Characterization and construction of sequentially consistent risk measures

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Abstract

In dynamic risk measurement the problem emerges of assessing the risk of a financial position at different times. Sufficient conditions are provided for conditional coherent risk measures, in order that the requirements of acceptance-, rejection- and sequential consistency are satisfied. It is shown that these conditions are often violated for standard methods of updating. A method is consequently proposed for constructing a sequentially consistent risk measure, which entails the modification of the set of probability measures used to obtain the risk assessment at an initial time. This is demonstrated for the coherent entropic risk measure and for the class of Choquet risk measures, which generalizes the well-known TVaR. Finally we consider the situation where the term of risk exposures is longer than the time horizon used in solvency assessment. Then, regulation such as Solvency II requires replacing the financial position itself with its fair value at the time horizon. We show that in this setting acceptance consistency can be preserved, though the same is not true about rejection consistency.

KEYWORDS: dynamic risk measures, sequential consistency, TVaR, Choquet risk measure, coherent entropic risk measure

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1 Introduction

The correct quantification of risks faced by financial institutions or insurance companies is a central task for both investors and regulators. *Risk measures* are essential tools for quantifying financial risks. Static risk measures, where uncertainty is resolved over a single period, have been extensively studied, see for example Artzner et al. (1999), Föllmer and Schied (2002), Frittelli and Gianin (2002), Goovaerts et al. (1984). Recent years have also seen an increasing interest in a dynamic approach to risk measurement, where several time periods are considered. In a dynamic setting, several issues emerge: the impact of available information on the risk assessment, the occurrence of intermediate payoffs, the time consistency among measurements of the same position at different points in time, and the assessment time horizon. Detlefsen and Scandolo (2005) introduced *conditional risk measures*, where the assessment outcome depends on new information becoming available. Riedel (2004), Weber (2006), Frittelli and Scandolo (2006), Artzner et al. (2007), Cheridito et al. (2006), among others, focused on risk measurements for stochastic processes. Roorda et al. (2005), Föllmer and Penner (2006), Gianin (2006), Tutsch (2006), Weber (2004), Roorda and Schumacher (2007) discussed different types of time consistency. In the literature on dynamic preferences, properties of time consistency were already studied by Koopmans (1960) and Epstein and Schneider (2003).

In the first part of the paper, we discuss the *time consistency* between assessments of the same financial position at several times. A risk measure satisfying appropriate time consistency can lead to more efficient capital management and reduce the risk of insolvency. A key notion in this area is that of *dynamic consistency*, see for example Föllmer and Penner (2006). It states that, if two positions are assessed in the same way in every future state, then should have the same assessment at the present time as well. Roorda and Schumacher (2007) proved that this requirement is equivalent to an attractive tower law property. However, in many cases dynamic consistency leads to a risk measure that produces very high capital requirement (see Tutsch (2006), Roorda and Schumacher (2008)). Furthermore, Kupper and Schachermayer (2008) prove that, under technical conditions, law-invariance (where the risk assessment depends only on the distribution of the position) and dynamic consistency reduce the class of possible risk measures to the entropic one. In this paper we focus on the weaker requirement of *sequential consistency* (Roorda and Schumacher (2007)), combining the ideas of *acceptance* and *rejection consistency* (Weber

(2004), Tutsch (2006)). This states:

a) A financial position cannot be considered acceptable at an initial time if it will be unacceptable in each successor state (*acceptance consistency*).

b) If the position is rejected in any state of nature at a future time point, then it should be rejected at an earlier time as well (*rejection consistency*).

We investigate sequential consistency for two of the standard ways of updating a coherent risk measure, discussed by Detlefsen and Scandolo (2005) and Tutsch (2008). The first update is obtained assuming that the new available information reduces the set of generalized scenarios that are used to construct the corresponding static risk measure. The second type of update assumes instead that new information does not influence this set. In both cases, we present sufficient conditions to ensure sequential consistency. Our results show that standard updates of a coherent risk measure (such as TVaR) often satisfy only the conditions for either acceptance or rejection consistency.

Consequently we provide a general method of constructing sequentially consistent dynamic risk measures, which requires modification of the risk measure used at the initial time. The technique is illustrated by building a sequentially consistent version of TVaR, which essentially coincides with the one proposed by Roorda and Schumacher (2008), for the coherent entropic risk measure recently introduced by Föllmer and Knispel (2011) and then extending the method to the general class of Choquet risk measures.

The last part of the paper concerns the time horizon of risk assessment. Even when exposure is to long term positions, the portfolio holder or the regulator is interested in determining the capital required at a future time point δ. For example, the impending Solvency II framework for European insurers requires that the safely invested capital corresponds to 99.5% VaR with 1 year time horizon. When the financial position expires before or at the time horizon δ, all the results of the first part apply. However, for longer term exposures, typical in insurance liabilities, the risk measure is applied to the fair value of the position at time δ, rather than to the position itself. This situation is outside the usual framework in the risk measures literature, as it essentially corresponds to risk measurement with an argument that changes over time, as the fair value at δ time units after measurement changes with new information. We show that even in this setting, acceptance consistency can still be preserved, but rejection consistency will in general not hold.

The paper is organized as follows. In Section 2 we review the notion of conditional coherent risk measures and discuss sequential consistency. In Section 3, we present technical conditions on the set of generalized scenarios to ensure acceptance and rejection consistency for two different types of update. In Section 4 a procedure for constructing a sequentially consistent risk measure is presented. Section 5 discusses the relation between sequential consistency and time horizon of risk assessment.

2 Conditional risk measures and time consistency

2.1 Conditional risk measures

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and define $\mathcal{X} = \mathcal{L}^\infty(\Omega, \mathcal{F}, \mathbb{P})$ the set of all bounded financial positions. Every inequality and equality involving elements of $\mathcal{X}$ is meant as holding $\mathbb{P}$-a.s. In order to take into account the role of new information, we introduce a non-trivial σ-algebra $\mathcal{G}$, such that $\{\emptyset, \Omega\} \subset \mathcal{G} \subset \mathcal{F}$. This means that at an intermediate time point before the expiry date of the portfolio, the investor or the regulator receives additional information $\mathcal{G}$. A re-assessment of the riskiness of the position at that time becomes of interest. The outcome of the new risk measurement $\rho_\mathcal{G}$ will depend on the information contained in $\mathcal{G}$, and $\rho_\mathcal{G}(X)$ will be a $\mathcal{G}$-measurable random variable. We will often refer to the starting time of the position as time 0 and the intermediate point in time, when the information is $\mathcal{G}$ is revealed, as time 1. Let $\mathcal{X}_\mathcal{G} := \mathcal{L}^\infty_\mathcal{G}(\Omega, \mathcal{G}, \mathbb{P})$ denote the set of all bounded random variables that are $\mathcal{G}$-measurable. Detlefsen and Scandolo (2005) introduce the following definition:

Definition 1. *A map $\rho_\mathcal{G} : \mathcal{X} \longrightarrow \mathcal{X}_\mathcal{G}$ is called a conditional convex risk measure if, for every $X, Y \in \mathcal{X}$, it satisfies the following properties:*

Monotonicity: *If $X \leq Y$, then $\rho_\mathcal{G}(X) \geq \rho_\mathcal{G}(Y)$.*

Conditional cash invariance: *If $Z \in \mathcal{X}_\mathcal{G}$, then $\rho(X + Z) = \rho(X) - Z$.*

Conditional convexity: $\rho(\lambda X + (1-\lambda)Y) \leq \lambda\rho(X) + (1-\lambda)\rho(Y)$ *for* $\lambda \in \mathcal{X}_{\mathcal{G}}$, $0 \leq \lambda \leq 1$.

Normalization: $\rho_{\mathcal{G}}(0) = 0$.

If it also satisfies

Conditional positive homogeneity: $\rho(\lambda X) = \lambda\rho(X)$ *for* $\lambda \in \mathcal{X}_{\mathcal{G}}$, $\lambda \leq 0$,

it is called a conditional coherent risk measure.

From the above properties, we can recover the definition of static coherent and convex risk measures introduced by Artzner et al. (1999) and Föllmer and Schied (2004), by simply substituting the σ-algebra $\mathcal{G}$ with the trivial one $\{\emptyset, \Omega\}$. In this case, we simply denote the risk measure $\rho(\cdot)$.

In the next sections we will make extensive use of the following sets:

$$\mathcal{M}_1(\mathbb{P}) := \{Q \text{ is a probability measure on } (\Omega, \mathcal{F}) \mid Q \ll \mathbb{P}\}$$

$$\mathcal{P}_{\mathcal{G}} := \{Q \in \mathcal{M}_1(\mathbb{P}) \mid Q \equiv \mathbb{P} \text{ on } \mathcal{G}\}.$$

Detlefsen and Scandolo (2005) proved that any risk measure of the form

$$\rho_{\mathcal{G}}(X) = ess \sup_{Q \in \mathcal{Q}_{\mathcal{G}}} E^Q[-X|\mathcal{G}]$$

for $\mathcal{Q}_{\mathcal{G}} \subseteq \mathcal{M}_1(\mathbb{P})$ is a conditional coherent risk measure.

A collection of conditional risk measures, with increasing level of information, is called *dynamic risk measure*. In our simple setting, the dynamic risk measure is given only by an unconditional and a conditional risk measure $(\rho, \rho_{\mathcal{G}})$. Unless otherwise specified, the conditional risk measure $\rho_{\mathcal{G}}(X)$ will be an *update* of ρ, meaning that $\rho_{\mathcal{G}} = \rho$ whenever $\mathcal{G} = \{\emptyset, \Omega\}$. There does not exist a unique update for a risk measure. For example, for a set of probability measures $\mathcal{Q} \subseteq \mathcal{M}_1(\mathbb{P})$, one can define the coherent risk measure:

$$\rho(X) = \sup_{Q \in \mathcal{Q}} E^Q[-X] \tag{1}$$

and the updates:

$$\hat{\rho}_{\mathcal{G}}(X) = ess \sup_{Q \in \hat{\mathcal{Q}}_{\mathcal{G}}} E^{Q}[-X \mid \mathcal{G}] \tag{2}$$

where $\hat{\mathcal{Q}}_{\mathcal{G}} \subseteq \{\mathcal{P}_{\mathcal{G}} \ \cap \ \mathcal{Q}\}$ and

$$\tilde{\rho}_{\mathcal{G}}(X) = ess \sup_{Q \in \mathcal{Q}} E^{Q}[-X \mid \mathcal{G}] \tag{3}$$

where the set $\mathcal{Q}$ remains unchanged over time. Update (3) is probably one of the simplest and most intuitive way of updating a risk measure, (Tutsch (2008)). Update (2), actually representing a class of possible updates, is more sophisticated and encompasses two key features of conditional risk measurement. First, the newly available information allows us to drop some probability measures, by the requirement $\hat{\mathcal{Q}}_{\mathcal{G}} \subseteq \mathcal{Q}$. Secondly, the property $Q \equiv \mathbb{P}$ on $\mathcal{G}$, means that at time 1 risk measurement proceeds so that the set of measures constructed is forward rather than backward looking. This way of updating a risk measure was used, among others, by Detlefsen and Scandolo (2005). In what follows we use extensively the two updates (2) and (3).

2.2 Examples of conditional risk measures

For an example consider the risk measure Tail Value at Risk (TVaR), that was proposed by Artzner et al. (1999) as a way to address the shortcomings of VaR. TVaR is a coherent risk measure and admits the following representation:

$$TVaR(X) = \sup_{Q \in \mathcal{Q}} E^{Q}[-X] \tag{4}$$

where

$$\mathcal{Q} := \{Q \in \mathcal{M}_1(\mathbb{P}) \mid \frac{dQ}{d\mathbb{P}} \leq \lambda^{-1}\}.$$

A possible update of type (2) for TVaR, proposed by Detlefsen and Scandolo (2005), corresponds to:

$$\widehat{TVaR}_{\mathcal{G}}(X) = ess \sup_{Q \in \hat{\mathcal{Q}}_{\mathcal{G}}} E^{Q}[-X \mid \mathcal{G}] \tag{5}$$

where

$$\hat{\mathcal{Q}}_{\mathcal{G}} := \{Q \in \mathcal{P}_{\mathcal{G}} \mid \frac{dQ}{d\mathbb{P}} \leq \lambda^{-1}\},$$

here $\hat{\mathcal{Q}}_{\mathcal{G}} = \{\mathcal{P}_{\mathcal{G}} \cap \mathcal{Q}\}$. Update (3) for TVaR corresponds to the conditional risk measure:

$$\widetilde{TVaR}_{\mathcal{G}}(X) = ess \sup_{Q \in \mathcal{Q}} E^{Q}[-X \mid \mathcal{G}]. \tag{6}$$

A conditional risk measure that is introduced in this paper is the *conditional coherent entropic risk measure*. This arises as a natural generalization of the static coherent entropic risk measure of Föllmer and Knispel (2011). In the unconditional case, this risk measure is defined as:

$$\rho^{e}(X) := \sup_{\mathcal{Q}} E^{Q}[-X] \tag{7}$$

where

$$\mathcal{Q} := \{Q \in \mathcal{M}_1(\mathbb{P}) \mid H(Q \mid \mathbb{P}) \leq c\}$$

and

$$H(Q \mid \mathbb{P}) = \begin{cases} E^{Q}[\ \log(\frac{dQ}{d\mathbb{P}})\ ] = E^{\mathbb{P}}[\frac{dQ}{d\mathbb{P}}\ \log(\frac{dQ}{d\mathbb{P}})\ ] & \text{if } Q \ll \mathbb{P} \\ +\infty \quad \text{otherwise} & \end{cases}$$

is the relative entropy of Q with respect to $\mathbb{P}$. A possible update, consistent with (2), is given by:

$$\hat{\rho}^{e}_{\mathcal{G}}(X) := ess \sup_{\hat{\mathcal{Q}}_{\mathcal{G}}} E^{Q}[-X|\mathcal{G}] \tag{8}$$

where

$$\hat{\mathcal{Q}}_{\mathcal{G}} := \{Q \in \mathcal{P}_{\mathcal{G}} \mid H_{\mathcal{G}}(Q \mid \mathbb{P}) \leq c\}$$

and

$$H_{\mathcal{G}}(Q \mid \mathbb{P}) := E^{Q}[\log(\frac{dQ}{d\mathbb{P}}) \mid \mathcal{G}] = E^{\mathbb{P}}[\ \frac{dQ}{d\mathbb{P}}\log(\frac{dQ}{d\mathbb{P}}) \mid \mathcal{G}]$$

is the conditional relative entropy of Q with respect to $\mathbb{P}$. To verify that (8) is in the class of updates (2), we have to check if $\hat{\mathcal{Q}}_{\mathcal{G}} \subseteq \{\mathcal{P}_{\mathcal{G}} \cap \mathcal{Q}\}$. The first inclusion $\hat{\mathcal{Q}}_{\mathcal{G}} \subseteq \mathcal{P}_{\mathcal{G}}$ is given. Now consider $Q \in \hat{\mathcal{Q}}_{\mathcal{G}}$:

$$H_{\mathcal{G}}(Q \mid \mathbb{P}) \leq c \quad \Rightarrow \quad E^{\mathbb{P}}[\ H_{\mathcal{G}}(Q \mid \mathbb{P})\ ] \leq c \quad \Rightarrow$$

$$E^{\mathbb{P}}[\ E^{Q}[\ \log(\frac{dQ}{d\mathbb{P}})|\mathcal{G}]\ ] \leq c \quad \Rightarrow \quad E^{Q}[\ \log(\frac{dQ}{d\mathbb{P}})\ ] = H(Q \mid \mathbb{P}) \leq c$$

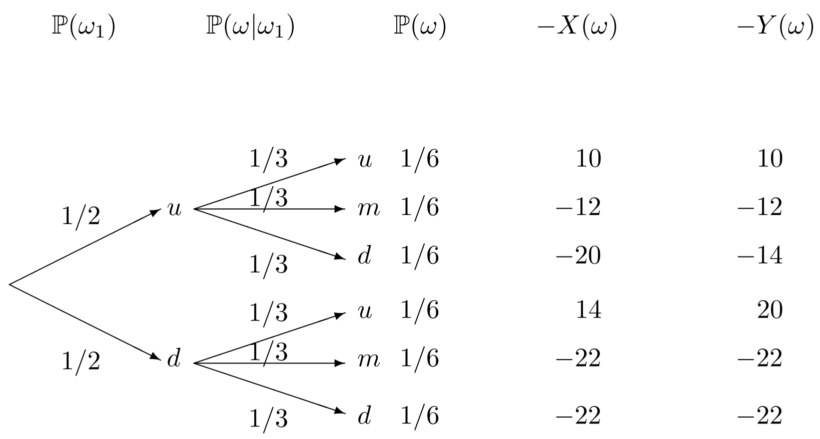

Figure 1: Probability distribution of $-X$ and $-Y$ under $\mathbb{P}$.

where we used $Q \equiv \mathbb{P}$ on $\mathcal{G}$.

2.3 Sequential consistency

It is reasonable to assume that a dynamic risk measure satisfies some notions of consistency. To illustrate this issue we present here two examples of inconsistency that are not desirable in a dynamic risk measure and that generally occur when we use updates (2) and (3).

Example 1. *Consider the dynamic risk measure ($TVaR, \widetilde{TVaR_{\mathcal{G}}}$) defined as in (4) and (6). Let $\Omega = \{uu, um, ud, du, dm, dd\}$ be the event space and $\mathbb{P}$ assign equal weight to every possible outcome as suggested by the binomial tree in figure 1. Set $\lambda = 2/3$. The set of probability measures $\mathcal{Q}$ is given by:*

$$\mathcal{Q} := \{Q \in \mathcal{M}_1(\mathbb{P}) \mid Q(\omega) \leq \frac{3}{2}\mathbb{P}(\omega) = \frac{1}{4} \quad \forall \, \omega \in \Omega\}.$$

For the financial position:

$$X = [-10, 12, 20, -14, 22, 22],$$

TVaR(X) is obtained assigning the highest admissible probability (i.e. 1/4) to the worst loss, then to the second worst one and so on until the probabilities used sum up to 1. Hence:

$$TVaR(X) = (14 + 10 - 12 - 20)\frac{1}{4} = -2 \leq 0.$$

For the conditional risk measure $\widetilde{TVaR_{\mathcal{G}}}$ we seek to maximize independently the *conditional expectations* $E^Q[-X|\ u]$ *and* $E^Q[-X|\ d]$ *over* $Q \in \mathcal{Q}$*. For the upper (u) branch, we set* $Q(uu|u) = 1,\ Q(um|u) = Q(ud|u) = 0$ *and* $Q(u) = \frac{1}{4}$*. Then it may be assumed that* $Q(du|d) = Q(dm|d) = Q(dd|d) = \frac{1}{3}$*. For the lower branch (d) we obtain* $\sup_{Q\in\mathcal{Q}} E^Q[-X|\ d] = 14$*. Therefore we have:*

$$\widetilde{TVaR}_{\mathcal{G}}(X)(\omega) = ess \sup_{Q\in\mathcal{Q}} E^Q[-X \mid \mathcal{G}] = \begin{cases} 10 \geq 0 & \text{if } \omega \in \{uu, um, ud\} \\ 14 \geq 0 & \text{if } \omega \in \{du, dm, dd\}. \end{cases}$$

Here, the position is acceptable at time 0 and 2 units of capital can be withdrawn from it. In contrast, at time 1, in both scenarios, the position is considered unacceptable and an amount of respectively 10 and 14 units of capital is required. This type of inconsistency, is particularly undesirable from the regulatory point of view as the risk holder may not be able to raise all the money needed (12 and 16 units in this example), leading to a possible insolvency risk. A good risk measure should detect the certainty of future capital needs, so that appropriate levels of capital can already be held at time 0.

The above example is close to the one used by Artzner et al. (2007) to illustrate a different type of inconsistency as follows:

Example 2 (Artzner et al. (2007))**.** *Here we consider update (2) for TVaR. Using the same setting than the previous example, we have*

$$\hat{\mathcal{Q}}_{\mathcal{G}} := \{Q \in \mathcal{P}_{\mathcal{G}} \mid Q(\omega|\omega') \leq \frac{3}{2}\mathbb{P}(\omega|\omega') = \frac{1}{2} \quad \forall\ \omega \in \Omega \text{ and } \omega' = u, d.\}$$

For the financial position:

$$Y = [-10, 12, 14, -20, 22, 22],$$

the risk measurement at time 0 is:

$$TVaR(Y) = (20 + 10 - 12 - 14)\frac{1}{4} = 1 \geq 0.$$

To calculate $\widehat{TVaR}_{\mathcal{G}}$*, we need to maximize the conditional probability of adverse outcomes, under the constraint* $Q(\omega|\omega') \leq \frac{1}{2}$*. The probability of the upper and lower branch is already fixed to be equal to* $\mathbb{P}$*, so* $Q(u) = Q(d) = \frac{1}{2}$ *for every*

$Q \in \hat{\mathcal{Q}}_{\mathcal{G}}$. *Then, we obtain:*

$$\widehat{TVaR}_{\mathcal{G}}(Y) = \begin{cases} (10-12)\frac{1}{2} = -1 \leq 0 & \text{if } \omega \in \{uu, um, ud\} \\ (20-22)\frac{1}{2} = -1 \leq 0 & \text{if } \omega \in \{du, dm, dd\}. \end{cases}$$

At time 0, the position Y is considered unacceptable and an amount of 1 unit of capital is required. At time 1, the position is considered acceptable in every state of the world and 1 unit of capital can actually be withdrawn. It means that TVaR penalizes a position that will anyway be accepted later on, requiring some capital that is not needed and that could be invested in a better way.

The same inconsistency holds for the coherent entropic risk measure $\rho^e(\cdot)$ and the update $\hat{\rho}^e_{\mathcal{G}}(\cdot)$ as it is shown in the following example.

Example 3. *Assume the same setting as in Example 1 and set $c = -\ln(2/3)$. For the financial position:*

$$Z = [-3, 14, 10, -9, 32, 32],$$

standard optimization techniques give $\rho^e(Z) = 0.6821 \geq 0$. The probability measure that attains the maximum in (7) is

$$Q = [0.2931,\ 0.0913,\ 0.1201,\ 0.4424,\ 0.0266,\ 0.0266].$$

For the conditional entropic risk measure, we have:

$$\hat{\rho}^e_{\mathcal{G}}(Z) = \begin{cases} -0.3483 \leq 0 & \text{if } \omega \in \{uu, um, ud\} \\ -0.3108 \leq 0 & \text{if } \omega \in \{du, dm, dd\}. \end{cases}$$

Again, at time 0 it is required to hold some capital, that in no-case will be asked at time 1.

To address such inconsistencies the notion of *sequential consistency* was proposed by Roorda and Schumacher (2007). It emerges as a combination of the two requirements of acceptance and rejection consistency, introduced by Weber (2004) for cash-flows and Tutsch (2006) for random variables.

Definition 2. *An unconditional and a conditional risk measure ρ and $\rho_{\mathcal{G}}$ are* *said to be sequentially consistent if, for every $X \in \mathcal{X}$, they satisfy:*

$$\rho_{\mathcal{G}}(X) \leq 0 \quad \Longrightarrow \quad \rho(X) \leq 0 \qquad \textit{acceptance consistency} \tag{9}$$

$$\rho_{\mathcal{G}}(X) \geq 0 \quad \Longrightarrow \quad \rho(X) \geq 0 \qquad \textit{rejection consistency.} \tag{10}$$

A risk measure satisfying (9) would not be subject to the inconsistencies seen in Example 2. Similarly, a risk measure satisfying (10) would avoid inconsistency faced in Example 1.

As stated by Roorda and Schumacher (2008) a dynamic risk measure $(\rho(X), \rho_{\mathcal{G}}(X))$ is sequentially consistent if and only if, for every $X \in \mathcal{X}$,

$$\inf_{\omega \in \Omega} \rho_{\mathcal{G}}(X) \;\leq\; \rho(X) \;\leq\; \sup_{\omega \in \Omega} \rho_{\mathcal{G}}(X). \tag{11}$$

This implies that the capital requirement at time 0 cannot be higher than the highest amount that could ever be asked in the future. On the other side, it cannot be smaller than the lowest amount of capital that would ever be required in the future.

3 Conditions for sequential consistency

3.1 Preliminaries

Before starting, we recall some notions that are essential for the next sections. Let $L^0(\overline{\mathbb{R}})$ be the space of extended random variables, i.e. of maps from Ω to $\overline{\mathbb{R}} := [-\infty, \infty]$.

Definition 3. *A set $\mathcal{Y} \subseteq L^0(\overline{\mathbb{R}})$ is upward directed if, for any two elements Y_1, $Y_2 \in \mathcal{Y}$, there is always a third one $Y \in \mathcal{Y}$ such that $Y \geq \max(Y_1, Y_2)$.*

For upward directed sets, the following result holds:

Lemma 1. *If $\mathcal{Y} \subseteq \mathcal{L}^0(\overline{\mathbb{R}})$ is upward directed, then*

$$E^{\mathbb{P}}[ess \sup \mathcal{Y}] = \sup_{Y \in \mathcal{Y}} E^{\mathbb{P}}[Y],$$

provided that the expectation exists.

The same holds if we replace the expectation with the expectation conditional to a σ-algebra $\mathcal{G} \subseteq \mathcal{F}$ (for a proof, see Detlefsen and Scandolo (2005)). In what follows we will extensively apply the above result to the set $\mathcal{C} := \{E^Q[-X \mid \mathcal{G}],\ Q \in \hat{\mathcal{Q}}_\mathcal{G}\}$. Here, for every $X \in \mathcal{X}$, each probability measure $Q \in \hat{\mathcal{Q}}_\mathcal{G}$ identifies a random variable $E^Q[-X \mid \mathcal{G}]$ and the essential supremum of $\mathcal{C}$ can be expressed as

$$ess \sup_{Q \in \hat{\mathcal{Q}}_\mathcal{G}} E^Q[-X|\mathcal{G}].$$

Following Detlefsen and Scandolo (2005), consider a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and a σ-algebra $\mathcal{G} \subseteq \mathcal{F}$. A regular conditional probability $Q_\mathcal{G}$ is defined as a map $Q_\mathcal{G} : (\Omega \times \mathcal{F}) \to [0, 1]$, that is, a version of the expected conditional value of $\mathbb{I}_A$ for any $A \in \mathcal{F}$ and a probability measure for $\omega \in \Omega$. For every $Q \in \mathcal{M}_1(\mathbb{P})$, the pasting probability $\mathbb{P}Q_\mathcal{G}$ is defined as

$$\mathbb{P}Q_\mathcal{G}(A) := E^\mathbb{P}[Q_\mathcal{G}(\cdot, A)] \qquad \forall A \in \mathcal{F},$$

where $Q_\mathcal{G}(\cdot, A)$ is a version of $E^Q[\mathbb{I}_A|\mathcal{G}]$.

In the case of a two-period binomial tree this concept becomes straightforward. The probability $\mathbb{P}Q_\mathcal{G}$ is obtained using $\mathbb{P}$ for the first period, from time 0 to time 1, and then switching to Q in the second one, from time 1 to time 2. The main property of pasting probability is:

$$E^{\mathbb{P}Q_\mathcal{H}}[X \mid \mathcal{G}] = E^\mathbb{P}[\ E^Q[X \mid \mathcal{H}] \mid \mathcal{G}] \qquad \text{for } \mathcal{H} \subseteq \mathcal{G},$$

for any σ-algebra $\mathcal{H}$ such that $\mathcal{H} \subseteq \mathcal{G} \subseteq \mathcal{F}$.

3.2 Conditions for sequential consistency

3.2.1 Sequential consistency conditions for update $\hat{\rho}_\mathcal{G}(\cdot)$

Consider two risk measures as in (1) and (2). For the dynamic risk measure $(\rho, \hat{\rho}_\mathcal{G})$, the following proposition holds:

Proposition 2. *(i) If, for every $X \in \mathcal{X}$, the set $\mathcal{C} := \{E^Q[-X \mid \mathcal{G}],\ Q \in \hat{\mathcal{Q}}_\mathcal{G}\}$ is upward directed, then the risk measures ρ and $\hat{\rho}_\mathcal{G}$ are rejection consistent.*

(ii) If, for every Q in $\mathcal{Q}$, the pasting probability $\mathbb{P}Q_{\mathcal{G}}$ is in $\hat{\mathcal{Q}}_{\mathcal{G}}$, then the risk measures ρ and $\hat{\rho}_{\mathcal{G}}$ are acceptance consistent.

Hence, if (i) and (ii) hold, then ρ and $\hat{\rho}_{\mathcal{G}}$ are sequentially consistent.

Proof. (i) Let $\hat{\rho}_{\mathcal{G}}(X) \geq 0$. Since the expected value of a positive random variable is again positive, we have:

$$\hat{\rho}_{\mathcal{G}}(X) = ess \sup_{Q \in \hat{\mathcal{Q}}_{\mathcal{G}}} E^Q[-X|\mathcal{G}] \geq 0 \qquad \Longrightarrow \qquad E^{\mathbb{P}}[\ ess \sup_{Q \in \hat{\mathcal{Q}}_{\mathcal{G}}} E^Q[-X|\mathcal{G}]\ ] \geq 0.$$

As the set C is upward directed, Lemma 1 leads to:

$$E^{\mathbb{P}}[\ ess \sup_{Q \in \hat{\mathcal{Q}}_{\mathcal{G}}} E^Q[-X \mid \mathcal{G}]\ ] = \sup_{Q \in \hat{\mathcal{Q}}_{\mathcal{G}}} E^{\mathbb{P}}[\ E^Q[-X \mid \mathcal{G}]\ ] \geq 0$$

but $Q \equiv \mathbb{P}$ on $\mathcal{G}$, therefore

$$E^{\mathbb{P}}[\ E^Q[-X|\mathcal{G}]\ ] = E^Q[\ E^Q[-X|\mathcal{G}]\ ] = E^Q[-X] \Longrightarrow \sup_{Q \in \hat{\mathcal{Q}}_{\mathcal{G}}} E^Q[-X] \geq 0.$$

Since $\hat{\mathcal{Q}}_{\mathcal{G}} \subseteq \mathcal{Q}$, we obtain:

$$\rho(X) = \sup_{Q \in \mathcal{Q}} E^Q[-X] \geq \sup_{Q \in \hat{\mathcal{Q}}_{\mathcal{G}}} E^Q[-X] \geq 0$$

as desired.
(ii) If $\hat{\rho}_{\mathcal{G}}(X) \leq 0$ then

$$E^Q[-X \mid \mathcal{G}] \leq 0 \qquad \forall Q \in \hat{\mathcal{Q}}_{\mathcal{G}}.$$

As $\mathbb{P}Q_{\mathcal{G}} \in \hat{\mathcal{Q}}_{\mathcal{G}}$ for every $Q \in \mathcal{Q}$:

$$E^{\mathbb{P}Q_{\mathcal{G}}}[-X \mid \mathcal{G}] \leq 0 \qquad \forall Q \in \mathcal{Q}.$$

By definition of the pasting probability:

$$0 \geq E^{\mathbb{P}Q_{\mathcal{G}}}[-X \mid \mathcal{G}] = E^{\mathbb{P}}[\ E^Q[-X \mid \mathcal{G}] \mid \mathcal{G}] = E^Q[-X \mid \mathcal{G}] \qquad \forall Q \in \mathcal{Q}.$$

Hence:

$$E^Q[-X] = E^Q[\ E^Q[-X \mid \mathcal{G}]\ ] \leq 0 \qquad \forall Q \in \mathcal{Q}$$

and

$$\rho(X) = \sup_{Q \in \mathcal{Q}} E^Q[-X] \leq 0$$

as desired. $\square$

Remark 1. *(i) is a technical condition ensuring that we can exchange the essential supremum and the expectation, where (ii), instead, requires that at time 0, we only use probability measures that will be also used for the risk assessment at time 1. In this way, we avoid measuring risk using probability measures that, in any case, will not even be considered when the information $\mathcal{G}$ is revealed.*

Remark 2. *Risk measures as in (1) and (2) generally fail acceptance consistency. Indeed, the proof of Prop. 2 cannot be applied in this case, because:*

$$ess \sup_{Q \in \hat{\mathcal{Q}}_{\mathcal{G}}} E^Q[-X|\mathcal{G}] \leq 0 \qquad \Longrightarrow \qquad \sup_{Q \in \hat{\mathcal{Q}}_{\mathcal{G}}} E^Q[-X] \leq 0$$

but generally,

$$\sup_{Q \in \mathcal{Q}} E^Q[-X] \geq \sup_{Q \in \hat{\mathcal{Q}}_{\mathcal{G}}} E^Q[-X] \leq 0,$$

so we cannot deduce

$$\rho(X) \leq 0.$$

Lemma 3. *$TVaR$ and $\widehat{TVaR}_{\mathcal{G}}$ are rejection consistent.*

Proof. To prove it, we need to verify that the set $\mathcal{C} := \{E^Q[-X \mid \mathcal{G}],\ Q \in \hat{\mathcal{Q}}_{\mathcal{G}}\}$ is upward directed , i.e. condition (i). Consider two probability measures Q' and Q'' in $\hat{\mathcal{Q}}_{\mathcal{G}}$ and define Q as

$$Q(B) = Q'(A \cap B) + Q''(A^c \cap B) \tag{12}$$

where the set $A \in \mathcal{G}$ is defined as

$$A := \{E^{Q'}[-X \mid \mathcal{G}] \geq E^{Q''}[-X \mid \mathcal{G}]\}.$$

It is not difficult to see that $Q \in \hat{\mathcal{Q}}_{\mathcal{G}}$. For every $C \in \mathcal{G}$

$$Q(C) = Q'(A \cap C) + Q''(A^c \cap C) = \mathbb{P}(A \cap C) + \mathbb{P}(A^c \cap C) = \mathbb{P}(C)$$

so $Q \equiv \mathbb{P}$ on $\mathcal{G}$. Similarly, for every $B \in \mathcal{F}$

$$Q(B) = Q'(A \cap B) + Q''(A^c \cap B) \leq \lambda^{-1}(\mathbb{P}(A \cap B) + \mathbb{P}(A^c \cap B)) \leq \lambda^{-1}\mathbb{P}(B)$$

so $\frac{dQ}{d\mathbb{P}} \leq \lambda^{-1}$ and

$$E^Q[-X \mid \mathcal{G}] = \mathbb{I}_A E^{Q'}[-X \mid \mathcal{G}] + \mathbb{I}_{A^c} E^{Q''}[-X \mid \mathcal{G}] \geq \max\{E^{Q'}[-X \mid \mathcal{G}],\ E^{Q''}[-X \mid \mathcal{G}]\}.$$

Therefore $\mathcal{C}$ is upward directed and $TVaR$ and $\widehat{TVaR}_{\mathcal{G}}$ are rejection consistent.

□

TVaR and the update $\widehat{TVaR}_{\mathcal{G}}$ do not satisfy sequential consistency because, as we have already seen in Example 2, they are acceptance inconsistent. It is immediate to verify that condition (*ii*) of Proposition 2 is not satisfied because the probability measure $Q^* \in \mathcal{Q}$ that maximizes $\sup_{Q\in\mathcal{Q}} E^Q[-X]$ in Example 2, is such that $\mathbb{P}Q^*_{\mathcal{G}}$ does not belong to $\hat{\mathcal{Q}}_{\mathcal{G}}$.

Lemma 4. *The coherent entropic risk measures $\rho^e(\cdot)$ and $\rho^e_{\mathcal{G}}(\cdot)$ are rejection consistent.*

Proof. Again, we only need to prove that the set $\mathcal{C} := \{E^Q[-X|\mathcal{G}], Q \in \hat{\mathcal{Q}}_{\mathcal{G}}\}$ is upward directed for every $X \in \mathcal{X}$. Following the steps of Lemma 3, we define a probability measure Q as in (12). We already know that $Q \equiv \mathbb{P}$ on $\mathcal{P}_{\mathcal{G}}$. By definition:

$$\begin{aligned} H_{\mathcal{G}}(Q|\mathbb{P}) &= E^Q[\log\frac{dQ}{d\mathbb{P}} \mid \mathcal{G}] \\ &= \mathbb{I}_A E^{Q'}[\log\frac{dQ'}{d\mathbb{P}}|\mathcal{G}] + \mathbb{I}_{A^c} E^{Q''}[\log\frac{dQ''}{d\mathbb{P}}|\mathcal{G}] \\ &\leq \max\{E^{Q'}[\log\frac{dQ'}{d\mathbb{P}}|\mathcal{G}], E^{Q''}[\log\frac{dQ''}{d\mathbb{P}}|\mathcal{G}]\} \leq c \end{aligned}$$

so that $Q \in \hat{\mathcal{Q}}_{\mathcal{G}}$ and the set is upward directed. □

Also $(\rho^e(\cdot), \rho^e_{\mathcal{G}}(\cdot))$ does not satisfy sequential consistency as follows from Example 3.

3.2.2 Sequential consistency conditions for update $\tilde{\rho}_{\mathcal{G}}(\cdot)$

We now discuss time consistency for risk measures where the set of probability measures is not updated when new information arrives, that is, the risk measures (1) and (3).

Proposition 5. *(i) The risk measures ρ and $\tilde{\rho}$ are acceptance consistent.*

(ii) If, for every $X \in \mathcal{X}$

(a) the set $\mathcal{C} := \{E^Q[-X \mid \mathcal{G}] \mid Q \in \mathcal{Q}\}$ is upward directed and $\mathbb{P}Q_{\mathcal{G}} \in \mathcal{Q}$ for every Q in $\mathcal{Q}$; or

(b) the supremum in the definition of $\tilde{\rho}$ is attained, i.e. :

$$\exists\, P^* \in \mathcal{Q}: \quad E^{P^*}[-X\,|\mathcal{G}] = ess \sup_{P \in \mathcal{Q}} E^P[-X\,|\mathcal{G}] \geq E^{P'}[-X\,|\mathcal{G}],\ \forall P' \in \mathcal{Q} \tag{13}$$

then ρ and $\tilde{\rho}$ are rejection consistent. Hence, if (a) or (b) hold, then ρ and $\tilde{\rho}$ are sequentially consistent.

Proof. (i) is proved by Tutsch (2008). Now we show that either of the conditions (a) and (b) implies rejection consistency. For (a) the proof follows the same steps of Prop. $2(i)$. For (b), condition (13) together with

$$\tilde{\rho}_{\mathcal{G}}(X) \geq 0,$$

imply that

$$\exists P^* \in \mathcal{Q} \qquad \text{such that} \qquad E^{P^*}[-X \mid \mathcal{G}] = \tilde{\rho}_{\mathcal{G}}(X) \geq 0.$$

Then,

$$E^{P^*}[E^{P^*}[-X \mid \mathcal{G}]] = E^{P^*}[-X] \geq 0.$$

As $P^* \in \mathcal{Q}$, we have:

$$\rho(X) = \sup_{Q \in \mathcal{Q}} E^Q[-X] \geq E^{P^*}[-X] \geq 0.$$

□

Remark 3. *Conditional risk measures with the additional property of being continuous from below admit a representation in terms of probability measures where the supremum is attained. Nevertheless, it is usually attained on a different set than $\mathcal{Q}$, such that condition (13) is not necessarily verified. For details see Bion-Nadal (2004). The situation becomes easier if we work in a setting where Ω is finite. In this case, the supremum is attained if the set $\mathcal{Q}$ is closed* *and convex and there exists a probability measure $P \in \mathcal{Q}$ such that*

$$PQ_{\mathcal{G}} \in \mathcal{Q} \quad \textit{for every} \;\; Q \in \mathcal{Q}. \tag{14}$$

An example of such a risk measure, satisfying sequential consistency on a finite probability space, was proposed by Roorda and Schumacher (2007). Define

$$STVaR(X) = \sup_{Q \in \mathcal{Q}'} E^Q[-X]$$

where

$$\mathcal{Q}' := \{Q \in \mathcal{M}_1(\mathbb{P}) \mid \frac{dQ}{d\mathbb{P}} \leq \lambda^{-1}, \; \frac{d\mathbb{P}Q_{\mathcal{G}}}{d\mathbb{P}} \leq \lambda^{-1}\}$$

and consider the update

$$\widetilde{STVaR}_{\mathcal{G}}(X) = ess \sup_{Q \in \mathcal{Q}'} E^Q[-X \mid \mathcal{G}].$$

When Ω is finite, the set $\mathcal{Q}'$ is a polytope and $\mathbb{P}$ satisfies condition (14), so, using convex analysis arguments, it is possible to show that the essential supremum is attained and the risk measure is sequentially consistent. A similar argument, where the supremum is attained, is used by Fasen and Svejda (2010) to construct a sequentially consistent version of distortion risk measures in a finite framework.

4 Constructing sequentially consistent risk measures

4.1 General construction

In the previous sections, conditions for the sequential consistency of dynamic risk measures were presented. However no risk measure considered actually satisfies these conditions on an in finite probability space. Now, drawing inspiration from Roorda and Schumacher (2007), we show that it is possible to slightly modify a dynamic risk measure in order to turn it into a sequentially consistent one. The method is applied to produce a sequentially consistent version of the coherent entropic risk measure as well as the class of coherent Choquet risk measure. A numerical example is given for TVaR.

Start again with a coherent risk measure as in (1). Suppose that we consider ρ suitable for our measurement purposes, but the update $\hat{\rho}_{\mathcal{G}}$ does not satisfy the conditions for sequential consistency required by Proposition 2. In order to construct a sequentially consistent risk measure, starting from the update $\hat{\rho}_{\mathcal{G}}$, we work backwards, defining a new unconditional risk measure as:

$$\hat{\rho}'(X) = \sup_{Q \in \hat{Q}'} E^{Q}[-X]$$

where

$$\hat{Q}' := \{Q \in \mathcal{M}_1(\mathbb{P}) \mid Q \in \mathcal{Q},\ \mathbb{P}Q_{\mathcal{G}} \in \hat{\mathcal{Q}}_{\mathcal{G}}\}.$$

Proposition 6. *If the set $\mathcal{C} := \{E^{Q}[-X \mid \mathcal{G}] \mid Q \in \hat{\mathcal{Q}}_{\mathcal{G}}\ \}$ is upward directed, then the risk measures $\hat{\rho}'(X)$ and $\hat{\rho}_{\mathcal{G}}(X)$ are sequentially consistent.*

Proof. From Prop. 2 they are sequentially consistent by construction. □

Remark 4. *Notice that $\rho(X)$ and $\hat{\rho}'(X)$ admit the same update (2), i.e.*

$$\hat{\rho}'_{\mathcal{G}}(X) = ess \sup_{Q \in \hat{\mathcal{Q}}'_{\mathcal{G}}} E^{Q}[-X \mid \mathcal{G}] = \hat{\rho}_{\mathcal{G}}(X)$$

where

$$\hat{\mathcal{Q}}'_{\mathcal{G}} := \{Q \in \mathcal{P}_{\mathcal{G}} \mid Q \in \hat{\mathcal{Q}}_{\mathcal{G}}\}.$$

Moreover, they are close in the sense that $\hat{\rho}'(X)$ requires, at time 0, all the conditions on the set of measures, required by $\rho(X)$, but in addition the conditions that will be required at time 1, when new information arrives. In other words, the condition $\mathbb{P}Q_{\mathcal{G}} \in \hat{\mathcal{Q}}_{\mathcal{G}}$ excludes, at time 0, probability measures that will not be used in the representation of the update. In this way, we avoid rejecting financial positions that would be accepted when the information in $\mathcal{G}$ is revealed.

Remark 5. *Once we have constructed the new unconditional risk measure*

$$\hat{\rho}'(X) = \sup_{Q \in \hat{Q}'} E^{Q}[-X]$$

where

$$\hat{Q}' := \{Q \in \mathcal{M}_1(\mathbb{P}) \mid Q \in \mathcal{Q},\ \mathbb{P}Q_{\mathcal{G}} \in \hat{\mathcal{Q}}_{\mathcal{G}}\},$$

we can easily see that, if the set $\mathcal{C} := \{E^{Q}[-X \mid \mathcal{G}] \mid Q \in \mathcal{Q}\ \}$ is upward directed, also the update (3)

$$\tilde{\rho}'_{\mathcal{G}}(X) = ess \sup_{Q \in \hat{Q}'} E^{Q}[-X|\mathcal{G}]$$

satisfies sequential consistency as it verifies all the conditions required in Prop. 5. We remark that in this case, the two updates are actually the same, ie:

$$ess \sup_{Q \in \hat{Q}'} E^Q[-X|\mathcal{G}] = ess \sup_{Q \in \hat{Q}'_{\mathcal{G}}} E^Q[-X|\mathcal{G}]$$

due to the structure of the probability measure sets $\hat{Q}'$ *and* $\hat{Q}'_{\mathcal{G}}$.

4.2 Examples of sequentially consistent risk measures

In this section, sequentially consistent versions of TVaR, coherent entropic, and Choquet risk measures are introduced.

We start from the static TVaR and the update $\widehat{TVaR}_{\mathcal{G}}$. From Example 2, we already know that this update fails acceptance consistency. As shown in Lemma 3, the set $\mathcal{C} := \{E^Q[-X \mid \mathcal{G}] \mid Q \in \hat{\mathcal{Q}}_{\mathcal{G}}\}$ is upward directed. Therefore, we can define a new unconditional risk measure, as:

$$\widehat{TVaR}'(X) = \sup_{Q \in \hat{Q}'} E^Q[-X]$$

where

$$\hat{Q}' := \{Q \in \mathcal{M}_1(\mathbb{P}) \,|\, Q \in \mathcal{Q},\, \mathbb{P}Q_{\mathcal{G}} \in \mathcal{Q}_{\mathcal{G}}\} = \{Q \in \mathcal{M}_1(\mathbb{P}) \,|\, \frac{dQ}{d\mathbb{P}} \leq \lambda^{-1}, \frac{d\mathbb{P}Q_{\mathcal{G}}}{d\mathbb{P}} \leq \lambda^{-1}\}.$$

From Prop. 6, $\widehat{TVaR}'$ and $\widehat{TVaR}_{\mathcal{G}}$ are sequentially consistent.

In the following example it is seen how the sequentially consistent version of TVaR solves the inconsistencies faced in the examples 1 and 2.

Example 4. *To see this, consider again the same setting as in Example 2, where* $\Omega = \{uu, um, ud, du, dd, dm\}$, $\mathbb{P}(\omega) = 1/6$ *for every* $\omega \in \Omega$ *and* $\lambda = 2/3$. *The set of probability measures considered at time 0 and time 1 are respectively:*

$$\hat{\mathcal{Q}}' := \{Q \in \mathcal{M}_1(\mathbb{P}) \mid Q(\omega) \leq \frac{3}{2}\mathbb{P}(\omega) = \frac{1}{4},$$

$$Q(\omega|\omega') \leq \frac{3}{2}\mathbb{P}(\omega|\omega') = \frac{1}{2} \quad \forall\, \omega \in \Omega,\ \omega' = \{u, d\}\}$$

and

$$\hat{\mathcal{Q}}_{\mathcal{G}} = \{Q \in \mathcal{P}_{\mathcal{G}} \mid Q(\omega|\omega') \leq \frac{3}{2}\mathbb{P}(\omega|\omega') = \frac{1}{2} \quad \forall\, \omega \in \Omega,\ \omega' = \{u, d\}\}.$$

For the financial position:

$$Y = [-10, 12, 14, -20, 22, 22],$$

we have

$$\widehat{TVaR}'(Y) = (10 - 12 + 20 - 22)\frac{1}{4} = -1 \neq TVaR(Y)$$

and

$$\widehat{TVaR}_{\mathcal{G}}(Y) = \begin{cases} (10-12)\frac{1}{2} = -1 & \text{if } \omega = uu, um, ud \\ (20-22)\frac{1}{2} = -1 & \text{if } \omega = du, dm, dd \end{cases}$$

Therefore, the acceptance inconsistency has been eliminated.

The new risk measure does not present the rejection inconsistency of Example 1 either. Indeed for the random variable:

$$X = \{-10, 12, 20, -14, 22, 22\},$$

we have

$$\widehat{TVaR}'(X) = (10 - 12 + 14 - 22)\frac{1}{4} = -\frac{5}{2}$$

while

$$\widehat{TVaR}_{\mathcal{G}}(X) = \begin{cases} (10-12)\frac{1}{2} = -1 & \text{if } \omega = uu, um, ud \\ (14-22)\frac{1}{2} = -4 & \text{if } \omega = du, dm, dd. \end{cases}$$

Now we show how Prop. 6 can be used to construct a sequentially consistent version of the coherent entropic risk measure. As shown in Lemma 4, the set $\mathcal{C} := \{E^Q[-X|\mathcal{G}], Q \in \hat{\mathcal{Q}}_{\mathcal{G}}\}$ is upward directed. To have sequential consistency we define a new risk measure

$$\rho'^{e}(X) := \sup_{\hat{Q}'} E^Q[-X] \tag{15}$$

where

$$\hat{Q}' := \{Q \in \mathcal{M}_1(\mathbb{P}) \mid H(Q|\mathbb{P}) \leq c \quad \text{and} \quad H_{\mathcal{G}}(\mathbb{P}Q_{\mathcal{G}}|\mathbb{P}) \leq c\}.$$

From Prop. 6 (15) and (8) are sequentially consistent.

The same procedure can be used to obtain a sequentially consistent version of Choquet risk measures, which can be seen as generalizations of TVaR. A similar result was obtained by Fasen and Svejda (2010) for Choquet risk measures in a finite setting. For a comprehensive discussion of Choquet risk measures we refer to Carlier and Dana (2003) and Tsanakas (2004). Here, we define a Choquet risk measure as:

$$\rho^C(X) = \sup_{Q \in \mathcal{Q}} \mathbb{E}^Q[-X],$$

where

$$\mathcal{Q} = \{Q \in \mathcal{M}_1(\mathbb{P}) \mid Q(A) \leq g(\mathbb{P}(A)) \quad \forall A \in \mathcal{F}\}$$

and $g : [0,1] \mapsto [0,1]$ is an increasing, concave function such that:

$$g(0) = 1; \qquad g(1) = 1.$$

A possible update for a Choquet risk measure is the following:

$$\hat{\rho}^C_{\mathcal{G}}(X) = \sup_{\hat{\mathcal{Q}}_{\mathcal{G}}} E^Q[-X|\mathcal{G}] \tag{16}$$

where

$$\hat{\mathcal{Q}}_{\mathcal{G}} := \{Q \in \mathcal{P}_{\mathcal{G}} \mid Q_{\mathcal{G}}(\cdot, A) \leq g(\mathbb{P}_{\mathcal{G}}(\cdot, A)) \quad \forall A \in \mathcal{F}\}$$

and $Q_{\mathcal{G}}(\cdot, A)$ is a version of $E^Q[\mathbb{I}_A|\mathcal{G}]$. To see that (16) belongs to the class of updates (2), we show that $\hat{\mathcal{Q}}_{\mathcal{G}} \subseteq \mathcal{Q}$. For every $Q \in \hat{\mathcal{Q}}_{\mathcal{G}}$ and for every $A \in \mathcal{F}$:

$$Q_{\mathcal{G}}(\cdot, A) \leq g(\mathbb{P}_{\mathcal{G}}(\cdot, A)) \qquad \Rightarrow \tag{17}$$

$$\mathbb{P}Q_{\mathcal{G}}(\cdot, A) = E^{\mathbb{P}}[Q_{\mathcal{G}}(\cdot, A)] \leq E^{\mathbb{P}}[g(\mathbb{P}_{\mathcal{G}}(\cdot, A))] \qquad \Rightarrow \tag{18}$$

$$Q(A) \leq g(E^{\mathbb{P}}[\mathbb{P}_{\mathcal{G}}(\cdot, A)]) = g(\mathbb{P}(A)), \tag{19}$$

where, in (18) we used the definition of pasting probabilities and (19) follows from $Q \in \mathcal{P}_{\mathcal{G}}$ and the Jensen's inequality. We have already seen from Example 1 that this kind of update generally is not acceptance consistent. Now, consider the new risk measure:

$$\hat{\rho}'^C(X) = \sup_{Q \in \hat{\mathcal{Q}}'} E^Q[-X] \tag{20}$$

where

$$\hat{\mathcal{Q}}' := \{Q \in \mathcal{M}_1(\mathbb{P}) \,|\, Q(A) \leq g(\mathbb{P}(A)) \quad \text{and} \quad Q_{\mathcal{G}}(\cdot, A) \leq g(\mathbb{P}_{\mathcal{G}}(\cdot, A)) \quad \forall A \in \mathcal{F}\},$$

the following result holds:

Lemma 7. *The risk measures (20) and (16) are sequentially consistent.*

Proof. We need to prove that the set $\mathcal{C} := \{E^Q[-X|\mathcal{G}],\ Q \in \hat{\mathcal{Q}}_{\mathcal{G}}\}$ is upward directed. To see it, consider again a probability measure Q and the set A defined as in (12), we only need to show that $Q \in \hat{\mathcal{Q}}_{\mathcal{G}}$. If $\mathbb{P}(A) = \{0, 1\}$ the proof is immediate. Assume now that $\mathbb{P}(A) \neq \{0, 1\}$, by the definition of Q' and Q'', for every $B \in \mathcal{F}$ we have:

$$\begin{aligned} Q(B) &= \mathbb{P}(A)Q'(B|A) + \mathbb{P}(A^c)Q''(B|A^c) \\ &\leq \mathbb{P}(A)g(\mathbb{P}(B|A)) + \mathbb{P}(A^c)g(\mathbb{P}(B|A^c)) \\ &\leq g(\mathbb{P}(A)\mathbb{P}(B|A) + \mathbb{P}(A^c)\mathbb{P}(B|A^c)) = g(\mathbb{P}(B)), \end{aligned}$$

where we used the concavity of $g(\cdot)$. It follows from Prop. 6 that $(\hat{\rho}'^C, \hat{\rho}^C_{\mathcal{G}})$ is sequentially consistent. □

5 The solvency time horizon in dynamic risk measurement

5.1 Sequential consistency in multiple periods

Here we briefly discuss the results of Sections 3 and 4 in a multi-period setting. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and $\{\mathcal{F}\}_{n\in[0,N]}$ a filtration with $N \in \mathbb{N}$ and $\mathcal{F} \equiv \mathcal{F}_N$. A dynamic coherent risk measure is then defined as a collection

$$(\rho_0, \rho_1, \ldots, \rho_{N-1}), \tag{21}$$

where:

$$\rho_n(X) := \rho_{\mathcal{F}_n}(X) = ess \sup_{Q\in\mathcal{Q}_n} E^Q[-X \mid \mathcal{F}_n] \qquad \forall n \in [0, \ldots, N-1].$$

for a certain set of measures $\mathcal{Q}_0, \ldots, \mathcal{Q}_{N-1}$.

The extension of the notion of sequential consistency to this setting is straightforward.

Definition 4. *The dynamic risk measure* $(\rho_0, \rho_1, \dots, \rho_{N-1})$ *is sequentially consistent if, for every* $X \in \mathcal{X}$*, it satisfies:*

(i) acceptance consistency

$$\rho_n(X) \leq 0 \quad \Longrightarrow \quad \rho_{n-1}(X) \ \leq 0 \qquad \forall n \in [0, N-1]; \quad \text{and} \tag{22}$$

(ii) rejection consistency

$$\rho_n(X) \geq 0 \quad \Longrightarrow \quad \rho_{n-1}(X) \geq 0 \qquad \forall n \in [0, N-1] \tag{23}$$

Again, given a static risk measure as in (1) we can define the updates:

$$\hat{\rho}_n(X) = ess \sup_{Q \in \hat{\mathcal{Q}}_n} E^Q[-X|\mathcal{F}_n] \tag{24}$$

and

$$\tilde{\rho}_n(X) = ess \sup_{Q \in \mathcal{Q}} E^Q[-X|\mathcal{F}_n] \tag{25}$$

where

$$\hat{\mathcal{Q}}_n \subseteq \{Q \in \mathcal{P}_n \ \cap \ Q \in \mathcal{Q}\}$$

and

$$\mathcal{P}_n := \{Q \in \mathcal{M}_1(\mathbb{P}) \mid Q \equiv \mathbb{P} \ \text{ on } \ \mathcal{F}_n\}$$

The results presented in Section 3 still hold. Specifically we have:

Corollary 8. *(i) If, for every* $X \in \mathcal{X}$*, the set* $\mathcal{C} := \{E^Q[-X \mid \mathcal{F}_n],\ Q \in \hat{\mathcal{Q}}_n\}$ *is upward directed for every* $n \in [0, \dots, N-1]$*, then the risk measure* $(\rho, \hat{\rho}_1, \dots, \hat{\rho}_{N-1})$ *is rejection consistent.*

(ii) If, for every Q *in* $\mathcal{Q}$*, the pasting probability* $\mathbb{P}Q_{\mathcal{F}_n}$ *is in* $\hat{\mathcal{Q}}_n$ *for every* $n \in [0, \dots, N-1]$*, then the risk measure* $(\rho, \hat{\rho}_1, \dots, \hat{\rho}_{N-1})$ *is acceptance consistent.*

Hence, if (*i*) *and* (*ii*) *hold, then* $(\rho, \hat{\rho}_1, \dots, \hat{\rho}_{N-1})$ *is sequentially consistent.*

Corollary 9. *(i) The dynamic risk measure* $(\rho, \tilde{\rho}_1, \dots, \tilde{\rho}_{N-1})$ *is acceptance consistent.*

(ii) If, for every $X \in \mathcal{X}$*,*

(a) *the set* $\mathcal{C} := \{E^Q[-X \mid \mathcal{F}_n] \mid Q \in \mathcal{Q}\}$ *is upward directed and* $\mathbb{P}Q_n \in \mathcal{Q}$ *for every* Q *in* $\mathcal{Q}$ *for every* $n \in [0, \ldots, N-1]$*, or*

(b) *the supremum in (25) is attained, i.e. :*

$$\exists\, P^* \in \mathcal{Q} \quad s.t. \quad E^{P^*}[-X \,|\mathcal{F}_n] = ess \sup_{P \in \mathcal{Q}} E^P[-X \,|\mathcal{F}_n] \geq E^{P'}[-X \,|\mathcal{F}_n] \qquad \forall P' \in \mathcal{Q} \tag{26}$$

then $(\rho, \tilde{\rho}_1, \ldots, \tilde{\rho}_{N-1})$ *is rejection consistent.*

Hence, if (a) or (b) hold, then $(\rho, \tilde{\rho}_1, \ldots, \tilde{\rho}_{N-1})$ *is sequentially consistent.*

Remark 6. *The procedure to build a sequentially consistent version of a dynamic coherent risk measure, is the same as the one seen in Section 4. We start from the update (24) at time* $N-1$ *and we proceed backwards, adding all the conditions that we need. For a random variable* X*, the new risk measure will be* $(\hat{\rho}'_n)_{n \in [0,N]}$*, where:*

$$\hat{\rho}'_n(X) = ess \sup_{Q \in \hat{\mathcal{Q}}'_n} E^Q[-X|\mathcal{F}_n]$$

and the set $\hat{\mathcal{Q}}'_n$ *is defined as:*

$$\hat{\mathcal{Q}}'_n := \{Q \in \mathcal{P}_n \mid \mathbb{P}Q_l \in \hat{\mathcal{Q}}'_l \quad \forall l \in \mathbb{N}, s.t. \quad n \leq l \leq N-1\}.$$

Note that, at the penultimate time, $\hat{\rho}'_{N-1}$ *coincides with* ρ_{N-1}*.*

5.2 Dynamic risk measures and solvency time horizon

Here we consider the effect of a solvency time horizon on risk measurement. Often regulatory capital requirements are specified in relation to a fixed time horizon, eg 1 year in insurance regulation such as Solvency II (or a much shorter horizon of 10 days, in banking under Basel II). When a portfolio contains long term liabilities (eg insurance contracts) that expire beyond the time horizon, the random terminal payoff has to be substituted with its (random) market consistent value at the time horizon. Valuation may be carried out either using "mark-to-market" replication arguments or, if that is not possible, using a "mark-to-model" cost of capital approach (see eg Wüthrich and Salzmann (2010)).

Here, we assume that a "mark-to-market" valuation is possible via a risk neutral measure Q^*. Hence, the position X is substituted with its price at the solvency time horizon δ. In insurance, this price is for example the price of reinsuring the position at time δ. In what follows, we will introduce a new risk measure that takes into account this aspect. Consider a random variable $X \in \mathcal{X}$ and define the functional:

$$\rho_0^\delta(X) := \sup_{P \in \mathcal{Q}} E^P \left[-E^{Q^*}[X \mid \mathcal{F}_\delta] \right]$$

for a certain pricing measure Q^*. In general, for $n \in [0, N-1]$,

$$\rho_n^\delta(X) := ess \sup_{P \in \mathcal{Q}_n} E^P \left[-E^{Q^*}[X|\mathcal{F}_{n+\delta}] \mid \mathcal{F}_n \right].$$

For the moment, we do not specify the set $\mathcal{Q}_n$ and thus what kind of update we will be using. Note that $\rho_n^\delta(X)$ is nothing but the application of a conditional coherent risk measure

$$\rho_n(\cdot) = ess \sup_{P \in \mathcal{Q}_n} E^P[- \cdot |\mathcal{F}_n]$$

to the conditional expectation of the position X under a certain probability measure Q^*. It is straightforward to prove that the conditional risk measure $\rho_n^\delta(X)$ is coherent.

5.3 Sequential consistency of $\rho^\delta(\cdot)$

We now consider whether the coherent risk measures

$$\rho_n^\delta(X) = \rho_n(E^{Q^*}[X \mid \mathcal{F}_{n+\delta}])$$

and

$$\rho_{n+1}^\delta(X) = \rho_{n+1}(E^{Q^*}[X \mid \mathcal{F}_{n+1+\delta}])$$

inherit some time consistency from ρ_n and ρ_{n+1}. For convenience, consider $\rho_0^\delta(X)$ and $\rho_1^\delta(X)$:

$$\rho_0^\delta(X) := \sup_{Q \in \mathcal{Q}} E^Q[\ E^{Q^*}[-X \mid \mathcal{F}_\delta]\ ] \tag{27}$$

and

$$\rho_1^\delta(X) := ess \sup_{Q \in \mathcal{Q}_1} E^Q[\ E^{Q^*}[-X \mid \mathcal{F}_{1+\delta}] \mid \mathcal{F}_1] \tag{28}$$

for certain sets $\mathcal{Q}$ and $\mathcal{Q}_1$.

Lemma 10. *If $\rho_0(X)$ and $\rho_1(X)$ are acceptance consistent, then so are $\rho_0^{\delta+1}(X)$ and $\rho_1^{\delta}(X)$.*

Proof. If $\rho_0(X)$ and $\rho_1(X)$ are acceptance consistent, then

$$\rho_1^{\delta}(X) = \rho_1(\ E^{Q^*}[X \mid \mathcal{F}_{1+\delta}]\ ) \leq 0 \quad \Longrightarrow \quad \rho_0(\ E^{Q^*}[X \mid \mathcal{F}_{1+\delta}]\ ) = \rho_0^{\delta+1}(X) \leq 0.$$

□

To establish consistency between ρ_0^{δ} and ρ_1^{δ}, we need some additional conditions on the set $\mathcal{Q}$ and the probability measure Q^*. In particular, we recall that a risk measure is *law-invariant*, if it assigns the same value to financial positions having the same distribution.

Proposition 11. *If the risk measures ρ_0 and ρ_1 are acceptance consistent, and either*

(i) The probability measure Q^* belongs to $\mathcal{Q}$ and $QQ_n^* \in \mathcal{Q}$ for every $Q \in \mathcal{Q}$ and for every $n \in [0, N-1]$; or

(ii) $Q^* \in \mathcal{M}_1(\mathbb{P})$ and the risk measure $\rho(\cdot)$ is coherent law-invariant and continuous from below,

then ρ_0^{δ} and ρ_1^{δ} are acceptance consistent.

Proof. (i) We already know that $\rho_1^{\delta}(X) \leq 0$, implies

$$\rho_0^{1+\delta}(X) = \sup_{Q\in\mathcal{Q}} E^Q[\ E^{Q^*}[-X \mid \mathcal{F}_{1+\delta}]\ ] \leq 0$$

so

$$E^Q[\ E^{Q^*}[-X \mid \mathcal{F}_{1+\delta}]\ ] \leq 0 \qquad \forall Q \in \mathcal{Q}$$

In particular we can choose $Q = RQ_\delta^* \in \mathcal{Q}$ for every $R \in \mathcal{Q}$ and obtain

$$E^{RQ_\delta^*}[\ E^{Q^*}[-X \mid \mathcal{F}_{1+\delta}]\ ] = E^R[\ E^{Q^*}[-X \mid \mathcal{F}_{\delta}]\ ] \leq 0 \qquad \forall R \in \mathcal{Q}$$

therefore

$$\rho_0^{\delta}(X) = \sup_{R\in\mathcal{Q}} E^R[\ E^{Q^*}[-X \mid \mathcal{F}_{\delta}]\ ] \leq 0.$$

(ii) The proof follows from Corollary 4.59 in Föllmer and Schied (2004), where is proved that ρ is monotone with respect to the second order stochastic dominance $\succeq$. From

$$E^{Q^*}[-X \mid \mathcal{F}_{1+\delta}] \succeq E^{Q^*}[-X \mid \mathcal{F}_{\delta}]$$

and Lemma 10, we obtain the acceptance consistency of ρ_0^δ and ρ_1^δ.

□

Therefore, acceptance consistency can still be valid when we substitute X with $E^{Q^*}[X|\mathcal{F}_\delta]$.

Remark 7. *The same does not hold for rejection consistency. Even if $\rho_0(X)$ and $\rho_1(X)$ are rejection consistent, this does not imply that $\rho_0^\delta(X)$ and $\rho_1^\delta(X)$ are as well. To see it, consider*

$$\rho_0^\delta(X) := \sup_{Q \in \mathcal{Q}} E^Q[\ E^{Q^*}[-X \mid \mathcal{F}_\delta]\ ] \tag{29}$$

and

$$\rho_1^\delta(X) := ess \sup_{Q \in \mathcal{Q}_1} E^Q[\ E^{Q^*}[-X \mid \mathcal{F}_{1+\delta}] \mid \mathcal{F}_1] \tag{30}$$

for certain sets $\mathcal{Q}$ and $\mathcal{Q}_1$. Again from the rejection consistency of $\rho_0(X)$ and $\rho_1(X)$, we have

$$\rho_1^\delta(X) \geq 0 \qquad \Longrightarrow \qquad \rho_0^{1+\delta(X)} \geq 0$$

but in general we do not have enough information to derive

$$\rho_0^{1+\delta}(X) \geq 0 \qquad \Longrightarrow \qquad \rho_0^\delta(X) \geq 0.$$

Then, if a position is rejected, this does not give enough information to reject also its conditional expectation, which is generally less volatile than the position itself.

6 Conclusions

We contribute to the discussion of the properties of dynamic risk measures, focusing on the time consistency of conditional coherent risk measures. Technical conditions are discussed to ensure *sequential consistency* for different types of updates. These requirements are generally not satisfied by coherent risk measures, such as e.g. TVaR. Hence, it becomes sometimes necessary to modify slightly the risk measure in order to obtain consistent dynamic risk measurements. This is achieved by an adjustment to the coherent risk measure set of generalized scenarios. The procedure amounts to excluding, a priori, probability measures that will not be taken into account, in any case, when new information is available. As an example, an application of this approach to TVaR, to the coherent entropic risk measure and to the class of Choquet risk measures is presented. Finally, we discuss the role of the solvency time horizon. When the position has a long term, solvency regulation often requires that risk is measured at an earlier time horizon. In this case, the argument of the risk measure is the position's fair value at that horizon. In this changed setting, acceptance consistency can be preserved, but in general we lose rejection consistency.

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