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Knizhnik-Zamolodchikov equation and extended symmetry for stable Hall states

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Abstract

We describe a n component abelian Hall fluid as a system of *composite bosons* moving in an average null field given by the external magnetic field and by the statistical flux tubes located at the position of the particles. The collective vacuum state, in which the bosons condense, is characterized by a Knizhnik-Zamolodchikov differential equation relative to a $\hat{U}(1)^n$ Wess-Zumino model. In the case of states belonging to Jain's sequences the Knizhnik-Zamolodchikov equation naturally leads to the presence of an $\hat{U}(1)\otimes\hat{SU}(n)$ extended algebra. Only the $\hat{U}(1)$ mode is charged while the $\hat{SU}(n)$ modes are neutral, in agreement with recent results obtained in the study of the edge states.

It is well known that a simple physical idea lies behind the different theoretical descriptions of the fractional quantum Hall effect (FQHE), namely that an Hall fluid should be described not in terms of the ordinary electrons but of the quasiparticles obtained by binding to them an appropriate number of vortices. This picture, clearly present in the Laughlin wave function[1], widely discussed and exploited by Wilczeck[2] and formulated in the framework of a Ginsburg-Landau model by several authors[3], finds a precise mathematical formulation in two dimensional Conformal Field Theory (2DCFT), where the non trivial braiding of two quasiparticles is realized as the exchange of the corresponding Vertex Operators[4], and in the Chern-Simons (CS) lagrangian approach[5], that gives the possibility of going beyond the mean field approximation, by evaluating the fluctuations of the CS field[6].

The most appealing realization of this physical idea has been put forward by Jain[7], who has suggested the possibility of looking at the FQHE for the electrons at filling $\nu = n/(2np \pm 1)$ as a manifestation of the integer effect for *composite fermions* obtained by attaching to each electron an even number of flux units opposite to the external magnetic field. While the most direct experimental support for this point of view derives from the observation that the most prominent Hall plateaux are seen at the fillings $\nu = n/(2n \pm 1)$ (principal sequence) and at $\nu = n/(4n \pm 1)$, further evidence has been gathered from the analysis of the energy gaps for such fillings[8] and from the study[9] of properties of the state at $\nu = 1/2$, the accumulation point of the principal sequence.

A very simple theoretical implementation of Jain's approach has been obtained by studying the motion of each quasiparticle in the presence of the external magnetic field and of infinitesimally thin statistical flux tubes located at the position of the other quasiparticles[10]. This analysis, that can be generalized to an arbitrary abelian Hall fluid, leads naturally to the Laughlin ground state wave function (gswf) and has the advantage of preserving in the general case the algebraic structures present in the case of filling $\nu = 1$, that is well understood in terms of a single particle description. As a consequence, in this framework, an easy proof has been given[10,11] of the important result[12,13] that the Laughlin gswf can be characterized as a highest weight state of the $W_{1+\infty}$ algebra of the area preserving non singular diffeomorfisms.

However in this paper, instead of promoting the electrons to *composite fermions*, we will turn them into *composite bosons* moving in an average null field made of the external magnetic field and the statistical flux tubes attached to the particles. In this approach, a different characterization of the ground state of a generic abelian Hall fluid can be obtained and typical properties of Jain's states can be unveiled. More specifically, for the simple case $\nu = 1/m$ the collective vacuum, in which the composite bosons condense, is characterized as a solution of the Knizhnik-Zamolodchikov (KZ) equation[14] for the correlators of a $\hat{U}(1)$ Wess-Zumino field, making contact with the analysis of the Hall effect in terms of 2DCFT. This result can be easily generalized to the generic n component abelian Hall fluid, where the KZ equation is relative to $\hat{U}(1)^n$ 2DCFT correlators. More interesting is the case of states belonging to Jain's sequences, where the KZ equation exhibits the presence of an $\hat{U}(1) \otimes \hat{SU}(n)$ extended algebra. This approach confirms in a simple way results obtained in the framework of representation theory of the $W_{1+\infty}$ algebra[15] and by studying the dynamics of the edge states in the presence of disorder[16]. It also gives a direct evidence that only the $\hat{U}(1)$ mode is charged while the remaining are neutral.

Let us start our discussion by recalling that in the $\nu = 1$ case the analytic part of the Laughlin wave function is easily obtained out the single particle states as a Slater determinant

$$\chi_{\nu=1}(z_1, z_2, \ldots, z_N) = \sum_P (-1)^P (b_1^\dagger)^{n_{i_1}} (b_2^\dagger)^{n_{i_2}} \ldots (b_N^\dagger)^{n_{i_N}} \chi_0 \,, \tag{1}$$

where $n_{i_1}, n_{i_2} \ldots n_{i_N}$ is a permutation, P, of the non negative integers smaller than the electron number N, $b_i^\dagger$ and b_i are creation and annihilation operators in the Fock-Bargmann representation:

$$b_i^\dagger = z_i \,, \qquad b_i = \partial_i \,, \tag{2}$$

and the vacuum state is defined by $b_i \chi_0 = 0$. Following Jain[7], one can describe the state at filling $\nu = 1/(2p+1) \equiv 1/m$ as a system, at the effective filling $\nu_{eff} = 1$, of *composite fermions* obtained by binding $-2p$ units of flux to each electron. By studying the motion of each composite fermion in the presence of the external magnetic field and of the other particles[10], one sees that the operators $b_i^\dagger$ and b_i are to be modified as follows:

$$b_i^\dagger = z_i \,, \quad b_i = \partial_i - 2p\partial_i \sum_{j \neq i} \ln(z_i - z_j) = \partial_i - 2p \sum_{j \neq i} \frac{1}{z_i - z_j} \,. \tag{3}$$

This implies that the Fock space is built out of an intrinsically collective vacuum χ_c defined by the condition $b_i \chi_c = 0$, and that the ground state has the same structure than the one given in eq. 1, namely:

$$\chi_{\frac{1}{m}}(z_1, z_2, \ldots, z_N) = \sum_P (-1)^P (b_1^\dagger)^{n_{i_1}} (b_2^\dagger)^{n_{i_2}} \ldots (b_N^\dagger)^{n_{i_N}} \chi_c = \prod_{i<j} (z_i - z_j)^m \,. \tag{4}$$

More generally, one can consider the case of a generic n component abelian Hall fluid characterized by a symmetric integer valued matrix K with odd diagonal elements[5]. Then the generalized Laughlin gswf, corresponding to the ground state of the composite fermions, is given by

$$\chi_K(\{z_i^I\}) = \prod_I \sum_{P_I} (-1)^{P_I} (b_1^{I\dagger})^{n^I_{i_1}} (b_2^{I\dagger})^{n^I_{i_2}} \ldots (b_N^{I\dagger})^{n^I_{i_{N_I}}} \chi_c \,, \tag{5}$$

where i labels the electrons in a given component, I the different components and $n^I_{i_1}, n^I_{i_2} \dots n^I_{i_{N_I}}$ is a permutation, P_I, of the sequence of non negative integers smaller than the number N_I of electrons in the component I. We have introduced the new operators

$$(b^I_i)^\dagger = z^I_i \,, \qquad b^I_i = \partial^I_i - \partial^I_i \sum{}' H_{IJ} \; \ln(z^I_i - z^J_j) = \partial^I_i - \sum{}' \frac{H_{IJ}}{z^I_i - z^J_j} \,, \tag{6}$$

where $H_{IJ} = K_{IJ} - \delta_{IJ}$, $\sum'$ means the sum on all values of j, J such that $z^J_j \neq z^I_i$ and the collective vacuum χ_c is defined by $b^I_i \chi_c = 0$.

As we have already mentioned, in this paper we follow a variation of this approach, by describing the system as made out of *composite bosons* rather than composite fermions. For example, in the case of filling $\nu = 1/(2p+1) \equiv 1/m$, we bind $-m$ units of flux to the electrons rather than $-2p$. Studying, along the same lines, the motion of the composite bosons one is lead to the introduction of the new set of operators

$$B^\dagger_i = z_i \,, \qquad B_i = \partial_i - m\, \partial_i \sum_{j \neq i} \ln(z_i - z_j) = \partial_i - m \sum_{j \neq i} \frac{1}{z_i - z_j} \,. \tag{7}$$

In this case the composite bosons will condense in a collective vacuum state, corresponding to the Laughlin gswf, defined by

$$B_i \chi_{\frac{1}{m}}(z_1, z_2, \dots, z_N) = 0 \,. \tag{8}$$

More generally, in the n component case, the composite bosons will condense in a collective vacuum defined by

$$B^I_i \chi_K(\{z^J_j\}) = 0 \,, \tag{9}$$

where the new set of operators is given by:

$$(B^I_i)^\dagger = z^I_i \,, \qquad B^I_i = \partial^I_i - \partial^I_i \sum{}' K_{IJ} \; \ln(z^I_i - z^J_j) = \partial^I_i - \sum{}' \frac{K_{IJ}}{z^I_i - z^J_j} \,. \tag{10}$$

In the case $\nu = 1/m$ the vacuum state condition, eq. 8, corresponds to the Knizhnik-Zamolodchikov (KZ) linear differential equation for the correlators of an abelian Wess-Zumino field of conformal weight $m/2$:

$$\left(\partial_i - m \sum_{j \neq i} \frac{1}{z_i - z_j} \right) \chi_{\frac{1}{m}}(z_1, z_2, \dots, z_N) = 0 \,. \tag{11}$$

Therefore to each particle we can associate an element of a local $\hat{U}(1)$ group of conformal weight $m/2$ that can be written as a Coulomb gas Vertex Operator (VO) $V_{\sqrt{m}}(z) =: \exp\left[i\sqrt{m}\phi(z)\right] :$, where $\phi(z)$ is a (properly compactified) free holomorphic scalar field, with the standard mode expansion

$$\phi(z) = q - ip\ln z + i\sum_{n\neq 0}\frac{a_n}{n}z^{-n}. \tag{12}$$

As a consequence, the Laughlin gswf can be expressed as a correlator of VO's:

$$\chi_{\frac{1}{m}}(z_1, z_2, \dots, z_N) = \left\langle V_{\sqrt{m}}(z_1)\dots V_{\sqrt{m}}(z_N)\right\rangle = \left\langle N\sqrt{m}\left|V_{\sqrt{m}}(z_1)\dots V_{\sqrt{m}}(z_N)\right|0\right\rangle, \tag{13}$$

where the Fock vacuum state, $\langle N\sqrt{m}|$, with momentum $N\sqrt{m}$, has been introduced to take into account momentum conservation. In the generic abelian case the equation defining the collective bosonic vacuum has the form

$$(\partial_i^I - \sum{}' \frac{K_{IJ}}{z_i^I - z_j^J})\chi_K(\{z_i^I\}) = 0\,. \tag{14}$$

The discussion relative to the one component case is easily generalized to the KZ equation for the n component Hall fluid, provided the K matrix is positive definite. We will make this assumption as only under this condition a completely consistent description on higher genus Riemann surfaces can be achieved[17]. This amounts to disregard Hall fluids for which the edge currents propagate in opposite directions[18]. For such fluids the quantization of Hall conductance is in general an open problem that, for the case of Jain's sequences, has been solved[19,16] by taking the presence of disorder explicitly into account. Under the positivity condition one can introduce a n-component vector of independent holomorphic fields $\vec{\phi}(z)$ and a set of n-component vectors $\vec{\beta}_I$ such that $\vec{\beta}_I\cdot\vec{\beta}_J = K_{IJ}$. Then by defining the VO's $V_{\vec{\beta}_I}(z) =: \exp[i\vec{\beta}_I\cdot\vec{\phi}(z)]:$ and the currents $J_a(z) = i\partial\phi_a(z)$ one obtains the following operator product expansion (OPE)

$$J_a(z)V_{\vec{\beta}_I}(w) \sim \frac{(\vec{\beta}_I)_a}{z-w}V_{\vec{\beta}_I}(w)\,. \tag{15}$$

Therefore one recognizes eq. 14 as the KZ equation for a $\hat{U}(1)^n$ Wess-Zumino model. The corresponding correlators will be given by

$$\chi_K(\{z_i^I\}) = \left\langle \prod_{I=1}^{n}\prod_{i=1}^{N_I} V_{\vec{\beta}_I}(\{z_i^I\})\right\rangle\,. \tag{16}$$

We turn now to the more interesting case relative to the fillings of the Jain's sequences, where out of the KZ equation will emerge the presence of an extended $\hat{U}(1)\otimes \hat{SU}(n)$ algebra. The structure of the K matrix for the Jain's sequences is given by $K_{IJ} = \delta_{IJ} + 2p$. Notice that for the Jain's sequences in all components there is the same number of electrons N_I. This derives from the condition $N_\Phi = K_{IJ}N_J$, where N_Φ is the number of units of external magnetic flux, that guarantees the cancellation, in the average, between the external magnetic field and the statistical field of the composite bosons.

The K matrix can be diagonalized by means of an orthogonal transformation, $K = O^T K_{diag} O$, where $K_{diag} = diag(1, \ldots, 1, 2np+1)$. The matrix element O_{IJ} with $I = 1, \ldots, n-1$ and $J = 1, \ldots, n$ are strictly related to the matrix elements of the diagonal generators of the $SU(n)$ group in the fundamental representation:

$$O_{aJ} = t^a_{JJ} \equiv (\vec{u}_J)_a \, , \qquad a = 1, \ldots, n-1, \tag{17}$$

(we have introduced the vectors $\vec{u}_J$ in order to simplify the notation). The remaining elements are given by $O_{nJ} = 1/\sqrt{n}$, $J = 1, \ldots, n$. The explicit form of the matrix t^a is given by

$$t^a = \frac{1}{\sqrt{a(a+1)}} diag(1, \ldots, 1, -a, 0, \ldots, 0) \, , \quad a = 1, \ldots n-1 \, . \tag{18}$$

Notice that t^a is traceless, as it should. We can then rewrite K_{IJ} as follows:

$$K_{IJ} = \frac{1}{\nu} + \vec{u}_I \cdot \vec{u}_J = \frac{1}{\nu} + t^a_{II} t^a_{JJ} \, , \tag{19}$$

and express eq. 14 in a form that makes manifest the presence of an extended symmetry, namely

$$(\partial^I_i - {\sum}' \frac{\nu^{-1}}{z^I_i - z^J_j} + {\sum}' \frac{t^a_{II} t^a_{JJ}}{z^I_i - z^J_j}) \chi_K(\{z^L_l\}) = 0 \, . \tag{20}$$

where ν is the filling factor relative to the Jain's states, $\nu = n/(2np+1)$. Notice that the numerator of the term appearing in the first sum of eq. 20 corresponds strictly to the $\hat{U}(1)$ case discussed above, see eq. 11. This suggest to perform an orthogonal transformation on the ϕ-fields:

$$\Phi_a = O_{aj} \phi_j \, , \quad a = 1, 2, \ldots, n-1 \, , \tag{21}$$

$$\Phi_+ = O_{nj} \phi_j = \frac{1}{\sqrt{n}} (\phi_1 + \phi_2 \ldots \phi_n) \, . \tag{22}$$

We can then rewrite the energy-momentum tensor and the VO's in terms of the new fields obtaining

$$T(z) = -\frac{1}{2} : \partial\vec{\phi}(z) \cdot \partial\vec{\phi}(z) := -\frac{1}{2} : \partial\Phi_+(z) \partial\Phi_+(z) : -\frac{1}{2} : \sum_a^{n-1} \partial\Phi_a(z) \partial\Phi_a(z) : , \tag{23}$$

and

$$V_{\vec{\beta}_I}(z) =: \exp[\vec{\beta}_I \cdot \vec{\phi}(z)] :=: e^{\frac{i}{\sqrt{\nu}} \Phi_+(z)} :: e^{i \vec{u}_I \cdot \vec{\Phi}(z)} : \equiv V_+(z) V_{\vec{u}_I}(z) \, . \tag{24}$$

We identify $i\partial\Phi_+(z)$ as a $\hat{U}(1)$ current and $J_a(z) = i\partial\Phi_a(z)$ as the diagonal currents of $\hat{SU}(n)$. Indeed

$$J_a(z) V_{\vec{u}_I}(w) \sim \frac{t^a_{II}}{z - w} V_{\vec{u}_I}(w) \, . \tag{25}$$

This identifies the coefficients t^a_{II} in the second sum of the KZ eq. 20 as the correct representation matrix elements. Although the non-diagonal terms do not appear in eq. 20, as they are absent in the energy-momentum tensor eq. 23, they are easily introduced as bilinear in the VO's[20]. As the roots of $SU(n)$ are of the form $\vec{\alpha} = \vec{u}_L - \vec{u}_M$, one has $J_{\vec{\alpha}} =: V_{\vec{u}_L} V^{\dagger}_{\vec{u}_M}$: and their action on the VO $V_{\vec{u}_I}$ is given by

$$J_{\vec{\alpha}}(z) V_{\vec{u}_I}(w) \sim \frac{\delta_{IM}}{z-w} V_{\vec{u}_L} \, . \tag{26}$$

In terms of the new fields the gswf characterized by the KZ equation takes the form

$$\chi_K(\{z_i^I\}) = \left\langle 0 \left| \prod_{I=1}^{n} \prod_{i=1}^{N_I} V_{\vec{u}_I}(z_i^I) \right| 0 \right\rangle \left\langle \sum_{I,i} \sqrt{\nu^{-1}} \left| \prod_{I=1}^{n} \prod_{i=1}^{N_I} V_{+}(z_i^I) \right| 0 \right\rangle . \tag{27}$$

Notice that the correlator factorizes and, while the $\hat{U}(1)$ factor carries the full information on the total momentum associated with the VO's, the $\hat{SU}(n)$ factor automatically satisfies the Coulomb gas neutrality condition. This implies, as it can be easily seen in terms of the equivalent plasma picture, that the $\hat{U}(1)$ mode is charged while the $\hat{SU}(n)$ modes are neutral. The same result can also be obtained by recalling that the 2DCFT momentum is only rescaled with respect to the physical charge or by explicitly evaluating the Hall conductance[17].

The difference between a generic n component Hall fluid and one relative to the Jain's sequences has important consequences on the structure of the theory on a non trivial compact Riemann surface and on the nature of the edge excitations. It has been shown that while in the generic case there are n sectors of charged edge excitations, in the Jain's case only the $\hat{U}(1)$ mode is charged while the remaining $n-1$ are neutral[15,16]. The same result has been obtained by analyzing the gswf's on a torus, in ref 17, where the correspondence between gswf's on genus one Riemann surface and edge states is discussed in the framework of 2DCFT, showing that both are characterized by the same integer lattice $\mathbf{Z}^n / K\mathbf{Z}^n$. The peculiar nature of the Jain's sequences is then related to the fact that the relative lattice can be recast in the much simpler form $\mathbf{Z}/(\det K)\mathbf{Z}$, in complete analogy to the case of a one-component fluid. Whether this characterization is physically relevant in explaining the observed prominence of Jain's sequences is an open and interesting problem.

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References

1. R. B. Laughlin, Phys. Rev. Lett. **50,** 1395 (1983).

2. See e.g. F. Wilczek, *Fractional Statistics and Anyon Superconductivity* , World Scientific Singapore (1990).

3. See M. Stone, *Quantum Hall Effect*, World Scientific Singapore (1992), Chap. 3 and reprints contained therein.

4. For a review see G. Cristofano, G. Maiella, R. Musto and F. Nicodemi, Nucl. Phys. **33C**, 119 (1993).

5. See e.g. J. Frölich and A. Zee, Nucl. Phys. **B364**, 517 (1991); X.G. Wen and A. Zee, Phys. Rev. **B 46**, 2290 (1992).

6. S. H. Simon and B. I. Halperin, Phys. Rev. **B 48** 17368 (1993), **B 50** 1807 (1994).

7. J. K. Jain, Phys. Rev. Lett. **65**, 199 (1989); Phys. Rev. **B 40**, 8079 (1989); **B 41** 7653 (1990).

8. R. R. Du, H. L. Stormer, D. C. Tsui, L. N. Pfeiffer and K. W. West, Phys. Rev. Lett. **70**, 2944 (1993).

9. See e.g. B. I. Halperin, P. A . Lee and N. Read, Phys. Rev. **B 47**, 7312 (1993); R. L. Willett, R. R. Ruel, M. A. Paalanen, K. W. West and L. N. Pfeiffer, Phys. Rev. **B 47**, 7344 (1993); J. K. Wang and V. J. Goldman, Phys. Rev. Lett. **67**, 74 (1991)

10. G. Cristofano, G. Maiella, R. Musto and F. Nicodemi, *FQHE and Jain's approach on the torus*, Int. J. Mod. Phys. **B**, in press.

11. M. Flohr and R. Varnhagen, Jour. Phys. **A27**, 3999 (1994); D. Karabali, Nucl. Phys. **FS B419**, 437 (1994).

12. A. Cappelli, C. A. Trugenberger and G. R. Zemba, Nucl. Phys. **B396** 465 (1993); Phys. Lett. **B 306** , 100 (1993); Phys. Rev. Lett. **72** 1902 (1994).

13. S. Iso, D. Karabali and B. Sakita, Phys. Lett. **B296** 143 (1992).

14. V. G. Knizhnik and A. B. Zamolodchikov, Nucl. Phys. **B247**, 83 (1984).

15. A. Cappelli, C. A. Trugenberger and G. R. Zemba, *Stable Hierarchical Quantum Hall Fluids as $W_{1+\infty}$ Minimal Models*, preprint UGVA-DPT 1995/01-879, DFTT 09/95, hep-th 9502021.

16. C.L. Kane and M. P. A. Fisher, *Impurity scattering and transport of fractional Quantum Hall edge states*, preprint cond-mat 9409028.

17. A. De Martino and R. Musto, *Abelian Hall Fluids and Edge States: a Conformal Field Theory Approach*, preprint DPS-NA-19/95; INFN-NA-IV-19/95; DFUPG-101/95; cond-mat 9505027.

18. see e.g. X. G. Wen, Int. J. Mod. Phys. **B6** (1992) 1711 and references contained therein.

19. C. L. Kane, M. P. A. Fisher and J. Polchinski, Phys. Rev. Lett. **72**, 4129 (1994).

20. P. Goddard and D. Olive, Int. J. Mod. Phys. **A1**, 303 (1986).