Low-energy theory and RKKY interaction for interacting quantum wires with Rashba spin-orbit coupling

Andreas Schulz[1], Alessandro De Martino[2], Philip Ingenhoven[1,3], and Reinhold Egger[1]
[1] *Institut für Theoretische Physik, Heinrich-Heine-Universität, D-40225 Düsseldorf, Germany*
[2] *Institut für Theoretische Physik, Universität zu Köln, Zülpicher Strasse 77, D-50937 Köln, Germany*
[3] *Institute of Fundamental Sciences, Massey University, Private Bag 11 222, Palmerston North, New Zealand*
(Dated: June 3, 2009)

We present the effective low-energy theory for interacting 1D quantum wires subject to Rashba spin-orbit coupling. Under a one-loop renormalization group scheme including all allowed interaction processes for not too weak Rashba coupling, we show that electron-electron backscattering is an irrelevant perturbation. Therefore no gap arises and electronic transport is described by a modified Luttinger liquid theory. As an application of the theory, we discuss the RKKY interaction between two magnetic impurities. Interactions are shown to induce a slower power-law decay of the RKKY range function than the usual 1D noninteracting $\cos(2k_Fx)/|x|$ law. Moreover, in the noninteracting Rashba wire, the spin-orbit coupling causes a twisted (anisotropic) range function with several different spatial oscillation periods. In the interacting case, we show that one special oscillation period leads to the slowest decay, and therefore dominates the RKKY interaction for large separation.

PACS numbers: 71.10.Pm, 85.75.-d, 73.63.-b

I. INTRODUCTION

Spin transport in one-dimensional (1D) quantum wires continues to be a topic of much interest in solid-state and nanoscale physics, offering interesting fundamental questions as well as technological applications.[1] Of particular interest to this field is the spintronic field effect transistor (spin-FET) proposal by Datta and Das,[2] where a gate-tunable Rashba spin-orbit interaction (SOI) of strength α allows for a purely electrical manipulation of the spin-dependent current. While the Rashba SOI arises from a structural inversion asymmetry[3,4,5] of the two- dimensional electron gas (2DEG) in semiconductor devices hosting the quantum wire, additional sources for SOI can be present. In particular, for bulk inversion asymmetric materials, the Dresselhaus SOI (of strength β) should also be taken into account. By tuning the Rashba SOI (via gate voltages) to the special point $\alpha = \beta$, the spin-FET was predicted to show a remarkable insensitivity to disorder,[6] see also Ref. 7. On top of these two, additional (though generally weaker) contributions may arise from the electric confinement fields forming the quantum wire. In this paper, we focus on the case of Rashba SOI and disregard all other SOI terms. This limit can be realized experimentally by applying sufficiently strong backgate voltages,[8,9,10,11] which create a large interfacial electric field and hence a significant and tunable Rashba SOI coupling α. The model studied below may also be relevant to 1D electron surface states of self-assembled gold chains.[12]

The noninteracting theory of such a "Rashba quantum wire" has been discussed in the literature,[13,14,15,16,17,18] and is summarized in Sec. II below. We here discuss electron-electron (e-e) interaction effects in the 1D limit, where only the lowest (spinful) band is occupied. The bandstructure at low energy scales is then characterized by two velocities,[19]

$$v_{A,B} = v_F(1 \pm \delta), \quad \delta(\alpha) \propto \alpha^4. \tag{1}$$

These reduce to a single Fermi velocity v_F in the absence of Rashba SOI ($\delta = 0$ for $\alpha = 0$), but they will be different for $\alpha \neq 0$, reflecting the broken spin $SU(2)$ invariance in a spin-orbit coupled system. The small-α dependence $\delta \propto \alpha^4$ follows for the model below and has also been reported in Ref. 20. Therefore, the velocity splitting (1) is typically weak. While a similar velocity splitting also happens in a magnetic Zeeman field (without SOI),[21] the underlying physics is different since time-reversal symmetry is not broken by SOI.

The bandstructure of a single-channel quantum wire with Rashba SOI should be obtained by taking into account at least the lowest two (spinful) subbands, since a restriction to the lowest subband alone would eliminate spin relaxation.[15,22,23] The problem in this truncated Hilbert space can be readily diagonalized, and yields two pairs of energy bands. When describing a single-channel quantum wire, one then keeps only the lower pair of these energy bands. We mention in passing that bandstructure effects in the presence of both Rashba SOI and magnetic fields have also been studied.[24,25,26,27,28] In addition, the possibility of a spatial modulation of the Rashba coupling was discussed,[29] but such phenomena will not be further considered here. Finally, disorder effects were addressed in Refs. 30,31.

For 1D quantum wires, it is well known that the inclusion of e-e interactions leads to a breakdown of Fermi liquid theory, and often implies Luttinger liquid (LL) behavior. This non-Fermi liquid state of matter has a number of interesting features, including the phenomenon of spin-charge separation.[32] Motivated mainly by the question of how the Rashba spin precession and Datta-Das oscillations in spin-dependent transport are affected by e-

e interactions, Rashba SOI effects on electronic transport in interacting quantum wires have been studied in recent papers.[15,20,22,33,34,35,36,37] In effect, however, all those works only took e-e forward scattering processes into account. Because of the Rashba SOI, one obtains a modified LL phase with broken spin-charge separation,[33,34] leading to a drastic influence on observables such as the spectral function or the tunneling density of states. Moroz *et al.* argued that e-e backscattering processes are irrelevant in the renormalization group (RG) sense, and hence can be omitted in a low-energy theory.[33,34] Unfortunately, their theory relies on an incorrect spin assignment of the subbands,[15,22] which then invalidates several aspects of their treatment of interaction processes.

The possibility that e-e backscattering processes become relevant (in the RG sense) in a Rashba quantum wire was raised in Ref. 38, where a spin gap was found under a weak-coupling two-loop RG scheme. If valid, this result has important consequences for the physics of such systems, and would drive them into a spin-density-wave type state. To establish the spin gap, Ref. 38 starts from a strict 1D single-band model and assumes both α and the e-e interaction as weak coupling constants flowing under the RG. Our approach below is different in that we include the Rashba coupling α from the outset in the single-particle sector, i.e., in a nonperturbative manner. We then consider the one-loop RG flow of all possible interaction couplings allowed by momentum conservation (for not too small α). This is an important difference to the scheme of Ref. 38, since the Rashba SOI eliminates certain interaction processes which become momentum-nonconserving. This mechanism is captured by our approach. The one-loop RG flow then turns out to be equivalent to a Kosterlitz-Thouless flow, and for the initial values realized in this problem, e-e backscattering processes are always irrelevant. Our conclusion is therefore that no spin gap arises because of SOI, and a modified LL picture is always sufficient. We mention in passing that in the presence of a magnetic field (which we do not consider), a spin gap can be present because of spin-nonconserving e-e "Cooper" scattering processes;[39,40] the effects of e-e forward scattering in Rashba wires with magnetic field were studied as well.[41,42,43,44] Below, we also provide estimates for the *renormalized* couplings entering the modified LL theory, see Eq. (26) below. When taking bare (instead of renormalized) couplings, we recover previous results.[22] Note that the SOI in carbon nanotubes[45] or graphene ribbons[46] leads to a similar yet different LL description. In particular, for (achiral) carbon nanotubes, the leading SOI does not break spin-charge separation.[45] We here only discuss Rashba SOI effects in semiconductor quantum wires in the absence of magnetic fields.

We apply our formalism to a study of the Ruderman-Kittel-Kasuya-Yosida (RKKY) interaction[47,48] between two spin-1/2 magnetic impurities, $\mathbf{\Sigma}_{1,2}$, separated by a distance x. The RKKY interaction is mediated by the conduction electrons in the quantum wire which are exchange-coupled (with coupling J) to the impurity spins. In the absence of both the e-e interaction and the SOI, one finds an isotropic exchange (Heisenberg) Hamiltonian,[48]

$$H_{\rm RKKY} = -J^2 F_{\rm ex}(x)\ \mathbf{\Sigma}_1 \cdot \mathbf{\Sigma}_2, \quad F_{\rm ex}(x) \propto \frac{\cos(2k_F x)}{|x|}, \tag{2}$$

where the $2k_F$-oscillatory RKKY range function $F_{\rm ex}(x)$ is specified for the 1D case. When the spin $SU(2)$ symmetry is broken by the SOI, spin precession sets in and the RKKY interaction is generally of a more complicated (twisted) form. For a noninteracting Rashba quantum wire, it has indeed been established[49,50,51] that the RKKY interaction becomes anisotropic and thus has a tensorial character. It can always be decomposed into an exchange (scalar) part, a Dzyaloshinsky-Moriya-like (vector) interaction, and an Ising- like (traceless symmetric tensor) coupling. On the other hand, in the presence of e-e interactions but without SOI, the range function has been shown[52] to exhibit a slow power-law decay, $F_{\rm ex}(x) \propto \cos(2k_F x)|x|^{-\eta}$, with an interaction- dependent exponent $\eta < 1$. The RKKY interaction in interacting quantum wires with SOI has not been studied before.

For the benefit of the focussed reader, we briefly summarize the main results of our analyis. The effective low-energy theory of an interacting Rashba quantum wire is given in Eq. (29), with the velocities (30) and the dimensionless interaction parameters (31). Previous theories did not fully account for the e-e backscattering processes, and the conspiracy of these processes with the broken $SU(2)$ invariance due to spin-orbit effects leads to $K_s < 1$ in Eq. (31). This in turn implies novel effects in the RKKY interaction of an interacting Rashba wire. In particular, the power-law decay exponent in an interacting Rashba wire, see Eq. (38), depends explicitly on both the interaction strength and on the Rashba coupling.

The structure of the remainder of this paper is as follows. In Sec. II, we discuss the bandstructure. Interaction processes and the one-loop RG scheme are discussed in Sec. III, while the LL description is provided in Sec. IV. The RKKY interaction mediated by an interacting Rashba quantum wire is then studied in Sec. V. Finally, we offer some conclusions in Sec. VI. Technical details can be found in the Appendix. Throughout the paper we use units where $\hbar = 1$.

II. SINGLE-PARTICLE DESCRIPTION

We consider a quantum wire electrostatically confined in the z- direction within the 2DEG (xz-plane) by a harmonic potential, $V_c(z) = m\omega^2 z^2/2$, where m is the effective mass. The noninteracting problem is then defined by the single-particle Hamiltonian[3,13,14,15,17]

$$H_{\rm sp} = \frac{1}{2m}\left(p_x^2 + p_z^2\right) + V_c(z) + \alpha\left(\sigma_z p_x - \sigma_x p_z\right), \tag{3}$$

where α is the Rashba coupling and the Pauli matrices $\sigma_{x,z}$ act in spin space. For $\alpha = 0$, the transverse problem is diagonal in terms of the familiar 1D harmonic oscillator eigenstates (Hermite functions) $H_n(z)$, with $n = 0, 1, 2, \ldots$ labeling the subbands (channels). Eigenstates of Eq. (3) have conserved longitudinal momentum $p_x = k$, and with the z-direction as spin quantization axis, $\sigma_z|\sigma\rangle = \sigma|\sigma\rangle$ with $\sigma = \uparrow, \downarrow = \pm$, the $\sigma_x p_z$ term implies mixing of adjacent subbands with associated spin flips. Retaining only the lowest ($n = 0$) subband from the outset thus excludes spin relaxation. We follow Ref. 15 and keep the two lowest bands, $n = 0$ and $n = 1$. The higher subbands $n \geq 2$ yield only tiny corrections, which can in principle be included as in Ref. 17. The resulting 4×4 matrix representing $H_{\rm sp}$ in this truncated Hilbert space is readily diagonalized and yields four energy bands. We choose the Fermi energy such that only the lower two bands, labeled by $s = \pm$, are occupied, and arrive at a reduced two-band model, where the quantum number $s = \pm$ replaces the spin quantum number. The dispersion relation is

$$E_s(k) = \omega + \frac{k^2}{2m} - \sqrt{\left(\frac{\omega}{2} + s\alpha k\right)^2 + \frac{m\omega\alpha^2}{2}}, \quad (4)$$

with eigenfunctions $\sim e^{ikx}\phi_{k,s}(z)$. The resulting asymmetric energy bands (4) are shown in Fig. 1. The transverse spinors (in spin space) are given by

$$\phi_{k,+}(z) = \begin{pmatrix} i\cos[\theta_+(k)]H_1(z) \\ \sin[\theta_+(k)]H_0(z) \end{pmatrix}, \quad (5)$$
$$\phi_{k,-}(z) = \begin{pmatrix} \sin[\theta_-(k)]H_0(z) \\ i\cos[\theta_-(k)]H_1(z) \end{pmatrix},$$

with k-dependent spin rotation angles (we take $0 \leq \theta_s(k) \leq \pi/2$)

$$\theta_s(k) = \frac{1}{2}\cot^{-1}\left(\frac{-2sk - \omega/\alpha}{\sqrt{2m\omega}}\right) = \theta_{-s}(-k). \quad (6)$$

As a result of subband mixing, the two spinor components of $\phi_{k,s}(z)$ carry a different z-dependence. They are therefore not just the result of a $SU(2)$ rotation. For $\alpha = 0$, we recover $\theta_s = \pi/2$, corresponding to the usual spin up and down eigenstates, with $H_0(z)$ as transverse wavefunction; the $s = +$ ($s = -$) component then describes the $\sigma = \downarrow$ ($\sigma = \uparrow$) spin eigenstate. However, for $\alpha \neq 0$, a peculiar implication of the Rashba SOI follows. From Eq. (6) we have $\lim_{k\to\pm\infty}\theta_s(k) = (1 \pm s)\pi/4$, such that both $s = \pm$ states have (approximately) spin $\sigma = \downarrow$ for $k \to \infty$ but $\sigma = \uparrow$ for $k \to -\infty$; the product of spin and chirality thus always approaches $\sigma{\rm sgn}(k) = -1$. Moreover, under the time-reversal transformation, $\mathcal{T} = i\sigma_y\mathcal{C}$ with the complex conjugation operator $\mathcal{C}$, the two subbands are exchanged,

$$e^{-ikx}\phi_{-k,-s}(z) = s\mathcal{T}[e^{ikx}\phi_{k,s}(z)], \quad E_{-s}(-k) = E_s(k). \quad (7)$$

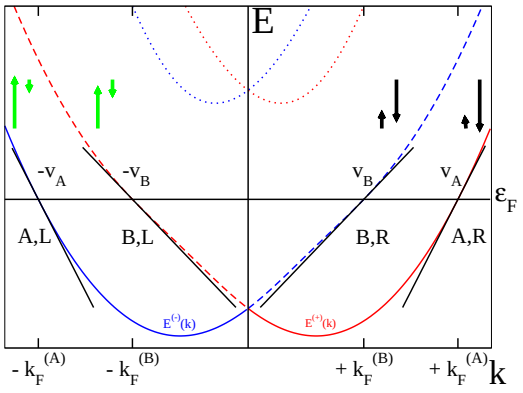

FIG. 1: (Color online) Schematic band structure (4) of a typical 1D Rashba quantum wire. The red/blue curves show the $s = \pm$ bands, and the dotted curves indicate the next subband (the Fermi energy ϵ_F is assumed below that band). For the low-energy description, we linearize the dispersion. It is notationally convenient to introduce bands A (solid lines) and B (dashed lines). Green and black arrows indicate the respective spin amplitudes (exaggerated). The resulting Fermi momenta are $\pm k_F^{(A,B)}$, with Fermi velocities $v_{A,B}$.

Time-reversal symmetry, preserved in the truncated description, makes this two-band model of a Rashba quantum wire qualitatively different from Zeeman-spin-split models.[21]

In the next step, since we are interested in the low-energy physics, we linearize the dispersion relation around the Fermi points $\pm k_F^{(A,B)}$, see Fig. 1, which results in two velocities v_A and v_B, see Eq. (1). The linearization of the dispersion relation of multi-band quantum wires around the Fermi level is known to be an excellent approximation for weak e-e interactions.[32] Explicit values for δ in Eq. (1) can be derived from Eq. (4), and we find $\delta(\alpha) \propto \alpha^4$ for $\alpha \to 0$, in accordance with previous estimates.[20] We mention that $\delta \lesssim 0.1$ has been estimated for typical geometries in Ref. 34. The transverse spinors $\phi_{ks}(z)$, Eq. (5), entering the low-energy description can be taken at $k = \pm k_F^{(A,B)}$, where the spin rotation angle (6) only assumes one of the two values

$$\theta_A = \theta_+\left(k_F^{(A)}\right), \quad \theta_B = \theta_-\left(k_F^{(B)}\right). \quad (8)$$

The electron field operator $\Psi(x, z)$ for the linearized two-band model with $\nu = A, B = +, -$ can then be expressed in terms of 1D fermionic field operators $\psi_{\nu,r}(x)$, where $r = R, L = +, -$ labels right- and left-movers,

$$\Psi(x, z) = \sum_{\nu,r=\pm} e^{irk_F^{(\nu)}x}\, \phi_{rk_F^{(\nu)},s=\nu r}(z)\, \psi_{\nu,r}(x), \quad (9)$$

with $\phi_{k,s}(z)$ specified in Eq. (5). Note that in the left-moving sector, band indices have been interchanged according to the labeling in Fig. 1.

In this way, the noninteracting second-quantized Hamiltonian takes the standard form for two inequivalent species of 1D massless Dirac fermions with different

velocities,

$$H_0 = -i \sum_{\nu,r=\pm} r v_\nu \int dx\ \psi^\dagger_{\nu,r} \partial_x \psi_{\nu,r}. \tag{10}$$

The velocity difference implies the breaking of the spin $SU(2)$ symmetry, a direct consequence of SOI. For $\alpha = 0$, the index ν coincides with the spin quantum number σ for left-movers and with $-\sigma$ for right-movers, and the above formulation reduces to the usual Hamiltonian for a spinful single-channel quantum wire.

III. INTERACTION EFFECTS

Let us now include e-e interactions in such a single-channel disorder-free Rashba quantum wire. With the expansion (9) and $\mathbf{r} = (x, z)$, the second-quantized two-body Hamiltonian

$$H_I = \frac{1}{2} \int d\mathbf{r}_1 d\mathbf{r}_2\ \Psi^\dagger(\mathbf{r}_1)\Psi^\dagger(\mathbf{r}_2) V(\mathbf{r}_1 - \mathbf{r}_2)\Psi(\mathbf{r}_2)\Psi(\mathbf{r}_1) \tag{11}$$

leads to 1D interaction processes. We here assume that the e-e interaction potential $V(\mathbf{r}_1 - \mathbf{r}_2)$ is externally screened, allowing to describe the 1D interactions as effectively local. Following standard arguments, for weak e-e interactions, going beyond this approximation at most leads to irrelevant corrections.[53] We then obtain the local 1D interaction Hamiltonian[54]

$$H_I = \frac{1}{2} \sum_{\{\nu_i, r_i\}} V_{\{\nu_i, r_i\}} \int dx\ \psi^\dagger_{\nu_1,r_1} \psi^\dagger_{\nu_2,r_2} \psi_{\nu_3,r_3} \psi_{\nu_4,r_4}, \tag{12}$$

where the summation runs over all quantum numbers $\nu_1, \ldots, \nu_4$ and $r_1, \ldots, r_4$ subject to momentum conservation,

$$r_1 k_F^{(\nu_1)} + r_2 k_F^{(\nu_2)} = r_3 k_F^{(\nu_3)} + r_4 k_F^{(\nu_4)}. \tag{13}$$

With the momentum transfer $q = r_1 k_F^{(\nu_1)} - r_4 k_F^{(\nu_4)}$ and the partial Fourier transform

$$\tilde{V}(q; z) = \int dx\ e^{-iqx} V(x, z) \tag{14}$$

of the interaction potential, the interaction matrix elements in Eq. (12) are given by

$$\begin{aligned} V_{\{\nu_i, r_i\}} \ &= \ \int dz_1 dz_2\ \tilde{V}(q; z_1 - z_2) \\ &\times \ \left[\phi^\dagger_{r_1 k_F^{(\nu_1)}, \nu_1 r_1} \cdot \phi_{r_4 k_F^{(\nu_4)}, \nu_4 r_4}\right](z_1) \\ &\times \ \left[\phi^\dagger_{r_2 k_F^{(\nu_2)}, \nu_2 r_2} \cdot \phi_{r_3 k_F^{(\nu_3)}, \nu_3 r_3}\right](z_2). \end{aligned} \tag{15}$$

Since the Rashba SOI produces a splitting of the Fermi momenta for the two bands, $\left|k_F^{(A)} - k_F^{(B)}\right| \simeq 2\alpha m$, the condition (13) eliminates one important interaction process available for $\alpha = 0$, namely interband backscattering (see below). This is a distinct SOI effect besides the broken spin $SU(2)$ invariance. Obtaining the complete "g-ology" classification[32] of all possible interaction processes allowed for $\alpha \neq 0$ is then a straightforward exercise. The corresponding values of the interaction matrix elements are generally difficult to evaluate explicitly, but in the most important case of a thin wire,

$$d \gg \frac{1}{\sqrt{m\omega}}, \tag{16}$$

where d is the screening length (representing, e.g., the distance to a backgate), analytical expressions can be obtained.[55] To simplify the analysis and allow for analytical progress, we therefore employ the thin-wire approximation (16) in what follows. In that case, we can neglect the z dependence in Eq. (14). Going beyond this approximation would only imply slightly modified values for the e-e interaction couplings used below. Using the identity

$$\begin{aligned} &\int dz\ \left[\phi^\dagger_{r k_F^{(\nu)}, \nu r} \cdot \phi_{r' k_F^{(\nu')}, \nu' r'}\right](z) = \\ &= \delta_{\nu\nu'}\delta_{rr'} + \cos(\theta_A - \theta_B)\delta_{\nu,-\nu'}\delta_{r,-r'}, \end{aligned} \tag{17}$$

where the angles $\theta_{A,B}$ were specified in Eq. (8), only two different values W_0 and W_1 for the matrix elements in Eq. (15) emerge. These nonzero matrix elements are

$$\begin{aligned} &V_{\nu r,\nu' r',\nu' r',\nu r} \equiv W_0 = \tilde{V}(q = 0), \\ &V_{\nu r,\nu' r',-\nu' -r',-\nu -r} \equiv W_1 \\ &= \cos^2(\theta_A - \theta_B)\ \tilde{V}\left(q = k_F^{(A)} + k_F^{(B)}\right). \end{aligned} \tag{18}$$

We then introduce 1D chiral fermion densities $\rho_{\nu r}(x) = :\psi^\dagger_{\nu r}\psi_{\nu r}:$, where the colons indicate normal-ordering. The interacting 1D Hamiltonian is $H = H_0 + H_I$ with Eq. (10) and

$$\begin{aligned} H_I \ &= \ \frac{1}{2} \sum_{\nu\nu', rr'} \int dx \Big([g_{2\parallel}\nu\delta_{\nu,\nu'} + g_{2\perp}\delta_{\nu,-\nu'}]\delta_{r,-r'} \\ &+ \ [g_{4\parallel}\nu\delta_{\nu,\nu'} + g_{4\perp}\delta_{\nu,-\nu'}]\delta_{r,r'} \Big)\ \rho_{\nu r}\rho_{\nu' r'} \\ &+ \ \frac{g_f}{2} \sum_{\nu r} \int dx\ \psi^\dagger_{\nu r}\psi^\dagger_{\nu,-r}\psi_{-\nu r}\psi_{-\nu,-r}. \end{aligned} \tag{19}$$

The e-e interaction couplings are denoted in analogy to the standard g-ology, whereby the g_4 (g_2) processes describe forward scattering of 1D fermions with equal (opposite) chirality $r = R, L = +, -$, and the labels $\parallel$, $\perp$, and f denote intraband, interband, and band flip processes, respectively. Since the bands $\nu = A, B = +, -$ are inequivalent, we keep track of the band index in the intraband couplings. The g_f term corresponds to intraband backscattering with band flip. The interband backscattering without band flip is strongly suppressed since it

does not conserve total momentum[56] and is neglected in the following. For $\alpha = 0$, the $g_{4,\parallel/\perp}$ couplings coincide with the usual ones[32] for spinful electrons, while g_f reduces to $g_{1\perp}$ and $g_{2,\parallel/\perp} \to g_{2,\perp/\parallel}$ due to our exchange of band indices in the left-moving sector. According to Eq. (18), the bare values of these coupling constants are

$$\begin{aligned} &g_{4\parallel\nu} = g_{4\perp} = g_{2\parallel\nu} = W_0, \\ &g_{2\perp} = W_0 - W_1, \quad g_f = W_1. \end{aligned} \tag{20}$$

The equality of the intraband coupling constants for the two bands is a consequence of the thin-wire approximation, which also eliminates certain exchange matrix elements.

The Hamiltonian $H_0 + H_I$ then corresponds to a specific realization of a general asymmetric two band-model, where the one-loop RG equations are known.[54,57] Using RG invariants, we arrive after some algebra at the two-dimensional Kosterlitz-Thouless RG flow equations,

$$\frac{d\bar{g}_2}{dl} = -\bar{g}_f^2, \quad \frac{d\bar{g}_f}{dl} = -\bar{g}_f \bar{g}_2, \tag{21}$$

for the rescaled couplings

$$\begin{aligned} \bar{g}_2 &= \frac{g_{2\parallel A}}{2\pi v_A} + \frac{g_{2\parallel B}}{2\pi v_B} - \frac{g_{2\perp}}{\pi v_F}, \\ \bar{g}_f &= \sqrt{\frac{1+\gamma}{2}}\,\frac{g_f}{\pi v_F}, \end{aligned} \tag{22}$$

where we use the dimensionless constant

$$\gamma = \frac{v_F^2}{v_A v_B} = \frac{1}{1-\delta^2} \geq 1. \tag{23}$$

As usual, the g_4 couplings do not contribute to the one-loop RG equations. The initial values of the couplings can be read off from Eq. (20),

$$\begin{aligned} \bar{g}_2(l=0) &= \frac{(\gamma-1)W_0 + W_1}{\pi v_F}, \\ \bar{g}_f(l=0) &= \sqrt{\frac{1+\gamma}{2}}\,\frac{W_1}{\pi v_F}. \end{aligned} \tag{24}$$

The solution of Eq. (21) is textbook material,[32] and $\bar{g}_f$ is known to be marginally irrelevant for all initial conditions with $|\bar{g}_f(0)| \leq \bar{g}_2(0)$. Using Eqs. (18) and (24), this implies with $\gamma \simeq 1 + \delta^2$ the condition

$$\tilde{V}(0) \geq \frac{1}{4}\cos^2(\theta_A - \theta_B)\; \tilde{V}\left(k_F^{(A)} + k_F^{(B)}\right), \tag{25}$$

which is satisfied for all physically relevant repulsive e-e interaction potentials. As a consequence, intraband backscattering processes with band flip, described by the coupling $\bar{g}_f$, are *always marginally irrelevant*, i.e., they flow to zero coupling as the energy scale is reduced, $\bar{g}_f^* = \bar{g}_f(l \to \infty) = 0$. Therefore *no gap arises*, and a modified LL model is the appropriate low-energy theory. We mention in passing that even if we neglect the velocity difference in Eq. (1), no spin gap is expected in a Rashba wire, i.e., the broken $SU(2)$ invariance in our model is not required to establish the absence of a gap.

The above RG procedure also allows us to extract *renormalized couplings* entering the low-energy LL description. The fixed-point value $\bar{g}_2^* = \bar{g}_2(l \to \infty)$ now depends on the Rashba SOI through γ in Eq. (23). With the interaction matrix elements $W_{0,1}$ in Eq. (18), it is given by

$$\bar{g}_2^* = \frac{\sqrt{[(\gamma-1)W_0 + W_1]^2 - (\gamma+1)W_1^2/2}}{\pi v_F}. \tag{26}$$

For $\alpha = 0$, we have $\gamma = 1$ and therefore $\bar{g}_2^* = 0$. The Rashba SOI produces the nonzero fixed-point value (26), reflecting the broken $SU(2)$ symmetry.

IV. LUTTINGER LIQUID DESCRIPTION

In this section, we describe the resulting effective low-energy Luttinger liquid (LL) theory of an interacting single-channel Rashba wire. Employing Abelian bosonization,[32] we introduce a boson field and its conjugate momentum for each band $\nu = A, B = +, -$. It is useful to switch to symmetric ("charge"), $\Phi_c(x)$ and $\Pi_c(x) = -\partial_x \Theta_c(x)$, and antisymmetric ("spin" for $\alpha = 0$), $\Phi_s(x)$ and $\Pi_s(x) = -\partial_x \Theta_s$, linear combinations of these fields and their momenta. The dual fields Φ and Θ then allow to express the electron operator from Eq. (9) and the "bosonization dictionary,"

$$\begin{aligned} \Psi(x,z) &= \sum_{\nu,r} \phi_{r k_F^{(\nu)},\nu r}(z)\,\frac{\eta_{\nu r}}{\sqrt{2\pi a}} \\ &\times\; e^{i r k_F^{(\nu)} x + i\sqrt{\pi/2}[r\Phi_c + \Theta_c + \nu r \Phi_s + \nu \Theta_s]}, \end{aligned} \tag{27}$$

where a is a small cutoff length and $\eta_{\nu r}$ are the standard Klein factors.[32,52,58] (To recover the conventional expression for $\alpha = 0$, due to our convention for the band indices in the left-moving sector, one should replace $\Phi_s, \Theta_s \to -\Theta_s, -\Phi_s$.) Using the identity (17), we can now express the 1D charge and spin densities,

$$\rho(x) = \int dz\, \Psi^\dagger \Psi, \quad \boldsymbol{S}(x) = \int dz\, \Psi^\dagger \frac{\boldsymbol{\sigma}}{2} \Psi, \tag{28}$$

in bosonized form. The (somewhat lengthy) result can be found in Appendix A.

The low-energy Hamiltonian is then taken with the fixed-point values for the interaction constants, i.e., backscattering processes are disregarded and only appear via the renormalized value of $\bar{g}_2^*$ in Eq. (26). Following standard steps, the kinetic term H_0 and the forward scattering processes then lead to the exactly solvable Gaussian field theory of a modified (extended) Luttinger liquid,

$$\begin{aligned} H &= \sum_{j=c,s} \frac{v_j}{2} \int dx\, \left(K_j \Pi_j^2 + \frac{1}{K_j} (\partial_x \Phi_j)^2 \right) \\ &+\; v_\lambda \int dx\, \left(K_\lambda \Pi_c \Pi_s + \frac{1}{K_\lambda} (\partial_x \Phi_c)(\partial_x \Phi_s) \right). \end{aligned} \tag{29}$$

Using the notations $\bar{g}_4 = W_0/\pi v_F$ and

$$\begin{aligned} y_\delta &= \frac{g^*_{2\|A} - g^*_{2\|B}}{4\pi v_F}, \\ y_\pm &= \frac{g^*_{2\|A} + g^*_{2\|B} \pm 2g^*_{2\perp}}{4\pi v_F}, \end{aligned}$$

where explicit (but lengthy) expressions for the fixed-point values $g^*_{2\|A/B}$ and $g^*_{2\perp}$ can be straightforwardly obtained from Eqs. (22) and (26), the renormalized velocities appearing in Eq. (29) are

$$\begin{aligned} v_c &= v_F\sqrt{(1+\bar{g}_4)^2 - y_+^2} \\ &\simeq v_F\sqrt{\left(1+\frac{W_0}{\pi v_F}\right)^2 - \left(\frac{2W_0 - W_1}{2\pi v_F}\right)^2}, \\ v_s &= v_F\sqrt{1-y_-^2} \simeq v_F, \\ v_\lambda &= v_F\sqrt{\delta^2 - y_\delta^2} \simeq v_F\delta\sqrt{1-\left(\frac{W_1}{4\pi v_F}\right)^2}. \end{aligned} \tag{30}$$

In the respective second equalities, we have specified the leading terms in $|\delta| \ll 1$, since the SOI-induced relative velocity asymmetry δ is small even for rather large α, see Eq. (1). The corrections to the quoted expressions are of $\mathcal{O}(\delta^2)$ and are negligible in practice. It is noteworthy that the "spin" velocity v_s is *not* renormalized for a Rashba wire, although it is well-known that v_s will be renormalized due to W_1 for $\alpha = 0$.[32] This difference can be traced to our thin-wire approximation (16). When releasing this approximation, there will be a renormalization in general. Finally, the dimensionless LL interaction parameters in Eq. (29) are given by

$$\begin{aligned} K_c &= \sqrt{\frac{1+\bar{g}_4 - y_+}{1+\bar{g}_4 + y_+}} \simeq \sqrt{\frac{2\pi v_F + W_1}{2\pi v_F + 4W_0 - W_1}}, \\ K_s &= \sqrt{\frac{1-y_-}{1+y_-}} \simeq 1 - \frac{\sqrt{W_0 W_1}}{\sqrt{2}\,\pi v_F}|\delta|, \\ K_\lambda &= \sqrt{\frac{\delta - y_\delta}{\delta + y_\delta}} \simeq \sqrt{\frac{4\pi v_F + W_1}{4\pi v_F - W_1}}, \end{aligned} \tag{31}$$

where the second equalities again hold up to contributions of $\mathcal{O}(\delta^2)$. When the $2k_F$ component of the interaction potential $W_1 = 0$, see Eq. (18), we obtain $K_s = K_\lambda = 1$, and thus recover the theory of Ref. 22. The broken spin $SU(2)$ symmetry is reflected in $K_s < 1$ when both $\delta \neq 0$ and $W_1 \neq 0$.

Since we arrived at a Gaussian field theory, Eq. (29), all low-energy correlation functions can now be computed analytically without further approximation. The linear algebra problem needed for this diagonalization is discussed in App. A.

V. RKKY INTERACTION

Following our discussion in Sec. I, we now investigate the combined effects of the Rashba SOI and the e-e interaction on the RKKY range function. We include the exchange coupling, $H' = J\sum_{i=1,2}\boldsymbol{\Sigma}_i \cdot \boldsymbol{S}(x_i)$, of the 1D conduction electron spin density $\boldsymbol{S}(x)$ to localized spin-1/2 magnetic impurities, separated by $x = x_1 - x_2$. The RKKY interaction $H_{\rm RKKY}$, describing spin-spin interactions between the two magnetic impurities, is then obtained by perturbation theory in J.[48] In the simplest 1D case (no SOI, no interactions), it is given by Eq. (2). In the general case, one can always express it in the form

$$H_{\rm RKKY} = -J^2\sum_{a,b} F^{ab}(x)\Sigma_1^a\Sigma_2^b, \tag{32}$$

with the range function now appearing as a tensor ($\beta = 1/k_BT$ for temperature T),

$$F^{ab}(x) = \int_0^\beta d\tau\ \chi^{ab}(x,\tau). \tag{33}$$

Here, the imaginary-time (τ) spin-spin correlation function appears,

$$\chi^{ab}(x,\tau) = \langle S^a(x,\tau)S^b(0,0)\rangle. \tag{34}$$

The 1D spin densities $S^a(x)$ (with $a = x,y,z$) were defined in Eq. (28), and their bosonized expression is given in App. A, which then allows to compute the correlation functions (34) using the unperturbed ($J = 0$) LL model (29). The range function thus effectively coincides with the static space-dependent spin susceptibility tensor. When spin $SU(2)$ symmetry is realized, $\chi^{ab}(x) = \delta^{ab}F_{\rm ex}(x)$, and one recovers Eq. (2), but in general this tensor is not diagonal. For a LL without Rashba SOI, $F_{\rm ex}(x)$ is as in Eq. (2) but with a slow power-law decay.[52]

If spin $SU(2)$ symmetry is broken, general arguments imply that Eq. (32) can be decomposed into three terms, namely (i) an isotropic exchange scalar coupling, (ii) a Dzyaloshinsky-Moriya (DM) vector term, and (iii) an Ising-like interaction,

$$\begin{aligned} H_{\rm RKKY}/J^2 &= -F_{\rm ex}(x)\boldsymbol{\Sigma}_1\cdot\boldsymbol{\Sigma}_2 - \boldsymbol{F}_{\rm DM}(x)\cdot(\boldsymbol{\Sigma}_1\times\boldsymbol{\Sigma}_2) \\ &\quad - \sum_{a,b} F^{ab}_{\rm Ising}(x)\Sigma_1^a\Sigma_2^b, \end{aligned} \tag{35}$$

where $F_{\rm ex}(x) = \frac{1}{3}\sum_a F^{aa}(x)$. The DM vector has the components

$$F^c_{\rm DM}(x) = \frac{1}{2}\sum_{a,b}\epsilon^{cab}F^{ab}(x),$$

and the Ising-like tensor

$$F^{ab}_{\rm Ising}(x) = \frac{1}{2}\left(F^{ab} + F^{ba} - \frac{2}{3}\sum_c F^{cc}\delta^{ab}\right)(x)$$

is symmetric and traceless. For a 1D noninteracting quantum wire with Rashba SOI, the "twisted" RKKY Hamiltonian (35) has recently been discussed,[49,50,51] and all range functions appearing in Eq. (35) were shown to decay $\propto |x|^{-1}$, as expected for a noninteracting system. Moreover, it has been emphasized[50] that there are different spatial oscillation periods, reflecting the presence of different Fermi momenta $k_F^{(A,B)}$ in a Rashba quantum wire.

Let us then consider the extended LL model (29), which includes the effects of both the e-e interaction and the Rashba SOI. The correlation functions (34) obey $\chi^{ba}(x,\tau) = \chi^{ab}(-x,-\tau)$, and since we find $\chi^{xz} = \chi^{yz} = 0$, the anisotropy acts only in the xy-plane. The four nonzero correlators are specified in App. A, where only the long-ranged $2k_F$ oscillatory terms are kept. These are the relevant correlations determining the RKKY interaction in the interacting quantum wire. We note that in the noninteracting case, there is also a "slow" oscillatory component, corresponding to a contribution to the RKKY range function $\propto \cos\left[\left(k_F^{(A)} - k_F^{(B)}\right)x\right]/|x|$. Remarkably, we find that this $1/x$ decay law is not changed by interactions. However, we will show below that interactions cause a slower decay of certain "fast" oscillatory terms, e.g., the contribution $\propto \cos(2k_F^{(B)}x)$. We therefore do not further discuss the "slow" oscillatory terms in what follows.

Collecting everything, we find the various range functions in Eq. (35) for the interacting case,

$$\begin{aligned} F_{\rm ex}(x) &= \frac{1}{6}\sum_\nu \Big[\big(1+\cos^2(2\theta_\nu)\big)\cos\left(2k_F^{(\nu)}x\right)F_\nu^{(1)}(x) \\ &+ \cos^2(\theta_A+\theta_B)\cos\left[(k_F^{(A)}+k_F^{(B)})x\right]F_\nu^{(2)}(x)\Big], \\ \boldsymbol{F}_{\rm DM}(x) &= \hat{e}_z\sum_\nu \frac{\nu}{2}\cos(2\theta_\nu)\sin\left(2k_F^{(\nu)}x\right)F_\nu^{(1)}(x), \quad (36) \\ F^{ab}_{\rm Ising}(x) &= \left[\frac{1}{2}\sum_\nu G_\nu^a(x) - F_{ex}(x)\right]\delta^{ab}, \end{aligned}$$

with the auxiliary vector

$$\boldsymbol{G}_\nu = \begin{pmatrix} \cos\left(2k_F^{(\nu)}x\right)F_\nu^{(1)}(x) \\ \cos^2(2\theta_\nu)\cos\left(2k_F^{(\nu)}x\right)F_\nu^{(1)}(x) \\ \cos^2(\theta_A+\theta_B)\cos\left[(k_F^{(A)}+k_F^{(B)})x\right]F_\nu^{(2)}(x) \end{pmatrix}.$$

The functions $F_\nu^{(1,2)}(x)$ follow by integration over τ from $\tilde{F}_\nu^{(1,2)}(x,\tau)$, see Eqs. (A1) and (A2) in App. A. This implies the respective decay laws for $a \ll |x| \ll v_F/k_BT$,

$$\begin{aligned} F_\nu^{(1)}(x) &\propto |a/x|^{-1+K_c+K_s+2\nu(1-K_c/K_\lambda^2)\frac{v_\lambda K_\lambda}{v_c+v_s}}, \quad (37) \\ F_\nu^{(2)}(x) &\propto |a/x|^{-1+K_c+1/K_s}. \end{aligned}$$

All those exponents approach unity in the noninteracting limit, in accordance with previous results.[49,50] Moreover, in the absence of SOI ($\alpha = \delta = 0$), Eq. (37) reproduces the known $|x|^{-K_c}$ decay law for the RKKY interaction in a conventional LL.[52]

Since $K_s < 1$ for an interacting Rashba wire with $\delta \neq 0$, see Eq. (31), we conclude that $F_\nu^{(1)}$ with $\nu = B$, corresponding to the slower velocity $v_B = v_F(1-\delta)$, leads to the slowest decay of the RKKY interaction. For large distance x, the RKKY interaction is therefore dominated by the $2k_F^{(B)}$ oscillatory part, and all range functions decay $\propto |x|^{-\eta_B}$ with the exponent

$$\eta_B = K_c + K_s - 1 - 2\left(1 - \frac{K_c}{K_\lambda^2}\right)\frac{v_\lambda K_\lambda}{v_c+v_s} < 1. \quad (38)$$

This exponent depends both on the e-e interaction potential and on the Rashba coupling α. The latter dependence also implies that electric fields are able to change the power-law decay of the RKKY interaction in a Rashba wire. The DM vector coupling also illustrates that the SOI is able to effectively induce off-diagonal couplings in spin space, reminiscent of spin precession effects. Also these RKKY couplings are $2k_F^{(B)}$ oscillatory and show a power-law decay with the exponent (38).

VI. DISCUSSION

In this paper, we have presented a careful derivation of the low-energy Hamiltonian of a homogeneous 1D quantum wire with not too weak Rashba spin-orbit interactions. We have studied the simplest case (no magnetic field, no disorder, single-channel limit), and in particular analyzed the possibility for a spin gap to occur because of electron-electron backscattering processes. The initial values for the coupling constants entering the one-loop RG equations were determined, and for rather general conditions, they are such that backscattering is marginally irrelevant and no spin gap opens. The resulting low-energy theory is a modified Luttinger liquid, Eq. (29), which is a Gaussian field theory formulated in terms of the boson fields $\Phi_c(x)$ and $\Phi_s(x)$ (and their dual fields). In this state, spin-charge separation is violated due to the Rashba coupling, but the theory still admits exact results for essentially all low-energy correlation functions.

Based on our bosonized expressions for the 1D charge and spin density, the frequency dependence of various susceptibilities of interest, e.g., charge- or spin-density wave correlations, can then be computed. As the calculation closely mirrors the one in Refs. 34,35, we do not repeat it here. One can then infer a "phase diagram" from the study of the dominant susceptibilities. According to our calculations, due to a conspiracy of the Rashba SOI and the e-e interaction, spin-density-wave correlations in the xy plane are always dominant for repulsive interactions.

We have studied the RKKY interaction between two magnetic impurities in such an interacting 1D Rashba

quantum wire. On general grounds, the RKKY interaction can be decomposed into an exchange term, a DM vector term, and a traceless symmetric tensor interaction. For a noninteracting wire, the corresponding three range functions have several spatial oscillation periods with a common overall decay $\propto |x|^{-1}$. We have shown that interactions modify this picture. The dominant contribution (characterized by the slowest power-law decay) to the RKKY range function is now $2k_F^{(B)}$ oscillatory for all three terms, with the same exponent $\eta_B < 1$, see Eq. (38). This exponent depends both on the interaction strength and on the Rashba coupling. This raises the intriguing possibility to tune the power-law exponent η_B governing the RKKY interaction by an electric field, since α is tunable via a backgate voltage. We stress again that interactions imply that a single spatial oscillation period (wavelength $\pi/k_F^{(B)}$) becomes dominant, in contrast to the noninteracting situation where several competing wavelengths are expected.

The above formulation also holds promise for future calculations of spin transport in the presence of both interactions and Rashba spin-orbit couplings, and possibly with disorder. Under a perturbative treatment of impurity backscattering, otherwise exact statements are possible even out of equilibrium. We hope that our work will motivate further studies along this line.

Acknowledgments

We wish to thank W. Häusler and U. Zülicke for helpful discussions. This work was supported by the SFB TR 12 of the DFG, and by the ESF network INSTANS.

APPENDIX A: BOSONIZATION FOR THE EXTENDED LUTTINGER LIQUID

In this appendix, we provide some technical details related to the evaluation of the spin-spin correlation function under the extended Luttinger theory (29). The exact calculation of such correlations is possible within the bosonization framework, and requires a diagonalization of Eq. (29).

The 1D charge and spin densities (28) can be written as the sum of slow and fast (oscillatory) contributions. Using Eq. (17), the bosonized form for the 1D charge density is

$$\begin{aligned}\rho(x) &= \sqrt{\frac{2}{\pi}}\,\partial_x\Phi_c - \frac{2i}{\pi a}\eta_{AR}\eta_{AL}\cos(\theta_A-\theta_B)\\ &\times \sin\left[\left(k_F^{(A)}+k_F^{(B)}\right)x+\sqrt{2\pi}\Phi_c\right]\cos(\sqrt{2\pi}\Theta_s).\end{aligned}$$

Similarly, using the identity

$$\begin{aligned}&\int dz\,\left[\phi^\dagger_{rk_F^{(\nu)},\nu r}\,\boldsymbol{\sigma}\,\phi_{r'k_F^{(\nu')},\nu' r'}\right](z) =\\ &\delta_{r,r'}\begin{pmatrix}\cos(\theta_A-\theta_B)\,\delta_{\nu,-\nu'}\\ -i\nu r\,\cos(\theta_A+\theta_B)\,\delta_{\nu,-\nu'}\\ \nu r\,\cos(2\,\theta_\nu)\,\delta_{\nu,\nu'}\end{pmatrix}+\\ &+\delta_{r,-r'}\begin{pmatrix}\delta_{\nu,\nu'}\\ -i\nu r\,\cos(2\,\theta_\nu)\,\delta_{\nu,\nu'}\\ \nu r\,\cos(\theta_A+\theta_B)\,\delta_{\nu,-\nu'}\end{pmatrix},\end{aligned}$$

the 1D spin density vector has the components

$$\begin{aligned}S^x(x) &= -i\frac{\eta_{AR}\eta_{BR}}{\pi a}\cos(\theta_A-\theta_B)\cos\left[\left(k_F^{(A)}-k_F^{(B)}\right)x+\sqrt{2\pi}\Phi_s\right]\sin(\sqrt{2\pi}\Theta_s)\\ &- i\frac{\eta_{AR}\eta_{AL}}{\pi a}\cos\left[\left(k_F^{(A)}+k_F^{(B)}\right)x+\sqrt{2\pi}\Phi_c\right]\sin\left[\left(k_F^{(A)}-k_F^{(B)}\right)x+\sqrt{2\pi}\Phi_s\right],\end{aligned}$$

$$\begin{aligned}S^y(x) &= i\frac{\eta_{AR}\eta_{BR}}{\pi a}\cos(\theta_A+\theta_B)\sin\left[\left(k_F^{(A)}-k_F^{(B)}\right)x+\sqrt{2\pi}\Phi_s\right]\sin(\sqrt{2\pi}\Theta_s)\\ &- i\sum_{\nu=A,B=+,-}\nu\frac{\eta_{\nu R}\eta_{\nu L}}{2\pi a}\cos(2\theta_\nu)\cos\left[2k_F^{(\nu)}x+\sqrt{2\pi}\left(\Phi_c+\nu\Phi_s\right)\right],\end{aligned}$$

$$\begin{aligned}S^z(x) &= \frac{1}{\sqrt{8\pi}}\left[\left(\cos 2\theta_A+\cos 2\theta_B\right)\partial_x\Theta_s+\left(\cos 2\theta_A-\cos 2\theta_B\right)\partial_x\Theta_c\right]\\ &- i\frac{\eta_{AR}\eta_{BL}}{\pi a}\cos(\theta_A+\theta_B)\cos\left[\left(k_F^{(A)}+k_F^{(B)}\right)x+\sqrt{2\pi}\Phi_c\right]\sin(\sqrt{2\pi}\Phi_s).\end{aligned}$$

Note that while $\partial_x\Phi_c$ is proportional to the (slow part of the) charge density, the (slow) spin density is determined by both c and s sectors.

Next we specify the nonzero components of the

imaginary-time spin-spin correlation function $\chi^{ab}(x,\tau)$, see Eq. (34). Using the above bosonized expressions, some algebra yields

$$\chi^{xx}(x,\tau) = \sum_\nu \frac{\cos\left(2k_F^{(\nu)}x\right)}{2(2\pi a)^2}\tilde{F}_\nu^{(1)}(x,\tau),$$

$$\chi^{yy}(x,\tau) = \sum_\nu \frac{\cos^2(2\theta_\nu)\cos\left(2k_F^{(\nu)}x\right)}{2(2\pi a)^2}\tilde{F}_\nu^{(1)}(x,\tau),$$

$$\chi^{zz}(x,\tau) = \sum_{\nu r} \frac{\cos^2(\theta_A+\theta_B)}{2(2\pi a)^2} \times \cos\left[\left(k_F^{(A)}+k_F^{(B)}\right)x\right]\tilde{F}_\nu^{(2)}(x,\tau),$$

and

$$\chi^{xy}(x,\tau) = \sum_\nu \frac{\nu\cos(2\theta_\nu)\sin\left(2k_F^{(\nu)}x\right)}{2(2\pi a)^2}\tilde{F}_\nu^{(1)}(x,\tau).$$

Here, the functions $\tilde{F}^{(1,2)}_{\nu=A,B=+,-}(x,\tau)$ are given by

$$\tilde{F}_\nu^{(1)}(x,\tau) = \prod_{j=1,2}\left|\frac{\beta u_j}{\pi a}\sin\left(\frac{\pi(u_j\tau-ix)}{\beta u_j}\right)\right|^{-\left(\Gamma^{(j)}_{\Phi_c\Phi_c}+\Gamma^{(j)}_{\Phi_s\Phi_s}+2\nu\Gamma^{(j)}_{\Phi_c\Phi_s}\right)}$$

and

$$\tilde{F}_\nu^{(2)}(x,\tau) = \prod_{j=1,2}\left|\frac{\beta u_j}{\pi a}\sin\left(\frac{\pi(u_j\tau-ix)}{\beta u_j}\right)\right|^{-\left(\Gamma^{(j)}_{\Phi_c\Phi_c}+\Gamma^{(j)}_{\Theta_s\Theta_s}\right)}\left[\frac{\sin\left(\frac{\pi(u_j\tau+ix)}{\beta u_j}\right)}{\sin\left(\frac{\pi(u_j\tau-ix)}{\beta u_j}\right)}\right]^{\nu\Gamma^{(j)}_{\Phi_c\Theta_s}}.$$

The dimensionless numbers $\Gamma^{(j)}$ appearing in the exponents follow from the straightforward (but lengthy) diagonalization of the extended LL Hamiltonian (29), where the u_j are the velocities of the corresponding normal modes. With the velocities (30) and the dimensionless Luttinger parameters (31), the result of this linear algebra problem can be written as follows. The normal-mode velocities u_1 and u_2 are

$$2u^2_{j=1,2} = v_c^2+v_s^2+2v_\lambda^2-(-1)^j\Big[(v_c^2-v_s^2)^2+ +4v_\lambda^2\left[v_cv_s\left(\frac{K_\lambda^2}{K_cK_s}+\frac{K_cK_s}{K_\lambda^2}\right)+v_c^2+v_s^2\right]\Big]^{1/2},$$

and the exponents $\Gamma^{(j=1,2)}$ appearing in $\tilde{F}_\nu^{(1,2)}(x,\tau)$ are given by

$$\Gamma^{(j)}_{\Phi_c\Phi_c} = \frac{(-1)^jK_cv_c}{u_j(u_1^2-u_2^2)}\left(v_s^2-u_j^2-\frac{K_\lambda^2v_\lambda^2v_s}{K_cK_sv_c}\right),$$

$$\Gamma^{(j)}_{\Phi_s\Phi_s} = \frac{(-1)^jK_sv_s}{u_j(u_1^2-u_2^2)}\left(v_c^2-u_j^2-\frac{K_\lambda^2v_\lambda^2v_c}{K_cK_sv_s}\right),$$

$$\Gamma^{(j)}_{\Phi_c\Phi_s} = \frac{(-1)^jK_\lambda v_\lambda}{u_j(u_1^2-u_2^2)}\left(v_\lambda^2-u_j^2-\frac{K_cK_sv_sv_c}{K_\lambda^2}\right),$$

$$\Gamma^{(j)}_{\Theta_s\Theta_s} = \frac{(-1)^jv_s}{K_su_j(u_1^2-u_2^2)}\left(v_c^2-u_j^2-\frac{K_cK_sv_\lambda^2v_c}{K_\lambda^2v_s}\right),$$

$$\Gamma^{(j)}_{\Phi_c\Theta_s} = \frac{(-1)^jv_\lambda}{u_1^2-u_2^2}\left(\frac{K_\lambda}{K_s}v_s+\frac{K_c}{K_\lambda}v_c\right).$$

Since $|\delta| \ll 1$, we now employ the simplified expressions for the velocities in Eq. (30) and the Luttinger liquid parameters in Eq. (31), which are valid up to $\mathcal{O}(\delta^2)$ corrections. In the interacting case, this yields for the normal-mode velocities simply $u_1 = v_c$ and $u_2 = v_s$. (In the noninteracting limit, the above equation instead yields $u_1 = v_A$ and $u_2 = v_B$, see Eq. (1).) Moreover, the exponents $\Gamma^{(j)}$ simplify to

$$\Gamma^{(1)}_{\Phi_c\Phi_c} = K_c, \quad \Gamma^{(2)}_{\Phi_c\Phi_c} = \Gamma^{(1)}_{\Phi_s\Phi_s} = \Gamma^{(1)}_{\Theta_s\Theta_s} = 0,$$

$$\Gamma^{(2)}_{\Phi_s\Phi_s} = K_s, \quad \Gamma^{(2)}_{\Theta_s\Theta_s} = 1/K_s,$$

$$\Gamma^{(1)}_{\Phi_c\Phi_s} = \frac{v_\lambda}{v_c^2-v_s^2}(K_\lambda v_c+K_cv_s/K_\lambda),$$

$$\Gamma^{(2)}_{\Phi_c\Phi_s} = -\frac{v_\lambda}{v_c^2-v_s^2}(K_\lambda v_s+K_cv_c/K_\lambda),$$

$$\Gamma^{(1,2)}_{\Phi_c\Theta_s} = \pm\Gamma^{(2)}_{\Phi_c\Phi_s}.$$

Collecting everything and taking the zero-temperature limit, the functions $\tilde{F}^{(1,2)}_{\nu=\pm}(x,\tau)$ take the form

$$\tilde{F}^{(1)}_{\nu}(x,\tau) = \left|\frac{v_c\tau - ix}{a}\right|^{-K_c - 2\nu v_\lambda \frac{K_\lambda v_c + K_c v_s/K_\lambda}{v_c^2 - v_s^2}} \times \left|\frac{v_s\tau - ix}{a}\right|^{-K_s + 2\nu v_\lambda \frac{K_\lambda v_s + K_c v_c/K_\lambda}{v_c^2 - v_s^2}}, \quad \text{(A1)}$$

and

$$\tilde{F}^{(2)}_{\nu}(x,\tau) = \left|\frac{v_c\tau - ix}{a}\right|^{-K_c} \left|\frac{v_s\tau - ix}{a}\right|^{-1/K_s} \times \left(\frac{(v_s\tau - ix)(v_c\tau + ix)}{(v_s\tau + ix)(v_c\tau - ix)}\right)^{-\nu \frac{v_\lambda(K_\lambda v_s + K_c v_c/K_\lambda)}{v_c^2 - v_s^2}}. \quad \text{(A2)}$$

The known form of the spin-spin correlations in a LL with $\alpha = 0$ is recovered by putting $v_\lambda \propto \delta = 0$.

[1] I. Zutic, J. Fabian, and S. Das Sarma, Rev. Mod. Phys. **76**, 323 (2004).
[2] S. Datta and B. Das, Appl. Phys. Lett. **56**, 665 (1990).
[3] Y.A. Bychkov and E.I. Rashba, J. Phys. C **17**, 6039 (1984).
[4] R. Winkler, Phys. Rev. B **62**, 4245 (2000).
[5] R. Winkler, *Spin-Orbit Coupling Effects in Two-Dimensional Electron and Hole Systems* (Springer, Berlin, 2003).
[6] J. Schliemann, J.C. Egues, and D. Loss, Phys. Rev. Lett. **90**, 146801 (2003).
[7] B. Andrei Bernevig, J. Orenstein, and S.-C. Zhang, Phys. Rev. Lett. **97**, 236601 (2006).
[8] J. Nitta, T. Akazaki, H. Takayanagi, and T. Enoki, Phys. Rev. Lett. **78**, 1335 (1997).
[9] G. Engels, J. Lange, Th. Schäpers, and H. Lüth, Phys. Rev. B **55**, R1958 (1997).
[10] D. Grundler, Phys. Rev. Lett. **84**, 6074 (2000).
[11] Y. Kato, R.C. Myers, A.C. Gossard, and D.D. Awschalom, Nature **427**, 50 (2004).
[12] J. Schäfer, C. Blumenstein, S. Meyer, M. Wisniewski, and R. Claessen, Phys. Rev. Lett. **101**, 236802 (2008).
[13] A.V. Moroz and C.H.W. Barnes, Phys. Rev. B **60**, 14272 (1999).
[14] A.V. Moroz and C.H.W. Barnes, Phys. Rev. B **61**, R2464 (2000).
[15] M. Governale and U. Zülicke, Phys. Rev. B **66**, 073311 (2002).
[16] E.A. de Andrada e Silva and G.C. La Rocca, Phys. Rev. B **67**, 165318 (2003).
[17] S.L. Erlingsson, J.C. Egues, and D. Loss, phys. stat. sol. (c) **3**, 4317 (2006).
[18] C.A. Perroni, D. Bercioux, V. Marigliano Ramaglia, and V. Cataudella, J. Phys.: Cond. Matt. **19**, 186227 (2007).
[19] A similar situation arises in chiral carbon nanotubes, see: A. De Martino, R. Egger, and A.M. Tsvelik, Phys. Rev. Lett. **97**, 076402 (2006).
[20] W. Häusler, Phys. Rev. B **63**, 121310(R) (2001).
[21] T. Kimura, K. Kuroki, and H. Aoki, Phys. Rev. B **53**, 9572 (1996).
[22] M. Governale and U. Zülicke, Solid State Comm. **131**, 581 (2004).
[23] T. Kaneko, M. Koshino, and T. Ando, Phys. Rev. B **78**, 245303 (2008).
[24] Y.V. Pershin, J.A. Nesteroff, and V. Privman, Phys. Rev. B **69**, 121306(R) (2004).
[25] J. Knobbe and Th. Schäpers, Phys. Rev. B **71**, 035311 (2005).
[26] R.G. Pereira and E. Miranda, Phys. Rev. B **71**, 085318 (2005).
[27] S. Debald and B. Kramer, Phys. Rev. B **71**, 115322 (2005).
[28] L. Serra, D. Sanchez, and R. Lopez, Phys. Rev. B **72**, 235309 (2005).
[29] X.F. Wang, Phys. Rev. B **69**, 035302 (2004).
[30] S. Kettemann, Phys. Rev. Lett. **98**, 176808 (2007).
[31] M. Scheid, M. Kohda, Y. Kunihashi, K. Richter, and J. Nitta, Phys. Rev. Lett. **101**, 266401 (2008).
[32] A.O. Gogolin, A.A. Nersesyan, and A.M. Tsvelik, *Bosonization and Strongly Correlated Systems* (Cambridge University Press, 1998); T. Giamarchi, *Quantum Physics in One Dimension* (Oxford University Press, 2004).
[33] A.V. Moroz, K.V. Samokhin, and C.H.W. Barnes, Phys. Rev. Lett. **84**, 4164 (2000).
[34] A.V. Moroz, K.V. Samokhin, and C.H.W. Barnes, Phys. Rev. B **62**, 16900 (2000).
[35] A. De Martino and R. Egger, Europhys. Lett. **56**, 570 (2001).
[36] A. Iucci, Phys. Rev. B **68**, 075107 (2003).
[37] W. Häusler, Phys. Rev. B **70**, 115313 (2004).
[38] V. Gritsev, G.I. Japaridze, M. Pletyukhov, and D. Baeriswyl, Phys. Rev. Lett. **94**, 137207 (2005).
[39] J. Sun, S. Gangadharaiah, and O.A. Starykh, Phys. Rev. Lett. **98**, 126408 (2007).
[40] S. Gangadharaiah, J. Sun, and O.A. Starykh, Phys. Rev. B **78**, 054436 (2008).
[41] Y. Yu, Y. Wen, J. Li, Z. Su, and S.T. Chui, Phys. Rev. B **69**, 153307 (2004).
[42] H.C. Lee and S.R. Eric Yang, Phys. Rev. B **72**, 245338 (2005).
[43] F. Cheng and G. Zhou, J. Phys.: Cond. Matt. **19**, 136215 (2007).
[44] P. Devillard, A. Crepieux, K.I. Imura, and T. Martin, Phys. Rev. B **72**, 041309(R) (2005).
[45] A. De Martino, R. Egger, K. Hallberg, and C.A. Balseiro, Phys. Rev. Lett. **88**, 206402 (2002).
[46] M. Zarea and N. Sandler, Phys. Rev. Lett. **99**, 256804 (2007).
[47] M.A. Ruderman and C. Kittel, Phys. Rev. **96**, 99 (1954); T. Kasuya, Prog. Theor. Phys. **16**, 45 (1956); K. Yosida, Phys. Rev. **106**, 893 (1957).
[48] C. Kittel, Sol. State Phys. **22**, 1 (1968); Y. Yafet, Phys. Rev. B **36**, 3948 (1987).
[49] H. Imamura, P. Bruno, and Y. Utsumi, Phys. Rev. B **69**, 121303(R) (2004).
[50] P. Lyu, N.-N. Liu, and C. Zhang, J. Appl. Phys. **102**, 103910 (2007).
[51] J. Simonin, Phys. Rev. Lett. **97**, 266804 (2006).
[52] R. Egger and H. Schoeller, Phys. Rev. B **54**, 16337 (1996); K. Hallberg and R. Egger, Phys. Rev. B **55**, R8646 (1997).

[53] Technically, one performs a gradient expansion in the relative coordinate $x_1 - x_2$. In this procedure, it is crucial to understand the products of fermionic operators in the sense of operator product expansions. Corrections to the leading terms are then irrelevant in the RG sense.

[54] O.A. Starykh, D.L. Maslov, W. Häusler, and L.I. Glazman, in *Interactions and Quantum Transport Properties of Lower Dimensional Systems*, Lecture Notes in Physics, Springer (Proceedings of the International WEH Workshop, July 1999, Hamburg).

[55] Note that Eq. (16) excludes the case of ultra-local contact interactions.

[56] It is in principle possible that such a process becomes important if a collective density readjustment between subbands takes place in the wire. However, this can only happen for almost equivalent subbands, see Ref. 54 for a detailed discussion. Here we assume that the SOI is strong enough to guarantee that such a readjustment does not occur.

[57] K.A. Muttalib and V.J. Emery, Phys. Rev. Lett. **57**, 1370 (1986).

[58] We adopt the convention $\eta_{\nu R}\eta_{\nu L}\eta_{-\nu,R}\eta_{-\nu,L} = 1$.