Correction: Exchange Option Under Jump-Diffusion Dynamics

RUGGERO CALDANA¹, GERALD H. L. CHEANG², CARL CHIARELLA³ & GIANLUCA FUSAI¹,4

¹ Department of Economics and Business Studies, Università del Piemonte Orientale, Novara, Italy, ² Centre for Industrial and Applied Mathematics, School of Information Technology and Mathematical Sciences, University of South Australia, Adelaide, Australia, ³ Finance Discipline Group, UTS Business School, University of Technology, Sydney, Australia, ⁴ Faculty of Finance, Cass Business School, City University, London, United Kingdom

(November 2013)

Abstract In this note, we provide the correct formula for the price of the European exchange option given in Cheang and Chiarella (2011) in a bi-dimensional jump diffusion model.

Key Words: exchange option, jump-diffusion.

Theorem 5.1 in Cheang and Chiarella (2011), page 259, gives a formula for the price of a European exchange option under jump diffusion dynamics. The formula is based on a wrong application of the change of numéraire from the risk-neutral to the spot measure. We amend the proof and provide the correct pricing formula for the exchange option.

Theorem 1: Suppose the asset prices follow the dynamics in formula (38) of Cheang and Chiarella (2011), and the continuous dividend rate for each asset is \( \xi_i \). Then when \( S_{1,t} = s_1 \) and \( S_{2,t} = s_2 \), the European exchange option price is

\[
C_{t}^{E}(s_1, s_2) = \sum_{k=0}^{\infty} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{e^{-(\tilde{\lambda}_1 + \tilde{\lambda}_2 + \tilde{\lambda})(T-t)}}{k! m! n!} \left[ s_1 e^{-(\tilde{\lambda}_1 + \tilde{\lambda}_2 + \tilde{\lambda})(T-t) + k \tilde{\kappa}_{11} + n \tilde{\kappa}_{12} + \frac{n^2}{2} \Phi(d_{1,t,k,m,n})} - s_2 e^{-(\tilde{\lambda}_1 + \tilde{\lambda}_2 + \tilde{\lambda})(T-t) + m \tilde{\kappa}_{21} + n \tilde{\kappa}_{22} + \frac{n^2}{2} \Phi(d_{2,t,k,m,n})} \right],
\]

where

\[
d_{1,t,k,m,n} = \frac{\ln \left( \frac{s_1}{s_2} \right) + (\xi_2 - \xi_1 - \tilde{\lambda}(\tilde{\kappa}_1 - \tilde{\kappa}_2) - \tilde{\lambda}_1 \tilde{\kappa}_1 + \tilde{\lambda}_2 \tilde{\kappa}_2)(T-t) + \mu_{k,m,n} + \frac{\sigma_{k,m,n}^2}{2}(T-t)}{\sigma_{k,m,n} \sqrt{T-t}},
\]

\[
d_{2,t,k,m,n} = d_{1,t,k,m,n} - \sigma_{k,m,n} \sqrt{T-t};
\]

Correspondence Address: Ruggero Caldana, Department of Economics and Business Studies, Università del Piemonte Orientale “Amedeo Avogadro”, via E. Perrone 18, Novara, Italy. Email: ruggero.caldana@eco.unipmn.it
with
\[ \mu_{k,m,n} = k(\bar{\alpha}_{11} + \delta_{11}^2/2) - m(\bar{\alpha}_{22} + \delta_{22}^2/2) + n(\bar{\alpha}_1 - \bar{\alpha}_2 + \delta_1^2/2 - \delta_2^2/2), \]
and
\[ \sigma_{k,m,n}^2 = \sigma^2 + \frac{k\delta_{11}^2}{T-t} + \frac{m\delta_{22}^2}{T-t} + \frac{n(\delta_1^2 + \delta_2^2 - 2\rho_1\delta_1\delta_2)}{T-t}, \quad \sigma^2 = \sigma_1^2 + \sigma_2^2 - 2\rho_1\sigma_1\sigma_2, \]
where \( \Phi \) is the standard normal probability distribution function.

**Proof:** Without loss of generality, we derive a formula for the exchange option price at time \( t = 0 \). The option price at \( t = 0 \) is then
\[
C_0^E(S_{1,0}, S_{2,0}) = \mathbb{E}_Q \left[ \frac{(S_{1,T} - S_{2,T})^+}{e^{rT}} \right] =

\begin{align*}
& S_{1,0}e^{-\xi_1T}\mathbb{E}_Q \left[ \exp \left[ -\frac{\sigma_1^2}{2} T + \sigma_1 \tilde{W}_{1,T} - \tilde{\lambda}_1 \tilde{Y}_1,T + \sum_{n=0}^{N_T} Y_{1,n} - \tilde{\lambda}_1 \tilde{K}_Z;T + \sum_{i=1}^{N_1,T} Z_{1,i} \right] \mathbb{I}_{\{S_{1,T}>S_{2,T}\}} \right] \\
& - S_{2,0}e^{-\xi_2T}\mathbb{E}_Q \left[ \exp \left[ -\frac{\sigma_2^2}{2} T + \sigma_2 \tilde{W}_{2,T} - \tilde{\lambda}_2 \tilde{Y}_2,T + \sum_{n=1}^{N_T} Y_{2,n} - \tilde{\lambda}_2 \tilde{K}_Z;T + \sum_{l=1}^{N_2,T} Z_{2,l} \right] \mathbb{I}_{\{S_{1,T}>S_{2,T}\}} \right].
\end{align*}

Using twice the change of numéraire from the risk neutral measure \( \mathbb{Q} \) to the spot measures \( \mathbb{Q}_1 \) (stock \( S_1 \) is taken as numéraire) and \( \mathbb{Q}_2 \) (stock \( S_2 \) is taken as numéraire), and conditioning on the number of idiosyncratic and common jumps the pricing formula of the exchange option requires the computation of \( \mathbb{Q}_1(A|N_1,T = k, N_2,T = m, N_T = n) \) and \( \mathbb{Q}_2(A|N_1,T = k, N_2,T = m, N_T = n) \), where \( A|N_1,T = k, N_2,T = m, N_T = n \) is the set defined as
\[
\left\{ \Xi_{k,m,n} > \ln \left( \frac{S_{2,0}}{S_{1,0}} \right) - \left( \xi_2 - \xi_1 - \frac{\sigma_1^2}{2} + \frac{\sigma_2^2}{2} - \tilde{\lambda}(\tilde{\kappa}_1 - \tilde{\kappa}_2) - \tilde{\lambda}_1 \tilde{K}_Z + \tilde{\lambda}_2 \tilde{K}_Z \right) T \right\}
\]
and
\[
\Xi_{k,m,n} = \sigma_1 \tilde{W}_{1,T} - \sigma_2 \tilde{W}_{2,T} + \sum_{i=0}^{k} Z_{1,i} - \sum_{l=0}^{m} Z_{2,l} + \sum_{j=0}^{n} (Y_{1,j} - Y_{2,j}).
\]

The proof in Cheang and Chiarella (2011) has to be corrected in the specification of the distribution of \( \Xi_{k,m,n} \) under \( \mathbb{Q}_1 \) and \( \mathbb{Q}_2 \). In particular to compute the distribution of \( Y \) under \( \mathbb{Q}_1 \) and \( \mathbb{Q}_2 \), we have to apply Theorem 3.1 of Cheang and Chiarella (2011), according to the following Radon–Nikodym derivatives
\[
\frac{d\mathbb{Q}_1}{d\mathbb{Q}} \bigg|_T = \exp \left[ -\frac{\sigma_1^2}{2} T + \sigma_1 \tilde{W}_{1,T} - \tilde{\lambda}_1 \tilde{Y}_1,T + \sum_{n=0}^{N_T} Y_{1,n} - \tilde{\lambda}_1 \tilde{K}_Z;T + \sum_{i=1}^{N_1,T} Z_{1,i} \right],
\]
and
\[
\frac{d\mathbb{Q}_2}{d\mathbb{Q}} \bigg|_T = \exp \left[ -\frac{\sigma_2^2}{2} T + \sigma_2 \tilde{W}_{2,T} - \tilde{\lambda}_2 \tilde{Y}_2,T + \sum_{n=1}^{N_T} Y_{2,n} - \tilde{\lambda}_2 \tilde{K}_Z;T + \sum_{l=1}^{N_2,T} Z_{2,l} \right].
\]

The parameter \( \gamma \) defined in Theorem 3.1 determines the distribution of the jump component \( Y \) through the following relation on the moment-generating function
\[
M_{\mathbb{Q}_1,Y}(u) = \frac{M_{\mathbb{Q},Y(u + \gamma)}}{M_{\mathbb{Q},Y(\gamma)}}, \quad i = 1, 2.
\]
Setting \( \gamma = [1, 0]^T \), Theorem 3.1 implies that the Wiener and the jump components, conditioned on the event \( N_{1,T} = k, N_{2,T} = m, N_T = n \), are normally distributed as

\[
\Xi_{k,m,n} \sim N((\sigma_1^2 - \rho \sigma_1 \sigma_2)T + n(\alpha_1 - \alpha_2 + \delta_1^2) - \rho \gamma \delta_1 \delta_2) + k(\alpha_{11} + \delta_{11}^2) - m\bar{\alpha}_{22} + \delta_{22}^2), \sigma_{k,m,n}^2(T).
\]

The Poisson process \( N_T \) has arrival intensity \( \hat{\lambda}_1 = \hat{\lambda}(1 + \hat{k}_1) \) and the Poisson process \( N_{1,T} \) has arrival intensity \( \hat{\lambda}_{Z_1} = \hat{\lambda}_1(1 + \hat{k}_{Z_1}) \) under \( \mathbb{Q}_1 \), with the intensity of \( N_{2,T} \) unchanged.

Similarly setting \( \gamma = [0, 1]^T \), it follows that the random variable \( \Xi_{T,k,m,n} \) is therefore normally distributed as

\[
\Xi_{k,m,n} \sim N((\rho \sigma_1 \sigma_2 - \sigma_2^2)T + n(\alpha_1 - \alpha_2 + \rho \gamma \delta_1 \delta_2 - \delta_2^2) + k(\alpha_{11} + \delta_{11}^2) - m\bar{\alpha}_{22} + \delta_{22}^2), \sigma_{k,m,n}^2(T).
\]

The Poisson process \( N_T \) has arrival intensity \( \hat{\lambda}_2 = \hat{\lambda}(1 + \hat{k}_2) \) and the Poisson process \( N_{2,T} \) has arrival intensity \( \hat{\lambda}_{Z_2} = \hat{\lambda}_2(1 + \hat{k}_{Z_2}) \) under \( \mathbb{Q}_2 \), and the intensity of \( N_{1,T} \) unchanged.

Straightforward computations as in Cheang and Chiarella (2011) conclude the proof. \( \Box \)

Table 1 provides numerical results. We consider nine different parameter scenarios (we also set \( \xi_1 = \xi_2 = 0, r = 0, t = 0 \) and \( T = 1 \)). Formula (1), row \( C_1^E \), has been computed truncating the triple sum to \( n = m = k = 25 \). We also provide the Monte Carlo estimate, row MC, obtained with \( N_{MC} = 10^7 \) random trials, implemented using a control variate method as described in Caldana and Fusai (2013). The row labeled C.I.L. gives the length of the 95% mean-centered Monte Carlo confidence interval. In all cases \( C_1^E \) matches the Monte Carlo solution up to the sixth digit.

<table>
<thead>
<tr>
<th>Scenario</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
</tr>
</thead>
<tbody>
<tr>
<td>( N_{1,0} )</td>
<td>100.00</td>
<td>100.00</td>
<td>96.00</td>
<td>100.00</td>
<td>96.00</td>
<td>100.00</td>
<td>96.00</td>
<td>100.00</td>
<td>100.00</td>
</tr>
<tr>
<td>( \delta_{1,0} )</td>
<td>0.10</td>
<td>0.10</td>
<td>0.10</td>
<td>0.15</td>
<td>0.15</td>
<td>0.15</td>
<td>0.15</td>
<td>0.10</td>
<td>0.10</td>
</tr>
<tr>
<td>( \sigma_1 )</td>
<td>0.10</td>
<td>0.10</td>
<td>0.10</td>
<td>0.10</td>
<td>0.10</td>
<td>0.10</td>
<td>0.10</td>
<td>0.10</td>
<td>0.10</td>
</tr>
<tr>
<td>( \rho )</td>
<td>0.90</td>
<td>0.90</td>
<td>0.90</td>
<td>0.50</td>
<td>0.50</td>
<td>0.50</td>
<td>0.50</td>
<td>0.90</td>
<td>0.90</td>
</tr>
<tr>
<td>( \lambda )</td>
<td>0.50</td>
<td>0.50</td>
<td>0.50</td>
<td>0.20</td>
<td>0.20</td>
<td>0.20</td>
<td>0.20</td>
<td>0.10</td>
<td>0.10</td>
</tr>
<tr>
<td>( \alpha_1 )</td>
<td>0.03</td>
<td>0.03</td>
<td>0.03</td>
<td>0.06</td>
<td>0.06</td>
<td>0.06</td>
<td>0.06</td>
<td>0.03</td>
<td>0.03</td>
</tr>
<tr>
<td>( \delta_1 )</td>
<td>0.10</td>
<td>0.10</td>
<td>0.10</td>
<td>0.03</td>
<td>0.03</td>
<td>0.03</td>
<td>0.03</td>
<td>0.03</td>
<td>0.03</td>
</tr>
<tr>
<td>( \rho_\nu )</td>
<td>-0.90</td>
<td>-0.90</td>
<td>-0.90</td>
<td>-0.80</td>
<td>-0.80</td>
<td>-0.80</td>
<td>-0.80</td>
<td>0.90</td>
<td>0.90</td>
</tr>
<tr>
<td>( \lambda_\nu )</td>
<td>0.50</td>
<td>0.50</td>
<td>0.50</td>
<td>0.20</td>
<td>0.20</td>
<td>0.20</td>
<td>0.20</td>
<td>0.10</td>
<td>0.10</td>
</tr>
<tr>
<td>( \alpha_{11} )</td>
<td>0.02</td>
<td>0.02</td>
<td>0.02</td>
<td>0.02</td>
<td>0.02</td>
<td>0.02</td>
<td>0.02</td>
<td>0.02</td>
<td>0.02</td>
</tr>
<tr>
<td>( \delta_{11} )</td>
<td>0.01</td>
<td>0.01</td>
<td>0.01</td>
<td>0.06</td>
<td>0.06</td>
<td>0.06</td>
<td>0.06</td>
<td>0.01</td>
<td>0.01</td>
</tr>
<tr>
<td>( \lambda_{11} )</td>
<td>0.10</td>
<td>0.10</td>
<td>0.10</td>
<td>0.10</td>
<td>0.10</td>
<td>0.10</td>
<td>0.10</td>
<td>0.10</td>
<td>0.10</td>
</tr>
<tr>
<td>( \alpha_{22} )</td>
<td>0.02</td>
<td>0.02</td>
<td>0.02</td>
<td>-0.07</td>
<td>-0.07</td>
<td>-0.07</td>
<td>-0.07</td>
<td>0.02</td>
<td>0.02</td>
</tr>
<tr>
<td>( \delta_{22} )</td>
<td>0.01</td>
<td>0.01</td>
<td>0.01</td>
<td>0.01</td>
<td>0.01</td>
<td>0.01</td>
<td>0.01</td>
<td>0.01</td>
<td>0.01</td>
</tr>
</tbody>
</table>

\( C_1^E \) prices the exchange option according to the analytical formula (1). MC displays the Monte Carlo estimate and C.I.L. gives the length of the 95% mean-centered Monte Carlo confidence interval.

Table 1. Exchange option values are computed for nine different scenarios. \( C_1^E \) prices the exchange option according to the analytical formula (1). MC displays the Monte Carlo estimate and C.I.L. gives the length of the 95% mean-centered Monte Carlo confidence interval.

References
