# On Perverse Equivalences and Rationality

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**Abstract.** We show that perverse equivalences between module categories of finite-dimensional algebras preserve rationality. As an application, we give a connection between some famous conjectures from the modular representation theory of finite groups, namely Broué's Abelian Defect Group conjecture and Donovan's Finiteness conjectures.

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Let $k$ be an algebraically closed field and let $A$ be a finite dimensional $k$-algebra. Denote by $k$-vect the category whose objects are finite dimensional $k$-vector spaces and whose morphisms are $k$-linear transformations between vector spaces, by $A$-mod the $k$-linear category of finitely generated (left) $A$-modules, and by $D^b(A)$ the bounded derived category of finitely generated $A$-modules. Recall that $D^b(A)$ is a $k$-linear, triangulated category.

In this article we will be dealing with additive, but possibly non $k$-linear functors between $k$-linear categories. Thus, if $\mathscr{F}: \mathcal{A} \to \mathcal{B}$ is a functor between $k$-linear categories, we will specify whether $\mathscr{F}$ is $k$-linear or merely additive.

Let $\sigma: k \to k$ be a field automorphism. If $V$ is a $k$-vector space, the $\sigma$-twist $V^\sigma$ of $V$ is the $k$-vector space which is equal to $V$ as a group, but where scalar multiplication is given by $\lambda \cdot x = \sigma^{-1}(\lambda)x$. Any $k$-linear map $f: V \to W$is also a $k$-linear map $f: V^\sigma \to W^\sigma$. Denote also by $\sigma: k$-vect $\to k$-vect the functor which sends an object $V$ to $V^\sigma$ and which is the identity on morphisms. Then $\sigma$ is an additive equivalence. We denote by $A^\sigma$ the $k$-algebra which equals $A^\sigma$ as a $k$-vector space and $A$ as a ring. The functor $\sigma$ on $k$-vect extends to an additive equivalence $\sigma: A$-mod $\to A^\sigma$-mod. However, $A$ and $A^\sigma$ are not necessarily Morita equivalent as $k$-algebras.

**Definition 1.** A $k$-linear equivalence $\mathscr{E}: A$-mod $\to A^\sigma$-mod is said to be a $\sigma$-*Morita equivalence* if $\mathscr{E}(V) \cong V^\sigma$ for all simple $A$-modules $V$. If there is a $\sigma$-Morita equivalence between $A$ and $A^\sigma$, then we say that $A$ and $A^\sigma$ are $\sigma$-*Morita equivalent.*

Note that if $\mathscr{E}: A$-mod $\to A^\sigma$-mod is a $\sigma$-Morita equivalence, then $\mathscr{E}$ induces a dimension preserving bijection between the set of isomorphism classes of simple $A$-modules and the set of isomorphism classes of simple $A^\sigma$-modules, whence $A$ and $A^\sigma$ are isomorphic as $k$-algebras.

Perverse equivalences were introduced by R. Rouquier and the first author in [5]. For a finite dimensional $k$-algebra $A$, denote by $\mathcal{S}_A$ a set of representatives for the isomorphism classes of simple $A$-modules. A $k$-linear equivalence of triangulated categories $\mathscr{F}\colon D^b(A) \to D^b(B)$, for $B$ a finite dimensional algebra, is said to be a *perverse equivalence* if the following holds :

There exists a non-negative integer $r$, a function $q\colon [0,r] \to \mathbb{Z}$, a filtration $\emptyset = \mathcal{S}_{-1} \subset \mathcal{S}_0 \subset \cdots \subset \mathcal{S}_r = \mathcal{S}_A$ and a filtration $\emptyset = \mathcal{S}'_{-1} \subset \mathcal{S}'_0 \subset \cdots \subset \mathcal{S}'_r = \mathcal{S}_B$ such that whenever $T$ is in $\mathcal{S}_i$, the composition factors of $H^{-j}(\mathscr{F}(T))$ are in $\mathcal{S}'_{i-1}$ for $j \neq q(i)$ and in $\mathcal{S}'_i$ for $j = q(i)$ (see [7, Definition 3.1.1]). The tuple $(r, q, \mathcal{S}, \mathcal{S}')$ is then called a perversity datum for $\mathscr{F}$. The algebras $A$ and $B$ are *perversely equivalent* if there exists a perverse equivalence $\mathscr{F}\colon D^b(A) \to D^b(B)$.

Trivial examples of perverse equivalences arise from Morita equivalences. Indeed, if $\mathscr{F}\colon A\text{-mod} \to B\text{-mod}$ is a $k$-linear equivalence, then the induced equivalence $\mathscr{F}\colon D^b(A) \to D^b(B)$ is a perverse equivalence with respect to the perversity datum $(r, q_0, \mathcal{S}, \mathcal{S}')$, where $r$ is any non-negative integer, $\mathcal{S}$ is any filtration on $\mathcal{S}_A$, as above, $\mathcal{S}'$ is the filtration on $\mathcal{S}_B$ induced by $\mathscr{F}$ and $q_0\colon [0,r] \to \mathbb{Z}$ is defined by $q(i) = 0$ for all $i$. The converse statement is the first part of the following lemma, which records basic properties of perverse equivalences.

**Lemma 2.** *[5, Lemma 3.69] Let $A$ and $B$ be finite-dimensional $k$-algebras, and let $\mathscr{F}\colon D^b(A) \to D^b(B)$ be a perverse equivalence with respect to the perversity datum $(r, q, \mathcal{S}, \mathcal{S}')$.*

1. *If $q = q_0$, then $\mathscr{F}$ restricts to a $k$-linear equivalence $A\text{-mod} \to B\text{-mod}$.*
2. *Any inverse equivalence $\mathscr{F}^{-1}\colon D^b(B) \to D^b(A)$ is a perverse equivalence with respect to the perversity datum $(r, -q, \mathcal{S}', \mathcal{S})$.*
3. *If $C$ is a finite-dimensional $k$-algebra, and $\mathscr{G}\colon D^b(B) \to D^b(C)$ is a perverse equivalence with respect to the perversity datum $(r, q', \mathcal{S}', \mathcal{S}'')$, then the composition $\mathscr{G} \circ \mathscr{F}\colon D^b(A) \to D^b(C)$ is a perverse equivalence with respect to the perversity datum $(r, q' + q', \mathcal{S}, \mathcal{S}'')$.*

Note that the functor $\sigma\colon A\text{-mod} \to A^\sigma\text{-mod}$ extends to an equivalence of triangulated categories $\sigma\colon D^b(A) \to D^b(A^\sigma)$ (again, not necessarily $k$-linear). Further, if $B$ is a finite dimensional $k$-algebra and $\mathscr{F}\colon D^b(A) \to D^b(B)$ is a $k$-linear exact functor, then $\sigma \circ \mathscr{F} \circ \sigma^{-1}\colon D^b(A^\sigma) \to D^b(B^\sigma)$ is $k$-linear and exact. If $\mathscr{F}\colon D^b(A) \to D^b(B)$ is a perverse equivalence with respect to perversity datum $(r, q, \mathcal{S}, \mathcal{S}')$, then we let $\mathcal{S}^\sigma$ and ${\mathcal{S}'}^\sigma$ be the filtrations defined by $T^\sigma \in \mathcal{S}_i^\sigma$ if and only if $T \in \mathcal{S}_i$ and ${T'}^\sigma \in \mathcal{S}_i'^\sigma$ if and only if $T' \in \mathcal{S}_i$, $-1 \leq i \leq r$. The following lemma is an immediate consequence of the definitions.

**Lemma 3.** *Let $A$ and $B$ be finite dimensional $k$-algebras and $\sigma\colon k \to k$ a field automorphism. If $\mathscr{F}\colon D^b(A) \to D^b(B)$ is a perverse equivalence with respect to the perversity datum $(r, q, \mathcal{S}, \mathcal{S}')$, then*

$$\sigma \circ \mathscr{F} \circ \sigma^{-1}\colon D^b(A^\sigma) \to D^b(B^\sigma)$$

*is a perverse equivalence with respect to the perversity datum $(r, q, \mathcal{S}^\sigma, {\mathcal{S}'}^\sigma)$.*

For a finite dimensional $k$-algebra $A$, denote by $K_0(A)$ the Grothendieck group of finitely generated $A$-modules with respect to short exact sequences and for a finitely generated $A$-module $X$ denote by $[X]$ the equivalence class of $X$ in $K_0(A)$; $K_0(A)$ is an abelian group, freely generated by $[V]$, $V \in \mathcal{S}_A$. If $B$ is a finite-dimensional $k$-algebra and $\mathscr{F}: D^b(A) \to D^b(B)$ is an exact functor (not necessarily $k$-linear), then we denote by $[\mathscr{F}]\colon K_0(A) \to K_0(B)$ the induced homomorphism, defined by $[\mathscr{F}]([X]) = \sum_i (-1)^i [H^i(\mathscr{F}(X))]$. We may interpret $[\mathscr{F}]$ as the homomorphism induced by $\mathscr{F}$ between the Grothendieck groups of the triangulated categories $D^b(A)$ and $D^b(B)$, once they are identified with $K_0(A)$ and $K_0(B)$ in the standard way. Hence if $\mathscr{G}\colon D^b(B) \to D^b(C)$ is also exact, for a finite dimensional $k$-algebra $C$, then $[\mathscr{G} \circ \mathscr{F}] = [\mathscr{G}] \circ [\mathscr{F}]$.

The following observation is the main result of this note.

**Proposition 4.** *Let $A$ and $B$ be finite dimensional $k$-algebras and $\sigma\colon k \to k$ a field automorphism. If $A$ and $B$ are perversely equivalent and $B$ and $B^\sigma$ are $\sigma$-Morita equivalent, then $A$ and $A^\sigma$ are $\sigma$-Morita equivalent.*

*Proof.* Let $\mathscr{F}: D^b(A) \to D^b(B)$ be a perverse equivalence with respect to perversity datum $(r, q, \mathcal{S}, \mathcal{S}')$ and let $\mathscr{E}\colon B\text{-mod} \to B^\sigma\text{-mod}$ be a $\sigma$-Morita equivalence. Denote also by $\mathscr{E}\colon D^b(B) \to D^b(B^\sigma)$ the equivalence induced by $\mathscr{E}$. Then the hypothesis on $\mathscr{E}$ implies that $\mathscr{E}$ is perverse with respect to the datum $(r, q_0, \mathcal{S}', \mathcal{S}'^\sigma)$. So by the above two lemmas, the composition

$$(\sigma \circ \mathscr{F} \circ \sigma^{-1})^{-1} \circ \mathscr{E} \circ \mathscr{F}\colon D^b(A) \to D^b(A^\sigma)$$

is a perverse equivalence with respect to the datum $(r, q_0, \mathcal{S}, \mathcal{S}^\sigma)$. By Lemma 2, it follows that $(\sigma \circ \mathscr{F} \circ \sigma^{-1})^{-1} \circ \mathscr{E} \circ \mathscr{F}$ restricts to a $k$-linear equivalence from $A$-mod to $A^\sigma$-mod.

It remains to show that $(\sigma \circ \mathscr{F} \circ \sigma^{-1})^{-1} \circ \mathscr{E} \circ \mathscr{F}$ is a $\sigma$-Morita equivalence. Let $V$ be a simple $A$-module. Then $(\sigma \circ \mathscr{F} \circ \sigma^{-1})^{-1} \circ \mathscr{E} \circ \mathscr{F}(V)$ is a simple $A^\sigma$-module, hence is completely determined by

$$\sum_i (-1)^i [H^i((\sigma \circ \mathscr{F} \circ \sigma^{-1})^{-1} \circ \mathscr{E} \circ \mathscr{F}(V))] = [(\sigma \circ \mathscr{F} \circ \sigma^{-1})^{-1} \circ \mathscr{E} \circ \mathscr{F}]([V]).$$

Each of the functors $\mathscr{E}$, $\mathscr{F}$ and $\sigma$ is exact, and by hypothesis $[\mathscr{E}] = [\sigma]$. Hence

$$[(\sigma \circ \mathscr{F} \circ \sigma^{-1})^{-1} \circ \mathscr{E} \circ \mathscr{F}] = [\sigma] \circ [\mathscr{F}]^{-1} \circ [\sigma]^{-1} \circ [\mathscr{E}] \circ [\mathscr{F}] = [\sigma].$$

Thus, $(\sigma \circ \mathscr{F} \circ \sigma^{-1})^{-1} \circ \mathscr{E} \circ \mathscr{F}(V) \cong V^\sigma$. The following commutative diagram may help the reader.

$$\begin{array}{ccc} D^b(A) & \xrightarrow{\mathscr{F}} & D^b(B) \\ \downarrow & & \downarrow \mathscr{E} \\ D^b(A^\sigma) & \xrightarrow{\sigma \circ \mathscr{F} \circ \sigma^{-1}} & D^b(B^\sigma) \end{array}$$

□

From now on let $p$ be a prime number. Suppose that $k = \bar{\mathbb{F}}_p$ is an algebraic closure of the field of $p$ elements and that $\sigma\colon k \to k$ is the Frobenius automorphism

$\lambda \to \lambda^p$. Recall from [3] that the *Morita Frobenius number* of a finite dimensional $k$-algebra $A$ is the least positive integer $m$ such that $A$ and $A^{\sigma^m}$ are Morita equivalent (as $k$-algebras). By [9], the Morita Frobenius number of $A$ is also the least positive integer $m$ such that a basic algebra of $A$ has an $\mathbb{F}_{p^m}$-form. Thus the Morita Frobenius number of $A$ is a measure of the rationality of $A$.

As a variation on the theme of Morita Frobenius numbers, for a finite dimensional $k$-algebra $A$, we define the *$\sigma$-Morita Frobenius number of $A$* to be the least positive number $m$ such that $A$ and $A^{\sigma^m}$ are $\sigma^m$-Morita equivalent. Note that some finite subfield of $k$ is a splitting field for $A$, whence the $\sigma$-Morita Frobenius number is defined. Also, clearly, the Morita Frobenius number of $A$ is always less than or equal to the $\sigma$-Morita Frobenius number of $A$. As an immediate consequence of Proposition 4 we obtain the following.

**Corollary 5.** *Let $k$ and $\sigma$ be as above and let $A$ and $B$ be finite dimensional $k$-algebras. If $A$ and $B$ are perversely equivalent, then the $\sigma$-Morita Frobenius number of $A$ is equal to the $\sigma$-Morita Frobenius number of $B$.*

We remark that there are no known examples of perversely equivalent (or even derived equivalent) algebras which do not have the same Morita Frobenius number.

The interest in Morita Frobenius numbers comes in part from the finiteness conjectures of Donovan in the local representation theory of finite groups (see [1, Conjecture M]). Let $P$ be a finite $p$-group. By a $P$-block we mean a block $A$ of the group algebra $kG$, $G$ a finite group, such that the defect groups of $A$ are isomorphic to $P$. Donovan's conjecture states that the number of Morita equivalence classes of $P$-blocks is bounded by a function that depends only on the order of $P$. A weak version of Donovan's conjecture states that the entries of the Cartan matrix of $P$-blocks are bounded by a function which depends only on the order of $P$. Both conjectures were inspired by Brauer's problem 22 ([4], see also [11]). In [9] the second author showed that the gap between Donovan's two conjectures is equivalent to the statement that the Morita Frobenius numbers of $P$-blocks are bounded by a function of $|P|$. This is now known as the rationality conjecture.

Let $G$ be a finite group, $P \leq G$, $A$ be a $P$-block of $kG$ and $B$ be the block of $kN_G(P)$ in Brauer correspondence with $B$. Recall that Broué's abelian defect group conjecture states that if $P$ is abelian, then $D^b(A)$ and $D^b(B)$ are equivalent (as $k$-linear triangulated categories). In [5] it is shown that many known instances of derived equivalences between $A$ and $B$ are compositions of perverse equivalences, for instance if $P$ is cyclic. It is further conjectured that if $G$ is a finite group of Lie type in characteristic $r \neq p$, and $P$ is abelian, then the derived equivalence between $A$ and $B$ predicted by Broué [2, §6] to arise from the complex of cohomology of Deligne-Lusztig varieties should be perverse (see [5], [7, §3.4.1], [6, Conjecture 1.3]).

The existence of (chains of) perverse equivalences between blocks and their Brauer correspondents has the following consequence for Donovan's conjecture.

**Theorem 6.** *Let $k = \bar{\mathbb{F}}_p$, $G$ be a finite group, $A$ be a block of $kG$, $P$ be a defect group of $A$ and $B$ be the block of $kN_G(P)$ in Brauer correspondence with $A$. Suppose that there exist finite dimensional $k$-algebras $A_0 := A, A_1, \cdots, A_n =: B$ such that $A_{i-1}$ is perversely equivalent to $A_i$, $1 \leq i \leq n$. Then the Morita Frobenius* *number of $A$ is at most $(|\mathrm{Out}(P)|_{p'})^2$, where* $\mathrm{Out}(P)$ *denotes the outer automorphism group of $P$ and* $|\mathrm{Out}(P)|_{p'}$ *denotes the $p'$-part of the order of* $\mathrm{Out}(P)$.

*Proof.* Let $a = |\mathrm{Out}(P)|_{p'}$. Since the Morita Frobenius number of $A$ is at most the $\sigma$-Morita number of $A$, by (repeated) applications of Corollary 5 it suffices to show that the $\sigma$-Morita Frobenius number of $B$ is at most $a^2$. By Külshammer's structure theorem for blocks with normal defect group [10], there exists a subgroup $E \le \mathrm{Aut}(P)$ of $p'$-order and an element $\alpha$ of $H^2(P \rtimes E, k^\times)$ such that $B$ is Morita equivalent to the twisted group algebra $k_\alpha(P \rtimes E)$ (as $k$-algebras). So, again by Proposition 5 it suffices to prove that the $\sigma$-Morita Frobenius number of $k_\alpha(P \rtimes E)$ is at most $a^2$. Note that since $E$ is a $p'$-group, $E$ is isomorphic to a subgroup of $\mathrm{Out}(P)$ and consequently $|E| \le a$.

Now, $k_\alpha(P \rtimes E)$ is isomorphic as $k$-algebra to an algebra of the form $kF\tilde{c}$, where $F$ is a central extension of $P \rtimes E$ by a cyclic group, say $Z$ of order $|E|$, and $\tilde{c} \in kZ$ is a central idempotent of $kF$ (see for example [12, §10.5]).

Let $k_0 \subset k$ be the splitting field of $x^{|E|^2-1}$ and let $a' = |k : \mathbb{F}_p|$. Then, $\tilde{c} \in k_0 Z$ and $kF\tilde{c} = k \otimes_{k_0} k_0 F\tilde{c}$. Since $P$ is normal in $F$, every simple $kF$-module is the inflation of a simple $k(F/P)$-module. On the other hand, since $|F/P| = |E|^2$, $k_0$ is a splitting field of $F/P$. Thus, $k_0 F\tilde{c}$ is a split $k_0$-algebra. It follows that $kF\tilde{c}$ and $(kF\tilde{c})^{\sigma^{a'}}$ are $\sigma^{a'}$- Morita equivalent. The result follows as $a' \le a^2$. □

**Remark.** The Morita Frobenius numbers of almost all unipotent blocks of quasi-simple finite groups of Lie type in non-describing characteristic have been determined in [8]. We hope that the approach to Morita Frobenius numbers via perverse equivalences outlined above will help in settling the remaining cases.

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