16.52	0.00
upper price bound	32.29	37.60	48.57	105.75	474.71

gamma -0.5, basis GSR 0.230					
multiple of basis GSR	1	2	4	10	infty
lower price bound	32.00	26.81	17.80	0.00	0.00
upper price bound	32.00	36.38	41.33	54.24	474.71

gamma 1, basis GSR 0.230					
multiple of basis GSR	1	2	4	10	infty
lower price bound	32.22	28.19	24.01	15.28	0.00
upper price bound	32.22	37.11	44.89	78.60	474.71

gamma -1, basis GSR 0.230					
multiple of basis GSR	1	2	4	10	infty
lower price bound	32.08	27.49	22.22	10.80	0.00
upper price bound	32.08	36.57	41.99	56.58	474.71

gamma 2, basis GSR 0.230					
multiple of basis GSR	1	2	4	10	infty
lower price bound	32.18	28.05	23.67	14.39	0.00
upper price bound	32.18	36.94	43.77	68.47	474.71

gamma -2, basis GSR 0.230					
multiple of basis GSR	1	2	4	10	infty
lower price bound	32.11	27.70	22.76	12.08	0.00
upper price bound	32.11	36.68	42.43	58.72	474.71

gamma 5, basis GSR 0.230					
multiple of basis GSR	1	2	4	10	infty
lower price bound	32.16	27.96	23.43	13.77	0.00
upper price bound	32.16	36.85	43.27	64.29	474.71

gamma -5, basis GSR 0.230					
multiple of basis GSR	1	2	4	10	infty
lower price bound	32.13	27.77	23.06	12.83	0.00
upper price bound	32.13	36.75	42.75	60.60	474.71

gamma 50, basis GSR 0.230					
multiple of basis GSR	1	2	4	10	infty
lower price bound	32.15	27.90	23.28	13.36	0.00
upper price bound	32.15	36.80	43.02	62.41	474.71

gamma -50, basis GSR 0.230					
multiple of basis GSR	1	2	4	10	infty
lower price bound	32.15	26.33	22.23	12.77	0.00
upper price bound	32.15	36.79	42.97	62.05	474.71

Figure 7. FTSE 100 equity index option price bounds implied by levels of CRRA Generalised Sharpe Ratios for different values of γ.

ity maximisation problem. The required numerical procedures were implemented in GAUSS[9].

Figure 7 summarises the option price bounds for different values of γ. It is striking how robust these results are with respect to changes in γ, particularly for low levels of Sharpe Ratio and for $|\gamma| \geq 1$. For example, at double the basis Sharpe Ratio (roughly 0.5 p.a.) the Cochrane and Saá-Requejo bounds are ($\gamma = -1$) $[27.49, 36.57]$, log-utility bounds ($\gamma = 1$) are [28.19,37.11], for $\gamma = 5$ the bounds are $[27.96, 36.85]$, for truncated bicubic utility ($\gamma = -5$) the bounds are $[27.77, 36.75]$ etc. Figure 8 describes the bounds implied by the negative exponential util-

[9] Code available from author's website.

ity. Again the results point at robustness of Generalised Sharpe Ratios; at double the basis Sharpe Ratio we obtain the bounds $[27.89, 36.81]$.

Exponential utility basis GSR 0.230					
multiple of basis GSR	1	2	4	10	infty
lower price bound	32.15	27.89	23.21	12.78	0.00
upper price bound	32.15	36.81	43.07	63.57	474.71

Figure 8. FTSE 100 equity index option price bounds implied by levels of CARA Generalised Sharpe Ratio

The price bounds appear to be largely invariant to the choice of utility function. An interesting open question is how tight would the bounds become with shorter rehedging intervals. Here we mean a limit with jumps; it is well known that in a diffusion limit with independent and identically distributed returns the bounds collapse to the Black-Scholes price.

5. Continuous time Brownian motion setting

In continuous time it is convenient to define a cumulative return on one unit of the numeraire invested in the bank account at the beginning and thereafter rolled over until time t

$$R_t = \exp\left(\int_0^t r_t dt\right).$$

The self-financing condition is written as

$$d\frac{V_t}{R_t} = d\frac{p_t}{R_t} + \frac{D_t}{R_t}dt, \tag{46}$$

and it is convenient to introduce the discounted gain process G

$$G_t = \frac{p_t}{R_t} + \int_0^t \frac{D_s}{R_s}ds.$$

Suppose that the discounted gain process is an Itô process with stochastic differential equation

$$dG_t = \mu_t dt + \sigma_t dB_t^P$$

where B_t^P stands for a vector of s uncorrelated Brownian motions under objective probability measure P.

The trick of risk-neutral pricing is to write dG_t as

$$dG_t = \sigma_t(\nu_t dt + dB_t^P)$$

and then set

$$dB_t^Q = \nu_t dt + dB_t^P \ .$$

The process ν_t is known as the *market price of risk*. It is a known result that the density process[10] m_t for the unconditional change of measure $m_T = \frac{dQ}{dP}$ under which B_t^Q is a martingale[11] is given as

$$m_t = \exp\left[-\frac{1}{2}\int_0^t ||\nu_s||^2 ds - \int_0^t \nu_s dB_s^P\right]. \tag{47}$$

By analogy to equation (42) we have

$$m_{t+dt|t} = \frac{m_{t+dt}}{m_t} = \exp[-\frac{1}{2}\int_t^{t+dt} ||\nu_s||^2 ds - \int_t^{t+dt} \nu_s dB_s^P], \tag{48}$$

that is the conditional change of measure is roughly speaking a lognormal variable.

5.1. Instantaneous no-good-deal restrictions

PROPOSITION 6. *The market price of risk ν_t does not admit Sharpe ratio of more than $h\sqrt{dt}$ between time t and $t + dt$ if and only if*

$$||\nu_t||^2 \leq h^2 \tag{49}$$

PROPOSITION 7. *The market price of risk ν_t does not admit certainty equivalent gain of more than adt for a utility function U from time t until time $t + dt$ if and only if*

$$\frac{1}{2}||\nu_t||^2 \leq A(V_t)a \tag{50}$$

where $A(V_t) = -\frac{U''(V_t)}{U'(V_t)}$ is the coefficient of absolute risk aversion.

---

[10] The density process m_t and the discount factor Λ_t used in Cochrane and Saá-Requejo are related through $\Lambda_t = \frac{m_t}{R_t}$.

[11] The no-good-deal restrictions derived in (49) guarantee that the Novikov condition

$$\mathrm{E}_0 \exp\left[\int_0^T ||\nu_t||^2 dt\right] < +\infty$$

is satisfied and hence the density process m_t is a martingale as required.

Proof. The proofs are stated in Appendix C. ■

Since our analysis was performed for small Sharpe Ratios and small certainty gains it is natural that the bounds in restrictions (49) and (50) correspond via (25). Proposition 7 shows that with Itô price processes instantaneous restrictions coincide for all utility functions and therefore for all Generalised Sharpe Ratios.

6. Time scaling of maximum attainable Sharpe Ratio

An interesting question is how the instantaneous no-good-deal restrictions affect availability of high Sharpe Ratios over a longer time horizon[12]. We will limit our attention to Hodges's Exponential Sharpe Ratio on the one hand and the CRRA family of Generalised Sharpe Ratios, on the other hand. For these two cases we have

$$\begin{aligned} \mathrm{E}_{t-1}\left[m_{t|t-1}\ln m_{t|t-1}\right] &\leq \frac{1}{2}h_E^2 \\ \mathrm{E}_{t-1}\left[m_{t|t-1}^{\frac{\gamma-1}{\gamma}}\right] &\leq \left(1+h_\gamma^2\right)^{\frac{1-\gamma}{2\gamma^2}}. \end{aligned}$$

Recall that the Arbitrage-Adjusted Sharpe Ratio (truncated quadratic utility) is a special case with $\gamma=-1$. For simplicity the risk-free interest rate is assumed to be 0.

PROPOSITION 8. *If the maximum Exponential Sharpe Ratio attainable over a short period dt is $h_E\sqrt{dt}$ then the maximum attainable Exponential Sharpe Ratio over T periods is $h_E\sqrt{T}$.*

Proof. The best attainable deal over time interval $[0,T]$ is bounded from above by

$$\mathrm{E}_0\left[m_T\ln m_T\right].$$

This expression can be written equivalently as

$$\mathrm{E}_0\left[m_T\ln m_{T-\triangle t}+m_{T-\triangle t}\frac{m_T}{m_{T-\triangle t}}\ln\frac{m_T}{m_{T-\triangle t}}\right]$$

[12] It is of course plausible that the actual upper bound on the long run Sharpe Ratios is *lower* than the one implied by the instantaneous Sharpe Ratio restrictions.

and using the law of iterated expectations we have

$$\mathrm{E}_0\left[m_T \ln m_T\right] =$$
$$= \mathrm{E}_0\left[\mathrm{E}_{T-\triangle t}\left[m_T \ln m_{T-\triangle t}\right] + m_{T-\triangle t}\mathrm{E}_{T-\triangle t}\left[m_{T|T-\triangle t}\ln m_{T|T-\triangle t}\right]\right] =$$
$$\leq \mathrm{E}_0\left[m_{T-\triangle t}\ln m_{T-\triangle t} + m_{T-\triangle t}\frac{1}{2}h_E^2\triangle t\right] =$$
$$= \frac{1}{2}h_E^2\triangle t + \mathrm{E}_0\left[m_{T-\triangle t}\ln m_{T-\triangle t}\right]$$

By induction then

$$\mathrm{E}_0\left[m_T \ln m_T\right] < \frac{1}{2}h_E^2 T$$

■

PROPOSITION 9. *If the maximum γ-Sharpe Ratio attainable over a short time dt is $h_\gamma\sqrt{dt}$ the maximum attainable γ-SR over T periods is $\sqrt{\exp[h_\gamma^2 T]-1}$.*

Proof. The best attainable deal over time interval $[0, T]$ is determined by

$$\mathrm{E}_0\left[m_T^{\frac{\gamma-1}{\gamma}}\right] = \mathrm{E}_0\left[m_{\triangle t|0}^{\frac{\gamma-1}{\gamma}} m_{2\triangle t|\triangle t}^{\frac{\gamma-1}{\gamma}}\dots m_{T-\triangle t|T-2\triangle t}^{\frac{\gamma-1}{\gamma}} m_{T|T-\triangle t}^{\frac{\gamma-1}{\gamma}}\right] =$$
$$= \mathrm{E}_0\left[m_{\triangle t|0}^{\frac{\gamma-1}{\gamma}}\mathrm{E}_{\triangle t}\left[m_{2\triangle t|\triangle t}^{\frac{\gamma-1}{\gamma}}\dots\mathrm{E}_{T-2\triangle t}\left[m_{T-\triangle t|T-2\triangle t}^{\frac{\gamma-1}{\gamma}}\mathrm{E}_{T-\triangle t}\left[m_{T|T-\triangle t}^{\frac{\gamma-1}{\gamma}}\right]\right]\right]\dots\right] \leq$$
$$\leq \left((1+h_\gamma^2\triangle t)^{\frac{T}{\triangle t}}\right)^{\frac{1-\gamma}{\gamma^2}} \to \left(\exp[h_\gamma^2 T]\right)^{\frac{1-\gamma}{\gamma^2}}$$

This also shows that all CRRA Generalised Sharpe Ratios have the same time scaling property. ■

Figure 9 compares the long run Sharpe Ratio restrictions implied by the maximum instantaneous Sharpe Ratio equal to 1. The instantaneous Exponential Sharpe Ratio provides a sharper bound on the attractiveness of a long term investment.

7. Limiting cases of good-deal price bounds

From the identity $\sigma_t\nu_t = \mu_t$ it follows that the market price of risk has a unique decomposition

$$\begin{aligned}\nu_t &= \eta_t + \psi_t\\ \eta_t &= \sigma_t^*(\sigma_t\sigma_t^*)^{-1}\mu_t\\ \eta_t^*\psi_t &= 0.\end{aligned}$$

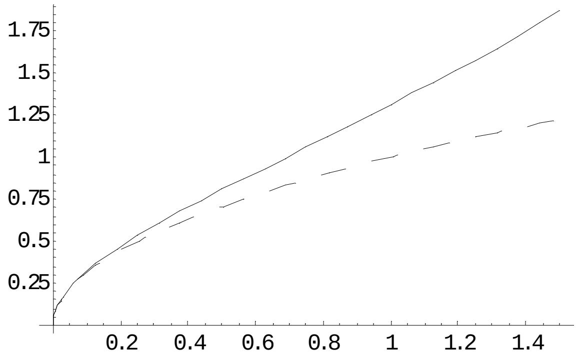

Figure 9. Maximum Exponential Sharpe Ratio (dashed line) and Arbitrage–Adjusted Sharpe Ratio (solid line) implied by instantaneous restrictions as a function of investment horizon. Instantaneous Sharpe Ratio limit set equal to 1.

From here we can see that

$$||\nu_t||^2 = ||\eta_t||^2 + ||\psi_t||^2 \tag{51}$$

and η can be naturally called the minimal market price of risk[13]. The minimal market price of risk naturally defines the minimal martingale measure via (47).

The following proposition asserts that the good-deal price bounds obtained from instantaneous state price restrictions lie between the unique price determined by the minimal martingale measure and the no-arbitrage super-replication bounds.

PROPOSITION 10. *Consider a contingent claim C_T and let us denote $C_{NA}^{\min}$ and $C_{NA}^{\max}$ respectively its no-arbitrage price bounds, $C_{NGD}^{\min}(h)$ and $C_{NGD}^{\max}(h)$ respectively its no-good-deal price bounds corresponding to maximum instantaneous Sharpe Ratio h, and C_0 its price determined by the minimal martingale measure. Then*

$$C_{NA}^{\min} \leq C_{NGD}^{\min}(h) \leq C_0 \leq C_{NGD}^{\max}(h) \leq C_{NA}^{\max}$$

and

$$\begin{aligned}
\lim_{||\psi||\to 0} C_{NGD}^{\min}(h) &= \lim_{||\psi||\to 0} C_{NGD}^{\max}(h) = C_0 \\
\lim_{h\to\infty} C_{NGD}^{\min}(h) &= C_{NA}^{\min} \\
\lim_{h\to\infty} C_{NGD}^{\max}(h) &= C_{NA}^{\max}
\end{aligned}$$

[13] The minimal market price of risk defines the minimal martingale measure via (), see Schweizer (1991).

Proof. The relationship between good-deal price bounds and no-arbitrage price bounds can be read off from Theorem 3.1.1 of El Karoui and Quenez (1995). As for the relationship with the minimal martingale measure, the martingale representation theorem under the minimal martingale measure allows us to write the contingent claim C_T uniquely as

$$\begin{aligned} C_T &= C_0 + \int_0^T \vartheta_t dG_t + \int_0^T \lambda_t dB_t, \\ \lambda_t \sigma_t^* &= 0. \end{aligned}$$

Using the Itô formula we find the expectation of C_T under an arbitrary equivalent martingale measure Q such that $\frac{dQ}{dP} = m_T$

$$\mathrm{E}_0 [m_T C_T] = C_0 - \mathrm{E}_0 \left[\int_0^T m_t \lambda_t^* \psi_t dt \right],$$

where

$$\begin{aligned} dm_t &= -m_t(\eta_t + \psi_t) dB_t \\ m_0 &= 1. \end{aligned}$$

Consequently the lower no-good-deal price bound is obtained as

$$C_{NGD}^{\min}(h) = \min_{||\psi_t||^2 \leq h_t^2 - ||\eta_t||^2} C_0 - \mathrm{E}_0 \left[\int_0^T m_t \lambda_t^* \psi_t dt \right] \leq C_0.$$

At the same time as $h_t \searrow ||\eta_t||$ we have $||\psi_t|| \to 0$ and $C_{NGD}^{\min}(h) \to C_0$. Analogous argument applies to the upper bound. ■

It is interesting to note that the minimal martingale measure has already been used to price non-redundant claims under stochastic volatility in Hofmann et al. (1992). For a closely related concept of local utility maximisation and neutral prices see Kallsen (2002).

8. Conclusions

The paper provides a generalisation of the incomplete market pricing technique of Cochrane and Saá-Requejo (2000) to good deals defined by an arbitrary (increasing) smooth utility function. We have derived the corresponding discount factor restrictions and linked these restrictions to the availability of Sharpe Ratios and Generalised Sharpe Ratios. In particular, we have extended the definition of the Sharpe Ratio from quadratic utility to the entire family of CRRA utility functions and given a number of numerical examples that demonstrate robustness of Generalised Sharpe Ratios. It is the author's conviction that the Generalised Sharpe Ratios, thanks to their ability to handle skewed asset returns, will become an indispensable performance evaluation tool for modern portfolio managers. Last but not least, we have shown that for Itô price processes the instantaneous good-deal price bounds coincide for all reward-for-risk measures.

Appendix

A. Proofs of Theorems 2 and 3

LEMMA 11. *Suppose* $|\Omega| < \infty$, *and* U *satisfies* $\lim_{x\to-\infty} \frac{x}{U(x)} = 0$. *Every unbounded sequence of desirable claims has a subsequence with a common direction, and this direction is strictly positive. Mathematically, if for a fixed* $a > 0$ *we have* $\mathrm{E}\left[U(V_0 + x_n)\right] \geq \mathrm{E}\left[U(V_0 + a)\right]$ *for all* n *and* $||x_n|| \to \infty$, *then there is* $z \geq 0$, $\Pr(z > 0) > 0$ *and a subsequence of* $\{x_n\}$ *such that* $\left\{\frac{x_n}{||x_n||}\right\} \to z$.

Proof. *Unit ball in a finite-dimensional space is compact therefore* $\left\{\frac{x_n}{||x_n||}\right\}$ *must have a convergent subsequence. Denote its limit* z. *By Lemma B.1 in* Černý and Hodges *(*2001*)* $z \geq 0$, $\Pr(z > 0) > 0$.

Proof of Theorem 2

1). By M_0 denote the subspace of marketed excess returns

$$M_0 \triangleq \{\theta X | \theta \in \mathbb{R}^n\}$$

For $Z \in M_0$ define $a(Z) \in \mathbb{R}$ implicitly from

$$U(V_0 + a(Z)) \triangleq \mathrm{E}\left[U(V_0 + Z)\right].$$

Set

$$\begin{aligned} \bar{a} &\triangleq \sup_{Z\in M_0} a(Z) \\ 0 &< \bar{a} \leq +\infty. \end{aligned}$$

By definition of supremum there is a sequence of marketed excess returns $\{Z_n\}$ such that $\{a(Z_n)\} \to \bar{a} > 0$. For large enough n we will have $a(Z_n) > \min(\frac{\bar{a}}{2}, 1)$ which means that $\{Z_n\}$ is a sequence of desirable claims. If $\{Z_n\}$ were unbounded, by Lemma 11 we could find an arbitrage excess return z and a subsequence $\left\{\frac{Z_n}{||Z_n||}\right\} \to z$. The marketed subspace M_0 is finite dimensional and therefore closed, furthermore $\frac{Z_n}{||Z_n||} \in M_0$, implying $z \in M_0$ which contradicts the no-arbitrage assumption. Thus $\{Z_n\}$ must be bounded. Then it must have a convergent subsequence $\{Z_n\} \to z \in M_0$. The function $a : \mathbb{R}^m \to \mathbb{R}$ is continuous which implies $a(z) = \bar{a} < +\infty$. We continue by proving two key results, 1.5a) and 1.5b).

1.5) Define

$$\begin{aligned} f(\theta, \lambda, p) &\triangleq \mathrm{E}\left[U\left((1+r)V_0 + \theta X + \lambda\left(Y - (1+r)p\right)\right)\right] \\ g(p) &\triangleq \max_{\theta,\lambda} f(\theta, \lambda, p). \end{aligned}$$

Property 1) guarantees existence of θ_{basis} such that

$$\max_{\theta} f(\theta, 0, .) = f(\theta_{basis}, 0, .).$$

Moreover, by the argument in the proof of Theorem 4.1 c) in Černý and Hodges Y has a unique price p_{basis} such that

$$g(p_{basis}) = \max_{\theta,\lambda} f(\theta, \lambda, p_{basis}) = f(\theta_{basis}, 0, p).$$

with

$$\mathrm{E}\left[(Y - (1+r)p_{basis})\, U'\left((1+r)V_0 + \theta_{basis}X\right)\right] = 0. \qquad (52)$$

Y is non-redundant if and only if it has a range of no-arbitrage prices $P_\infty(Y) \triangleq (p_{-\infty}, p_\infty)$ such that $p_{-\infty} < p_\infty$ (this is a consequence of Theorem 4.1 in Černý and Hodges). From (52) we know that $p_{basis} \in (p_{-\infty}, p_\infty)$ in cases U1) and U2), and $p_{basis} \in [p_{-\infty}, p_\infty]$ in the case U3). We claim that 1.5a) $g(p)$ is strictly decreasing on $(p_{-\infty}, p_{basis}]$ and strictly increasing on $[p_{basis}, p_\infty)$,and 1.5b) $g(p)$ is continuous on $P_\infty(Y) \cup p_{basis}$.

1.5a) i) For $p_{-\infty} < p < p_{basis}$ we have $g(p) > g(p_{basis})$. To show this define $h(\lambda) \triangleq f(\theta_{basis}, \lambda, p)$. Then by virtue of (52)

$$h'(0) = (1+r)\left(p_{basis} - p\right) \mathrm{E}\left[U'\left((1+r)V_0 + \theta_{basis}X\right)\right] > 0. \qquad (53)$$

by strict monotonicity in cases U1) U2). In case U3) (53) still holds, because by assumption U3) in Definition 1 $\mathrm{E}\left[U'\left((1+r)V_0 + \theta_{basis}X\right)\right] = 0$ would imply $\theta_{basis}X > 0$,which would mean arbitrage among basis assets.

By Theorem 1.30 in Beavis and Dobbs (1990) U is continuously differentiable and therefore h is continuously differentiable. A Taylor expansion of the form

$$h(\lambda) = h(0) + \lambda h'(\xi\lambda) \text{ for } 0 < \xi < 1$$

shows that for sufficiently small $\lambda > 0$ we have $h(\lambda) > h(0)$ and consequently

$$g(p) \geq h(\lambda) > h(0) = g(p_{basis}).$$

1.5a) ii) Now take $p_{-\infty} < p_1 < p_2 < p_{basis}$. By 1. and 2. there are θ_2 and λ_2 such that

$$g(p_2) = f(\theta_2, \lambda_2, p_2). \tag{54}$$

We claim $\lambda_2 > 0$, arguing by contradiction. $\lambda_2 \leq 0$ together with monotonicity of U imply that $f(\theta_2, \lambda_2, p_2)$ is non-decreasing in p_2

$$g(p_2) = f(\theta_2, \lambda_2, p_2) \leq f(\theta_2, \lambda_2, p_{basis}) \leq g(p_{basis})$$

which contradicts 1.5a) i). With $\lambda_2 > 0$ $f(\theta_2, \lambda_2, p_2)$ is a strictly decreasing function of p_2 (in case U3 we again appeal to Definition 1) and therefore

$$g(p_2) = f(\theta_2, \lambda_2, p_2) < f(\theta_2, \lambda_2, p_1) \leq g(p_1).$$

The proof for $p_{basis} < p_1 < p_2 < p_\infty$ proceeds symmetrically.

1.5b) Take $p_{-\infty} < p \leq p_{basis}$. Assume by contradiction $\lim_{p_n \to p+} g(p_n) < g(p)$. By 1) there must be θ, λ such that $g(p) = f(\theta, \lambda, p)$. Since f is continuous in p we have $g(p) = \lim_{p_n \to p+} f(\theta, \lambda, p_n) \leq \lim_{p_n \to p+} g(p_n)$, a contradiction. Assume now $\lim_{p_n \to p-} g(p_n) = g(p) + \delta$ with $\delta > 0$. By 1) there is a sequence $\{\theta_n, \lambda_n\}$ such that $g(p_n) = f(\theta_n, \lambda_n, p_n)$, and by 1.5a) ii) $\lambda_n > 0$. Fix $\varepsilon > 0$ such that $p - \varepsilon > p_{-\infty}$. For sufficiently large n we have $p_n > p - \varepsilon$ and $g(p_n) > g(p)$ and hence $g(p) < f(\theta_n, \lambda_n, p_n) < f(\theta_n, \lambda_n, p - \varepsilon)$. Therefore $\{\theta_n X + \lambda_n (Y - (1+r)(p-\varepsilon)\}$ define a sequence of desirable claims. If this sequence were unbounded by Lemma 11 a subsequence would have a strictly positive common direction implying arbitrage in the market with excess returns $X, Y - (1+r)(p-\varepsilon)$, which would contradict $p - \varepsilon > p_{-\infty}$. Hence the sequence of desirable claims must be bounded, without loss of generality this implies $\{\theta_n, \lambda_n\}$ bounded. Consequently $\{\theta_n, \lambda_n\}$ has a convergent subsequence with limit θ, λ. Since f is a continuous function of θ and λ we have

$$\liminf_{p_n \to p-} g(p_n) = \liminf_{p_n \to p-} f(\theta_n, \lambda_n, p_n) < \lim_{p_n \to p-} f(\theta_n, \lambda_n, p - \varepsilon) = f(\theta, \lambda, p - \varepsilon)$$

Note that the sequence $\{\theta_n, \lambda_n\}$ is independent of the choice of ε and therefore θ, λ can be chosen independently of ε. This means for any small ε we have

$$g(p) + \delta = \liminf_{p_n \to p-} g(p_n) < f(\theta, \lambda, p - \varepsilon)$$

For fixed θ and λ, $f(\theta,\lambda,p-\varepsilon)$ is a continuous function of ε hence

$$g(p)+\delta \le f(\theta,\lambda,p).$$

Finally, by definition $f(\theta,\lambda,p)\le g(p)$ which contradicts $\delta>0$.

2,3) In cases U1,2) by virtue of (52) $p_{basis}\in(p_{-\infty},p_{\infty})$ and $P_{\bar{a}}(Y)$ is non-empty for $\bar{a}\ge a_{basis}$. In case U3) it may happen that $p_{basis}=p_{\infty}$ (or $p_{basis}=p_{-\infty}$) and then $P_{a_{basis}}(Y)$ is empty. However, in such case 1.5a,b) imply that $g(p)$ is continuous and decreasing on $(p_{-\infty},p_{\infty}]$,hence $P_{\bar{a}}(Y)$ has non-empty interior for $\bar{a}>a_{basis}$. Convexity is a direct consequence of 1.5a).

4) One cannot have a mis-priced redundant asset and $a(X,Y-(1+r)p)$ finite in either of the three cases U1,2,3). Thus redundant assets command a unique price. For Y non-redundant the claim follows directly from 1.5a,b).

5) Again, this is a direct consequence of 1.5a,b). In the case U1) the absence of good deals already implies the absence of arbitrage (see Lemma B.3 in Černý and Hodges) and therefore $P_{\bar{a}}(Y)=\{p|g(p)\le \mathrm{E}\left[U\left((1+r)V_0+\bar{a}\right)\right]\}$ which is a closed interval, necessarily strictly inside the open no-arbitrage price region.

6) By virtue of 1.5a,b). $g(p)$ is continuous, and therefore finite-valued, on $(p_{-\infty},p_{basis}]$, thus for any $p\in(p_{-\infty},p_{basis}]$ there is $\bar{a}<\infty$ such that $g(p)\le \mathrm{E}\left[U\left((1+r)V_0+\bar{a}\right)\right]$ and $p\in P_{\bar{a}}(Y)$. Similarly for the interval $[p_{basis},p_{\infty})$.

7) In cases U2,3) absence of good deals allows for some arbitrage opportunities, but these arbitrage opportunities lie on the boundary of the positive orthant (see Lemma B.2 in Černý and Hodges), consequently the no-good-deal price range may include the points $p_{-\infty}$ and p_{∞}, but nothing beyond these points.

Proof of Theorem 3.

1) By Theorem 2, part 1) there is market portfolio $z\in M_0$ such that

$$a_{basis}=\sup_{Z\in M_0} a(Z)=a(z).$$

Function $f:\mathbb{R}^m\to\mathbb{R}$,

$$f(Z)\triangleq \mathrm{E}\left[U(V_0+Z)\right] \tag{55}$$

is convex and continuous therefore the upper level set $K\triangleq\{Z|f(Z)\ge a_{basis}\}$ is convex and closed. Furthermore, the interior points of K do not intersect M_0. By Theorem 1.13 in Beavis and Dobbs (1990) there is a hyperplane that separates K and M_0, in other words there is a linear functional ζ on $\mathbb{R}^m$ such that

$$\begin{aligned}\zeta(M_0) &= 0 \\ \zeta(K) &\ge 0.\end{aligned} \tag{56}$$

Continuity of ζ implies

$$f(z+\triangle V)\leq f(a_{basis})=f(z) \tag{57}$$
$$\text{for all } \triangle V \text{ such that } \zeta(\triangle V)=0. \tag{58}$$

From (57) we deduce that ζ is strictly positive with probability 1. By contradiction if ζ is not strictly positive with probability 1 then there is $\triangle V \in \mathbb{R}^m$ such that $\triangle V \geq 0, \triangle V \neq 0$ and $\zeta(\triangle V)=0$. Function f is strictly increasing, hence with this choice of $\triangle V$ we have $f(z+\triangle V) > f(z)$ which contradicts (57). Define a complete market pricing rule p

$$p(Y)\triangleq \frac{\sum_{i=1}^m \zeta_i Y_i}{(1+r)\sum_{i=1}^m \zeta_i}.$$

By virtue of (56) p prices correctly all the basis excess returns, and by construction it prices correctly also the risk-free security. p is strictly positive with probability 1, which implies no arbitrage in the completed market. Finally, by virtue of (57) the completed market does not admit good deals. Conversely, if there is no good deal in the completed market, there cannot be a good deal among basis excess returns. Finally, in the case U1) a complete market with $a_{basis}<\infty$ implies the absence of arbitrage (see Lemma B.3 in Černý and Hodges) and hence the condition $m>0$ is not necessary.

2) The same procedure as in part 1) proves existence of a non-negative complete market price rule p consistent with basis assets, consequently we have

$$P_{\bar{a}}(Y)\subseteq \tilde{P}_{\bar{a}}(Y).$$

From the arbitrage theorem

$$\left\{\frac{\mathrm{E}[mY]}{1+r}\middle|\, \mathrm{E}[mX]=0, m>0\right\}=(p_{-\infty},p_{\infty})$$

and from the continuity

$$\tilde{P}_{\bar{a}}(Y)\subseteq\left\{\frac{\mathrm{E}[mY]}{1+r}\middle|\, \mathrm{E}[mX]=0, m\geq 0\right\}\subseteq[p_{-\infty},p_{\infty}]. \tag{59}$$

From the extension theorem $\tilde{P}_{\bar{a}}(Y)$ cannot contain prices inside $P_{\infty}(Y)$ and outside $P_{\bar{a}}(Y)$, which together with (59) gives

$$\tilde{P}_{\bar{a}}(Y)\subseteq P_{\bar{a}}(Y)\cup\{p_{-\infty}\}\cup\{p_{\infty}\}.$$

B. Arbitrage-adjusted Sharpe Ratio

Suppose the excess return X has a piecewise absolutely continuous cumulative distribution function F. From (34)

$$h_A(X) = \frac{1}{\min_\lambda \int_{-\infty}^{\frac{1}{\lambda}} (1 - 2\lambda x + \lambda^2 x^2) dF(x)} - 1 \text{ for } \lambda > 0. \tag{60}$$

Let us examine the optimisation in the denominator. The integral is well defined as long as $\int_{-\infty}^{0} x^2 dF(x)$ is finite, thus a necessary and sufficient condition for its existence is finite variance of $X_- \triangleq -\min(X, 0)$. Let us now calculate the formal derivatives with respect to λ

$$\frac{\partial}{\partial \lambda} \int_{-\infty}^{\frac{1}{\lambda}} (1 - 2\lambda x + \lambda^2 x^2) dF(x) \; = \; 2 \int_{-\infty}^{\frac{1}{\lambda}} (-x + \lambda x^2) dF(x) \tag{61}$$

$$\frac{\partial^2}{\partial \lambda^2} \int_{-\infty}^{\frac{1}{\lambda}} (1 - 2\lambda x + \lambda^2 x^2) dF(x) \; = \; +2 \int_{-\infty}^{\frac{1}{\lambda}} x^2 dF(x). \tag{62}$$

By §7.3 Theorem 11 in Widder (1989) the interchanges of differentiation and integration are warranted. Equation (61) implies that with $\int_{-\infty}^{+\infty} x dF(x) > 0$ (60) attains global maximum at $\lambda^* > 0$. When $\int_{-\infty}^{+\infty} x dF(x) < 0$ truncation proceeds from the other end, formally we apply the procedure above to $-X$.

If we realise that $\frac{1}{\lambda}$ corresponds to x_{cap}, the first order condition (61) implies

$$x_{\text{cap}} \int_{-\infty}^{x_{\text{cap}}} x dF(x) = \int_{-\infty}^{x_{\text{cap}}} x^2 dF(x), \tag{63}$$

which can be restated in terms of the capped distribution as follows

$$\begin{aligned} x_{\text{cap}} \left[\int_{-\infty}^{x_{\text{cap}}} x dF(x) + x_{\text{cap}}(1 - F(x_{\text{cap}})) \right] \; &= \; \int_{-\infty}^{x_{\text{cap}}} x^2 dF(x) + x_{\text{cap}}^2 (1 - F(x_{\text{cap}})), \\ x_{\text{cap}} \mathrm{E}\left[\min(X, x_{\text{cap}})\right] \; &= \; \mathrm{E}\left[(\min(X, x_{\text{cap}}))^2 \right]. \end{aligned}$$

The same trick can be used to show that (60) is in fact equal to the Sharpe Ratio of the capped distribution.

Our task now is to evaluate (63) for a lognormally distributed return. Let us write

$$X = e^{\mu + \sigma Z} - e^r,$$

where Z is a standard normal variable, r is risk-free rate of return, expected risky return is $e^{\mu + \frac{s^2}{2}}$ and the variance of risky return is

$e^{2\mu+2s^2} - e^{2\mu+s^2}$. We first recall an auxiliary result

$$\int_{-\infty}^{z_{\text{cap}}} e^{\alpha+\beta z} d\Phi(z) = e^{\alpha+\frac{\beta^2}{2}} \Phi(z_{\text{cap}} - \beta),$$

which follows easily by direct integration or by referring to Black-Scholes formula. We apply this result repeatedly with

$$z_{\text{cap}} = \frac{\ln(x_{\text{cap}} + e^r) - \mu}{\sigma}$$

to obtain

$$\int_{-\infty}^{x_{\text{cap}}} x dF_X(x) = \int_{-\infty}^{z_{\text{cap}}} (e^{\mu+\sigma z} - e^r) d\Phi(z) =$$
$$= e^{\mu+\frac{\sigma^2}{2}} \Phi\left(z_{\text{cap}} - \sigma\right) - e^r \Phi(z_{\text{cap}})$$
$$\int_{-\infty}^{x_{\text{cap}}} x^2 dF_X(x) = \int_{-\infty}^{z_{\text{cap}}} (e^{\mu+\sigma z} - e^r)^2 d\Phi(z) =$$
$$= e^{2\mu+2\sigma^2} \Phi\left(z_{\text{cap}} - 2\sigma\right) - 2e^{r+\mu+\frac{\sigma^2}{2}} \Phi\left(z_{\text{cap}} - \sigma\right) + e^{2r} \Phi(z_{\text{cap}}).$$

The first order condition therefore reads

$$x_{\text{cap}} \left[e^{\mu+\frac{\sigma^2}{2}} \Phi\left(z_{\text{cap}}\right) - e^r \Phi(z_{\text{cap}}) \right] =$$
$$= e^{2\mu+2\sigma^2} \Phi\left(z_{\text{cap}} - 2\sigma\right) - 2e^{r+\mu+\frac{\sigma^2}{2}} \Phi\left(z_{\text{cap}} - \sigma\right) + e^{2r} \Phi(z_{\text{cap}}).$$

To solve it one has to perform a straightforward numerical search over $x_{\max}$.

C. Continuous time limit

Proof of Proposition 6

Recall from (48)

$$m_{t+dt|t} = \exp[-\frac{1}{2} \int_t^{t+dt} ||\nu_s||^2 ds - \int_t^{t+dt} \nu_s dB_s^P].$$

Assuming that ν_s is constant in the time interval $[t, t + dt]$ we have

$$m_{t+dt|t} = \exp[-\frac{1}{2} ||\nu_t||^2 dt] \exp\left(-\nu_t dB_t\right),$$

where dB_t is distributed normally with mean 0 and variance dt. From the moment generating function of normal distribution we have

$$\mathrm{E}_t \left[\exp\left(-2\nu_t dB_t\right)\right] = \exp(2||\nu_t||^2 dt)$$

and consequently

$$\mathrm{E}_t\left[m^2_{t+dt|t}\right] = \exp(||\nu_t||^2 dt) = 1 + ||\nu_t||^2 dt + o(dt) \tag{64}$$

We can write the restriction (19) as

$$\mathrm{E}_t\left[m^2_{t+dt|t}\right] \leq 1 + h^2 dt$$

and evaluate the left hand side using the expression (64) to obtain

$$\begin{aligned} 1 + ||\nu_t||^2 dt + o(dt) &\leq 1 + h^2 dt \\ ||\nu_t||^2 &\leq h^2. \end{aligned}$$

Proof of Proposition 7

Define process z_τ as follows

$$z_\tau \equiv m_{t+\tau|t} = 1 - \int_t^{t+\tau} z_s \nu_s dB_s,$$

that is z_τ represents the conditional change of measure starting at time t. We know from (14) that the optimal wealth satisfies

$$V_{t+dt} = I(\lambda z_{dt}),$$

and that λ is found from the condition

$$\mathrm{E}_t\left[z_{dt} I(\lambda z_{dt})\right] = (1 + r_t dt)V_t + o(dt)$$

Assuming that ν_s is constant in the interval $[t, t+dt]$ and using the Itô's formula we find

$$\mathrm{E}_t\left[z_{dt} I(\lambda z_{dt})\right] = I(\lambda) + \left(\lambda I'(\lambda) + \frac{1}{2}\lambda^2 I''(\lambda)\right) ||\nu_t||^2 dt + o(dt)$$

and hence

$$(1 + r_t dt)V_t = I(\lambda) + \left(\lambda I'(\lambda) + \frac{1}{2}\lambda^2 I''(\lambda)\right) ||\nu_t||^2 dt + o(dt) \tag{65}$$

Now we use the Itô's formula again to find $\mathrm{E}_t\left[U(V_{t+dt})\right]$

$$\begin{aligned} \frac{d^2}{dz^2} U\left(I(\lambda z)\right) &= \lambda^2 I'(\lambda z) + \lambda^3 z I''(\lambda z) \\ \mathrm{E}_t\left[U(V_{t+dt})\right] &= U[I(\lambda)] + \frac{1}{2}\left(\lambda^2 I'(\lambda) + \lambda^3 I''(\lambda)\right) ||\nu_t||^2 dt + o(dt) \end{aligned} \tag{66}$$

The good-deal restriction is

$$\mathrm{E}_t\left[U(V_{t+dt})\right] \leq U\left((1 + r_t dt)V_t + a dt\right)$$

Substituting from expression (65) and using Taylor expansion we obtain

$$\mathrm{E}_t\left[U(V_{t+dt})\right] \leq U\left(I(\lambda)\right) + U'\left(I(\lambda)\right)\left(\left(\lambda I'(\lambda) + \frac{1}{2}\lambda^2 I''(\lambda)\right)||\nu_t||^2 + a\right)dt + o(dt)$$

Finally, substitution for $\mathrm{E}_t\left[U(V_{t+dt})\right]$ from equation (66) shows that the good-deal restriction becomes

$$-\frac{1}{2}\lambda I'(\lambda)||\nu||^2 \leq a + O(dt). \tag{67}$$

Differentiating both sides of the identity $U'[I(\lambda)] = \lambda$ we obtain

$$\begin{aligned} U''[I(\lambda)]I'(\lambda) &= 1 \\ -\lambda I'(\lambda) &= -\frac{U'[I(\lambda)]}{U''[I(\lambda)]}. \end{aligned}$$

Since equation (65) implies $I(\lambda) = V_t + O(dt)$ we have

$$-\lambda I'(\lambda) = \frac{1}{A(V_t)} + O(dt)$$

and the good-deal restriction (67) is shown to be of the form

$$\frac{1}{2}||\nu_t||^2 \leq A(V_t)a.$$

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