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Boundary S-matrix for the Gross-Neveu Model [1]

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Abstract

We study the scattering theory for the Gross-Neveu model on the half-line. We find the reflection matrices for the elementary fermions, and by fusion we compute the ones for the two-particle bound-states, showing that they satisfy non-trivial bootstrap consistency conditions. We also compute more general reflection matrices for the Gross-Neveu model and the nonlinear sigma model, and argue that they correspond to the integrable boundary conditions we identified in our previous paper [5].

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1 Introduction

After the seminal paper by Ghoshal and Zamolodchikov [1] (see also [2]) on integrable quantum field theories with boundary, a lot of work has been done extending it, especially in the analysis of different models. In this paper we study one of these extensions, namely the $O(N)$ Gross-Neveu (GN) model [3] on the half-line, which is closely related to the boundary $O(N)$ non-linear sigma (nlσ) model studied by Ghoshal in [4]. Recently [5] we have found new integrable boundary conditions (bc's) for the GN and nlσ models, based on the microscopic (lagrangian) description of these models. Here we find general (diagonal) solutions for the boundary Yang-Baxter equation and propose a one-to-one correspondence between these solutions and the boundary conditions we found in [5].

This paper is organized as follows. In the next section we briefly review the GN and nlσ models, and their exact S-matrices; in section 3 we find the exact reflection matrices for the GN model with the physically interesting free (Neumann) and fixed boundary conditions; in section 4 we solve the boundary Yang-Baxter equation for the more general boundary conditions we found in our previous paper [5], and establish the correspondence between those boundary conditions and the reflection matrices we find; in the final section we present our conclusions. We also have an appendix where we write the exact amplitudes for the scattering between bound-states of elementary fermions in the GN model [7, 8]. Their computation is a simple exercise in fusion, but since we need their explicit form in this paper, we give the results in the appendix.

2 The Gross-Neveu Model

The GN model is defined by the following lagrangian

$$\mathcal{L}_{gn} = \bar{\psi} i \partial\!\!\!/ \psi + \frac{g^2}{2} (\bar{\psi}\psi)^2 \ , \tag{2.1}$$

where ψ is a N-component massless Majorana fermion in the fundamental representation of $O(N)$. In the above equation $\bar{\psi}\psi$ should be understood as $\sum_{i=1}^{N} \bar{\psi}^i \psi^i$ and so on. In this paper we will refer to the $O(N)$ GN model simply as GN model. It is useful to write the GN model lagrangian in light-cone coordinates, where it reads

$$\mathcal{L}_{gn} = 2\psi_+^i i\partial_- \psi_+^i + 2\psi_-^i i\partial_+ \psi_-^i + 2g^2 (\psi_+^i \psi_-^i)^2 \ . \tag{2.2}$$

The particle spectrum of the GN model [6] is composed by the $O(N)$ vector multiplet of elementary particles included in the lagrangian (the "elementary fermions") and a set of $O(N)$ multiplets (scalar and higher rank antisymmetric tensors) of increasing mass, which can be thought of as bound-states of a number of elementary fermions[1]. In what follows we will restrict our considerations to the sector of the theory containing only the elementary fermions (denoted by the symbols A_i, $i = 1, \ldots, N$) and the two-fermion bound-states: the isoscalar particle (denoted by B), corresponding to the bound-state in the isoscalar channel of the S-matrixof elementary fermions, and the antisymmetric multiplet (denoted by B_{ij}), corresponding to the bound-state in the antisymmetric channel.

The exact S-matrixfor the elementary fermions was found by Zamolodchikov and Zamolodchikov in [7] and we quote it here for further reference. The Faddeev-Zamolodchikov algebra is

$$A_i(\theta_1)A_j(\theta_2) = \delta_{ij}\sigma_1(\theta_{12})\sum_{k=1}^{n} A_k(\theta_2)A_k(\theta_1) + \\ + \sigma_2(\theta_{12})A_j(\theta_2)A_i(\theta_1) + \sigma_3(\theta_{12})A_i(\theta_2)A_j(\theta_1)\,, \tag{2.3}$$

where $\theta_{12} = \theta_1 - \theta_2$. The $\sigma_i(\theta)$ are given by

$$\sigma_1(\theta) = -\frac{i\lambda}{i\pi - \theta}\sigma_2(\theta) \qquad , \qquad \sigma_3(\theta) = -\frac{i\lambda}{\theta}\sigma_2(\theta)\,, \tag{2.4}$$

and

$$\sigma_2(\theta) = \frac{\sinh\theta + i\sin\lambda}{\sinh\theta - i\sin\lambda}\,\sigma_2^0(\theta)\,, \tag{2.5}$$

with

$$\sigma_2^0(\theta) = \frac{\Gamma(\frac{\lambda}{2\pi} - \frac{i\theta}{2\pi})\,\Gamma(\frac{1}{2} - \frac{i\theta}{2\pi})}{\Gamma(\frac{1}{2} + \frac{\lambda}{2\pi} - \frac{i\theta}{2\pi})\,\Gamma(-\frac{i\theta}{2\pi})}\frac{\Gamma(\frac{1}{2} + \frac{\lambda}{2\pi} + \frac{i\theta}{2\pi})\,\Gamma(1 + \frac{i\theta}{2\pi})}{\Gamma(1 + \frac{\lambda}{2\pi} + \frac{i\theta}{2\pi})\,\Gamma(\frac{1}{2} + \frac{i\theta}{2\pi})}\,, \tag{2.6}$$

where we introduced $\lambda = 2\pi/(N-2)$. The pole at $\theta = i\lambda$ in the CDD prefactor in equation 2.5 gives rise, by bootstrap, to the full exact spectrum of bound-states.

The S-matrixelements for B and B_{ij} particles can be obtained from the elementary fermion S-matrixby fusion, using the identity [7]

$$A_i(\theta - i\lambda/2)\,A_j(\theta + i\lambda/2) = \delta_{ij}\,B(\theta) + \sqrt{(N-4)}\,B_{ij}(\theta)\ . \tag{2.7}$$

A partial list of these amplitudes can be found in the appendix.

The GN model is closely related to the $O(N)$ nlσ model; they share very similar properties, and in particular the elementary particles in both models have the same S-matrix, up to a CDD factor. The $O(N)$ nlσ model is defined by the following lagrangian

$$\mathcal{L}_{nl\sigma} = \frac{1}{2g_0}\partial_\mu\vec{\mathbf{n}}\cdot\partial^\mu\vec{\mathbf{n}}\ , \tag{2.8}$$

where $\vec{\mathbf{n}}$ is a vector in N-dimensional space, subject to the constraint $\vec{\mathbf{n}}\cdot\vec{\mathbf{n}} = 1$. The exact S-matrixfor the elementary particles in the nlσ model is given by equations 2.3-2.6 with σ_2 substituted by σ_2^0 [10], and consequently there are no bound-states in this model.

[1]In fact, the spectrum contains also kink states, associated to the spontaneous breaking of the chiral symmetry [9].

3 Reflection Matrices

In this section we will compute the reflection matrices for the GN model with free and fixed boundary conditions, following Ghoshal's analysis for the nlσ model [4]. Before we proceed let us note that what Ghoshal means by fixed boundary condition is not what we meant by Dirichlet boundary condition in our previous paper [5] (and in this one). Ghoshal's condition corresponds to leaving only *one* component of the field $\vec{\mathbf{n}}$ fixed at the boundary, while by Dirichlet boundary condition it should be understood that the field $\vec{\mathbf{n}}$ is fixed at the boundary, $\vec{\mathbf{n}}(x,t)|_{x=0} = \vec{\mathbf{n}}_0$.

Since the S-matrixof the GN model is, up to a CDD factor, the same as the S-matrixof the $O(N)$ nlσ model, the boundary Yang-Baxter equation (BYBE) for the two models is exactly the same, and so the reflection matrix of the GN model is given by the one of the nlσ model multiplied by the appropriate CDD factors[2]. This indicates that there is a one-to-one correspondence between the integrable boundary conditions for the GN and nlσ model. Similarly to the bulk case, the different physics of these two models resides in these CDD prefactors.

Let us summarize the results of the analysis of integrable boundary conditions for the GN model [5]. The action of the boundary GN model is given by

$$S_{bgn} = \int_{-\infty}^{0} dx \int_{-\infty}^{\infty} dt\ \mathcal{L}_{gn} + \int_{-\infty}^{\infty} dt\ \mathcal{L}_b\,, \tag{3.1}$$

where $\mathcal{L}_b$ is the boundary action. As we have shown in [5] the boundary lagrangian

$$\mathcal{L}_b = \sum_{i=1}^{N} \frac{i}{2}\epsilon_i \psi_+^i \psi_-^i\ , \tag{3.2}$$

where $\epsilon_i = \pm 1$, preserves the integrability of the GN model at the quantum level. The boundary condition derived from this action is

$$\psi_+^i|_{x=0} = \epsilon_i \psi_-^i|_{x=0}\ . \tag{3.3}$$

Borrowing the terminology from the nlσ model, we will refer to bc's with all $\epsilon_i = +1$ or all $\epsilon_i = -1$ respectively as Neumann and Dirichlet bc's. Therefore, up to index reshuffling, we have $N+1$ inequivalent boundary conditions, which have to correspond to different solutions of the boundary Yang-Baxter equation. Due to the fact that the boundary interaction does not involve any flavor-changing terms, we should be able to find *diagonal* solutions for the BYBE, which will be done in section 4. In this section, following Ghoshal analysis, we exhibit solutions for the free and fixed boundary conditions, which serves as a warm-up for the more general case.

[2] This will give us a minimal solution, without any extra poles in the physical strip.

3.1 Free Boundary Condition

As it follows from our discussion above, we should assume that the reflection amplitude for the elementary fermions is given by

$$R_i^j(\theta) \equiv R(\theta)\, \delta_i^j\,. \tag{3.4}$$

Physically it means that we can not change the 'flavor' of the fermion by scattering it off the boundary and that the amplitude of scattering does not depend on the index i. For the nlσ model this ansatz corresponds to no interactions on the boundary [4]. In the case of GN model, it corresponds both to the bc's in equation 3.3 with all ϵ_i equal to "+" or all ϵ_i equal to "−", the difference between the two cases lying in a CDD factor. Due to the similarity between GN and nlσ model S-matrices, $R(\theta)$ can be written as

$$R(\theta) = f(\theta)\, R_0(\theta)\,, \tag{3.5}$$

where $R_0(\theta)$ is the reflection amplitude for the nlσ model with free bc computed by Ghoshal in [4]:

$$R_0(\theta) = \frac{\Gamma(\frac{1}{2}+\frac{\lambda}{4\pi}-\frac{i\theta}{2\pi})\,\Gamma(1+\frac{i\theta}{2\pi})\,\Gamma(\frac{3}{4}+\frac{\lambda}{4\pi}+\frac{i\theta}{2\pi})\,\Gamma(\frac{1}{4}-\frac{i\theta}{2\pi})}{\Gamma(\frac{1}{2}+\frac{\lambda}{4\pi}+\frac{i\theta}{2\pi})\,\Gamma(1-\frac{i\theta}{2\pi})\,\Gamma(\frac{3}{4}+\frac{\lambda}{4\pi}-\frac{i\theta}{2\pi})\,\Gamma(\frac{1}{4}+\frac{i\theta}{2\pi})}\,. \tag{3.6}$$

The prefactor $f(\theta)$ is fixed by unitarity and boundary crossing-unitarity, which generally read

$$R_i^k(\theta)\, R_k^j(-\theta) = \delta_i^j \qquad , \qquad K^{ij}(\theta) = S_{i'j'}^{ji}(2\theta)\, K^{i'j'}(-\theta)\,, \tag{3.7}$$

where $K^{ij}(\theta) = C^{ii'} R_{i'}^j(\frac{i\pi}{2}-\theta)$, and C^{ij} is the charge conjugation matrix. Equations 3.7 imply that $f(\theta)$ should satisfy

$$f(\theta) f(-\theta) \;=\; 1\,, \tag{3.8}$$

$$f(\frac{i\pi}{2}-\theta) \;=\; \frac{\sinh 2\theta + i\sin\lambda}{\sinh 2\theta - i\sin\lambda}\; f(\frac{i\pi}{2}+\theta)\,. \tag{3.9}$$

The minimal solution of 3.8, 3.9 can be found by elementary methods. In fact, there are two minimal solutions, with rather different physical properties. They are given by

$$f(\theta) = \Phi(\theta)\, \frac{\sinh\frac{1}{2}(\theta+\frac{i\lambda}{2})\, \sinh\frac{1}{2}(\theta-\frac{i\lambda}{2}-\frac{i\pi}{2})}{\sinh\frac{1}{2}(\theta-\frac{i\lambda}{2})\, \sinh\frac{1}{2}(\theta+\frac{i\lambda}{2}+\frac{i\pi}{2})}\,, \tag{3.10}$$

and

$$f(\theta) = \Phi(\theta)\, \frac{\sinh\frac{1}{2}(\theta+\frac{i\lambda}{2})\, \sinh\frac{1}{2}(\theta-\frac{i\lambda}{2}+\frac{i\pi}{2})}{\sinh\frac{1}{2}(\theta-\frac{i\lambda}{2})\, \sinh\frac{1}{2}(\theta+\frac{i\lambda}{2}-\frac{i\pi}{2})}\,, \tag{3.11}$$

differing by the CDD factor

$$F_{\text{CDD}}(\theta) = \frac{\tanh(\frac{i\pi}{4}-\frac{i\lambda}{4}+\frac{\theta}{2})}{\tanh(\frac{i\pi}{4}-\frac{i\lambda}{4}-\frac{\theta}{2})}\,. \tag{3.12}$$

The first solution exhibits only a simple pole in the physical strip, at $\theta = i\lambda/2$, corresponding to the bound-state pole at $\theta = i\lambda$ in the bulk S-matrix. The second solution exhibits an additional simple pole at $\theta = i\pi/2 - i\lambda/2$, meaning that the boundary state for this solution contains a zero-rapidity single-particle contribution by the particle B [1].

In equations 3.10, 3.11 $\Phi(\theta)$ is a prefactor satisfying

$$\Phi(\theta)\,\Phi(-\theta) = 1 \qquad , \qquad \Phi(\frac{i\pi}{2} - \theta) = -\,\Phi(\frac{i\pi}{2} + \theta)\,. \tag{3.13}$$

The above equations are exactly those for the reflection matrix of the Ising model [1]. Since we do not expect any free parameters in our reflection matrices (the boundary term in the lagrangian 3.2 having no free parameters), we pick up the minimal solutions corresponding precisely to the boundary conditions $\psi_+|_{x=0} = \pm\psi_-|_{x=0}$ for Ising fermions:

$$\Phi_+ = -i\coth\left(\frac{i\pi}{4} - \frac{\theta}{2}\right) \qquad , \qquad \Phi_- = i\tanh\left(\frac{i\pi}{4} - \frac{\theta}{2}\right)\,. \tag{3.14}$$

Therefore, we propose that $R(\theta)$ in equation 3.5 with Φ_+ corresponds to Neumann bc and with Φ_- to Dirichlet bc.

As we noticed before, the reflection amplitude $R(\theta)$ has a pole at $\theta = \frac{i\lambda}{2}$, corresponding to the bound-state pole at $\theta = i\lambda$ in the bulk S-matrix. Since the particles B and B_{ij} can be interpreted as bound-states of A_iA_j, we can use the boundary-bootstrap equation to compute the reflection amplitudes for them (see also [11]), assuming that the boundary has no structure and consequently that the only non vanishing reflection factors are $R_B^B(\theta)$ and $R_{B_{ij}}^{B_{ij}}(\theta)$. Recall that in general, if the particle A_b can be interpreted as a bound-state of $A_{a_1}A_{a_2}$ (corresponding to a pole in the bulk scattering amplitude at $\theta = iu^b_{a_1a_2}$), the boundary S-matrixelements for A_b can be obtained by taking the appropriate residue at the bound-state pole of the two-particle boundary S-matrix$R^{a_1a_2}_{a_1a_2}(\theta_1,\theta_2)$ [1]:

$$f^b_{a_1a_2}\,R^c_b(\theta) = f^c_{c_1c_2}\,R^{b_2}_{a_2}(\theta - i\bar{u}^{a_1}_{a_2b})\,S^{b_1c_2}_{a_1b_2}(2\theta + i\bar{u}^{a_2}_{ba_1} - i\bar{u}^{a_1}_{a_2b})\,R^{c_1}_{b_1}(\theta + i\bar{u}^{a_2}_{ba_1})\,. \tag{3.15}$$

where $\bar{u} = \pi - u$ and $f^b_{a_1a_2}$ are the three-particle on-shell couplings defined by the residue of the bulk S-matrixat the bound state pole:

$$S^{c_1c_2}_{a_1a_2}(\theta) \simeq \frac{\theta - iu^b_{a_1a_2}}{i\,f^b_{a_1a_2}}\,f^{c_1c_2}_b\,. \tag{3.16}$$

Notice that the fused reflection amplitude is manifestly unitary, and we need only check boundary crossing-unitarity. A straightforward bootstrap computation gives

$$R^B_B(\theta) = -\frac{(i\lambda + 2\theta)(i\pi + 2\theta)}{(2\theta)(i\pi - 2\theta)}R(\theta_-)\,R(\theta_+)\sigma_2(2\theta)\,, \tag{3.17}$$

$$R^{B_{ij}}_{B_{ij}}(\theta) = -\frac{i\lambda + 2\theta}{2\theta}R(\theta_-)\,R(\theta_+))\sigma_2(2\theta)\,, \tag{3.18}$$

where $\theta_\pm = \theta \pm \frac{i\lambda}{2}$.

We can check the consistency of the bootstrap computation by verifying that the appropriate boundary crossing-unitarity equation is satisfied by these reflection amplitudes. It can be written easily, but one should be careful with factors coming from charge conjugation (see the appendix in [12]). The final result is

$$K^{BB}(\theta) = K^{BB}(-\theta) S_{BB}^{BB}(2\theta) + \frac{N(N-1)}{2} K^{B_{ij}B_{ij}}(-\theta) S_{B_{ij}B_{ij}}^{BB}(2\theta) \tag{3.19}$$

where $K^{BB}(\theta) = R_B^B(\frac{i\pi}{2} - \theta)$ and $K^{B_{ij}B_{ij}}(\theta) = -2R_{B_{ij}}^{B_{ij}}(\frac{i\pi}{2} - \theta)$. By using the bulk amplitudes listed in the appendix, equation 3.19 can be easily shown to be satisfied.

3.2 Fixed Boundary Condition

Now let us consider the case where the first $N-1$ fermions satisfy the "+" bc and the N-th fermion satisfies the "−" bc. In terms of reflection matrices, this situation is described by the ansatz

$$\begin{aligned} R_i^i(\theta) &\equiv R_1(\theta)\,, \qquad i = 1, \ldots, N-1\,, \\ R_N^N(\theta) &\equiv R_2(\theta)\,, \end{aligned} \tag{3.20}$$

which is the same as the one for fixed bc in the nlσ model considered by Ghoshal, and therefore we follow his analysis closely. The amplitudes 3.20 can be written as

$$\begin{aligned} R_1(\theta) &= f(\theta)\, R_1^0(\theta)\,, \\ R_2(\theta) &= f(\theta)\, R_2^0(\theta)\,, \end{aligned} \tag{3.21}$$

where $f(\theta)$ is given by 3.10 and R_1^0 and R_2^0 are the amplitudes for the nlσ model. From the BYBE Ghoshal found

$$X(\theta) \equiv \frac{R_1^0(\theta)}{R_2^0(\theta)} = \frac{i\pi - 2\theta}{i\pi + 2\theta}\,. \tag{3.22}$$

and by solving unitarity and crossing-unitarity

$$R_1^0(\theta) = -\frac{\Gamma(\frac{1}{2} + \frac{\lambda}{4\pi} - \frac{i\theta}{2\pi})\Gamma(1 + \frac{i\theta}{2\pi})\,\Gamma(\frac{1}{4} + \frac{\lambda}{4\pi} + \frac{i\theta}{2\pi})\,\Gamma(\frac{3}{4} - \frac{i\theta}{2\pi})}{\Gamma(\frac{1}{2} + \frac{\lambda}{4\pi} + \frac{i\theta}{2\pi})\,\Gamma(1 - \frac{i\theta}{2\pi})\,\Gamma(\frac{1}{4} + \frac{\lambda}{4\pi} - \frac{i\theta}{2\pi})\,\Gamma(\frac{3}{4} + \frac{i\theta}{2\pi})}\,,$$

$$R_2^0(\theta) = -\frac{\Gamma(\frac{1}{2} + \frac{\lambda}{4\pi} - \frac{i\theta}{2\pi})\,\Gamma(1 + \frac{i\theta}{2\pi})\,\Gamma(\frac{1}{4} + \frac{\lambda}{4\pi} + \frac{i\theta}{2\pi})\,\Gamma(-\frac{1}{4} - \frac{i\theta}{2\pi})}{\Gamma(\frac{1}{2} + \frac{\lambda}{4\pi} + \frac{i\theta}{2\pi})\,\Gamma(1 - \frac{i\theta}{2\pi})\,\Gamma(\frac{1}{4} + \frac{\lambda}{4\pi} - \frac{i\theta}{2\pi})\,\Gamma(-\frac{1}{4} + \frac{i\theta}{2\pi})}\,.$$

Using these reflection amplitudes we can compute, as before, the reflection factors for the two-particle bound-states. Notice that in this case we can obtain B by fusing fermions satisfying "+" bc:

$$R_B^B(\theta) = R_1(\theta_-)\, R_1(\theta_+)\, [(N-1)\, \sigma_1(2\theta) + \sigma_2(2\theta) + \sigma_3(2\theta)] + R_1(\theta_-)\, R_2(\theta_+)\sigma_1(2\theta)\;; \tag{3.23}$$

but we can also write B as the fusion of the N-th fermion, which satisfies "−" bc:

$$R_B^B(\theta) = R_2(\theta_-)\,R_2(\theta_+)\,[\sigma_1(2\theta)+\sigma_2(2\theta)+\sigma_3(2\theta)] + R_2(\theta_-)\,R_1(\theta_+)\,(N-1)\,\sigma_1(2\theta)\ . \tag{3.24}$$

These two expressions for $R_B^B(\theta)$ have to be equal, and this provides a non-trivial consistency condition for the boundary-bootstrap:

$$\begin{aligned} X(\theta_-)\,X(\theta_+)\,[(N-1)\,\sigma_1(2\theta)+\sigma_2(2\theta)+\sigma_3(2\theta)] + X(\theta_-)\,\sigma_1(2\theta) = \\ = [\sigma_1(2\theta)+\sigma_2(2\theta)+\sigma_3(2\theta)] + (N-1)\,\sigma_1(2\theta)\,X(\theta_+)\,. \end{aligned} \tag{3.25}$$

As we have checked, equation 3.25 turns out to be an identity. Notice that in this equation all information needed is the ratio R_1/R_2, which is fixed by the BYBE, and the ratio between bulk S-matrixelements (CDD factors cancel out) which is fixed by the bulk Yang-Baxter equation; in other words, it depends only on the $O(N)$ structure and not at all on CDD factors. This is quite surprising, since the consistency check is meaningful only if bound-states exist, which instead depends crucially on the presence of the CDD factor.

The explicit expression for $R_B^B(\theta)$ is

$$R_B^B(\theta) = -\frac{(i\lambda+2\theta)(i\pi-i\lambda+2\theta)}{(2\theta)(i\pi-i\lambda-2\theta)}R_1(\theta_-)R_1(\theta_+)\sigma_2(2\theta)\,. \tag{3.26}$$

Similar bootstrap computations give the reflection amplitudes for the antisymmetric tensor components,

$$R_{B_{ij}}^{B_{ij}}(\theta) = -\frac{(i\lambda+2\theta)}{2\theta}R_1(\theta_-)R_1(\theta_+)\sigma_2(2\theta)\,,\quad i,j\neq N\,, \tag{3.27}$$

$$R_{B_{iN}}^{B_{iN}}(\theta) = -\frac{(i\lambda+2\theta)(i\pi-i\lambda+2\theta)}{(2\theta)(i\pi-i\lambda-2\theta)}R_1(\theta_-)R_1(\theta_+)\sigma_2(2\theta)\,. \tag{3.28}$$

These amplitudes satisfy unitarity automatically. In this case boundary crossing-unitarity reads

$$\begin{aligned} K^{BB}(\theta) = K^{BB}(-\theta)S_{BB}^{BB}(2\theta) + \frac{(N-1)(N-2)}{2}K^{B_{ij}B_{ij}}(-\theta)S_{B_{ij}B_{ij}}^{BB}(2\theta) + \\ + (N-1)K^{B_{iN}B_{iN}}(-\theta)S_{B_{iN}B_{iN}}^{BB}(2\theta)\ . \end{aligned} \tag{3.29}$$

and we have checked that it is indeed satisfied.

4 General Boundary Conditions

For the general boundary condition 3.3 let us assume that for $i = 1,\ldots,M$, $\epsilon_i = +1$ and for $i = M+1,\ldots,N$, $\epsilon_i = -1$, which is, up to index reshuffling, the most general boundary condition we have to consider. Since, as before, the boundary action has no term involving flavor change, we have to assume that the reflection matrix is diagonal. Therefore we have the following ansatz for the non-vanishing elements of the reflection matrix[3]

$$R_i^i(\theta) = R_1(\theta)\,, \qquad\qquad R_a^a(\theta) = R_2(\theta)\ . \tag{4.1}$$

Similarly as in previous case, the BYBE fixes the ratio $R_1(\theta)/R_2(\theta)$. Consider first the factorization of the two-particle reflection process $|A_a(\theta)\,A_i(\theta')\rangle_{in} \to |A_i(-\theta')\,A_a(-\theta)\rangle_{out}$; it gives the following equation:

$$\begin{aligned}\sigma_2(\theta_{12})\,R_1(\theta_1)\,\sigma_3(\bar\theta_{12})\,R_1(\theta_2) + \sigma_3(\theta_{12})\,R_2(\theta_1)\,\sigma_2(\bar\theta_{12})\,R_1(\theta_2) = \\ = R_2(\theta_2)\,R_1(\theta_1)\,\sigma_2(\bar\theta_{12})\,R_1(\theta_1)\,\sigma_3(\theta_{12}) + R_2(\theta_2)\,\sigma_3(\bar\theta_{12})\,R_2(\theta_1)\,\sigma_2(\theta_{12})\,,\end{aligned} \tag{4.2}$$

where $\bar\theta_{12} = \theta_1 + \theta_2$. By dividing this expression by $R_2(\theta_1)\,R_2(\theta_2)\,\sigma_2(\theta_{12})\,\sigma_2(\bar\theta_{12})$ and taking the limit $\theta_1 \to \theta_2$ we get a differential equation for $X(\theta) = R_1(\theta)/R_2(\theta)$,

$$\frac{d}{d\theta}X(\theta) = \frac{X^2(\theta)-1}{2\theta}\ , \tag{4.3}$$

whose solutions are

$$X(\theta) = \frac{C-\theta}{C+\theta}\,, \qquad \text{and} \qquad X(\theta) = 1\ , \tag{4.4}$$

C being an arbitrary integration constant. The solution $X(\theta) = 1$ corresponds to Neumann *or* Dirichlet boundary conditions, since in these cases $R_1(\theta) = R_2(\theta)$, which we have analyzed in the previous section[4].

Consider now the factorization of the process $|A_a(\theta)\,A_a(\theta')\rangle_{in} \to |A_a(-\theta')\,A_a(-\theta)\rangle_{out}$. It gives the following equation:

$$\begin{aligned}&[M\sigma_1(\theta_{12}) + \sigma_2(\theta_{12}) + \sigma_3(\theta_{12})]\,R_1(\theta_1)\,\sigma_1(\bar\theta_{12}\,R_2(\theta_2) + \\ &\quad + \sigma_1(\theta_{12})\,R_2(\theta_1)\left[(N-M)\sigma_1(\bar\theta_{12}) + \sigma_2(\bar\theta_{12}) + \sigma_3(\bar\theta_{12})\right]R_2(\theta_2) = \\ &= R_1(\theta_2)\left[M\sigma_1(\bar\theta_{12}) + \sigma_2(\bar\theta_{12}) + \sigma_3(\bar\theta_{12})\right]R_1(\theta)\,\sigma_1(\theta_{12}) + \\ &\quad + R_1(\theta_2)\,\sigma_1(\bar\theta_{12})\,R_2(\theta_1)\,[(N-M)\sigma_1(\theta_{12}) + \sigma_2(\theta_{12}) + \sigma_3(\theta_{12})]\ .\end{aligned} \tag{4.5}$$

If we plug $X(\theta)$ in this expression, the final, compact result is that C is fixed to be

$$C = -\frac{i\pi}{2}\frac{N-2M}{N-2}\ . \tag{4.6}$$

By analyzing the BYBE for the other processes we find that 4.6 is the unique consistent solution. Notice that by taking $M = N-1$ in equation 4.6, we obtain $C = i\pi/2$ and, for

[3]From now on we will use letters in the middle of the alphabet $(i, j, ...)$ for fermions labeled from 1 to M, and letters in the beginning of the alphabet $(a, b, ...)$ for fermions labeled from $M+1$ to N.

[4]Recall that the Neumann and Dirichlet cases will only differ by overall CDD factors.

the nlσ , we recover Ghoshal's results for fixed bc. All that is left to do is to compute the prefactors for the reflection amplitudes, by using unitarity and boundary crossing-unitarity 3.7, which will fix $R_1(\theta)$ and $R_2(\theta)$ up to CDD factors. These conditions read explicitly

$$R_1(\theta)R_1(-\theta)=1 \qquad , \qquad R_2(\theta)R_2(-\theta)=1\,, \tag{4.7}$$

$$R_1(\frac{i\pi}{2}-\theta)=\left[\frac{i\lambda-2\theta}{i\lambda+2\theta}\right]\left[\frac{i\lambda(N-M-1)-2\theta}{i\lambda(N-M-1)+2\theta}\right]\sigma_I(2\theta)\,R_1(\frac{i\pi}{2}+\theta)\,, \tag{4.8}$$

where $\sigma_I(\theta)=N\sigma_1(\theta)+\sigma_2(\theta)+\sigma_3(\theta)$. The minimal solution of equations 4.8 is given by

$$\begin{aligned} R_1(\theta)=-f(\theta)R_0(\theta)\,&\frac{\Gamma(\frac14+\frac{\lambda}{4\pi}+\frac{i\theta}{2\pi})\,\Gamma(\frac34+\frac{\lambda}{4\pi}-\frac{i\theta}{2\pi})\,\Gamma(\frac14+\frac{\lambda}{4\pi}(N-M-1)+\frac{i\theta}{2\pi})}{\Gamma(\frac14+\frac{\lambda}{4\pi}-\frac{i\theta}{2\pi})\,\Gamma(\frac34+\frac{\lambda}{4\pi}+\frac{i\theta}{2\pi})\,\Gamma(\frac14+\frac{\lambda}{4\pi}(N-M-1)-\frac{i\theta}{2\pi})}\times\\ &\times\frac{\Gamma(\frac34+\frac{\lambda}{4\pi}(N-M-1)-\frac{i\theta}{2\pi})}{\Gamma(\frac34+\frac{\lambda}{4\pi}(N-M-1)+\frac{i\theta}{2\pi})}\,, \end{aligned} \tag{4.9}$$

and

$$\begin{aligned} R_2(\theta)=-f(\theta)R_0(\theta)\,&\frac{\Gamma(\frac14+\frac{\lambda}{4\pi}+\frac{i\theta}{2\pi})\,\Gamma(\frac34+\frac{\lambda}{4\pi}-\frac{i\theta}{2\pi})\,\Gamma(\frac14+\frac{\lambda}{4\pi}(N-M-1)+\frac{i\theta}{2\pi})}{\Gamma(\frac14+\frac{\lambda}{4\pi}-\frac{i\theta}{2\pi})\,\Gamma(\frac34+\frac{\lambda}{4\pi}+\frac{i\theta}{2\pi})\,\Gamma(\frac14+\frac{\lambda}{4\pi}(N-M-1)-\frac{i\theta}{2\pi})}\times\\ &\times\frac{\Gamma(-\frac14+\frac{\lambda}{4\pi}(N-M-1)-\frac{i\theta}{2\pi})}{\Gamma(-\frac14+\frac{\lambda}{4\pi}(N-M-1)+\frac{i\theta}{2\pi})}\,, \end{aligned} \tag{4.10}$$

where $f(\theta)$ is given in equation 3.10 and R_0 in equation 3.6. The same amplitudes, but with $f\,R_0$ replaced by R_0, apply to the nlσ model.

Notice that R_2 has a pole at $\theta=-\frac{i\lambda}{4}(N-2M)$, which is in the physical strip for $N/2<M\leq N-1$. As argued by Ghoshal, this pole signals the presence of one-particle contributions in the boundary state. Since upon the substitution $M\to N-M$, the ratio $X(\theta)\to 1/X(\theta)$ and R_1 and R_2 get interchanged, for $1<M<N/2$ it will be R_1 to exhibit this pole in the physical strip.

It is interesting to notice also that for $M=N$, R_1 reduces to the second minimal solution for the free boundary condition, equation 3.11.

The amplitudes for the two-particle bound-states can be computed by boundary-bootstrap, and they read explicitly

$$R_B^B(\theta)=-\frac{(i\lambda+2\theta)(2C-i\lambda+2\theta)}{(2\theta)(2C-i\lambda-2\theta)}R_1(\theta_-)R_1(\theta_+)\sigma_2(2\theta)\,, \tag{4.11}$$

$$R_{B_{ij}}^{B_{ij}}(\theta)=-\frac{(i\lambda+2\theta)}{2\theta}R_1(\theta_-)R_1(\theta_+)\sigma_2(2\theta)\,, \tag{4.12}$$

$$R_{B_{ab}}^{B_{ab}}(\theta)=-\frac{(i\lambda+2\theta)(2C-i\lambda+2\theta)(2C+i\lambda+2\theta)}{(2\theta)(2C-i\lambda-2\theta)(2C+i\lambda-2\theta)}R_1(\theta_-)R_1(\theta_+)\sigma_2(2\theta)\,, \tag{4.13}$$

$$R_{B_{ia}}^{B_{ia}}(\theta)=-\frac{(i\lambda+2\theta)(2C-i\lambda+2\theta)}{(2\theta)(2C-i\lambda-2\theta)}R_1(\theta_-)R_1(\theta_+)\sigma_2(2\theta)\,, \tag{4.14}$$

In this more general case, the appropriate bootstrap consistency condition corresponding to the generalization of equation 3.25 is

$$\begin{aligned} &X(\theta_-)X(\theta_+)[M\sigma_1(2\theta)+\sigma_2(2\theta)+\sigma_3(2\theta)]+X(\theta_-)(N-M)\sigma_1(2\theta)= \\ &=[(N-M)\sigma_1(2\theta)+\sigma_2(2\theta)+\sigma_3(2\theta)]+M\sigma_1(2\theta)X(\theta_+)\,. \end{aligned} \tag{4.15}$$

It is very easy to see that if $M=N-1$ this reduces to equation 3.25. As before we should stress that bootstrap consistency requires only information obtained from the BYBE, via the ratio $X(\theta)$, and the fact that the elementary fermions form isoscalar bound-states. Finally, we have also explicitly checked that the reflection amplitudes listed above satisfy the appropriate boundary crossing-unitarity.

5 Conclusions

In this paper we have computed the minimal boundary S-matrixfor the Gross-Neveu model, extending Ghoshal's analysis of the nonlinear σ model. We found general (diagonal) solutions for the boundary Yang-Baxter equation for both models and connected them to the boundary conditions proposed recently in our paper [5]. We also proved that the solutions presented in this paper are consistent with the boundary-bootstrap. This seems to indicate that the boundary contains a lot of information of the bulk theory. It would be interesting to investigate how much we can learn about bulk integrable models *starting* from the boundary. As a natural follow-up to this work it would be interesting to study the boundary Yang-Baxter equation in general and see if there are non-diagonal solutions, that is flavor-changing scattering off the boundary. There should be possible, then to find associated microscopic boundary conditions for the elementary fields.

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A Appendix

In this appendix we list the S-matrixelements for scattering processes involving two-particle bound-states (the isoscalar and the first antisymmetric tensor), that we need to write down the boundary crossing-unitarity equation 3.19, 3.29. These amplitudes are obtained from those of elementary fermions by fusion.

Recall that if the scattering amplitude of two particles A_a and A_b has a simple pole in the physical strip at, say, $\theta = iu_{ab}^c$, with residue given by :

$$S_{ab}^{a'b'}(\theta) \simeq \frac{i f_{ab}^c f_c^{a'b'}}{\theta - iu_{ab}^c}\,, \tag{A.1}$$

then the scattering of the bound-state particle A_c with all other particles in the theory can be obtained by the bootstrap equation [10]

$$f_{ab}^c S_{cd}^{c'd'}(\theta) = f_{a_1 b_1}^{c'} S_{bd}^{b_1 d_1}(\theta - i\bar{u}_{b\bar{c}}^{\bar{a}})\, S_{ad_1}^{a_1 d'}(\theta + i\bar{u}_{\bar{c}a}^{\bar{b}})\,, \tag{A.2}$$

where $\bar{u} \equiv \pi - u$. In the case of GN model, we get

$$\begin{aligned}
S_{BB}^{BB}(\theta) &= \frac{\theta(i\pi-\theta)\left[\lambda(\pi-3\lambda)+\theta(i\pi-\theta)\right]-2\pi\lambda^2(\lambda-\pi))}{\theta(i\pi-\theta)(\theta-i\lambda)(i\pi-\theta-i\lambda)}\sigma_2(\tilde{\theta}_+)\,\sigma_2(\tilde{\theta}_-)\,\sigma_2^2(\theta)\,,\\
S_{BB}^{B_{ij}B_{ij}}(\theta) &= \frac{2i(N-4)\lambda^3}{(i\pi-\theta)(\theta-i\lambda)(i\pi-\theta-i\lambda)}\sigma_2(\tilde{\theta}_+)\,\sigma_2(\tilde{\theta}_-)\,\sigma_2^2(\theta)\,,\\
S_{B_{ij}B_{ij}}^{BB}(\theta) &= \frac{i(N-4)\lambda^3}{2(i\pi-\theta)(\theta-i\lambda)(i\pi-\theta-i\lambda)}\sigma_2(\tilde{\theta}_+)\,\sigma_2(\tilde{\theta}_-)\,\sigma_2^2(\theta)\,,\\
S_{BB_{ij}}^{BB_{ij}}(\theta) &= \frac{\theta(i\pi-\theta)+\lambda(\pi-3\lambda)}{(\theta-i\lambda)(i\pi-\theta-i\lambda)}\sigma_2(\tilde{\theta}_+)\,\sigma_2(\tilde{\theta}_-)\,\sigma_2^2(\theta)\,,\\
S_{BB_{ij}}^{B_{ij}B}(\theta) &= \frac{-i(N-4)\lambda^3}{\theta(\theta-i\lambda)(i\pi-\theta-i\lambda)}\sigma_2(\tilde{\theta}_+)\,\sigma_2(\tilde{\theta}_-)\,\sigma_2^2(\theta)\quad,
\end{aligned}$$

where $\tilde{\theta}_\pm = \theta \pm i\lambda$. Unitarity follows directly from the fusion procedure and crossing symmetry is satisfied with charge conjugation matrix elements $C^{BB} = 1$ and $C^{B_{ij}B_{ij}} = -2$.

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