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Solving the quantum nonlinear Schrödinger equation with δ-type impurity

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We establish the exact solution of the nonlinear Schrödinger equation with a delta-function impurity, representing a pointlike defect which reflects and transmits. We solve the problem both at the classical and the second quantized levels. In the quantum case the Zamolodchikov–Faddeev algebra, familiar from the case without impurities, is substituted by the recently discovered reflection-transmission (RT) algebra, which captures both particle–particle and particle–impurity interactions. The off-shell quantum solution is expressed in terms of the generators of the RT algebra and the exact scattering matrix of the theory is derived. © *2005 American Institute of Physics.* [DOI: 10.1063/1.1842353]

I. INTRODUCTION

Impurity problems arise in different areas of quantum field theory and are essential for understanding a number of phenomena in condensed matter physics. At the experimental side, the recent interest in pointlike impurities (defects) is triggered by the great progress in building nanoscale devices.

The interaction of quantum fields with impurities represents in general a hard and yet unsolved problem, but there are relevant achievements[1–11] in the case of integable systems in $1+1$ space–time dimensions. The study[12–19] of the special case of purely reflecting impurities (boundaries) indicates factorized scattering theory[20–24] as the most efficient method for dealing with this kind of problem. The method provides on-shell information about the system and allows to derive the exact scattering matrix. The goal of the present paper is to extend this framework, exploring the possibility to recover off-shell information and to reconstruct the quantum fields, generating the above scattering matrix. We test this possibility on one of the most extensively studied integrable systems—the nonlinear Schrödinger (NLS) model.[25–32] More precisely, we are concerned below with the NLS model coupled to a delta-function impurity. The basic tool of our investigation is a specific exchange algebra,[6,7] called reflection-transmission (RT) algebra. The RT algebra is a generalization of the Zamolodchikov–Faddeev (ZF)[21,23] algebra used in the case without defects. The RT algebra is originally designed for the construction of the total scattering operator from the fundamental scattering data, namely the two-body bulk scattering matrix and the reflec-

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tion and transmission amplitudes of a single particle interacting with the defect. In what follows we demonstrate that in the NLS model the same algebra allows to reconstruct the corresponding off-shell quantum field as well. Being the first exactly solvable example with nontrivial bulk scattering matrix, the NLS model sheds some light on the interplay between pointlike impurities, integrability, and symmetries. In this respect our solution clarifies a debated question about the Galilean invariance of the bulk scattering matrix.

After introducing the model in Sec. II, we establish the solution, both at the classical (Sec. II B) and second-quantized (Sec. III) levels. We do this in detail, clarifying the basic properties of the solution. In Sec. IV we derive from the off-shell quantum field the total scattering matrix of the model, showing that it coincides with the one obtained directly from factorized scattering. In Sec. V we indicate some generalizations. Our conclusions and ideas about further developments are also collected there. Appendixes A and B are devoted to the proofs of some technical results.

We present below the analysis of the so-called δ-type impurity. A wider class of defects, interacting with the NLS model and preserving its integrability, can be treated in a similar way.[33] We have chosen to focus here on the particular δ-type defect in order to keep the length of the proofs reasonable, referring to Ref. 33 for a more physically oriented treatment of the general case (without detailed proofs).

II. INTRODUCING AN IMPURITY IN THE NLS MODEL

We start by recalling some well-known results about the NLS model without impurity. The reason for this is twofold: first, because this is a good guide to tackle the problem with impurity and second, because the central piece of the solution of the NLS model, the Rosales expansion,[34,35] can be adapted to the impurity case.

A. The model to solve

The field theoretic version of NLS is described by a classical complex field $\Phi(t,x)$ whose equation of motion reads

$$(i\partial_t + \partial_x^2)\Phi(t,x) = 2g|\Phi(t,x)|^2\Phi(t,x). \tag{2.1}$$

The corresponding action takes the form

$$\mathcal{A}_{\mathrm{NLS}} = \int_{\mathbb{R}} \mathrm{d}t \int_{\mathbb{R}} \mathrm{d}x (i\overline{\Phi}(t,x)\partial_t\Phi(t,x) - |\partial_x\Phi(t,x)|^2 - g|\Phi(t,x)|^4), \tag{2.2}$$

and, being in particular invariant under time translation, ensures the conservation of the energy

$$\mathcal{E}_{\mathrm{NLS}} = \int_{\mathbb{R}} \mathrm{d}x(|\partial_x\Phi(t,x)|^2 + g|\Phi(t,x\ |^4). \tag{2.3}$$

The latter is non-negative for $g \geqslant 0$.

It is well-known that this is a nonrelativistic integrable model[36] (see also Ref. 30 for a review) and an explicit solution for the field was given by Rosales in Ref. 34,

$$\Phi(t,x) = \sum_{n=0}^{\infty} (-g)^n \Phi^{(n)}(t,x), \tag{2.4}$$

where

$$\Phi^{(n)}(t,x) = \int_{\mathbb{R}^{2n+1}} \prod_{\substack{i=1\\ j=0}}^{n} \frac{\mathrm{d}p_i}{2\pi}\frac{\mathrm{d}q_j}{2\pi}\overline{\lambda}(p_1)\cdots\overline{\lambda}(p_n)\lambda(q_n)\cdots\lambda(q_0)\frac{e^{i\Sigma_{j=0}^{n}(q_jx-q_j^2t)-i\Sigma_{i=1}^{n}(p_ix-p_i^2t)}}{\prod_{i=1}^{n}(p_i-q_{i-1})(p_i-q_i)} \tag{2.5}$$

and the overbar denotes complex conjugation.

The level $n=0$ is the linear part of the field corresponding to the free Schrödinger equation. It was argued in Ref. 32 that this solution is well-defined for a large class of functions λ [containing the Schwarz space $\mathcal{S}(\mathbb{R})$] and an upper bound for g was given for the series (2.4) to converge uniformly in x. It also represents a physical field since it vanishes as $x\to\pm\infty$. In the same paper, the authors considered NLS on the half-line $\mathbb{R}^+$, which can be seen as the model on the whole line in the presence of a purely reflecting impurity sitting at the origin. Therefore, the latter represents a particular case of the model with transmitting and reflecting impurity at $x=0$ we wish to contemplate in this paper. They gave the following action:

$$\mathcal{A}_R = \int_{\mathbb{R}}\mathrm{d}t\int_{\mathbb{R}^+}\mathrm{d}x(i\overline{\Phi}(t,x)\partial_t\Phi(t,x) - |\partial_x\Phi(t,x)|^2 - g|\Phi(t,x)|^4) - \eta\int_{\mathbb{R}}\mathrm{d}t|\Phi(t,0)|^2,$$

where $\eta\in\mathbb{R}$ is the parameter controlling the boundary condition

$$\lim_{x\to 0^+}(\partial_x - \eta)\Phi(t,x) = 0. \tag{2.6}$$

In our case, since the impurity is allowed to reflect and transmit, we must take the $\mathbb{R}^-$ part into account and we are led to work with the following action:

$$\mathcal{A}_{\mathrm{RT}} = \mathcal{A}_+ + \mathcal{A}_- + \mathcal{A}_0, \tag{2.7}$$

where

$$\mathcal{A}_\pm = \int_{\mathbb{R}}\mathrm{d}t\int_{\mathbb{R}^\pm}\mathrm{d}x(i\overline{\Phi}(t,x)\partial_t\Phi(t,x) - |\partial_x\Phi(t,x)|^2 - g|\Phi(t,x)|^4), \tag{2.8}$$

$$\mathcal{A}_0 = -2\eta\int_{\mathbb{R}}\mathrm{d}t|\Phi(t,0)|^2. \tag{2.9}$$

The form of $\mathcal{A}_{\mathrm{RT}}$ shows the particular status of the origin $x=0$ where the impurity sits. Again, the invariance of the action under time translations ensures the conservation of the energy,

$$\mathcal{E}_{\mathrm{RT}} = \int_{\mathbb{R}^-\oplus\mathbb{R}^+}\mathrm{d}x(|\partial_x\Phi(t,x)|^2 + g|\Phi(t,x)|^4) + 2\eta|\Phi(t,0)|^2. \tag{2.10}$$

It is positive for $g\geqslant 0, \eta\geqslant 0$, which is what we assume in the rest of this paper. We will see that η characterizes the transmission and reflection properties of the impurity. Using the variational principle, one deduces the equation of motion and the boundary conditions for the field: $\Phi(t,x)$ must be the solution of NLS on $\mathbb{R}^-$ and $\mathbb{R}^+$, continuous at $x=0$ and satisfy a "jump condition" at the origin. It must also vanish at infinity as a physical field.

Definition 2.1: The nonlinear Schrödinger model with a transmitting and reflecting impurity at the origin is described by the following boundary problem for the field $\Phi(t,x)$:

$$(i\partial_t + \partial_x^2)\Phi(t,x) - 2g|\Phi(t,x)|^2\Phi(t,x) = 0, \quad x\neq 0, \tag{2.11}$$

$$\lim_{x\to 0^+}\{\Phi(t,x) - \Phi(t,-x)\} = 0, \tag{2.12}$$

$$\lim_{x\to 0^+}\{(\partial_x\Phi)(t,x)-(\partial_x\Phi)(t,-x)\}-2\eta\Phi(t,0)=0, \tag{2.13}$$

$$\lim_{x\to\pm\infty}\Phi(t,x)=0. \tag{2.14}$$

B. Explicit solution

As announced, the Rosales solution[34] can be adapted suitably to solve the problem of definition 2.1. Since (2.4) is a solution of NLS on $\mathbb{R}$, it is easy to devise a solution for (2.11). Starting from two copies of (2.4) and (2.5), one based on a function λ_+ and the other on a function λ_-, denoted $\Phi_+(t,x)$ and $\Phi_-(t,x)$, respectively, we define

$$\Phi(t,x)=\begin{cases}\Phi_+(t,x), & x>0,\\ \Phi_-(t,x), & x<0,\\ \frac{1}{2}(\Phi_+(t,0)+\Phi_-(t,0)), & x=0.\end{cases} \tag{2.15}$$

It is clearly solution of (2.11) for $x\neq 0$ and from the vanishing of $\Phi_\pm(t,x)$ as $x\to\pm\infty$, (2.14) is also satisfied. However, there is no reason why, in general, $\Phi(t,x)$ so defined should satisfy the boundary conditions (2.12) and (2.13). In order to satisfy these conditions, we parametrize λ_+, λ_- as follows:

$$\begin{pmatrix}\lambda_+(p)\\ \lambda_-(p)\end{pmatrix}=\begin{pmatrix}1 & T(p)\\ T(-p) & 1\end{pmatrix}\begin{pmatrix}\mu_+(p)\\ \mu_-(p)\end{pmatrix}+\begin{pmatrix}R(p) & 0\\ 0 & R(-p)\end{pmatrix}\begin{pmatrix}\mu_+(-p)\\ \mu_-(-p)\end{pmatrix}, \tag{2.16}$$

where

$$T(p)=\frac{p}{p+i\eta},\quad R(p)=\frac{-i\eta}{p+i\eta},\quad p\in\mathbb{R}, \tag{2.17}$$

and $\mu_\pm(p)$ are arbitrary Schwarz test functions. Then, the functions $\lambda_\pm(p)$ satisfy

$$\lambda_\pm(p)=T(\pm p)\lambda_\mp(p)+R(\pm p)\lambda_\pm(-p),\qquad \forall\, p\in\mathbb{R} \tag{2.18}$$

which follows from the identities

$$R(p)R(-p)+T(p)T(-p)=1\quad\text{and}\quad T(p)R(-p)+R(p)T(-p)=0,\qquad \forall\, p\in\mathbb{R}. \tag{2.19}$$

These relations plus a particular choice for the form of $\mu_\pm$ will be essential in the proof of the theorem 2.2 below.

Anticipating the quantum case, if we interpret λ_+ (respectively, λ_-) as a wave packet, (2.18) shows that each wave packet in $\mathbb{R}^+$ (respectively, $\mathbb{R}^-$) is equivalent to the superimposition of a transmitting part coming from $\mathbb{R}^-$ (respectively, $\mathbb{R}^+$) and a reflected part in $\mathbb{R}_+$ (respectively, $\mathbb{R}_-$). This physical interpretation will show up in the next section when we construct a Fock representation of the creation and annihilation operators.

We are now in position to state the main result of this section whose lengthy proof we defer until Appendix A.

Theorem 2.2: Let μ_+, μ_- *be given by*

$$\mu_\pm(k)=\pm\frac{\mu_0(\pm k)+(k\mp i\eta)\mu_1(k)}{k\mp i\eta+1}, \tag{2.20}$$

where μ_0, μ_1 are arbitrary Schwartz functions, μ_1 being even and let $\Phi_+(t,x)$, $\Phi_-(t,x)$ be given by the Rosales expansion (2.4) and (2.5) with λ replaced by λ_+ and λ_-, respectively. Then, $\Phi(t,x)$ as defined in (2.15) satisfies the boundary conditions (2.12) and (2.13), i.e.,

$$\lim_{x\to 0^+}\{\Phi(t,x)-\Phi(t,-x)\}=0,$$

$$\lim_{x\to 0^+}\{(\partial_x\Phi)(t,x)-(\partial_x\Phi)(t,-x)\}-2\eta\Phi(t,0)=0.$$

With this result, we can say that $\Phi(t,x)$ rewritten as

$$\Phi(t,x)=\theta(x)\Phi_+(t,x)+\theta(-x)\Phi_-(t,x), \tag{2.21}$$

where $\theta(x)$ is the Heaviside function defined here to be $\frac{1}{2}$ at $x=0$, is the classical solution of the nonlinear Schrödinger model with impurity as given in definition 2.1.

We want to emphasize that these boundary conditions decouple for the nonlinear part of the field (as shown in Appendix A) and this is due to the reflection-transmission property (2.18) satisfied by λ_+ and λ_-. This already gives a good hint that the construction of a local field from the quantum counterparts of λ_+, λ_- is achievable, as we now explain.

III. QUANTIZATION OF THE SYSTEM

In this section, we move on to the construction and resolution of the quantized version of NLS with impurity. As we mentioned earlier, the crucial ingredient is the RT algebra which encodes the properties of the impurity.

A. Reflection-transmission algebra

Here we rely on the constructions developed in Ref. 7 and recast them in the particular context of the scalar nonlinear Schrödinger model (no internal degrees of freedom, special form of the exchange matrix and of the generators, see also Ref. 11).

We consider the associative algebra with identity element **1** and two sets of generators, $\{a_\alpha(p), a_\alpha^\dagger(p); p\in\mathbb{R}, \alpha=\pm\}$ and $\{r(p), t(p); p\in\mathbb{R}\}$, called the bulk and defect (reflection and transmission) generators. The label $\alpha=\pm$ refers to the half-line $\mathbb{R}^\pm$ with respect to the impurity (in practice it will indicate where the particle is created or annihilated). Introducing the measurable function $S:\mathbb{R}\times\mathbb{R}\to\mathbb{C}$ defined by

$$S(p)=\frac{p-ig}{p+ig} \tag{3.1}$$

the $\mathcal{S}$-matrix is defined in our context by

$$\mathcal{S}=\sum_{\alpha_1,\alpha_2=\pm}\mathcal{S}_{\alpha_1\alpha_2}(p_1,p_2)E_{\alpha_1\alpha_1}\otimes E_{\alpha_2\alpha_2}, \tag{3.2}$$

where $\mathcal{S}_{\alpha_1\alpha_2}(p_1,p_2)=S(\alpha_1p_1-\alpha_2p_2)$ and $(E_{\alpha\beta})_{\sigma\gamma}=\delta_{\alpha\sigma}\delta_{\beta\gamma}$. It is easy to check that $\mathcal{S}$ satisfies the unitarity condition and the quantum Yang–Baxter equation

$$\mathcal{S}_{12}(p_1,p_2)\mathcal{S}_{21}(p_2,p_1)=\mathbb{1}\otimes\mathbb{1}, \tag{3.3}$$

$$\mathcal{S}_{12}(p_1,p_2)\mathcal{S}_{13}(p_1,p_3)\mathcal{S}_{23}(p_2,p_3)=\mathcal{S}_{23}(p_2,p_3)\mathcal{S}_{13}(p_1,p_3)\mathcal{S}_{12}(p_1,p_2). \tag{3.4}$$

Our defect generators $r(p)$, $t(p)$ are related to $r_\alpha^\beta(p)$, $t_\alpha^\beta(p)$ defined in Ref. 7 by

$$r_\alpha^\beta(p)=\delta_\alpha^\beta r(\alpha p)\quad\text{and } t_\alpha^\beta(p)=\epsilon_\alpha^\beta t(\alpha p)\text{ with }\epsilon=\begin{pmatrix}0&1\\1&0\end{pmatrix}. \tag{3.5}$$

All this setup gives rise to a particular RT algebra whose defining relations then read as follows.

(i) Bulk exchange relations,

$$a_{\alpha_1}(p_1)a_{\alpha_2}(p_2) - S(\alpha_2 p_2 - \alpha_1 p_1)a_{\alpha_2}(p_2)a_{\alpha_1}(p_1) = 0, \tag{3.6}$$

$$a^\dagger_{\alpha_1}(p_1)a^\dagger_{\alpha_2}(p_2) - S(\alpha_2 p_2 - \alpha_1 p_1)a^\dagger_{\alpha_2}(p_2)a^\dagger_{\alpha_1}(p_1) = 0, \tag{3.7}$$

$$a_{\alpha_1}(p_1)a^\dagger_{\alpha_2}(p_2) - S(\alpha_1 p_1 - \alpha_2 p_2)a^\dagger_{\alpha_2}(p_2)a_{\alpha_1}(p_1) = 2\pi\delta(p_1 - p_2)[\delta^{\alpha_2}_{\alpha_1}\mathbf{1} + \epsilon^{\alpha_2}_{\alpha_1}t(\alpha_1 p_1)] + 2\pi\delta(p_1 + p_2)\delta^{\alpha_2}_{\alpha_1}r(\alpha_1 p_1). \tag{3.8}$$

(ii) Defect exchange relations,

$$[r(p_1), r(p_2)] = 0, \tag{3.9}$$

$$[t(p_1), t(p_2)] = 0, \tag{3.10}$$

$$[t(p_1), r(p_2)] = 0. \tag{3.11}$$

(iii) Mixed exchange relations,

$$a_{\alpha_1}(p_1)r(p_2) = S(p_2 - p_1)S(p_2 + p_1)r(p_2)a_{\alpha_1}(p_1), \tag{3.12}$$

$$r(p_1)a^\dagger_{\alpha_2}(p_2) = S(p_1 - p_2)S(p_1 + p_2)a^\dagger_{\alpha_2}(p_2)r(p_1), \tag{3.13}$$

$$a_{\alpha_1}(p_1)t(p_2) = S(p_2 - p_1)S(p_2 + p_1)t(p_2)a_{\alpha_1}(p_1), \tag{3.14}$$

$$t(p_1)a^\dagger_{\alpha_2}(p_2) = S(p_1 - p_2)S(p_1 + p_2)a^\dagger_{\alpha_2}(p_2)t(p_1). \tag{3.15}$$

(iv) Finally, the defect generators are required to satisfy unitarity conditions,

$$t(p)t(-p) + r(p)r(-p) = \mathbf{1}, \tag{3.16}$$

$$t(p)r(-p) + r(p)t(-p) = 0, \tag{3.17}$$

which amount to implement the physical energy conservation when reflection and transmission occur.

Since we aim at second quantize a physical system, we now turn to the Fock representation of this algebraic setup as it is presented in Ref. 7. What we need is to represent the generators $\{a_\alpha(p), a^\dagger_\alpha(p), r(p), t(p), p \in \mathbb{R}\}$ as operator-valued distributions acting on a common invariant subspace of a Hilbert space, $\mathcal{F}$, to be defined. We should also identify a normalizable vacuum state Ω annihilated by a_α and cyclic with respect to $a^\dagger_\alpha$. Applying the general construction of Ref. 7, we know that each such Fock representation is characterized by two numerical matrices $\mathcal{T}(p)$ and $\mathcal{R}(p)$. Here we take

$$\mathcal{T}(p) = \begin{pmatrix} 0 & T(p) \\ T(-p) & 0 \end{pmatrix}, \quad \mathcal{R}(p) = \begin{pmatrix} R(p) & 0 \\ 0 & R(-p) \end{pmatrix} \tag{3.18}$$

with T, R given in (2.17). Now consider

$$\mathcal{L} = \bigoplus_{\alpha=\pm} L^2(\mathbb{R}) \tag{3.19}$$

endowed with the usual scalar product

$$\langle \varphi, \psi \rangle = \int_{\mathbb{R}} \mathrm{d}p \sum_{\alpha=\pm} \overline{\varphi_\alpha}(p) \psi_\alpha(p), \tag{3.20}$$

which makes it a Hilbert space for the associated norm denoted $\|\cdot\|$. Then, the n-particle subspace $\mathcal{H}^{(n)}$ is the subspace of the n-fold tensor product $\mathcal{L}^{\otimes n}$ defined as follows. If $\varphi^{(n)} \in \mathcal{L}^{\otimes n}$, we identify it with the column whose entries are $\varphi^{(n)}_{\alpha_1,\ldots,\alpha_n}$. Then explicitly, $\mathcal{H}^{(0)} = \mathcal{C}$ and for $n \geqslant 1$, $\varphi^{(n)} \in \mathcal{H}^{(n)}$ if and only if

$$\varphi^{(n)} \in \mathcal{L}^{\otimes n},$$

$$\varphi^{(n)}_{\alpha_1\cdots\alpha_n}(p_1, \ldots, p_n) = T(\alpha_n p_n) \varphi^{(n)}_{\alpha_1\cdots\alpha_{n-1},-\alpha_n}(p_1, \ldots, p_{n-1}, p_n) + R(\alpha_n p_n) \varphi^{(n)}_{\alpha_1\cdots\alpha_{n-1}\alpha_n}(p_1, \ldots, p_{n-1}, -p_n), \tag{3.21}$$

$$\begin{aligned} n > 1,\ \varphi^{(n)}_{\alpha_1\cdots\alpha_i\alpha_{i+1}\cdots\alpha_n}(p_1, \ldots, p_i, p_{i+1}, \ldots, p_n) &= S(\alpha_i p_i - \alpha_{i+1} p_{i+1}) \\ &\times \varphi^{(n)}_{\alpha_1\cdots\alpha_{i+1}\alpha_i\cdots\alpha_n}(p_1, \ldots, p_{i+1}, p_i, \ldots, p_n), \quad 1 < i < n-1. \end{aligned} \tag{3.22}$$

The Fock space is $\mathcal{F} = \oplus_{n=0}^{\infty} \mathcal{H}^{(n)}$ and the common invariant subspace is the finite particle space $\mathcal{D}$ spanned by the linear combination of sequences $\varphi = (\varphi^{(0)}, \varphi^{(1)}, \ldots, \varphi^{(n)}, \ldots)$ with $\varphi^{(n)} \in \mathcal{H}^{(n)}$ and $\varphi^{(n)} = 0$ for n large enough. $\mathcal{D}$ is dense in $\mathcal{F}$. We extend the scalar product, again denoted by $\langle \cdot, \cdot \rangle$, to $\mathcal{F}$,

$$\begin{aligned} \forall\, \varphi, \psi \in \mathcal{F}, \quad \langle \varphi, \psi \rangle &= \sum_{n=0}^{\infty} \langle \varphi^{(n)}, \psi^{(n)} \rangle \\ &= \sum_{n=0}^{\infty} \int_{\mathbb{R}^n} \mathrm{d}p_1 \cdots \mathrm{d}p_n \sum_{\alpha_1,\ldots,\alpha_n=\pm} \overline{\varphi}_{\alpha_1\cdots\alpha_n}(p_1, \ldots, p_n) \psi_{\alpha_1\cdots\alpha_n}(p_1, \ldots, p_n). \end{aligned}$$

The unit norm vacuum state is $\Omega = (1, 0, \ldots, 0, \ldots)$ and belongs to $\mathcal{D}$.

Now, we can define the action of the smeared bulk operators $\{a(\mathfrak{f}), a^\dagger(\mathfrak{f}); \mathfrak{f} \in \oplus_{\alpha=\pm} C_0^\infty(\mathbb{R})\}$ on $\mathcal{D}$ as follows:

$$a(\mathfrak{f})\Omega = 0, \tag{3.23}$$

and for any $\varphi^{(n)} \in \mathcal{H}^{(n)}$,

$$[a(\mathfrak{f})\varphi]^{(n-1)}_{\alpha_1\cdots\alpha_{n-1}}(p_1, \ldots, p_{n-1}) = \sqrt{n} \int_{-\infty}^{\infty} \frac{\mathrm{d}p}{2\pi} \sum_{\alpha=\pm} \overline{\mathfrak{f}_\alpha}(p) \varphi^{(n)}_{\alpha\alpha_1\cdots\alpha_{n-1}}(p, p_1, \ldots, p_{n-1}), \tag{3.24}$$

$$[a^\dagger(\mathfrak{f})\varphi]^{(n+1)}_{\alpha_1\cdots\alpha_{n+1}}(p_1, \ldots, p_{n+1}) = \sqrt{n+1} [P^{(n+1)} \mathfrak{f} \otimes \varphi^{(n)}]_{\alpha_1\cdots\alpha_{n+1}}(p_1, \ldots, p_{n+1}), \tag{3.25}$$

where $P^{(n)}$ is the orthogonal projector in $\mathcal{L}^{\otimes n}$ defined in Ref. 7. For completeness, the explicit form of (3.25) is given in Appendix B. These operators are bounded on each $\mathcal{H}^{(n)}$,

$$\forall\, \varphi \in \mathcal{H}^{(n)}, \quad \|a(\mathfrak{f})\varphi\| \leqslant \sqrt{n} \|\mathfrak{f}\| \, \|\varphi\|, \quad \|a^\dagger(\mathfrak{f})\varphi\| \leqslant \sqrt{n+1} \|\mathfrak{f}\| \, \|\varphi\|. \tag{3.26}$$

In particular, they are continuous in the smearing function $\mathfrak{f}$. Finally, they satisfy

$$\forall\, \varphi, \psi \in \mathcal{D}, \quad \langle \varphi, a(\mathfrak{f})\psi \rangle = \langle a^\dagger(\mathfrak{f})\varphi, \psi \rangle. \tag{3.27}$$

The defect generators are represented as multiplicative operators on $\mathcal{D}$, preserving the bulk particle number,

$$[r(p)\varphi]^{(n)}_{\alpha_1\cdots\alpha_n}(p_1,\ldots,p_n) = S(p-\alpha_1 p_1)\cdots S(p-\alpha_n p_n)R(p)S(\alpha_n p_n + p)\cdots S(\alpha_1 p_1 + p) \times \varphi^{(n)}_{\alpha_1\cdots\alpha_n}(p_1,\ldots,p_n), \tag{3.28}$$

$$[t(p)\varphi]^{(n)}_{\alpha_1\cdots\alpha_n}(p_1,\ldots,p_n) = S(p-\alpha_1 p_1)\cdots S(p-\alpha_n p_n)T(p)S(\alpha_n p_n + p)\cdots S(\alpha_1 p_1 + p) \times \varphi^{(n)}_{\alpha_1\cdots\alpha_n}(p_1,\ldots,p_n). \tag{3.29}$$

It follows then that r and t have nonvanishing vacuum expectation values

$$\langle \Omega, r(p)\Omega\rangle = R(p), \quad \langle \Omega, t(p)\Omega\rangle = T(p). \tag{3.30}$$

Introducing finally the operator-valued distributions $a_\alpha(p)$, $a^\dagger_\alpha(p)$ as

$$a(\mathfrak{f}) = \int_{\mathbb{R}} \frac{dp}{2\pi}\sum_{\alpha=\pm} \overline{\mathfrak{f}_\alpha}(p) a_\alpha(p), \quad a^\dagger(\mathfrak{f}) = \int_{\mathbb{R}} \frac{dp}{2\pi}\sum_{\alpha=\pm} a^\dagger_\alpha(p)\mathfrak{f}_\alpha(p) \tag{3.31}$$

one can check that the defining relations of the RT algebra are satisfied on $\mathcal{D}$. The operators $a, a^\dagger$ will be referred to as annihilation and creation operators, respectively. Implementing the automorphism ϱ defined in Ref. 7 for which we know that it is realized by the identity operator for any Fock representation, we get the quantum analog of the reflection-transmission property (2.18)

$$a_\alpha(p) = \epsilon^\beta_\alpha t(\alpha p) a_\beta(p) + \delta^\beta_\alpha r(\alpha p) a_\beta(-p), \tag{3.32}$$

$$a^\dagger_\alpha(p) = \epsilon^\alpha_\beta a^\dagger_\beta(p) t(\beta p) + \delta^\alpha_\beta a^\dagger_\beta(-p) r(-\beta p). \tag{3.33}$$

B. The question of operator domains

From the above it appears that the natural domain to start with is $\mathcal{D}$. Actually, it is much too big for practical calculations and we would like to work on a dense subspace of $\mathcal{D}$ which would play the role of the standard formal "state space," a basis of which is usually denoted by $|k_1,\ldots,k_n\rangle$, $k_1 > \cdots > k_n$. As a first step, we define

$$\mathcal{D}^0_0 = \mathbb{C},$$

$$\mathcal{D}^n_0 = \{a^\dagger_{\alpha_1}(\mathfrak{f}_1)\cdots a^\dagger_{\alpha_n}(\mathfrak{f}_n)\Omega; \mathfrak{f}_i \in C^\infty_0(\mathbb{R}), \quad \alpha_i = \pm, i=1,\ldots,n\}, \quad n \geqslant 1. \tag{3.34}$$

One can check that $\mathcal{D}^n_0$ is dense in $\mathcal{H}^{(n)}$, i.e., Ω is cyclic with respect to $a^\dagger_\alpha$. The corresponding domain $\mathcal{D}_0$, dense in $\mathcal{D}$, is the linear space of sequences $\varphi=(\varphi^{(0)},\varphi^{(1)},\ldots,\varphi^{(n)},\ldots)$ with $\varphi^{(n)} \in \mathcal{D}^n_0$ and $\varphi^{(n)}=0$ for n large enough. $\mathcal{D}_0$ is stable under the action of $a_\alpha(\mathfrak{f})$ and $a^\dagger_\alpha(\mathfrak{f})$. Finally, since T, R, and S are bounded, C^∞-functions, $\mathcal{D}^n_0 \subset C^\infty_0(\mathbb{R}^n)$. Now in order to formulate the desired properties of the quantum field in the next paragraph, we introduce a partial ordering relation on $C^\infty_0(\mathbb{R})$ by

$$f > g \Leftrightarrow \forall\, x \in \mathrm{supp}(f), \quad \forall\, y \in \mathrm{supp}(g), \quad |x| > |y|, \tag{3.35}$$

which extends naturally to $C^\infty_0(\mathbb{R}^\alpha)$, $\alpha=\pm$. Let us introduce

$$\tilde{a}^\dagger_\alpha(t,x) = \int_{\mathbb{R}} \frac{dp}{2\pi} a^\dagger_\alpha(p) e^{-ipx+ip^2 t}, \quad (t,x) \in \mathbb{R}^2,$$

$$\tilde{a}^{\dagger}_{\alpha}(t,f) = \int_{\mathbb{R}} \mathrm{d}x\, \tilde{a}^{\dagger}_{\alpha}(t,x) f(x), \quad f \in C_0^{\infty}(\mathbb{R}). \tag{3.36}$$

Now, fix $t \in \mathbb{R}$ and $\alpha_1, \dots, \alpha_n$ and define (vect standing for "linear span of") $\tilde{\mathcal{D}}_0^0 = \mathbb{C}$ and for $n \geqslant 1$,

$$\tilde{\mathcal{D}}^n_{0,\alpha_1\cdots\alpha_n} = \mathrm{vect}\{\tilde{a}^{\dagger}_{\alpha_1}(t,f_{1,\alpha_1})\cdots\tilde{a}^{\dagger}_{\alpha_n}(t,f_{n,\alpha_n})\Omega;\ f_{1,\alpha_1} \succ \cdots \succ f_{n,\alpha_n}, f_{i,\alpha_i} \in C_0^{\infty}(\mathbb{R}^{\alpha_i}),$$
$$0 \notin \mathrm{supp}(f_{i,\alpha_i}),\ i = 1, \dots, n\} \tag{3.37}$$

then the following theorem holds.

Theorem 3.1: $\forall t \in \mathbb{R}$, $\forall \alpha_1, \dots, \alpha_n = \pm$, $\tilde{\mathcal{D}}^n_{0,\alpha_1\cdots\alpha_n}$ *is dense in* $\mathcal{H}^{(n)}$.

Proof: We only need to consider $n \geqslant 1$. The proof relies on two known results of standard analysis. First, the Fourier transform of a C^{∞}-function with compact support is real analytic (i.e., a Gevrey class 1 function). Second, a real analytic function vanishing on a given open subset U of an open connected set O, vanishes on the whole of O (see, e.g., Ref. 37).

Here, it suffices to show that $\tilde{\mathcal{D}}^n_{0,\alpha_1\cdots\alpha_n}$ is dense in $\mathcal{D}_0^n$ for any $t \in \mathbb{R}$ so let us consider the matrix element

$$\tilde{A}_{t,\varphi,\alpha_1\cdots\alpha_n}(x_1, \dots, x_n) = \langle \varphi^{(n)}, \tilde{a}^{\dagger}_{\alpha_1}(t,x_1)\cdots\tilde{a}^{\dagger}_{\alpha_n}(t,x_n)\Omega\rangle, \tag{3.38}$$

where $\varphi^{(n)} \in \mathcal{D}_0^n$ is arbitrary. To prove the statement, we now have to show that

$$\tilde{A}_{t,\varphi,\alpha_1\cdots\alpha_n}(x_1, \dots, x_n) = 0, \quad \forall\, |x_1| > \cdots > |x_n| > 0, \quad x_i \in \mathbb{R}^{\alpha_i}, \quad i = 1, \dots, n \tag{3.39}$$

implies $\varphi^{(n)} = 0$. From (3.36), we get

$$\tilde{A}_{t,\varphi,\alpha_1\cdots\alpha_n}(x_1, \dots, x_n) = \int_{\mathbb{R}^n} \prod_{j=1}^{n} \frac{\mathrm{d}p_j}{2\pi} e^{-ip_jx_j + itp_j^2} \langle \varphi^{(n)}, a^{\dagger}_{\alpha_1}(p_1)\cdots a^{\dagger}_{\alpha_n}(p_n)\Omega\rangle, \tag{3.40}$$

which shows that $\tilde{A}_{t,\varphi,\alpha_1\cdots\alpha_n}$ is the Fourier transform of a C^{∞}-function with compact support and is therefore real analytic. Condition (3.39) amounts to saying that $\tilde{A}_{t,\varphi,\alpha_1\cdots\alpha_n}$ vanishes on the set

$$U_{\alpha_1\cdots\alpha_n} = \{x \in \mathbb{R}^n\ s.t.\ |x_1| > \cdots > |x_n| > 0,\ x_i \in \mathbb{R}^{\alpha_i},\ i = 1, \dots, n\}. \tag{3.41}$$

$U_{\alpha_1\cdots\alpha_n}$ being an open subset of (the open and connected space) $\mathbb{R}^n$, we conclude that $\tilde{A}_{t,\varphi,\alpha_1\cdots\alpha_n}$ vanishes on $\mathbb{R}^n$. This gives in turn that

$$\langle \varphi^{(n)}, a^{\dagger}_{\alpha_1}(p_1)\cdots a^{\dagger}_{\alpha_n}(p_n)\Omega\rangle = 0, \quad \forall\, p_j \in \mathbb{R}, \quad j = 1, \dots, n, \tag{3.42}$$

or, equivalently, from the cyclicity of Ω with respect to $a^{\dagger}$

$$\varphi^{(n)}_{\alpha_1\cdots\alpha_n}(p_1, \dots, p_n) = 0, \quad \forall\, p_j \in \mathbb{R}, \quad j = 1, \dots, n. \tag{3.43}$$

Now using the properties (3.21) and (3.22) satisfied by $\varphi^{(n)}$, we get

$$\varphi^{(n)}_{\alpha_1\cdots\alpha_n}(p_1, \dots, p_n) = 0, \quad \forall\, p_j \in \mathbb{R}, \quad \forall\, \alpha_j = \pm, \quad j = 1, \dots, n \tag{3.44}$$

that is $\varphi^{(n)} = 0$. ■

This theorem will prove to be fundamental in the sequel to derive the required properties of the quantum field operator. Indeed, it will be enough to perform all calculations only on states in

$$\widetilde{\mathcal{D}}_0^{n,\alpha} = \widetilde{\mathcal{D}}^n_{0,\underbrace{\alpha\cdots\alpha}_{n}} \quad \text{with } \alpha = \pm \tag{3.45}$$

and conclude for the whole domain $\mathcal{D}$ by a continuity argument.

Lemma 3.2: Let $f_{1,\alpha_1} \succ \cdots \succ f_{n,\alpha_n}$ *and* $h_{1,\beta_1} \succ \cdots \succ h_{n,\beta_n}$*, then*

$$\langle \widetilde{a}^\dagger_{\alpha_1}(t,f_{1,\alpha_1}) \cdots \widetilde{a}^\dagger_{\alpha_n}(t,f_{n,\alpha_n})\Omega, \widetilde{a}^\dagger_{\beta_1}(t,h_{1,\beta_1}) \cdots \widetilde{a}^\dagger_{\beta_n}(t,h_{n,\beta_n})\Omega\rangle = \prod_{j=1}^n \delta_{\alpha_j\beta_j}\langle f_{j,\alpha_j}, h_{j,\beta_j}\rangle. \tag{3.46}$$

In particular, for $\varphi \in \widetilde{D}^n_{0,\alpha_1\cdots\alpha_n}$ *represented as*

$$\varphi = \sum_{\beta\in B} \widetilde{a}^\dagger_{\alpha_1}(t,f^\beta_{1,\alpha_1}) \cdots \widetilde{a}^\dagger_{\alpha_n}(t,f^\beta_{n,\alpha_n}), \quad f^\beta_{1,\alpha_1} \succ \cdots \succ f^\beta_{n,\alpha_n}, \quad \forall\, \beta \in B, \tag{3.47}$$

where *B is a finite set, one has* $\|\varphi\| = \|\Sigma_{\beta\in B} f^\beta_{1,\alpha_1} \otimes \cdots \otimes f^\beta_{n,\alpha_n}\|$.

Proof: To get (3.46), one uses an induction on n and combines (3.36), (3.27), (3.8), and (3.23) together with the support conditions on the smearing functions. Using a contour integral argument, these support conditions imply that all the contributions arising from the RT algebra vanish except for the usual δ- term producing the right-hand side. Equation (3.47) is a mere consequence of (3.46). ∎

Remark: It is important to realize that the n particle space $\mathcal{H}^{(n)}$ is the central piece in this construction and that, on this space, any operation we have considered (scalar product, creation operator, Fourier transform) is continuous in the smearing functions. Since $C_0^\infty(\mathbb{R})$ is dense in $\mathcal{S}(\mathbb{R})$, the Schwarz space, we can extend the above (especially the definition of $\mathcal{D}_0^n$) to smearing functions in $\mathcal{S}(\mathbb{R})$.

C. Quantum field

We start by defining $\Phi(t,f)$ as

$$\Phi(t,f) = \int_{\mathbb{R}} \mathrm{d}x \sum_{\alpha=\pm} \overline{f_\alpha}(x)\Phi_\alpha(t,x), \quad f \in \mathcal{C} \text{ where } \mathcal{C} = \bigoplus_{\alpha=\pm} C_0^\infty(\mathbb{R}^\alpha). \tag{3.48}$$

f is viewed as a column vector $f = \binom{f_+}{f_-}$ with $f_\alpha \in C_0^\infty(\mathbb{R}^\alpha)$ and $0 \notin \mathrm{supp}(f_\alpha)$. Following the standard argument of Ref. 29, we replace $\lambda_\alpha(p), \overline{\lambda_\alpha}(p)$ in the Rosales expansion of the classical field (2.4) and (2.5) by the operators $a_\alpha(p), a^\dagger_\alpha(p)$ in order to define

$$\Phi_\alpha(t,x) = \sum_{n=0}^\infty (-g)^n \Phi^{(n)}_\alpha(t,x), \quad g > 0 \tag{3.49}$$

and

$$\Phi^{(n)}_\alpha(t,x) = \int_{\mathbb{R}^{2n+1}} \prod_{\substack{i=1\\j=0}}^n \frac{\mathrm{d}p_i}{2\pi}\frac{\mathrm{d}q_j}{2\pi} a^\dagger_\alpha(p_1)\cdots a^\dagger_\alpha(p_n) a_\alpha(q_n)\cdots a_\alpha(q_0) \frac{e^{i\Sigma_{j=0}^n(q_jx - q_j^2t) - i\Sigma_{i=1}^n(p_ix - p_i^2t)}}{\prod_{i=1}^n (p_i - q_{i-1} - i\alpha\varepsilon)(p_i - q_i - i\alpha\varepsilon)}, \tag{3.50}$$

where we used an $i\varepsilon$ prescription depending on $\alpha = \pm$.

We now have several requirements to meet for our quantum theory to be well defined. We must give a precise meaning to $\Phi_\alpha(t,x)$, show that the canonical commutation relations as well as the boundary conditions (2.12) and (2.13) hold in a sense we shall make precise and that $\Phi_\alpha(t,x)$ is indeed the quantum solution we look for.

We start by associating $\Phi_\alpha(t,x)$ with the quadratic form defined on $\mathcal{D} \times \mathcal{D}$ by

$$(\varphi,\psi)\mapsto\langle\varphi,\Phi_\alpha(t,x)\psi\rangle, \tag{3.51}$$

$\mathcal{D}$ containing only finite particle vectors, it is enough to investigate $\langle\varphi,\Phi_\alpha^{(n)}(t,x)\psi\rangle$ for arbitrary n.

Proposition 3.3: $\forall\ n\geqslant 0$, $\forall\ \varphi,\psi\in\mathcal{D}$, $(t,x)\mapsto\langle\varphi,\Phi_\alpha^{(n)}(t,x)\psi\rangle$ *is a* C^∞ *function.*

Proof: The proof is the same as in Ref. 32. ■

We define the conjugate $\Phi_\alpha^\dagger(t,x)$ again as a quadratic form on $\mathcal{D}\times\mathcal{D}$ by

$$\langle\varphi,\Phi_\alpha^\dagger(t,x)\psi\rangle=\langle\Phi_\alpha(t,x)\varphi,\psi\rangle. \tag{3.52}$$

It has the same smoothness properties and from (3.27), we get

$$\Phi_\alpha^{\dagger(n)}(t,x)=\int_{\mathbb{R}^{2n+1}}\prod_{\substack{i=1\\j=0}}^{n}\frac{\mathrm{d}p_i}{2\pi}\frac{\mathrm{d}q_j}{2\pi}a_\alpha^\dagger(q_0)\cdots a_\alpha^\dagger(q_n)a_\alpha(p_n)\cdots a_\alpha(p_1)$$
$$\times\frac{e^{-i\Sigma_{j=0}^n(q_jx-q_j^2t)+i\Sigma_{i=1}^n(p_ix-p_i^2t)}}{\prod_{i=1}^n(p_i-q_{i-1}+i\alpha\varepsilon)(p_i-q_i+i\alpha\varepsilon)}. \tag{3.53}$$

Defining the smeared version

$$\Phi^\dagger(t,f)=\int_{\mathbb{R}}\mathrm{d}x\sum_{\alpha=\pm}\Phi_\alpha^\dagger(t,x)f_\alpha(x),\quad f\in\mathcal{C} \tag{3.54}$$

we conclude that $\Phi(t,f)$ and $\Phi^\dagger(t,f)$ are understood as quadratic forms on the domain $\mathcal{D}$ and are related by

$$\langle\varphi,\Phi_\alpha^\dagger(t,f)\psi\rangle=\langle\Phi_\alpha(t,f)\varphi,\psi\rangle. \tag{3.55}$$

To get true quantum fields, we need to show that these quadratic forms give rise to operators on $\mathcal{D}$. This requires the following two lemmas.

Lemma 3.4: $\forall\ \varphi,\psi\in\mathcal{D}$,

(i) *For* $h_{1,\alpha}>\cdots>h_{n,\alpha}$,

$$\langle\varphi,\Phi_\alpha(t,f_\alpha)\tilde{a}_\alpha^\dagger(t,h_{1,\alpha})\cdots\tilde{a}_\alpha^\dagger(t,h_{n,\alpha})\Omega\rangle=\sum_{j=1}^n\langle f_\alpha,h_{j,\alpha}\rangle$$
$$\times\langle\varphi,\tilde{a}_\alpha^\dagger(t,h_{1,\alpha})\cdots\widehat{\tilde{a}_\alpha^\dagger(t,h_{j,\alpha})}\cdots\tilde{a}_\alpha^\dagger(t,h_{n,\alpha})\Omega\rangle, \tag{3.56}$$

where the hatted symbol is omitted.

(ii) *For* $h_\alpha>f_\alpha$,

$$\langle\varphi,\Phi_\alpha^\dagger(t,f_\alpha)\tilde{a}_\alpha^\dagger(t,h_\alpha)\psi\rangle=\langle\varphi,\tilde{a}_\alpha^\dagger(t,h_\alpha)\Phi_\alpha^\dagger(t,f_\alpha)\psi\rangle. \tag{3.57}$$

(iii) *For* $f_\alpha>h_{j,\alpha}$, $j=1,\ldots,n$,

$$\langle\varphi,\Phi_\alpha^\dagger(t,f_\alpha)\tilde{a}_\alpha^\dagger(t,h_{1,\alpha})\cdots\tilde{a}_\alpha^\dagger(t,h_{n,\alpha})\Omega\rangle=\langle\varphi,\tilde{a}_\alpha^\dagger(t,f_\alpha)\tilde{a}_\alpha^\dagger(t,h_{1,\alpha})\cdots\tilde{a}_\alpha^\dagger(t,h_{n,\alpha})\Omega\rangle. \tag{3.58}$$

Proof: One just has to apply the order by order technique developed in Ref. 29. The latter heavily relied on the ZF algebra satisfied by the creation and annihilation operators. Here, one must take care in addition of the many contributions of the defect generators but it is remarkable that the RT algebra satisfied by the bulk and defect operators leads to the same results (using the support requirements of the smearing functions and the conditions $g>0$, $\eta>0$, all the defect contributions vanish). One realizes in these manipulations, especially in (3.58), that the contributions of Φ, $\Phi^\dagger$ on $\widetilde{\mathcal{D}}_0^{n,\alpha}$ are carried by the zeroth order corresponding to the linear problem (it is the Fourier transform of a, $a^\dagger$). ■

Lemma 3.5: Given $\varphi_\alpha \in \widetilde{\mathcal{D}}_0^{n,\alpha}$, $\psi_\alpha \in \widetilde{\mathcal{D}}_0^{n+1,\alpha}$ and $f_\alpha \in C_0^\infty(\mathbb{R}^\alpha)$, the quadratic form (3.51) satisfies the following boundedness condition:

$$|\langle \varphi_\alpha, \Phi_\alpha(t,f_\alpha)\psi_\alpha\rangle| \leqslant (n+1)\|f_\alpha\|\,\|\varphi_\alpha\|\,\|\psi_\alpha\|. \tag{3.59}$$

Proof: The proof is similar to that given in Ref. 32 and uses lemmas 3.2 and 3.4(i). ■

From the Riesz lemma and theorem 3.1, we conclude that $\Phi_\alpha(t,f_\alpha):\mathcal{H}^{(n+1)} \to \mathcal{H}^{(n)}$ is a bounded operator for any $n \geqslant 0$. Thus, it defines an operator on the common invariant domain $\mathcal{D}$. The same holds for $\Phi_\alpha^\dagger(t,f_\alpha)$, $\mathcal{H}^{(n)} \to H^{(n+1)}$ by (3.55). We can therefore collect our results in the following theorem.

Theorem 3.6: $\Phi(t,f)$, $\Phi^\dagger(t,f):\mathcal{D}\to\mathcal{D}$ *are Hermitian conjugate, linear operators and satisfy*

$$\Phi(t,f)\Omega = 0, \quad \Phi^\dagger(t,f)\Omega = \tilde{a}^\dagger(t,f)\Omega. \tag{3.60}$$

Finally, we will have a *nonrelativistic quantum field* if we prove the canonical commutation relations for Φ, $\Phi^\dagger$.

Theorem 3.7: $\{\Phi(t,f), \Phi^\dagger(t,f), f\in\mathcal{C}\}$ *realize a Fock representation of the equal time canonical commutation relations on* $\mathcal{D}$,

$$[\Phi(t,f_1),\Phi(t,f_2)] = 0 = [\Phi^\dagger(t,f_1),\Phi^\dagger(t,f_2)], \tag{3.61}$$

$$[\Phi(t,f_1),\Phi^\dagger(t,f_2)] = \langle f_1,f_2\rangle. \tag{3.62}$$

Proof: We know that it suffices to compute the commutators on $\widetilde{\mathcal{D}}_0^{n,+}$ or $\widetilde{\mathcal{D}}_0^{n,-}$ for arbitrary n and then extend the results by continuity to $\mathcal{H}^{(n)}$ and by linearity to $\mathcal{D}$. From theorem 3.6, we get that (i)–(iii) of lemma 3.4 hold as operator equalities. Let us start with the first commutator. It is made out of four parts,

$$\begin{aligned}[\Phi(t,f_1),\Phi(t,f_2)] = {} & [\Phi_+(t,f_{1,+}),\Phi_+(t,f_{2,+})] + [\Phi_+(t,f_{1,+}),\Phi_-(t,f_{2,-})] + [\Phi_-(t,f_{1,-}),\Phi_+(t,f_{2,+})] \\ & + [\Phi_-(t,f_{1,-}),\Phi_-(t,f_{2,-})]. \end{aligned}\tag{3.63}$$

The first and fourth parts of the right-hand side are easily seen to be zero from (i) of lemma 3.4 One has for $\alpha=\pm$,

$$\begin{aligned}\Phi_\alpha(t,f_{1,\alpha})\Phi_\alpha(t,f_{2,\alpha})\tilde{a}_\alpha^\dagger(t,h_{1,\alpha})\cdots\tilde{a}_\alpha^\dagger(t,h_{n,\alpha})\Omega = {} & \sum_{j=1}^{n}\sum_{\substack{k=1\\k\neq j}}^{n}\langle f_{2,\alpha},h_{j,\alpha}\rangle\langle f_{1,\alpha},h_{k,\alpha}\rangle\tilde{a}_\alpha^\dagger(t,h_{1,\alpha})\cdots \\ & \times\widehat{\tilde{a}_\alpha^\dagger(t,h_{j,\alpha})}\cdots\widehat{\tilde{a}_\alpha^\dagger(t,h_{k,\alpha})}\cdots\tilde{a}_\alpha^\dagger(t,h_{n,\alpha})\Omega,\end{aligned}\tag{3.64}$$

which is symmetric under the exchange of f_1 and f_2 implying the vanishing of the commutators. As for the mixed terms, one can check that

$$\Phi_\alpha(t,f_{i,\alpha})\tilde{a}_{-\alpha}^\dagger(t,h_{1,-\alpha})\cdots\tilde{a}_{-\alpha}^\dagger(t,h_{n,-\alpha})\Omega = 0, \quad i=1,2 \tag{3.65}$$

implying the vanishing of the second and third commutators on $\widetilde{\mathcal{D}}_0^{n,-\alpha}$ and hence on $\mathcal{D}$. Now the vanishing of $[\Phi^\dagger(t,f_1),\Phi^\dagger(t,f_2)]$ on $\mathcal{D}$ is obtained by Hermitian conjugation. This proves (3.61).

Equation (3.62) is obtained as follows. Again, we split the commutator into four parts. Now given a state in $\widetilde{\mathcal{D}}_0^{n,\alpha}$, we assume $h_{k,\alpha} \succ f_{2,\alpha} \succ h_{k+1,\alpha}$ for some k and using lemma 3.4, we compute for $\alpha=\pm$,

$$\Phi_\alpha(t,f_{1,\alpha})\Phi_\alpha^\dagger(t,f_{2,\alpha})\tilde{a}_\alpha^\dagger(t,h_{1,\alpha})\cdots\tilde{a}_\alpha^\dagger(t,h_{n,\alpha})\Omega = \langle f_{1,\alpha},f_{2,\alpha}\rangle\tilde{a}_\alpha^\dagger(t,h_{1,\alpha})\cdots\tilde{a}_\alpha^\dagger(t,h_{n,\alpha})\Omega + \mathfrak{S} \tag{3.66}$$

and

$$\Phi_\alpha^\dagger(t,f_{2,\alpha})\Phi_\alpha(t,f_{1,\alpha})\tilde{a}_\alpha^\dagger(t,h_{1,\alpha})\cdots\tilde{a}_\alpha^\dagger(t,h_{n,\alpha})\Omega = \mathfrak{S}, \tag{3.67}$$

where $\mathfrak{S}$ is

$$\sum_{j=1}^{n}\langle f_{1,\alpha},h_{j,\alpha}\rangle\tilde{a}_\alpha^\dagger(t,h_{1,\alpha})\cdots\widehat{\tilde{a}_\alpha^\dagger(t,h_{j,\alpha})}\cdots\tilde{a}_\alpha^\dagger(t,h_{k,\alpha})\tilde{a}_\alpha^\dagger(t,f_{2,\alpha})\tilde{a}_\alpha^\dagger(t,h_{k+1,\alpha})\cdots\tilde{a}_\alpha^\dagger(t,h_{n,\alpha})\Omega.$$

This gives

$$[\Phi_+(t,f_{1,+}),\Phi_+^\dagger(t,f_{2,+})] + [\Phi_-(t,f_{1,-}),\Phi_-^\dagger(t,f_{2,-})] = \langle f_{1,+},f_{2,+}\rangle + \langle f_{1,-},f_{2,-}\rangle = \langle f_1,f_2\rangle, \tag{3.68}$$

i.e., the desired contribution. It is then straightforward using (3.65) to verify that the mixed terms do not contribute

$$[\Phi_+(t,f_{1,+}),\Phi_-^\dagger(t,f_{2,-})] = [\Phi_-(t,f_{1,-}),\Phi_+^\dagger(t,f_{2,+})] = 0.$$

■

Now we prove that Ω is cyclic with respect to $\Phi^\dagger$ and that $\Phi(t,x)$ is the solution of the quantum nonlinear Schrödinger equation with impurity. Extending the partial ordering $\succ$ to functions in $\mathcal{C}$ as follows:

$$\text{for } f,g\in\mathcal{C}, \quad f\succ g \Leftrightarrow f_\alpha \succ g_\alpha, \quad \alpha=\pm, \tag{3.69}$$

one can prove the following theorems.

Theorem 3.8: *The space*

$$\mathcal{H}_0^{(n)} = \text{vect}\{\Phi^\dagger(t,f_1)\cdots\Phi^\dagger(t,f_n)\Omega; \quad f_i\in\mathcal{C}, \quad i=1,\ldots,n, \quad f_n\succ\ldots\succ f_1\} \tag{3.70}$$

is dense in $\mathcal{H}^{(n)}$.

Proof: Let $\varphi^{(n)}\in\mathcal{H}^{(n)}$ and suppose

$$\langle\varphi^{(n)},\Phi^\dagger(t,f_1)\cdots\Phi^\dagger(t,f_n)\Omega\rangle = 0, \quad \forall f_n\succ\cdots\succ f_1.$$

Then, it is true in particular for $f_{i,-}=0$, $i=1,\ldots,n$ but in that case, we have

$$\Phi^\dagger(t,f_1)\cdots\Phi^\dagger(t,f_n)\Omega = \tilde{a}_+^\dagger(t,f_{n,+})\cdots\tilde{a}_+^\dagger(t,f_{1,+})\Omega$$

which implies $\varphi^{(n)}=0$ since $\tilde{\mathcal{D}}_0^{n,+}$ is dense in $\mathcal{H}^{(n)}$. ■

Theorem 3.9: *The quantum field Φ is solution of the quantum nonlinear Schrödinger equation with impurity, i.e., it satisfies*

$$(i\partial_t+\partial_x^2)\langle\varphi,\Phi(t,x)\psi\rangle = 2g\langle\varphi,:\Phi\Phi^\dagger\Phi:(t,x)\psi\rangle \tag{3.71}$$

and the following boundary conditions:

$$\lim_{x\to0^+}\langle\varphi,\{\Phi_+(t,x)-\Phi_-(t,-x)\}\psi\rangle = 0, \tag{3.72}$$

$$\lim_{x\to0^+}\partial_x\langle\varphi,\{\Phi_+(t,x)+\Phi_-(t,-x)\}\psi\rangle = 2\eta\lim_{x\to0}\langle\varphi,\Phi(t,x)\psi\rangle, \tag{3.73}$$

$$\lim_{x\to\pm\infty}\langle\varphi,\Phi(t,x)\psi\rangle = 0, \tag{3.74}$$

for any φ, $\psi \in \mathcal{D}$.

Proof: Inspired by the classical case, we split the field as follows:

$$\Phi(t,x) = \theta(x)\Phi_+(t,x) + \theta(-x)\Phi_-(t,x). \tag{3.75}$$

The main difficulty here is to specify a normal ordering prescription for the analog of the cubic term. We adopt the prescription detailed in Ref. 32 for the normal ordering denoted $:\cdots:$ and apply it to Φ_α, $\alpha=\pm$. Then following Ref. 32 (theorem 5), one gets that the quantum field Φ_α is solution of the nonlinear Schrödinger equation on the half-line $\mathbb{R}^\alpha$: for all $\varphi, \psi \in \mathcal{D}$,

$$(i\partial_t + \partial_x^2)\langle \varphi, \Phi_\alpha(t,x)\psi\rangle = 2g\langle \varphi, :\Phi_\alpha\Phi_\alpha^\dagger\Phi_\alpha:(t,x)\psi\rangle. \tag{3.76}$$

The situation is now similar to the classical case and we have to check the quantum analog of (2.12)–(2.14). The idea lies again in realizing that Eqs. (3.72)–(3.74) can be cast into a zeroth-order/linear problem. Following the line of argument of Ref. 32 (theorem 6), one shows that given $\varphi, \psi \in \mathcal{D}$, there exists $\chi \in \mathcal{H}^{(1)}$ such that $\langle \varphi, \Phi(t,f)\psi\rangle = \langle \Omega, \Phi(t,f)\chi\rangle$ and χ is independent of f. This gives in particular $\langle \varphi, \Phi_\alpha(t,f_\alpha)\psi\rangle = \langle \Omega, \Phi_\alpha(t,f_\alpha)\chi\rangle$, $\alpha=\pm$ and we can compute

$$\langle \varphi, \Phi_\alpha(t,x)\psi\rangle = \langle \tilde{a}_\alpha^\dagger(t,x)\Omega, \chi\rangle = \int_{\mathbb{R}} \frac{\mathrm{d}p}{2\pi} e^{ipx-ip^2t}\chi_\alpha(p). \tag{3.77}$$

Then, Eqs. (3.72) and (3.73) are easily obtained using the property (3.21) satisfied by χ. Finally, since $\chi_\alpha \in L^2(\mathbb{R})$, $\langle \varphi, \Phi_\alpha(t,x)\psi\rangle$ as a function of x is also in $L^2(\mathbb{R})$ and therefore vanishes at infinity. Noting that $\lim_{x\to\pm\infty}\langle \varphi, \Phi(t,x)\psi\rangle = \lim_{x\to\pm\infty}\langle \varphi, \Phi_\pm(t,x)\psi\rangle$, we get (3.74). ■

We have finally achieved the goal of this section: we have explicitly constructed off-shell local fields for the quantum nonlinear Schrödinger system on the line in the presence of a transmitting and reflecting impurity. As mentioned in Ref. 7, this remained a challenging open problem for which we brought an answer here. In other words, the quantum inverse scattering method remains valid in the presence of an impurity provided that the ZF algebra is replaced by the RT algebra.

IV. SCATTERING THEORY

Scattering theory in the presence of an impurity was studied on general grounds in Ref. 7 by introducing the RT algebra which, being a generalization of the ZF and boundary algebras, is believed to prove fundamental also in the study of off-shell correlations functions and symmetries for $1+1$-dimensional integrable systems with impurity.

In this section, we aim at giving some credit to this in the context of the nonlinear Schrödinger model. Indeed from the above results, we can get some insight in the correlations functions of the theory. The correlations functions vanish unless they involve the same number of Φ and $\Phi^\dagger$ and for a given $2n$-point function, we need at most the first $(n-1)$-order terms in the Rosales expansion of the field. This reads

$$\begin{aligned}&\langle \Omega, \Phi(t_1,x_1)\cdots\Phi(t_n,x_n)\Phi^\dagger(t_{n+1},x_{n+1})\cdots\Phi^\dagger(t_{2n},x_{2n})\Omega\rangle \\ &\quad= \sum_{\substack{K\leqslant n-1\\ L\leqslant n-1}} g^{K+L}\langle \Omega, \Phi^{(k_1)}(t_1,x_1)\cdots\Phi^{(k_n)}(t_n,x_n)\Phi^{\dagger(l_1)}(t_{n+1},x_{n+1})\cdots\Phi^{\dagger(l_n)}(t_{2n},x_{2n})\Omega\rangle,\end{aligned} \tag{4.1}$$

where $K=\Sigma_{i=1}^n k_i$ and $L=\Sigma_{i=1}^n l_i$ and the sum runs over all n-uplets $(k_1,\ldots,k_n)$, $(l_1,\ldots,l_n)\in\mathbb{Z}_+^n$ such that $K,L\leqslant n-1$.

One has, for example (with $t_{12}=t_1-t_2$, $x_{12}=x_1-x_2$, and $\tilde{x}_{12}=x_1+x_2$),

$$\langle \Omega, \Phi(t_1,x_1)\Phi^*(t_2,x_2)\Omega\rangle = \int_{-\infty}^{+\infty} \frac{\mathrm{d}p}{2\pi} e^{-ip^2 t_{12}}\{\theta(x_1)\,\theta(x_2)[e^{ipx_{12}} + R(p)e^{ip\tilde{x}_{12}}] + \theta(-x_1)\,\theta(-x_2)[e^{ipx_{12}} + \overline{R}(p)e^{ip\tilde{x}_{12}}] + \theta(x_1)\,\theta(-x_2)T(p)e^{ipx_{12}} + \theta(-x_1)\,\theta(x_2)\overline{T}(p)e^{ipx_{12}}\}. \tag{4.2}$$

More importantly, using the Haag–Ruelle approach suitably, we can relate off-shell and asymptotic theories and, doing so, fill the gap of our quantum field theory. Indeed, on the one hand, we know from Ref. 7 that the Fock representation of the RT algebra generates the asymptotic states of a general integrable theory with impurity with corresponding S-matrix. On the other hand, in this paper we constructed off-shell local time-dependent fields whose behavior as $t \to \pm\infty$ we would like to know.

A. Asymptotic theory

The first step is to characterize wave packets for the free Schrödinger equation which take into account the presence of the impurity at $x=0$. We adopt the following setup. For $\mathfrak{f} \in C_0^\infty(\mathbb{R})$, we define

$$f^{\,t}(x) = \int_{\mathbb{R}} \frac{\mathrm{d}p}{2\pi} \mathfrak{f}(p) e^{ipx - ip^2 t}. \tag{4.3}$$

We transpose the partial ordering (3.35) to functions of the variable p.

Definition 4.1: Given $n, m \geqslant 1$, consider two sets of functions

$$\mathfrak{H}_n = \{\mathfrak{h}_{i,\alpha_i} \in C_0^\infty(\mathbb{R}^{\alpha_i}), \quad i = 1, \ldots, n\} \quad \text{and } \mathfrak{G}_m = \{\mathfrak{g}_{i,\beta_i} \in C_0^\infty(\mathbb{R}^{-\beta_i}), \quad i = 1, \ldots, m\}, \tag{4.4}$$

where the functions obey the following order prescriptions:

$$\mathfrak{h}_{1,\alpha_1} \succ \cdots \succ \mathfrak{h}_{n,\alpha_n}, \quad \mathfrak{g}_{m,\beta_m} \succ \cdots \succ \mathfrak{g}_{1,\beta_1}. \tag{4.5}$$

We also define

$$h^{\theta}_{i,\alpha_i}(x) = \theta(\alpha_i x) h^{t}_{i,\alpha_i}(x), \quad g^{\theta}_{i,\beta_i}(x) = \theta(\beta_i x) g^{t}_{i,\beta_i}(x). \tag{4.6}$$

By construction, $h^{\theta}_{i,\alpha_i}(x)$ represent wave packets in $\mathbb{R}^{\alpha_i}$ moving away from the impurity towards $\alpha_i\infty$ while $g^{\theta}_{i,\beta_i}(x)$ represent wave packets in $\mathbb{R}^{\beta_i}$ moving towards the impurity. One already understands that they will be relevant for the so-called "out" and "in" states, respectively. In fact, this is the main theorem of this section for which we need some preliminary results.

From the preceding section, we know the exchange and commutation properties of $\Phi^\dagger$ and $\tilde{a}^\dagger$ smeared with ordered functions in the variable x. Here, our wave packets were constructed from ordered functions in p but we made no assumption as to their ordering in x. Therefore, we must include all the possibilities and this requires the use of the permutation group of n elements $\mathfrak{S}_n$. For $\sigma \in \mathfrak{S}_n$, $\pi \in \mathfrak{S}_m$, $n, m \geqslant 2$, we introduce

$$\theta^{\sigma}_{h}(\alpha_1 x_1, \ldots, \alpha_n x_n) = \prod_{\substack{i,j=1\\ i<j}}^{n} \theta(\alpha_{\sigma_i} x_{\sigma_i} - \alpha_{\sigma_j} x_{\sigma_j}), \tag{4.7}$$

$$\theta^{\pi}_{g}(\beta_1 x_1, \ldots, \beta_m x_m) = \prod_{\substack{i,j=1\\ i>j}}^{m} \theta(\beta_{\pi_i} x_{\pi_i} - \beta_{\pi_j} x_{\pi_j}), \tag{4.8}$$

satisfying

$$\sum_{\sigma\in\mathfrak{S}_n}\theta_h^\sigma(\alpha_1x_1,\ldots,\alpha_nx_n)=1=\sum_{\pi\in\mathfrak{S}_m}\theta_g^\pi(\beta_1x_1,\ldots,\beta_mx_m). \tag{4.9}$$

Lemma 4.2: Given any two sets of functions in $\mathfrak{H}_n$ and $\mathfrak{G}_m$,

(i) *The following limits hold:*

$$\lim_{t\to+\infty}\|h^\theta_{1,\alpha_1}\otimes\cdots\otimes h^\theta_{n,\alpha_n}-h^t_{1,\alpha_1}\otimes\cdots\otimes h^t_{n,\alpha_n}\|=0,$$

$$\lim_{t\to-\infty}\|g^\theta_{1,\beta_1}\otimes\cdots\otimes g^\theta_{m,\beta_m}-g^t_{1,\beta_1}\otimes\cdots\otimes g^t_{m,\beta_m}\|=0. \tag{4.10}$$

(ii) *Let e_n be the identity of $\mathfrak{S}_n$ and let us define*

$$H^\sigma_{\alpha_1\cdots\alpha_n}(x_1,\ldots,x_n)=h^\theta_{1,\alpha_1}(x_1)\cdots h^\theta_{n,\alpha_n}(x_n)\,\theta_h^\sigma(\alpha_1x_1,\ldots,\alpha_nx_n),$$

$$G^\pi_{\beta_1\cdots\beta_m}(x_1,\ldots,x_m)=g^\theta_{1,\beta_1}(x_1)\cdots g^\theta_{m,\beta_m}(x_m)\,\theta_g^\pi(\beta_1x_1,\ldots,\beta_mx_m). \tag{4.11}$$

Then

$$\lim_{t\to+\infty}\|H^\sigma_{\alpha_1\cdots\alpha_n}\|=0,\quad \lim_{t\to-\infty}\|G^\pi_{\beta_1\cdots\beta_m}\|=0\quad \textit{for all }\sigma\neq e_n,\quad \pi\neq e_m. \tag{4.12}$$

(iii) *The following estimate is valid for any $F\in L^2(\mathbb{R}^n)$,*

$$\left\|\int_{\mathbb{R}^n}\mathrm{d}x_1\cdots\mathrm{d}x_nF(x_1,\ldots,x_n)\tilde{a}^\dagger_{\alpha_1}(t,x_1)\cdots\tilde{a}^\dagger_{\alpha_n}(t,x_n)\right\|\leqslant\sqrt{n!}\|F\|. \tag{4.13}$$

Proof: The ideas are the same as those detailed in Ref. 32 from theorem 7 onwards and rest especially on the use of the weak limit

$$\lim_{t\to\pm\infty}\frac{e^{itk}}{k\pm i\varepsilon}=0. \tag{4.14}$$

We just stress again that in our case all the above holds thanks to the use of the RT algebra and by paying careful attention to the support conditions encoded in (4.5). ■

We are now in position to identify the asymptotic behavior of the field as $t\to\pm\infty$.

Theorem 4.3: *The following limits hold in the strong sense in the Fock space $\mathcal{F}$:*

$$\lim_{t\to+\infty}\Phi^\dagger(t,h^\theta_{1,\alpha_1})\cdots\Phi^\dagger(t,h^\theta_{n,\alpha_n})\Omega=a^\dagger_{\alpha_1}(\mathfrak{h}_{1,\alpha_1})\cdots a^\dagger_{\alpha_n}(\mathfrak{h}_{n,\alpha_n})\Omega, \tag{4.15}$$

$$\lim_{t\to-\infty}\Phi^\dagger(t,g^\theta_{1,\beta_1})\cdots\Phi^\dagger(t,g^\theta_{m,\beta_m})\Omega=a^\dagger_{\beta_1}(\mathfrak{g}_{1,\beta_1})\cdots a^\dagger_{\beta_m}(\mathfrak{g}_{m,\beta_m})\Omega. \tag{4.16}$$

Proof: We note first that from (3.75) one gets $\Phi^\dagger(t,h^\theta_{i,\alpha_i})=\Phi^\dagger_{\alpha_i}(t,h^\theta_{i,\alpha_i})$ and $\Phi^\dagger(t,g^\theta_{i,\beta_i})=\Phi^\dagger_{\beta_i}(t,g^\theta_{i,\beta_i})$ so that

$$\Phi^\dagger(t,h^\theta_{i,\alpha_i})\Omega=\tilde{a}^\dagger_{\alpha_i}(t,h^\theta_{i,\alpha_i})\Omega\quad\text{and }\Phi^\dagger(t,g^\theta_{i,\beta_i})\Omega=\tilde{a}^\dagger_{\beta_i}(t,g^\theta_{i,\beta_i})\Omega. \tag{4.17}$$

Moreover, for $\mathfrak{f}_\alpha\in C_0^\infty(\mathbb{R}^\alpha)$, one has

$$a^\dagger_\alpha(\mathfrak{f}_\alpha)=\tilde{a}^\dagger_\alpha(t,f^t_\alpha). \tag{4.18}$$

Collecting all this, theorem 4.3 is proved for $n=m=1$ using (i), and (iii) of lemma (4.2),

$$\|\Phi^\dagger(t,f_\alpha^\theta)\Omega - a_\alpha^\dagger(\mathfrak{f}_\alpha)\Omega\| = \|\tilde{a}_\alpha^\dagger(t,f_\alpha^\theta)\Omega - \tilde{a}_\alpha^\dagger(t,f_\alpha^t)\Omega\| \leqslant \|f_\alpha^\theta - f_\alpha^t\|, \tag{4.19}$$

$\mathfrak{f}$ playing the role of $\mathfrak{h}$ or $\mathfrak{g}$. Now we want to compute the left-hand sides of Eqs. (4.15) and (4.16) for $n,m \geqslant 2$. We give details for Eq. (4.15),

$$\begin{aligned}\Phi^\dagger(t,h_{1,\alpha_1}^\theta)\cdots\Phi^\dagger(t,h_{n,\alpha_n}^\theta)\Omega &= \sum_{\sigma\in\mathfrak{S}_n}\int_{\mathbb{R}^n} \mathrm{d}x_1\cdots\mathrm{d}x_n\, H_{\alpha_1\cdots\alpha_n}^\sigma(x_1,\ldots,x_n)\Phi_{\alpha_1}^\dagger(t,x_1)\cdots\Phi_{\alpha_n}^\dagger(t,x_n)\Omega \\ &= \sum_{\sigma\in\mathfrak{S}_n}\int_{\mathbb{R}^n} \mathrm{d}x_1\cdots\mathrm{d}x_n\, H_{\alpha_1\cdots\alpha_n}^\sigma(x_1,\ldots,x_n)\tilde{a}_{\alpha_{\sigma_1}}^\dagger(t,x_{\alpha_{\sigma_1}})\cdots\tilde{a}_{\alpha_{\sigma_n}}^\dagger(t,x_{\alpha_{\sigma_n}})\Omega \\ &= \tilde{a}_{\alpha_1}^\dagger(t,h_{1,\alpha_1}^\theta)\cdots\tilde{a}_{\alpha_n}^\dagger(t,h_{n,\alpha_n}^\theta)\Omega + \sum_{\substack{\sigma\in\mathfrak{S}_n\\ \sigma\neq e_n}}\int_{\mathbb{R}^n} \mathrm{d}x_1\cdots\mathrm{d}x_n\, H_{\alpha_1\cdots\alpha_n}^\sigma(x_1,\ldots,x_n) \\ &\quad\times\{\tilde{a}_{\alpha_{\sigma_1}}^\dagger(t,x_{\alpha_{\sigma_1}})\cdots\tilde{a}_{\alpha_{\sigma_n}}^\dagger(t,x_{\alpha_{\sigma_n}})\Omega - \tilde{a}_{\alpha_1}^\dagger(t,h_{1,\alpha_1}^\theta)\cdots\tilde{a}_{\alpha_n}^\dagger(t,h_{n,\alpha_n}^\theta)\Omega\},\end{aligned} \tag{4.20}$$

where we used point (iii) of lemma 3.4 and (3.61) for $\Phi^\dagger$ in the second equality. Applying (4.13) then gives

$$\begin{aligned}&\|\Phi^\dagger(t,h_{1,\alpha_1}^\theta)\cdots\Phi^\dagger(t,h_{n,\alpha_n}^\theta)\Omega - a_{\alpha_1}^\dagger(\mathfrak{h}_{1,\alpha_1})\cdots a_{\alpha_n}^\dagger(\mathfrak{h}_{n,\alpha_n})\Omega\| \leqslant \sqrt{n!}\|h_{1,\alpha_1}^\theta\otimes\cdots\otimes h_{n,\alpha_n}^\theta - h_{1,\alpha_1}^t\otimes\cdots \\ &\quad\otimes h_{n,\alpha_n}^t\| + 2\sqrt{n!}\sum_{\substack{\sigma\in\mathfrak{S}_n\\ \sigma\neq e_n}}\|H_{\alpha_1\cdots\alpha_n}^\sigma\|,\end{aligned} \tag{4.21}$$

implying (4.15) by points (i)–(ii) of lemma 4.2 Similar computations give

$$\begin{aligned}&\|\Phi^\dagger(t,g_{1,\beta_1}^\theta)\cdots\Phi^\dagger(t,g_{m,\beta_m}^\theta)\Omega - a_{\beta_1}^\dagger(\mathfrak{g}_{1,\beta_1})\cdots a_{\beta_m}^\dagger(\mathfrak{g}_{m,\beta_m})\Omega\| \leqslant \sqrt{m!}\|g_{1,\beta_1}^\theta\otimes\cdots\otimes g_{m,\beta_m}^\theta - g_{1,\beta_1}^t \\ &\quad\otimes\cdots\otimes g_{m,\beta_m}^t\| + 2\sqrt{m!}\sum_{\substack{\pi\in\mathfrak{S}_m\\ \neq e_m}}\|G_{\beta_1\cdots\beta_m}^\pi\|,\end{aligned} \tag{4.22}$$

proving (4.16). ∎

B. Scattering matrix

Now that we have identified the natural "free" dynamics approached by our interacting field as $t\to\pm\infty$, we are left with the verification of asymptotic completeness allowing the construction of a unitary S-matrix. We emphasize here that our "in" and "out" spaces are slightly different from those exhibited in Ref. 7 because of our ordering involving absolute values, so that we must recheck their properties.

Proposition 4.4: Let

$$\mathcal{F}^{\mathrm{in}} = \mathrm{vect}\{\Omega, a_{\beta_1}^\dagger(\mathfrak{g}_{1,\beta_1})\cdots a_{\beta_m}^\dagger(\mathfrak{g}_{m,\beta_m})\Omega,\ \beta_i = \pm,\ i=1,\ldots,m,\ m\geqslant 1\}, \tag{4.23}$$

$$\mathcal{F}^{\mathrm{out}} = \mathrm{vect}\{\Omega, a_{\alpha_1}^\dagger(\mathfrak{h}_{1,\alpha_1})\cdots a_{\alpha_n}^\dagger(\mathfrak{h}_{n,\alpha_n})\Omega,\ \alpha_i = \pm,\ i=1,\ldots,n,\ n\geqslant 1\}, \tag{4.24}$$

where $\mathfrak{h}_{i,\alpha_i}$ and $\mathfrak{g}_{j,\beta_j}$ run over $\mathfrak{H}_n$ and $\mathfrak{G}_m$.

Then, $\mathcal{F}^{\mathrm{in}}$ and $\mathcal{F}^{\mathrm{out}}$ are separately dense in $\mathcal{F}$.

Proof: We deal with $\mathcal{F}^{\mathrm{in}}$. Again, it is sufficient to consider the matrix element,

$$A_{t,\varphi,\beta_1\cdots\beta_p}(p_1,\ldots,p_m)=\langle\varphi^{(n)},a^\dagger_{\beta_1}(t,p_1)\cdots a^\dagger_{\beta_m}(t,p_m)\Omega\rangle, \tag{4.25}$$

where $\varphi^{(n)}\in\mathcal{H}^{(n)}$ is arbitrary and to show that

$$A_{t,\varphi,\beta_1\cdots\beta_p}(p_1,\ldots,p_m)=0,\quad \forall\,|p_1|<\cdots<|p_m|,\quad p_i\in\mathbb{R}^{-\beta_i},\quad \beta_i=\pm,\quad i=1,\ldots,m \tag{4.26}$$

implies $\varphi^{(n)}=0$. From the cyclicity of Ω with respect to $a^\dagger$, (4.26) gives

$$\varphi^{(n)}_{\beta_1\cdots\beta_p}(p_1,\ldots,p_m)=0,\quad \forall\,|p_1|<\cdots<|p_m|,\quad p_i\in\mathbb{R}^{-\beta_i},\quad \beta_i=\pm,\quad i=1,\ldots,m \tag{4.27}$$

and in view of the properties of $\varphi^{(n)}\in\mathcal{H}^{(n)}$, this implies in turn

$$\varphi^{(n)}_{\beta_1\cdots\beta_p}(p_1,\ldots,p_m)=0,\quad \forall\,p_i\in\mathbb{R},\quad \beta_i=\pm,\quad i=1,\ldots,m, \tag{4.28}$$

i.e., $\varphi^{(n)}=0$. The case of $\mathcal{F}^{\text{out}}$ is similar. ■

We turn to the definition of the scattering operator **S** of our theory.

Proposition 4.5: Take functions in $\mathfrak{H}_n$ *and let* **S**: $\mathcal{F}^{\text{out}}\to\mathcal{F}^{\text{in}}$ *act as follows:*

$$\mathbf{S}\Omega=\Omega\quad \textit{and}\ \mathbf{S}{:}a^\dagger_{\alpha_1}(\mathfrak{h}_{1,\alpha_1})\cdots a^\dagger_{\alpha_n}(\mathfrak{h}_{n,\alpha_n})\Omega\mapsto a^\dagger_{\alpha_n}(\hat{\mathfrak{h}}_{n,\alpha_n})\cdots a^\dagger_{\alpha_1}(\hat{\mathfrak{h}}_{1,\alpha_1})\Omega \tag{4.29}$$

$$\textit{where}\quad \hat{\mathfrak{h}}_{i,\alpha_i}(p)=\mathfrak{h}_{i,\alpha_i}(-p)\in\mathfrak{G}_n. \tag{4.30}$$

Then **S** *is invertible and* **S**, $\mathbf{S}^{-1}$ *are unitary operators acting on* $\mathcal{F}$.

Proof: From the definitions (4.29) and (4.30), one deduces immediately that $\mathbf{S}^{-1}$ is well defined. Then, it is straightforward, albeit lengthy, to check that

$$\begin{aligned}&\langle\mathbf{S}a^\dagger_{\alpha_1}(\mathfrak{h}_{1,\alpha_1})\cdots a^\dagger_{\alpha_n}(\mathfrak{h}_{n,\alpha_n})\Omega,\mathbf{S}a^\dagger_{\gamma_1}(\mathfrak{f}_{1,\gamma_1})\cdots a^\dagger_{\gamma_n}(\mathfrak{f}_{n,\gamma_n})\Omega\rangle\\ &\quad=\langle a^\dagger_{\alpha_1}(\mathfrak{h}_{1,\alpha_1})\cdots a^\dagger_{\alpha_n}(\mathfrak{h}_{n,\alpha_n})\Omega,a^\dagger_{\gamma_1}(\mathfrak{f}_{1,\gamma_1})\cdots a^\dagger_{\gamma_n}(\mathfrak{f}_{n,\gamma_n})\Omega\rangle.\end{aligned} \tag{4.31}$$

In evaluating the left-hand side, one just has to notice that all the contributions coming from the defect generators vanish due to the support properties of the smearing functions and one is left with what would be obtained by using the ZF algebra. Then, it is just a matter of changing the variables into their opposite to get the right-hand side.

Next, following the line of argument given in Ref. 19, one extends **S** to $\mathcal{F}^{\text{out}}$ by linearity, preserving unitarity. This gives rise to bounded linear operators which one can uniquely extend by continuity to the whole of $\mathcal{F}$. We note that this last step is allowed by the asymptotic completeness property satisfied by $\mathcal{F}^{\text{out}}$ and $\mathcal{F}^{\text{in}}$ (cf. Proposition 4.4). The case of $\mathbf{S}^{-1}$ is similar. ■

Refering now to Ref. 7 we finish the description of our scattering theory by defining the correspondence between in and out states and the asymptotic states identified in theorem 4.3 (correspondence already anticipated in our calling $\mathcal{F}^{\text{out}}$ and $\mathcal{F}^{\text{in}}$ the "in" and "out" spaces),

$$|\mathfrak{g}_{1,\beta_1};\ldots;\mathfrak{g}_{m,\beta_m}\rangle^{\text{in}}=a^\dagger_{\beta_1}(\mathfrak{g}_{1,\beta_1})\cdots a^\dagger_{\beta_m}(\mathfrak{g}_{m,\beta_m})\Omega, \tag{4.32}$$

$$|\mathfrak{h}_{1,\alpha_1};\ldots;\mathfrak{h}_{n,\alpha_n}\rangle^{\text{out}}=a^\dagger_{\alpha_1}(\mathfrak{h}_{1,\alpha_1})\cdots a^\dagger_{\alpha_n}(\mathfrak{h}_{n,\alpha_n})\Omega. \tag{4.33}$$

Transition amplitudes are therefore easily computable from

$$^{\text{out}}\langle \mathfrak{h}_{1,\alpha_1}; \ldots ; \mathfrak{h}_{n,\alpha_n} | \mathfrak{g}_{1,\beta_1}; \ldots ; \mathfrak{g}_{m,\beta_m} \rangle^{\text{in}} = \langle a^\dagger_{\alpha_1}(\mathfrak{h}_{1,\alpha_1}) \cdots a^\dagger_{\alpha_n}(\mathfrak{h}_{n,\alpha_n})\Omega, a^\dagger_{\beta_1}(\mathfrak{g}_{1,\beta_1}) \cdots a^\dagger_{\beta_m}(\mathfrak{g}_{m,\beta_m})\Omega \rangle \tag{4.34}$$

and using (3.27), (3.8), (3.13), (3.15), and (3.23). One recovers for transition amplitudes that they vanish unless $n=m$ as expected for an integrable system where particle production does not occur. As an example, we derive in our context the one and two particle transition amplitudes obtained in Ref. 6. We start with the computation of the correlators,

$$\langle a^\dagger_\alpha(p)\Omega, a^\dagger_\beta(q)\Omega \rangle = \delta^\beta_\alpha \delta(p-q) + \epsilon^\beta_\alpha \delta(p-q) T(\alpha p) + \delta^\beta_\alpha \delta(p+q) R(\alpha p) \tag{4.35}$$

and

$$\begin{aligned}
&\langle a^\dagger_{\alpha_1}(p_1) a^\dagger_{\alpha_2}(p_2)\Omega, a^\dagger_{\beta_1}(q_1) a^\dagger_{\beta_2}(q_2)\Omega \rangle \\
&\quad = S(\alpha_1 p_1 - \beta_1 q_1)[\delta^{\beta_1}_{\alpha_2} + \epsilon^{\beta_1}_{\alpha_2} T(\alpha_2 p_2)][\delta^{\beta_2}_{\alpha_1} + \epsilon^{\beta_2}_{\alpha_1} T(\alpha_1 p_1)] \\
&\qquad \times \delta(p_2 - q_1)\,\delta(p_1 - q_2) + S(\alpha_1 p_1 - \beta_1 q_1)[\delta^{\beta_1}_{\alpha_2} R(\alpha_2 p_2)][\delta^{\beta_2}_{\alpha_1} + \epsilon^{\beta_2}_{\alpha_1} T(\alpha_1 p_1)] \\
&\qquad \times \delta(p_2 + q_1)\,\delta(p_1 - q_2) + S(\alpha_1 p_1 - \beta_1 q_1)[\delta^{\beta_1}_{\alpha_2} + \epsilon^{\beta_1}_{\alpha_2} T(\alpha_2 p_2)][\delta^{\beta_2}_{\alpha_1} R(\alpha_1 p_1)]\delta(p_2 - q_1)\,\delta(p_1 + q_2) \\
&\qquad + S(\alpha_1 p_1 - \beta_1 q_1)[\delta^{\beta_1}_{\alpha_2} R(\alpha_2 p_2)][\delta^{\beta_2}_{\alpha_1} R(\alpha_1 p_1)]\delta(p_2 + q_1)\,\delta(p_1 + q_2) \\
&\qquad + [\delta^{\beta_1}_{\alpha_1} + S(\alpha_1 p_1 - \beta_2 q_2) S(\alpha_1 p_1 + \beta_2 q_2)\epsilon^{\beta_1}_{\alpha_1} T(\alpha_1 p_1)][\delta^{\beta_2}_{\alpha_2} + \epsilon^{\beta_2}_{\alpha_2} T(\alpha_2 p_2)]\delta(p_1 - q_1)\,\delta(p_2 - q_2) \\
&\qquad + S(\alpha_1 p_1 - \beta_2 q_2) S(\alpha_1 p_1 + \beta_2 q_2)[\delta^{\beta_1}_{\alpha_1} R(\alpha_1 p_1)][\delta^{\beta_2}_{\alpha_2} + \epsilon^{\beta_2}_{\alpha_2} T(\alpha_2 p_2)]\delta(p_1 + q_1)\,\delta(p_2 - q_2) \\
&\qquad + [\delta^{\beta_1}_{\alpha_1} + S(\alpha_1 p_1 - \beta_2 q_2) S(\alpha_1 p_1 + \beta_2 q_2)\epsilon^{\beta_1}_{\alpha_1} T(\alpha_1 p_1)][\delta^{\beta_2}_{\alpha_2} R(\alpha_2 p_2)]\delta(p_1 - q_1)\,\delta(p_2 + q_2) \\
&\qquad + S(\alpha_1 p_1 - \beta_2 q_2) S(\alpha_1 p_1 + \beta_2 q_2)[\delta^{\beta_1}_{\alpha_1} R(\alpha_1 p_1)][\delta^{\beta_2}_{\alpha_2} R(\alpha_2 p_2)]\delta(p_1 + q_1)\,\delta(p_2 + q_2)
\end{aligned} \tag{4.36}$$

We note that the result for the two-particle correlator differs from that obtained in Ref. 6 by the appearance of two S coefficients in the four last terms. This is due to the fact that we started with a more general RT algebra where the defect generators do not necessarily obey the linear relations used in Ref. 6. For the one-particle amplitudes, there are two possibilities according to the relative signs of the in and out states

$$^{\text{out}}\langle \mathfrak{h}_\pm, \mathfrak{g}_\pm \rangle^{\text{in}} = \begin{cases} \displaystyle\int_0^\infty \frac{\mathrm{d}p}{2\pi} \overline{\mathfrak{h}_+}(p) R(p) \mathfrak{g}_+(-p), \\ \displaystyle\int_{-\infty}^0 \frac{\mathrm{d}p}{2\pi} \overline{\mathfrak{h}_-}(p) R(-p) \mathfrak{g}_-(-p), \end{cases} \tag{4.37}$$

$$^{\text{out}}\langle \mathfrak{h}_\pm | \mathfrak{g}_\mp \rangle^{\text{in}} = \begin{cases} \displaystyle\int_0^\infty \frac{\mathrm{d}p}{2\pi} \overline{\mathfrak{h}_+}(p) T(p) \mathfrak{g}_-(p), \\ \displaystyle\int_{-\infty}^0 \frac{\mathrm{d}p}{2\pi} \overline{\mathfrak{h}_-}(p) T(-p) \mathfrak{g}_+(p). \end{cases} \tag{4.38}$$

One clearly sees the particle-impurity interaction through the reflection coefficient R for a final and an initial state on the same half-line and through the transmission coefficient T otherwise, as expected. The particle–particle interaction through the bulk interaction coefficient S shows up in the 2^4 different two-particle amplitudes. As an illustration, we compute four such amplitudes gathered into two generic expressions:

$$
\begin{aligned}
{}^{\mathrm{out}}\langle \mathfrak{h}_{1,\pm};\mathfrak{h}_{2,\pm}|\mathfrak{g}_{1,\pm};\mathfrak{g}_{2,\pm}\rangle^{\mathrm{in}} = \int_{\mathbb{R}^\pm}\frac{\mathrm{d}p_1}{2\pi}\int_{\mathbb{R}^\pm}\frac{\mathrm{d}p_2}{2\pi}&\big(\overline{\mathfrak{h}_{1,\pm}}(p_1)\overline{\mathfrak{h}_{2,\pm}}(p_2)R(\pm p_2)S(\pm p_1 \pm p_2)R(\pm p_1)\mathfrak{g}_{1,\pm}(-p_2)\\
&\times \mathfrak{g}_{2,\pm}(-p_1) + \overline{\mathfrak{h}_{1,\pm}}(p_1)\overline{\mathfrak{h}_{2,\pm}}(p_2)R(\pm p_1)S(\pm p_1 \pm p_2)S(\pm p_1 \mp p_2)R(\pm p_2)\\
&\times \mathfrak{g}_{1,\pm}(-p_1)\mathfrak{g}_{2,\pm}(-p_2)\big)
\end{aligned}
\tag{4.39}
$$

and

$$
\begin{aligned}
{}^{\mathrm{out}}\langle \mathfrak{h}_{1,\pm};\mathfrak{h}_{2,\pm}|\mathfrak{g}_{1,\pm};\mathfrak{g}_{2,\mp}\rangle^{\mathrm{in}} = \int_{\mathbb{R}^\pm}\frac{\mathrm{d}p_1}{2\pi}\int_{\mathbb{R}^\pm}\frac{\mathrm{d}p_2}{2\pi}&\big(\overline{\mathfrak{h}_{1,\pm}}(p_1)\overline{\mathfrak{h}_{2,\pm}}(p_2)R(\pm p_2)S(\pm p_1 \pm p_2)T(\pm p_1)\\
&\times \mathfrak{g}_{1,\pm}(-p_2)\mathfrak{g}_{2,\mp}(p_1) + \overline{\mathfrak{h}_{1,\pm}}(p_1)\overline{\mathfrak{h}_{2,\pm}}(p_2)R(\pm p_1)\\
&\times S(\pm p_1 \pm p_2)S(\pm p_1 \mp p_2)T(\pm p_2)\mathfrak{g}_{1,\pm}(-p_1)\mathfrak{g}_{2,\mp}(p_2)\big).
\end{aligned}
\tag{4.40}
$$

More complex transition amplitudes contain the same building blocks namely R, T, and S, which shows that the corresponding processes involve a succession of particle-impurity and particle–particle interactions as expected from the factorized scattering occurring in this integrable model.

V. DISCUSSION AND CONCLUSIONS

We have analyzed above the NLS model interacting with a δ-type impurity, establising the exact classical and quantum solutions. We have shown that an appropriate RT algebra and its Fock representation allow to construct not only the scattering operator, but also the off-shell quantum field $\Phi(t,x)$. As already mentioned in the introduction, these results can be extended[33] to a whole class of point-like defects, substituting (3.72) and (3.73) by the impurity boundary conditions

$$
\lim_{x\downarrow 0}\begin{pmatrix}\langle \varphi,\Phi(t,x)\psi\rangle\\ \partial_x\langle \varphi,\Phi(t,x)\psi\rangle\end{pmatrix} = \alpha\begin{pmatrix} a & b\\ c & d\end{pmatrix}\lim_{x\uparrow 0}\begin{pmatrix}\langle \varphi,\Phi(t,x)\psi\rangle\\ \partial_x\langle \varphi,\Phi(t,x)\psi\rangle\end{pmatrix},
\tag{5.1}
$$

where

$$
\{a,\ldots,d \in \mathbb{R},\ \alpha \in \mathbb{C}: ad - bc = 1,\ \overline{\alpha}\alpha = 1,\}.
\tag{5.2}
$$

In absence of impurity bound states, namely in the domain

$$
\begin{aligned}
&a + d + \sqrt{(a-d)^2+4} \leqslant 0, \quad b < 0,\\
&c(a+d)^{-1} \geqslant 0, \quad b = 0,\\
&a + d - \sqrt{(a-d)^2+4} \geqslant 0, \quad b > 0,
\end{aligned}
\tag{5.3}
$$

one can treat the model closely following the δ-impurity case, because the corresponding reflection and transmission matrices $\mathcal{R}$ and $\mathcal{T}$ have the same analytic properties as (3.18).

We would like to comment finally on the symmetry content of the solution derived in the paper. It is quite obvious that impurities break down Galilean (Lorentz) invariance of the *total* scattering matrix **S**. However, since the *bulk* scattering matrix $\mathcal{S}$ describes the scattering away from the impurity, some authors[1–5] have assumed that $\mathcal{S}$ preserves these symmetries and that the breaking in **S** is generated exclusively by the reflection and transmission coefficients $\mathcal{R}$ and $\mathcal{T}$. This assumption however, combined with the conditions of factorized scattering, implies[1,5] that $\mathcal{S}$ is constant, which is too restrictive. In fact, one is left with a few systems of limited physical interest. In order to avoid this negative result, a consistent factorized scattering theory was developed in Refs. 6 and 7, which does not necessarily assume that $\mathcal{S}$ is Galilean (Lorentz) invariant. Since the impurity NLS model considered above is the first concrete application of this framework with nontrivial bulk scattering, the lesson from it is quite instructive. Focusing on $\mathcal{S}$ (3.2), we see that Galilean invariance is broken by the entries which describe the scattering of two incoming particles localized for $t \to -\infty$ on the different half-lines $\mathbb{R}_-$ and $\mathbb{R}_+$ respectively. Indeed, these entries depend on k_1+k_2 and not on k_1-k_2. An intuitive explanation for this breaking is that before such particles scatter, one of them must necessarily cross the impurity. The nontrivial transmission is therefore the origin of the symmetry breaking in $\mathcal{S}$. This conclusion agrees with the observation that in systems which allow only reflection (e.g., models on the half-line), one can have[14–19] both Galilean (Lorentz) invariant and nonconstant bulk scattering matrices.

The issue of internal symmetries in the presence of impurities has been partially addressed in Refs. 8 and 11. In particular, the role of the reflection and transmission elements of the RT algebra as symmetry generators has been established. However, this question deserves further investigation. It will be interesting in this respect to extend the analysis[38] of the SU(N)–NLS model on the half-line to the impurity case. Work is in progress on this aspect.

Let us conclude by observing that the concept of RT algebra indeed represents a powerful tool for solving the NLS model with impurities. We are currently exploring the possibility to apply this algebraic framework also to the quantization of other integrable systems with defects.

APPENDIX A: PROOF OF THEOREM 2.2

First, notice that (2.12) and (2.13) translate into

$$\lim_{x\to 0^+} \{\Phi_+(t,x) - \Phi_-(t,-x)\} = 0, \tag{A1}$$

$$\lim_{x\to 0^+} \{(\partial_x\Phi_+)(t,x) - (\partial_x\Phi_-)(t,-x)\} - 2\eta\Phi(t,0) = 0, \tag{A2}$$

which we are going to check order by order in the Rosales expansion. The idea is to introduce the one-to-one correspondence

$$\beta_\pm(p) = \tfrac{1}{2}\{\lambda_+(p) \pm \lambda_-(-p)\}, \quad p \in \mathbb{R} \tag{A3}$$

and it is not difficult to check that

$$\beta_\alpha(p) = B_\alpha(p)\beta_\alpha(-p), \quad \text{with } B_\alpha(p) = \alpha\frac{p - i\alpha\eta}{p + i\eta}, \quad \alpha = \pm\,. \tag{A4}$$

Take $n=0$ corresponding to the linear problem. One gets

$$\lim_{x\to 0^+} \{\Phi_+^{(0)}(t,x) - \Phi_-^{(0)}(t,-x)\} = \int_{\mathbb{R}} \frac{\mathrm{d}p}{2\pi}\beta_-(p)e^{-ip^2t},$$

$$\lim_{x\to 0^+} \{(\partial_x\Phi_+^{(0)})(t,x) - (\partial_x\Phi_-^{(0)})(t,-x)\} - 2\eta\Phi^{(0)}(t,0) = \int_{\mathbb{R}} \frac{\mathrm{d}p}{2\pi}(ip-\eta)\beta_+(p)e^{-ip^2t},$$

which vanish using the properties (A4). It is interesting to note that the time-dependent phase e^{-ip^2t}, being even in p, does not play any role in the vanishing of the previous expressions. It will be the same in the following as we shall see.

For $n \geqslant 1$, we start by changing variables in the Rosales expansion according to $(p_1,\ldots,p_n,q_n,\ldots,q_0) \to (k_1,\ldots,k_{2n-1},-k_{2n}\ldots,-k_0)$ and we use the one-to-one correspondence (A3) to rewrite the left-hand side of (A1) as

$$\lim_{x\to 0^+}\{\Phi_+^{(n)}(0,x)-\Phi_-^{(n)}(0,-x)\} = \sum_{\alpha_0,\ldots,\alpha_{2n}=\pm}\left(1-\prod_{i=0}^{2n}\alpha_i\right)\int_{\mathbb{R}^{2n+1}}\prod_{i=0}^{2n}\frac{\mathrm{d}k_i}{2\pi}\overline{\beta}_{\alpha_1}(k_1)\cdots\overline{\beta}_{\alpha_{2n-1}}(k_{2n-1})$$
$$\times\beta_{\alpha_{2n}}(-k_{2n})\cdots\beta_{\alpha_0}(-k_0)\frac{e^{-i\Sigma_{j=0}^{2n}k_j^2t}}{\prod_{j=1}^{2n}(k_j+k_{j-1})}. \tag{A5}$$

In view of the linear case, we "B_α-symmetrize" the integrand of the previous integral for each k_i. Introducing

$$B_\alpha^\sigma(p) = \begin{cases} 1 & \text{for } \sigma = + , \\ B_\alpha(p) & \text{for } \sigma = - , \end{cases} \tag{A6}$$

this reads

$$\frac{1}{2^{2n+1}}\sum_{\sigma_0,\ldots,\sigma_{2n}=\pm}\frac{B_{\alpha_1}^{\sigma_1}(k_1)\cdots B_{\alpha_{2n-1}}^{\sigma_{2n-1}}(k_{2n-1})B_{\alpha_{2n}}^{\sigma_{2n}}(-k_{2n})\cdots B_{\alpha_0}^{\sigma_0}(-k_0)}{\prod_{j=1}^{2n}(\sigma_jk_j+\sigma_{j-1}k_{j-1})}\overline{\beta}_{\alpha_1}(k_1)\cdots\overline{\beta}_{\alpha_{2n-1}}(k_{2n-1})$$
$$\times\beta_{\alpha_{2n}}(-k_{2n})\cdots\beta_{\alpha_0}(-k_0)e^{-i\Sigma_{j=0}^{2n}k_j^2t}$$

which we rewrite as

$$\frac{1}{2^{2n+1}}\sum_{\sigma_0,\ldots,\sigma_{2n}=\pm}B_{\alpha_1}^{\sigma_1}(k_1)\cdots B_{\alpha_{2n-1}}^{\sigma_{2n-1}}(k_{2n-1})B_{\alpha_{2n}}^{\sigma_{2n}}(-k_{2n})\cdots B_{\alpha_0}^{\sigma_0}(-k_0)\prod_{j=1}^{2n}(\sigma_{j-1}k_{j-1}-\sigma_jk_j)$$
$$\times\frac{\overline{\beta}_{\alpha_1}(k_1)\cdots\overline{\beta}_{\alpha_{2n-1}}(k_{2n-1})\beta_{\alpha_{2n}}(-k_{2n})\cdots\beta_{\alpha_0}(-k_0)}{\prod_{j=1}^{2n}(k_{j-1}^{2n}-k_j^2)}e^{-i\Sigma_{j=0}^{2n}k_j^2t}.$$

Let us concentrate on the part depending on the σ's. Developing explicitly the sum over σ_{2n}, one gets

$$\frac{1}{2^{2n+1}}\sum_{\sigma_0,\ldots,\sigma_{2n-1}=\pm}B_{\alpha_1}^{\sigma_1}(k_1)\cdots B_{\alpha_{2n-1}}^{\sigma_{2n-1}}(k_{2n-1})B_{\alpha_{2n-2}}^{\sigma_{2n-2}}(-k_{2n-2})\cdots B_{\alpha_0}^{\sigma_0}(-k_0)\prod_{j=1}^{2n-1}(\sigma_{j-1}k_{j-1}-\sigma_jk_j)$$
$$\times\left(\delta_{\alpha_{2n},+}\frac{2k_{2n}}{k_{2n}+i\eta}-\delta_{\alpha_{2n},-}2k_{2n}\right)$$

Collecting all the pieces depending on k_{2n}, one gets a function proportional to

$$\frac{k_{2n}}{k_{2n-2}^2-k_{2n}^2}\left(\frac{\beta_+(-k_{2n})}{k_{2n}+i\eta}-\beta_-(-k_{2n})\right). \tag{A7}$$

Now taking μ_+,μ_- as in (2.20) it is not hard to see that the function in parentheses in (A7) is identically zero, implying the vanishing of (A5).

The case of the jump condition is treated in complete analogy. Indeed, in evaluating the term proportional to η in (A2) in terms of $\beta_\pm$, all one must do is to replace $(1-\Pi_{i=0}^{2n}\alpha_i)$ in (A5) by $(1+\Pi_{i=0}^{2n}\alpha_i)$. The rest of the argument implies therefore that

$$\Phi^{(n)}(0,0)=0, \quad n \geqslant 1. \tag{A8}$$

As for the term involving derivatives of the field, an analogous treatment produces the following integrand:

$$\frac{1}{2^{2n+1}} \sum_{\sigma_0,\ldots,\sigma_{2n}=\pm} B_{\alpha_1}^{\sigma_1}(k_1)\cdots B_{\alpha_{2n-1}}^{\sigma_{2n-1}}(k_{2n-1}) B_{\alpha_{2n}}^{\sigma_{2n}}(-k_{2n})\cdots B_{\alpha_0}^{\sigma_0}(-k_0) \prod_{j=1}^{2n}(\sigma_{j-1}k_{j-1}-\sigma_j k_j)$$
$$\times\left(\sum_{j=0}^{2n} i\sigma_j k_j\right)\frac{\overline{\beta}_{\alpha_1}(k_1)\cdots\overline{\beta}_{\alpha_{2n-1}}(k_{2n-1})\beta_{\alpha_{2n}}(-k_{2n})\cdots\beta_{\alpha_0}(-k_0)}{\prod_{j=1}^{2n}(k_{j-1}^2-k_j^2)} e^{-i\Sigma_{j=0}^{2n}k_j^2 t}.$$

This time, one must develop the sum for σ_{2n} and σ_{2n-1}. This produces the function (A7) but in the variable k_{2n-1} and we know it vanishes. This leads to

$$\lim_{x\to 0^+}\{(\partial_x\Phi_+^{(n)})(0,x)-(\partial_x\Phi_-^{(n)})(0,-x)\}=0, \quad n \geqslant 1. \tag{A9}$$

As already mentioned, we see that the continuity and the jump condition of the field hold for any time t. Put another way, they are conserved in time and this is due to the dispersion relation of the free Schrödinger equation (being quadratic in k_j, it is not affected by all the symmetrizations $k_j \to -k_j$ involved in the proof).

It is remarkable that the jump condition actually decouples for the nonlinear terms ($n \geqslant 1$) as seen from (A8) and (A9). This is also true for the continuity which, combined with (A8) shows that

$$\Phi_-^{(n)}(0,0)=\Phi_+^{(n)}(0,0)=0, \quad n \geqslant 1.$$

■

APPENDIX B: EXPLICIT FORM OF THE ACTION OF THE CREATION OPERATOR

The projector $P^{(n)}$ is constructed in Ref. 7 in terms of the generators of the Weyl group associated to the root system of the classical Lie algebra B_n and of their representation on $\mathcal{L}^{\otimes n}$. In our context, we get for $f \in \mathcal{C}$ and $\varphi^{(n-1)} \in \mathcal{H}^{(n-1)}$,

$$\begin{aligned}[a^\dagger(\mathfrak{f})\varphi]_{\alpha_1\cdots\alpha_n}^{(n)}(p_1,\ldots,p_n) = \frac{1}{2\sqrt{n}}\sum_{k=1}^{n} & S(\alpha_{k-1}p_{k-1}-\alpha_k p_k)\cdots S(\alpha_1 p_1-\alpha_k p_k)(f_{\alpha_k}(p_k) \\ & + C_k(\alpha_1 p_1,\ldots,\alpha_n p_n)[T(\alpha_k p_k)f_{-\alpha_k}(p_k)+R(\alpha_k p_k)f_{\alpha_k}(-p_k)]) \\ & \times \varphi^{(n-1)}_{\alpha_1\cdots\hat{\alpha_k}\cdots\alpha_n}(p_1,\ldots,p_n),\end{aligned} \tag{B1}$$

where we have defined

$$C_k(p_1,\ldots,p_n)=S(p_k-p_1)\cdots S(\widehat{p_k-p_k})\cdots S(p_k-p_n)S(p_n+p_k)\cdots S(\widehat{p_k+p_k})\cdots S(p_1+p_k).$$

All the hatted symbols must be omitted.

One recognizes the reflected and transmitted structure inside the square brackets of (B1) which, combined with all the S matrices, ensures the properties (3.21) and (3.22) required for the functions of $\mathcal{H}^{(n)}$.

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