Approximate pricing of swaptions in affine and quadratic models

Anna Maria Gambaro*, Ruggero Caldana† and Gianluca Fusai‡

Abstract

This paper proposes new bounds on the prices of European-style swaptions for affine and quadratic interest rate models. These bounds are computable whenever the joint characteristic function of the state variables is known. In particular, our lower bound involves the computation of a one-dimensional Fourier transform independently of the swap length. In addition, we control the error of our method by providing a new upper bound on swaption price that is applicable to all considered models. We test our bounds on different affine models and on a quadratic Gaussian model. We also apply our procedure to the multiple-curve framework. The bounds are found to be accurate and computationally efficient.[1]

JEL classification codes: G12, G13.

KEYWORDS: Pricing, swaptions, affine-quadratic models, Fourier transform, bounds.

1 Introduction

The accurate pricing of swaption contracts is fundamental in interest rate markets. Swaptions are among the most liquid over-the-counter (OTC) derivatives and are largely used for hedging purposes. Many applications also require efficient computation of swaption prices, such as calibration, estimation of risk metrics and credit and debit value adjustment (CVA and DVA) valuation. In the calibration of interest rate models, a large number of swaptions with different

*Dipartimento di Statistica e Metodi Quantitativi, Università Milano Bicocca (DiSMeQ), Via Bicocca degli Arcimboldi 8, 20126 Milano, Italy, Email: a.gambaro@campus.unimib.it (*Corresponding Author*).

†Accenture S.p.A., Via Maurizio Quadrio 17, 20154 Milano, Italy, Email: ruggero.caldana@gmail.com.

‡Faculty of Finance, Cass Business School, City University, 106 Bunhill Row, London, EC1Y 8TZ, UK and Dipartimento di Studi per l'Economia e l'Impresa, Università del Piemonte Orientale "A. Avogadro", via E. Perrone 18 28100 Novara, Italy, Email: Gianluca.Fusai.1@city.ac.uk.

[1] *Aknowledgements:* The authors are indebted to Laura Ballotta and the participants of XVI Workshop on Quantitative Finance, (Università degli studi di Parma) and the conference Challenges in Derivatives Markets, (Technische Universität München) for useful conversations and constructive comments. We would like to thank Laura Ballotta for useful feedback. All remaining errors are ours. We acknowledge financial support by Universitá degli Studi del Piemonte Orientale, Gruppo Nazionale per l'Analisi Matematica, la Probabilitá e le loro Applicazioni (GNAMPA).

maturities, swap lengths and strikes are priced using iterative procedures aimed at fitting market quotations. Similarly, in the estimation of risk metrics for a portfolio of swaptions, if a full revaluation setting is used and millions of possible scenarios are considered, a fast pricing algorithm is essential to obtain results in a reasonable time. In addition, the Basel III accords introduced the CVA and DVA charge for OTC contracts, and for the simplest and most popular kind of interest rate derivative, i.e. interest rate swap, the two adjustments can be estimated by pricing a portfolio of forward start European swaptions (see Brigo and Masetti (2005)). Hence, the appeal of a fast and exact closed-form solution for the swaption pricing problem is explained.

The famous Jamshidian (1989) formula is applicable only when the short rate depends on a single stochastic factor while for multi-factor interest rate models, several approximate methods have been developed in the literature. Munk (1999) approximates the price of an option on a coupon bond by a multiple of the price of an option on a zero-coupon bond with time to maturity equal to the stochastic duration of the coupon bond. The method of Schrager and Pelsser (2006) is based on approximating the affine dynamics of the swap rate under the relevant swap measure. These methods are fast but not very accurate for out-of-the-money options. The method of Collin-Dufresne and Goldstein (2002) is based on an Edgeworth expansion of the density of the swap rate and requires a time-consuming calculation of the moments of the coupon bond and it provides reliable estimation only for a low volatility level. An estimation of the error of the Collin-Dufresne and Goldstein (2002) has been provided in Zheng (2013). Singleton and Umantsev (2002) (henceforth S&U) introduce the idea of approximating the exercise region in the space of the state variables. This method has the advantage of producing accurate results across a wide range of strikes, in particular for out-of-the-money swaptions. However, it does not allow a simple extension to general affine interest rate models because it requires the knowledge of the joint probability density function of the state variables in the closed form. Kim (2014) generalizes and simplifies the S&U method. Up to now, Kim's method seems to be the most efficient proposed in the literature. Nevertheless, Kim's method requires the calculation of as many Fourier transforms as the number of cash flows in the underlying swap, which implies that the run time of the algorithm increases with the swap length. Moreover, none of these papers discusses the direction of the error, i.e. whether the price is overestimated or underestimated. Further, except for Collin-Dufresne and Goldstein (2002), none of the methods proposed in the literature is able to estimate or control the approximation error. Recently, a lower and an upper bound on swaption prices was proposed in Nunes and Prazeres (2014), but these are applicable only to Gaussian models.

Similar to S&U and Kim, we propose a lower bound that is based on an approximation of the exercise region via an event set defined through a function of the model factors. Our pricing formula consists of the valuation of an option on the approximate exercise region and requires a single Fourier transform. Our procedure gives a new perspective with respect to existing methods, such as those of S&U and Kim. Indeed, we prove that their approximations are also lower bounds to the swaption price. To the best of our knowledge, this has not been reported previously. Moreover, we develop methods to control the approximation error by deriving a new upper bound on swaption prices.

Finally, we extend the lower and upper bounds to multiple-curve models that reflect the presence of various interest curves in the market after the 2007 crisis. Multiple-curve interest rate models are widely discussed in the literature (see, among others Ametrano and Bianchetti (2009), Morini (2009) and recently Moreni and Pallavicini (2014) and Fanelli (2016)). In particular, we concentrate on the affine multiple-curve model developed in Moreni and Pallavicini (2014). To the best of our knowledge, none of the approximated methods previously described for pricing swaptions has been developed for a multiple-curve interest rate framework.

The paper is organized as follows. Section 2 introduces a general formula for the lower bound on swaption prices based on an approximation of the exercise region. In addition, the popular methods of S&U and Kim are proved to be included in our setting. Then we apply the general lower bound formula to the case of affine models and Gaussian quadratic interest rate models and we find an efficient algorithm to calculate analytically the approximated swaption price. In section 3, the new upper bound is presented for affine-quadratic models. Section 4 extends the previously described bounds to a multiple-curve model. Section 5 shows the results of numerical tests. Conclusions and remarks are presented in the last section.

2 Lower bound on swaption prices

In this section, we discuss the general pricing formula for a receiver European-style swaption and the approximations presented in S&U and Kim. In particular, we prove that these approximations are lower bounds.

A European swaption is a contract that gives the right to its owner to enter into an underlying interest rate swap, i.e. it is a European option on a swap rate. It can be equivalently interpreted as an option on a portfolio of zero-coupon bonds (or as an option on a coupon bond). Let t be the current date, T the option expiration date, $T_1, ..., T_n$ the underlying swap payment dates (by construction $t < T < T_1 < ... < T_n$) and R the fixed rate of the swap. The payoff of a receiver swaption is

$$\left(\sum_{h=1}^{n} w_h P(T,T_h) - 1\right)^+ ,$$

where $w_h = R\,(T_h - T_{h-1})$ for $h = 1,..,n-1$, $w_n = R\,(T_n - T_{n-1}) + 1$, and $P(T,T_h)$ is the price at time T of a zero-coupon bond expiring at time T_h. The time t no-arbitrage price is the risk-neutral expected value of the discounted payoff,

$$C(t) = \mathbb{E}_t\left[e^{-\int_t^T r(\mathbf{X}(s))ds}\left(\sum_{h=1}^{n} w_h P(T,T_h) - 1\right)^+\right] \tag{1}$$

where $r(\mathbf{X}(s))$ is the short rate at time s, and $\mathbf{X}(s)$ denotes the state vector at time s of a multi-factor stochastic model. The price formula (1) after a change of measure to the T-forward measure becomes

$$C(t) \;=\; P(t,T)\,\mathbb{E}_t^T\left[\left(\sum_{h=1}^{n} w_h P(T,T_h) - 1\right) I(\mathcal{A})\right] \tag{2}$$

with I denoting the indicator function, and $\mathcal{A}$ is the exercise region, which is seen as a subset of the space events Ω,

$$\mathcal{A} = \{\omega \in \Omega : \sum_{h=1}^{n} w_h P(T,T_h) \geq 1\}.$$

By changing the measure of each expected value from the T forward measure to the T_h measure, the pricing formula in expression (2) can be written as

$$C(t) = \sum_{h=1}^{n} w_h P(t,T_h)\,\mathbb{P}_t^{T_h}[\mathcal{A}] - P(t,T)\,\mathbb{P}_t^{T}[\mathcal{A}]$$

where $\mathbb{P}_t^S[\mathcal{A}]$ denotes the time t probability of the exercise set $\mathcal{A}$ under the S-forward measure. S&U and Kim replace the exercise set $\mathcal{A}$ in the above formula by a new set $\mathcal{G}$ that makes the computation of the swaption price much simpler, and then their approximated pricing formula reads as (see Singleton and Umantsev (2002) and Kim (2014) for further details)

$$C_{\mathcal{G}}(t) = \sum_{h=1}^{n} w_h P(t,T_h)\,\mathbb{P}_t^{T_h}[\mathcal{G}] - P(t,T)\,\mathbb{P}_t^{T}[\mathcal{G}]. \tag{3}$$

The choice of the approximated exercise region is made so that the above probabilities can be computed by performing $n+1$ Fourier inversions, where n is the number of payments in the underlying swap. We can now show that $C_{\mathcal{G}}(t)$ is a lower bound approximation to the true price. Indeed, we observe that for any event set $\mathcal{G} \subset \Omega$:

$$\mathbb{E}_t^T\left[\left(\sum_{h=1}^n w_h P(T,T_h)-1\right)^+\right] \geq \mathbb{E}_t^T\left[\left(\sum_{h=1}^n w_h P(T,T_h)-1\right)^+ I(\mathcal{G})\right]$$
$$\geq \mathbb{E}_t^T\left[\left(\sum_{h=1}^n w_h P(T,T_h)-1\right) I(\mathcal{G})\right]$$

Then by discounting we obtain:

$$C(t) \geq LB_{\mathcal{G}}(t) := P(t,T)\,\mathbb{E}_t^T\left[\left(\sum_{h=1}^n w_h P(T,T_h)-1\right) I(\mathcal{G})\right], \tag{4}$$

i.e. $LB_{\mathcal{G}}(t)$ is a lower bound to the swaption price for all possible sets $\mathcal{G}$. Using the same change of measures as in S&U and Kim, it immediately follows that

$$LB_{\mathcal{G}}(t) = C_{\mathcal{G}}(t).$$

Therefore, the approximated pricing formula presented in S&U and Kim are indeed lower bounds. This was not previously noted. In particular, our new framework allows us to control the approximation error by providing an upper bound. In addition, we show how to speed up the computation of the formula (4) by performing a single Fourier transform. This allows a reduction of the computational cost, mainly when we have to price swaptions written on long-maturity swaps.

2.1 Affine and Gaussian quadratic models

In affine and quadratic interest rate models, the price at T of a zero-coupon bond with expiration T_h can be written as the exponential of a quadratic form of the state variables,

$$P(T,T_h) = e^{\mathbf{X}^\top(T) C_h \mathbf{X}(T) + \mathbf{b}_h^\top \mathbf{X}(T) + a_h} \tag{5}$$

for $\mathbf{X}(T)$ a d-dimensional state vector and $a_h = A(T-T_h)$, $\mathbf{b}_h = \mathbf{B}(T-T_h)$ and $C_h = C(T-T_h)$ functions of the payment date T_h, which are model specific. Fixing a date T_h, $\mathbf{b}_h$ is a d-dimensional vector and C_h is a $d \times d$ symmetric matrix.

From the literature (Ahn et al. (2002), Leippold and Wu (2012) and Kim (2014)), we know that if the risk-neutral dynamics of the state variates are Gaussian, then the functions $A(\tau)$, $\mathbf{B}(\tau)$ and $C(\tau)$ are the solution of a system of ordinary differential equations with initial condition $A(0) = 0$, $\mathbf{B}(0) = \underline{0}$, $C(0) = 0_{d\times d}$. Affine models can be obtained by forcing C_h to be a null matrix. For affine models, under certain regularity conditions, the functions $A(\tau)$ and $\mathbf{B}(\tau)$ are the solution of a system of $d+1$ ordinary differential equations that are completely determined by the specification of the risk-neutral dynamics of the short rate (see Duffie and Kan (1996) and Duffie, Pan and Singleton (2000) for further details). The solutions of these equations are known in closed form for most common affine models.

From Duffie, Pan and Singleton (2000) and Kim (2014), we know that the quadratic T-forward joint characteristic function of the model factors $\mathbf{X}$ has the form

$$\begin{aligned}\Phi(\boldsymbol{\lambda}, \Lambda) &= \mathbb{E}_t^T\left[e^{\boldsymbol{\lambda}^\top \mathbf{X}(T)+\mathbf{X}(T)^\top \Lambda \mathbf{X}(T)}\right] \\ &= e^{\tilde{A}(T-t,\boldsymbol{\lambda},\Lambda)-A(T-t)+(\tilde{\mathbf{B}}(T-t,\boldsymbol{\lambda},\Lambda)-\mathbf{B}(T-t))^\top \mathbf{X}(t)+\mathbf{X}(t)^\top(\tilde{C}(T-t,\boldsymbol{\lambda},\Lambda)-C(T-t))\mathbf{X}(t)}\end{aligned} \tag{6}$$

where $\boldsymbol{\lambda} \in \mathbb{C}^d$ and Λ is a complex $d \times d$ symmetric matrix. If $\mathbf{X}(t)$ is a Gaussian quadratic process (or an affine process, i.e. Λ, $\tilde{C}$ and C are null matrices), the functions $\tilde{A}(\tau, \boldsymbol{\lambda}, \Lambda)$, $\tilde{\mathbf{B}}(\tau, \boldsymbol{\lambda}, \Lambda)$ and $\tilde{C}(\tau, \boldsymbol{\lambda}, \Lambda)$ are the solutions of the same ODE system of the zero-coupon bond functions, but with initial conditions $\tilde{A}(0, \boldsymbol{\lambda}, \Lambda) = 0$, $\tilde{\mathbf{B}}(0, \boldsymbol{\lambda}, \Lambda) = \boldsymbol{\lambda}$, and $\tilde{C}(0, \boldsymbol{\lambda}, \Lambda) = \Lambda$.

In the case of a quadratic model, it is convenient to define the approximate exercise region $\mathcal{G}$ using a quadratic form of the state vector,

$$\mathcal{G} = \{\omega \in \Omega : \mathbf{X}(T)^\top \Gamma\, \mathbf{X}(T) + \boldsymbol{\beta}^\top \mathbf{X}(T) \geq k\},$$

where Γ is a constant $d \times d$ symmetric matrix, $\boldsymbol{\beta} \in \mathbb{R}^d$ and $k \in \mathbb{R}$.

Proposition 2.1. The lower bound to the European swaption price for quadratic interest rate models is given by the following formula:

$$LB(t) = \max_{k\in\mathbb{R},\, \boldsymbol{\beta}\in\mathbb{R}^d,\, \Gamma\in Sym_d(\mathbb{R})} LB_{\boldsymbol{\beta},\Gamma}(k;t), \tag{7}$$

where

$$LB_{\beta,\Gamma}(k;t) \quad = \quad P(t,T)\,\frac{e^{-\delta k}}{\pi}\int_0^{+\infty} Re\left(e^{-i\gamma k}\psi(\delta+i\gamma)\right)d\gamma, \tag{8}$$

and

$$\psi(z) = \left(\sum_{h=1}^{n} w_h e^{a_h}\,\Phi\left(\mathbf{b}_h + z\boldsymbol{\beta}, C_h + z\Gamma\right) - \Phi\left(z\boldsymbol{\beta}, z\Gamma\right)\right)\frac{1}{z}, \tag{9}$$

with $\psi(z)$ defined for $Re(z) > 0$ for receiver swaptions and for $Re(z) < 0$ for payer swaptions. The integral in formula (8) must be interpreted as a Cauchy principal value integral and δ is a positive or negative constant for receiver or payer swaptions, respectively.

Proof: See Appendix A.

For two-factor affine interest rate models, Singleton and Umantsev (2002) propose to approximate the exercise boundary of an option on a coupon bond with a straight line that closely matches the exercise boundary where the conditional density of the model factors is concentrated. Kim (2014) improves on the S&U idea and considers three different types of approximation for the exercise region. We choose its approximation "A" because it appears to be the most accurate.[2] In the approximation "A", the approximated exercise region is obtained by a first-order Taylor expansion of the coupon bond price, which is defined as

$$B(\mathbf{X}(T)) = \sum_{h=1}^{n} w_h P(T, T_h), \tag{10}$$

around the point on the true exercise boundary where the density function of the model factors is largest. Moreover, Kim (2014) extends his approximation "A" to Gaussian quadratic interest rate models using a second-order Taylor expansion of the coupon bond. In this way, the optimization of the lower bound (formula (7)), which can be very expensive, is not performed. It is instead replaced by a preliminary search of the parameters Γ, $\boldsymbol{\beta}$ and k, which are chosen via the Taylor expansion of the coupon bond price.

In particular, for affine models, the first-order Taylor expansion of the coupon bond is a tangent hyperplane approximation. In fact, the approximated exercise boundary is defined as

$$\boldsymbol{\beta}^\top \mathbf{X}(T) + \alpha = 0,$$

with

$$\alpha = -\nabla B(\mathbf{X}^*)^\top \mathbf{X}^*, \quad \boldsymbol{\beta} = \nabla B(\mathbf{X}^*) \quad \text{and} \quad k = -\alpha. \tag{11}$$

Hence, it is a tangent hyperplane to the true exercise boundary at the point, $\mathbf{X}(T) = \mathbf{X}^*$, where the density function of the model factors is the largest. In order to calculate the point $\mathbf{X}^*$, we use the equation (2.20) of Kim (2014). A two-dimensional visualization of the approximate exercise region is shown in Figure 1.

[Figure 1 approximately here]

[2] The three approximations presented in Kim (2014) are lower bounds, as proved in section 2. Therefore, the most precise is the one that produces the highest price, which was not discussed in the Kim paper.

Once Γ, $\boldsymbol{\beta}$ and k are found, the Kim approximation requires the computation of $n+1$ forward probability $\mathbb{P}_t^{T_h}[\mathcal{G}]$, as in formula (3). This is done by performing $n+1$ one-dimensional Fourier inversions. In contrast, our lower bound is calculated as in formula (8), i.e. performing a single one-dimensional Fourier transform with respect to the parameter k.

3 Upper bound on swaption price

In this section, we define a new upper bound to swaption prices that is applicable to all affine and quadratic interest rate models. First of all, it is straightforward to see that for a lower bound defined by a generic approximated exercise set $\mathcal{G}$, the (undiscounted) approximation error is

$$\begin{aligned} & \frac{1}{P(t,T)}\left(C(t) - \widehat{LB}(t)\right) \\ = \;& \mathbb{E}_t^T[(B(\mathbf{X}(T)) - 1)^+] - \mathbb{E}_t^T[(B(\mathbf{X}(T)) - 1)I(\mathcal{G})] \\ = \;& \mathbb{E}_t^T[(B(\mathbf{X}(T)) - 1)^+ I(\mathcal{G}^c)] + \mathbb{E}_t^T[(1 - B(\mathbf{X}(T)))^+ I(\mathcal{G})] \\ = \;& \Delta_1 + \Delta_2, \end{aligned}$$

where $B(\mathbf{X}(T))$ is the coupon bond price defined as in formula (10). The previous formula for the approximation error is valid also for payer swaptions. In general, Δ_1 and Δ_2 are not explicitly computable. However, we can provide upper bounds ϵ_1 and ϵ_2 to them. Hence, an upper bound to the swaption price easily follows:

$$UB(t) = \widehat{LB}(t) + P(t,T)\,(\epsilon_1 + \epsilon_2), \tag{12}$$

for $\epsilon_1 \geq \Delta_1$ and $\epsilon_2 \geq \Delta_2$.

For every set of strikes $(K_1, ..., K_n)$ such that $\sum_{h=1}^n K_h = 1$, upper bounds to the errors are

$$\Delta_1 \leq \epsilon_1 = \sum_{h=1}^n \mathbb{E}_t^T[(w_h P(T,T_h) - K_h)^+ \, I(\mathcal{G}^c)], \tag{13}$$

$$\Delta_2 \leq \epsilon_2 = \sum_{h=1}^n \mathbb{E}_t^T[(K_h - w_h P(T,T_h))^+ \, I(\mathcal{G})], \tag{14}$$

where $P(T,T_h)$ is the price at time T of the zero-coupon bond with maturity T_h. However, without a proper choice of the strikes $(K_1, ..., K_n)$, the approximations can be very rough and so we want to find the values of $(K_1, ..., K_n)$ that reduce the error without performing a time-consuming multidimensional numerical minimization. Given that

$$
\begin{aligned}
(B(\mathbf{X}(T))-1)^{+} &= B(\mathbf{X}(T))\left(1-\frac{1}{B(\mathbf{X}(T))}\right)^{+} \\
&= \sum_{h=1}^{n} w_h P(T,T_h)\left(1-\frac{1}{B(\mathbf{X}(T))}\right)^{+} \\
&= \sum_{h=1}^{n}\left(w_h P(T,T_h)-\frac{w_h P(T,T_h)}{B(\mathbf{X}(T))}\right)^{+} \qquad (15)
\end{aligned}
$$

as $B(\mathbf{X}(T)) > 0$ and $w_h P(T,T_h) > 0 \ \forall \mathbf{X}(T)$, we note that the following equality holds:

$$
\mathbb{E}_t^T[(B(\mathbf{X}(T))-1)^{+}\, I(\mathcal{G}^c)] = \sum_{h=1}^{n} \mathbb{E}_t^T[(w_h P(T,T_h) - K_h(\mathbf{X}(T)))^{+}\ I(\mathcal{G}^c)],
$$

for

$$
K_h(\mathbf{X}(T)) = \frac{w_h P(T,T_h)}{B(\mathbf{X}(T))}.
$$

By similar reasoning, we also have:

$$
\mathbb{E}_t^T[(1-B(\mathbf{X}(T)))^{+}\, I(\mathcal{G})] = \sum_{h=1}^{n} \mathbb{E}_t^T[(K_h(\mathbf{X}(T)) - w_h P(T,T_h))^{+}\ I(\mathcal{G})].
$$

Hence, if in formula (13) and (14), we choose the strikes $(K_1, ..., K_n)$ in the following way:

$$
K_h = K_h(\mathbf{X}^*) = w_h\ P(T,T_h)|_{\mathbf{X}(T)=\mathbf{X}^*}, \qquad (16)
$$

then the equalities $\epsilon_1 = \Delta_1$ and $\epsilon_2 = \Delta_2$ hold in $\mathbf{X}(T) = \mathbf{X}^*$, the point on the true exercise boundary where the density function of the model factors is largest. The computation of $\mathbf{X}^*$ is explained in section 2.1.

This allows us to avoid a multidimensional optimization with respect to $(K_1, ...K_n)$.

3.1 Affine and Gaussian quadratic models

The following proposition explains how to compute the quantities ϵ_1 and ϵ_2 defined in expressions (13) and (14), and hence the upper bound in formula (12), using the Fourier Transform method.

Proposition 3.1. The upper bound to the European swaption price for quadratic interest rate models is given by the following formula:

$$
UB(t) = \widehat{LB}(t) + P(t,T)\,(\epsilon_1(-\alpha) + \epsilon_2(-\alpha)) \qquad (17)
$$

where

$$\epsilon_1(k) = \frac{1}{2\pi^2}\int_0^{+\infty} d\gamma\, Re\left(\int_{-\infty}^{+\infty} d\omega \sum_{h=1}^{n} w_h e^{a_h}\, e^{-(\delta+i\gamma)k} e^{-(\eta+i\omega)k_h}\, \psi_h(\delta+i\gamma, \eta+i\omega)\right),$$

$$\epsilon_2(k) = -\frac{1}{2\,\pi^2}\int_0^{+\infty} d\gamma\, Re\left(\int_{-\infty}^{+\infty} d\omega \sum_{h=1}^{n} w_h e^{a_h}\, e^{(\delta-i\gamma)k} e^{(\eta-i\omega)k_h}\, \psi_h(-\delta+i\gamma, -\eta+i\omega)\right),$$

and

$$\psi_h(z,y) = -\frac{\Phi\left(z\boldsymbol{\beta}+(y+1)\mathbf{b}_h, z\Gamma+(y+1)C_h\right)}{zy(y+1)}, \tag{18}$$

where $\widehat{LB}(t)$ is given in Proposition 2.1, $k_h = log(K_h) - log(w_h) - a_h$, K_h are defined in equation (16) and $\Phi(\boldsymbol{\lambda}, \Lambda)$ is defined in equation (6). The upper bound formula is valid for both receiver and payer swaptions. If $Re(z) < 0$ and $Re(y) > 0$, $\psi_h(z,y)$ is the double Fourier transform of

$$\mathbb{E}_t^T[(e^{\mathbf{b}_h{}^\top\mathbf{X}+\mathbf{X}^\top C_h\mathbf{X}} - e^{k_h})^+\, I(\mathbf{X}^\top\Gamma\mathbf{X} + \boldsymbol{\beta}^\top\mathbf{X} < k)],$$

and if $Re(z) > 0$ and $Re(y) < -1$, $\psi_h(z,y)$ is the transform of

$$\mathbb{E}_t^T[(e^{k_h} - e^{\mathbf{b}_h{}^\top\mathbf{X}+\mathbf{X}^\top C_h\mathbf{X}})^+\, I(\mathbf{X}^\top\Gamma\mathbf{X} + \boldsymbol{\beta}^\top\mathbf{X} > k)],$$

with $\delta > 0$, $\eta > 1$ constants.

Proof: See Appendix B.

We note some important mathematical features of the swaption pricing problem in the affine interest rate model case. In this set up, C_h and Γ are null matrices, which simplifies the upper bound formula. The coupon bond $B(\mathbf{X}(T))$ seen as a function of the model factors $\mathbf{X}(T)$ is convex as it is a positive linear combination of convex functions, the ZCBs. In fact, the zero-coupon price seen as a function of the state vector, i.e. $P(T, T_h) = e^{\mathbf{b}_h^\top\mathbf{X}(T)+a_h}$, is a convex function because it is composed of convex monotone functions, the exponential, and a linear function of $\mathbf{X}$. Thus, the convexity of the sub-level $\{B(\mathbf{X}(T)) \le 1\}$ ensues from the previous argument.

Choosing the tangent hyperplane approximation as the lower bound and resorting to the hyperplane separation theorem, it follows immediately that the approximate exercise region is included in the true region, as graphically illustrated in Figure 2 for a two-factor case,

$$\mathcal{G} = \{\boldsymbol{\beta}^\top\mathbf{X} + \alpha \ge 0\} \subseteq \{B(\mathbf{X}(T)) \ge 1\},$$

provided that α and $\boldsymbol{\beta}$ are defined as in formula (11).

[Figure 2 approximately here]

Hence, the separation theorem guarantees that Δ_2 is zero, which allows us to compute only the term ϵ_1 in Proposition 3.1.

It is possible to show that for one-factor affine interest rate models, the upper bound coincides with the Jamshidian (1989) formula.

4 Bounds for affine Gaussian specification

For the affine Gaussian model, the lower bound can be calculated analytically as follows:

$$LB_{\boldsymbol{\beta}}(k;t) = P(t,T)\,\omega\left(\sum_{h=1}^{n} w_h\, e^{a_h+\mathbf{b}_h^\top \boldsymbol{\mu}+\frac{1}{2}V_h+\frac{1}{2}d_h^2}\, N(\omega\,(d_h-d)) - N(-\omega\, d)\right),$$

where $\omega = 1$ for receiver swaptions and $\omega = -1$ for payer swaptions. The upper bound formula can be simplified to

$$\epsilon_1(k) = \int_{-\infty}^{d} dz\, \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} \sum_{h=1}^{n} w_h e^{a_h} \left(e^{M_h+\frac{V_h}{2}}\, N\left(\frac{M_h - \log Y_h + V_h}{\sqrt{V_h}}\right) - Y_h N\left(\frac{M_h - \log Y_h}{\sqrt{V_h}}\right)\right),$$

where $d = \frac{k-\boldsymbol{\beta}^\top \boldsymbol{\mu}}{\sqrt{\boldsymbol{\beta}^\top V \boldsymbol{\beta}}}$, $d_h = \mathbf{b}_h^\top \mathbf{v}$, $V_h = \mathbf{b}_h^\top (V - \mathbf{v}\mathbf{v}^\top)\mathbf{b}_h$, $\mathbf{v} = \frac{V\boldsymbol{\beta}}{\sqrt{\boldsymbol{\beta}^\top V \boldsymbol{\beta}}}$, $M_h = \mathbf{b}_h^\top \boldsymbol{\mu} + z\mathbf{b}_h^\top \mathbf{v}$, $Y_h = \frac{K_h}{w_h e^{a_h}}$ and $\mu = \mathbb{E}_t^T[\mathbf{X}(T)]$ and $V = Var_t(\mathbf{X}(T))$ are the mean and covariance matrix of the variable $\mathbf{X}(T)$ that is multivariate normal under the T-forward measure. $N(x)$ represents the standard Gaussian cumulative distribution function. Proofs of the simplified bounds are in Appendix C and D.

5 Approximate pricing of swaption in a multiple-curve framework

In this section, we extend the previously described lower and upper bounds to multiple-curve models, which better reflect the real behaviour of the interest rate market after the 2007 crisis. The (payer) swaption formula in the multi-curve framework becomes

$$C(t) = P(t,T)\,\mathbb{E}_t^T\left[\left(\sum_{j=1}^{n} P(T,T_j)\, x\,(F^x(T,T_j,x) - K)\right)^+\right] \tag{19}$$

where $x = T_j - T_{j-1}$ is the tenor $\forall j = 1, ..., n$ and $T_0 = T$. $F^x(t, T, x)$ is the fair rate of a FRA contract written on the Libor rate between $T - x$ and T and tenor x (usually $x =$ 1M, 3M, 6M or 12M). $P(t, T)$ is the price at time t of a risk-free zero-coupon bond with maturity T.

We test the lower and upper bounds with reference to the Gaussian specification of the multiple-curve model presented in Moreni and Pallavicini (2014). In this model, the FRA rate and the risk-free ZCB price have affine forms. The Markovian-affine representation of the FRA rate is

$$\log\left(\frac{1 + x\, F^x(t, T, x)}{1 + x\, F^x(0, T, x)}\right) = G(t, T, x)^\top\, \mathbf{X}(t) + a(t, T, x), \tag{20}$$

where $a(t, T, x)$ is a deterministic coefficient, $G(t, T, x)$ is a deterministic d-dimensional vector and $\mathbf{X}(t)$ is a vector of the Markovian process and it is multivariate normal. A similar Markovian representation can be obtained for the ZCB price:

$$\log\left(P(t, T)\frac{P(0, t)}{P(0, T)}\right) = -G(t, T)^\top\, \mathbf{X}(t) + a(t, T), \tag{21}$$

where $a(t, T)$ is a deterministic coefficient and $G(t, T)$ is a deterministic d-dimensional vector.

More model details are given in Appendix H.

5.1 Lower bound formula applied to a multi-curve weighted Gaussian model

Using the Markovian representation of the FRA rate and of the risk-free ZCBs in the swaption pricing formula (19), we obtain:

$$C(t) = P(t, T)\, \mathbb{E}_t^T\left[\left(\sum_{j=1}^{n} w_{1j}\, e^{(G_{1j})^\top \mathbf{X}(T) + a_{1j}} - w_{2j}\, e^{(G_{2j})^\top \mathbf{X}(T) + a_{2j}}\right) \mathrm{I}(\mathcal{A})\right],$$

where

$\mathcal{A}$ is the exercise region and is in the form

$$\mathcal{A} = \{\omega \in \Omega : \sum_{j=1}^{n} w_{1j}\, e^{(G_{1j})^\top \mathbf{X}(T) + a_{1j}} - w_{2j}\, e^{(G_{2j})^\top \mathbf{X}(T) + a_{2j}} > 0\},$$

$w_{1j} = \frac{P(t, T_j)}{P(t, T)}(1 + x\, F^x(t, T_j, x))$ and $w_{2j} = \frac{P(t, T_j)}{P(t, T)}(1 + xK)$,

$G_{1j} = G(T, T_j, x) - G(T, T_j)$ and $G_{2j} = -G(T, T_j)$,

$a_{1j} = a(T, T_j, x) + a(T, T_j)$ and $a_{2j} = a(T, T_j)$.

If we substitute the set $\mathcal{A}$ with any other event set $\mathcal{G} \in \Omega$, we obtain a lower bound of the true price. In the affine class models, it is convenient to define the set $\mathcal{G}$ using a linear function of the state variates,

$$\mathcal{G} = \{\omega \in \Omega : \boldsymbol{\beta}^\top \mathbf{X}(T) \geq k\},$$

with $\boldsymbol{\beta}$ and α defined in formula (11). The lower bound is provided in the following proposition.

Proposition 5.1. The lower bound to the European swaption price, for the multiple-curve weighted Gaussian model, is given by the following formula:

$$\widehat{LB}(t) = \max_{k \in \mathbb{R},\, \beta \in \mathbb{R}^d} LB_\beta(k;t). \tag{22}$$

For fixed parameters k and β, the lower bound is

$$\begin{aligned} LB_\beta(k;t) &= P(t,T)\,\omega \sum_{j=1}^{n} \left(w_{1j}\ \exp\left((G_{1j})^\top \mu + a_{1j} + \frac{1}{2} V_{1j}^G + \frac{1}{2}(d_{1j})^2 \right) \mathcal{N}(\omega\,(d_{1j} - d)) \right. \\ &- \left. w_{2j}\ \exp\left((G_{2j})^\top \mu + a_{2j} + \frac{1}{2} V_{2j}^G + \frac{1}{2}(d_{2j})^2 \right) \mathcal{N}(\omega\,(d_{2j} - d)) \right), \end{aligned} \tag{23}$$

where $\omega = -1$ for receiver swaption and $\omega = 1$ for payer swaption, $d = \frac{k - \boldsymbol{\beta}^\top \boldsymbol{\mu}}{\sqrt{\boldsymbol{\beta}^\top V \boldsymbol{\beta}}}$, $d_{ij} = (G_{ij})^\top v$ for $i = 1,2$ and $j = 1,...,d$, $v = \frac{V\beta}{\sqrt{\beta^\top V \beta}}$, $V_{ij}^G = (G_{ij})^\top (V - v\,v^\top)\,G_{ij}$ for $i = 1,2$ and $j = 1,...,d$ and $\mu = \mathbb{E}_t^T[\mathbf{X}(T)]$ and $V = Var_t(\mathbf{X}(T))$ are the mean and covariance matrix of the variable $\mathbf{X}(T)$, which is multivariate normal under the T-forward measure.

Proof: See Appendix E.

5.2 Upper bound formula applied to a multi-curve weighted Gaussian model

In a multiple-curve framework, the swaption price can also be written as

$$C(t) = P(t,T)\,\mathbb{E}_t^T\left[(B_1(\mathbf{X}(T)) - B_2(\mathbf{X}(T)))^+ \right] \tag{24}$$

where

$$\begin{aligned} B_1(\mathbf{X}(T)) &= \sum_{j=1}^{n} P(T,T_j)\,(1 + x\,F^x(T,T_j,x)) = \sum_{j=1}^{n} w_{1j}\, e^{(G_{1j})^\top \mathbf{X}(T) + a_{1j}}, \\ B_2(\mathbf{X}(T)) &= (1 + x\,K) \sum_{j=1}^{n} P(T,T_j) = \sum_{j=1}^{n} w_{2j}\, e^{(G_{2j})^\top \mathbf{X}(T) + a_{2j}}. \end{aligned}$$

Hence, the (undiscounted) approximation error of the lower bound defined in Proposition 5.1 is

$$\begin{aligned}
& \frac{1}{P(t,T)}\left(C(t) - \widehat{LB}(t)\right) \\
= \;& \mathbb{E}_t^T[(B_1(\mathbf{X}(T)) - B_2(\mathbf{X}(T)))^+ I(\mathcal{G}^c)] + \mathbb{E}_t^T[(B_2(\mathbf{X}(T)) - B_1(\mathbf{X}(T)))^+ I(\mathcal{G})] \\
= \;& \Delta_1 + \Delta_2.
\end{aligned}$$

The previous equality holds for both receiver and payer swaptions. Applying the same reasoning as in the single-curve case, we find that the upper bound is

$$UB(t) = \hat{LB}(t) + P(t,T)(\epsilon_1 + \epsilon_2), \tag{25}$$

where ϵ_1 and ϵ_2 are the upper bounds for Δ_1 and Δ_2 and their expressions are as follows:

$$\begin{aligned}
\epsilon_1 &= \sum_{j=1}^{n} \mathbb{E}_t^T[P(T,T_j)\,(1 + x\,F^x(T,T_j,x) - K_j)^+\, I(\mathcal{G}^c)] \\
&= \sum_{j=1}^{n} \mathbb{E}^T\left[\left(w_{1j}\, e^{G_{1j}^\top \mathbf{X}(T) + a_{1j}} - \tilde{w}_{2j}\, e^{G_{2j}^\top \mathbf{X}(T) + a_{2j}}\right)^+ \; I(\mathcal{G}^c)\right], && (26) \\
\epsilon_2 &= \sum_{j=1}^{n} \mathbb{E}_t^T[P(T,T_j)\,(K_j - 1 - x\,F^x(T,T_j,x))^+\, I(\mathcal{G})] \\
&= \sum_{j=1}^{n} \mathbb{E}^T\left[\left(\tilde{w}_{2j}\, e^{G_{2j}^\top \mathbf{X}(T) + a_{2j}} - w_{1j}\, e^{G_{1j}^\top \mathbf{X}(T) + a_{1j}}\right)^+ \; I(\mathcal{G})\right], && (27)
\end{aligned}$$

where $\tilde{w}_{2j} = \frac{P(t,T_j)}{P(t,T)} K_j$ and

$$K_j = 1 + x\,F(T,T_j,x)|_{\mathbf{X}(T)=\mathbf{X}^*}, \tag{28}$$

where $\mathbf{X}^*$ is the point on the true exercise boundary (i.e. $B_1(\mathbf{X}(T)) - B_2(\mathbf{X}(T) = 0))$ where the density function of the model factors is largest.

Proposition 5.2. The upper bound to the European swaption price for the multiple-curve weighted Gaussian model is given by the following formula:

$$UB(t) = \widehat{LB}(t) + P(t,T)\,(\epsilon_1(-\alpha) + \epsilon_2(-\alpha)), \tag{29}$$

where

$$
\begin{aligned}
\epsilon_1(k) &= \int_{-\infty}^{d} dz\, \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} \sum_{j=1}^{n} w_{1j}\, e^{a_{1j}+M_{1j}+\frac{1}{2}V_{1j}^G}\, N\left(d_{1j}\right) - \tilde{w}_{2j}\, e^{a_{2,j}+M_{2j}+\frac{1}{2}V_{2j}^G}\, N\left(d_{2j}\right), \\
d_{1j} &= \frac{\log\left(\frac{w_{1j}}{\tilde{w}_{2j}}\right) + M_{1j} + a_{1j} - M_{2j} - a_{2j} + V_{1j}^G - Cov_j}{\sqrt{V_{1j}^G + V_{2j}^G - 2Cov_j}}, \\
d_{2j} &= d_{1j} - \sqrt{V_{1j}^G + V_{2j}^G - 2Cov_j}, \\
\epsilon_2(k) &= \int_{d}^{+\infty} dz\, \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} \sum_{j=1}^{n} \tilde{w}_{2j}\, e^{a_{2j}+M_{2j}+\frac{1}{2}V_{2j}^G}\, N\left(\delta_{1j}\right) - w_{1j}\, e^{a_{1j}+M_{1j}+\frac{1}{2}V_{1j}^G}\, N\left(\delta_{2j}\right), \\
\delta_{1j} &= \frac{-\log\left(\frac{w_{1j}}{\tilde{w}_{2j}}\right) - M_{1j} - a_{1j} + M_{2j} + a_{2j} + V_{2j}^G - Cov_j}{\sqrt{V_{1j}^G + V_{2j}^G - 2Cov_j}}, \\
\delta_{2j} &= \delta_{1j} - \sqrt{V_{1j}^G + V_{2j}^G - 2Cov_j},
\end{aligned}
$$

and $\widehat{LB}(t)$ is given in Proposition 5.1, $d = \frac{k-\beta^\top\mu}{\sqrt{\beta^\top V\beta}}$, $V_{ij}^G = G_{ij}^\top(V - vv^\top)G_{ij}$ and $Cov_j = G_{1j}^\top(V - vv^\top)G_{2j}$ for $i = 1, 2$ and $j = 1, ..., d$, $M_{ij} = G_{ij}^\top\mu + zG_{ij}^\top v$ for $i = 1, 2$ and $j = 1, ..., d$ $v = \frac{V\beta}{\sqrt{\beta^\top V\beta}}$, and $\mu = \mathbb{E}_t^T[\mathbf{X}(T)]$ and $V = Var_t(\mathbf{X}(T))$ are the mean and covariance matrix of the variable $\mathbf{X}(T)$, which is multivariate normal under the T-forward measure and $N(x)$ is the standard Gaussian cumulative distribution function. The upper bound formula holds for both receiver and payer swaption.

Proof: See Appendix F.

6 Numerical results

For each model, we fix a set of parameters and we calculate a matrix of swaption prices with different maturities, swap lengths and three different strikes, i.e. ATMF (at-the-money forward), ITMF (0.85 × ATMF for affine models and ATMF - 0.75% for the quadratic model) and OTMF (1.15 × ATMF for affine models and ATMF + 0.75% for the quadratic model). This is a common choice in the literature (see, for instance, Schrager and Pelsser (2006), Singleton and Umantsev (2002) and Kim (2014)). The description and values of the parameters for each model are reported, respectively, in Appendix G and I. The tested models are a three-factor affine Gaussian model, a two-factor affine Cox, Ingersoll and Ross (CIR) model, a two-factor affine Gaussian model with double exponential jumps, a two-factor Gaussian quadratic model and a two-factor affine multiple-curve Gaussian model.

Monte Carlo is used as a benchmark for the computation of the true swaption price. The 97.5% mean-centred Monte Carlo confidence interval is used as a measure of the accuracy. For the affine three-factor Gaussian model, we add as a benchmark the lower bound proposed in Nunes and Prazeres (2014), which is extremely accurate.

For the affine three-factor Gaussian model and the Gaussian multi-curve model, the lower bounds are obtained via the closed formula described in sections 4 and 5.1. Kim's prices are calculated using the closed price formula for the T-forward probabilities (formula (3.9) and (3.16), Kim (2014)). For the two-factor CIR model, the Gaussian model with jumps and the Gaussian quadratic model, the integrals involved in the lower bound and in Kim's method are evaluated by a Gauss-Kronrod quadrature rule using Matlab's built-in function **quadgk**.

The Matlab function **quadgk** is also used for the integral appearing in the upper bound formula for the three-factor Gaussian model and for the Gaussian multi-curve model (see section 4 and 5.2). For the two-factor CIR model, the Gaussian model with jumps and the Gaussian quadratic model, the upper bound formula requires the calculus of double integrals that are evaluated using Matlab's function **quad2d**, an iterative algorithm that divides the integration region into quadrants and approximates the integral over each quadrant by a two-dimensional Gauss quadrature rule.

Another important fact is that our lower bound formula is suitable for use as a control variate to reduce the Monte Carlo simulation error. The approximated formula is easily implemented in a Monte Carlo scheme and turns out to be very effective. In this way, the simulation error is considerably reduced.

Numerical results obtained with parameters reported in Appendix I are shown in Tables 1-5. Computational time for each pricing method is also given in Table 7.

6.1 Test with random parameters

In this section, we test the robustness of the bounds' approximation to parameter changes. We use 100 randomly simulated parameters for the two-factor CIR model. The model parameters are independent and uniformly distributed within a reasonable range, which is shown in Appendix I.

For each set of simulated parameters, we calculate a matrix of swaption prices with different maturities and swap lengths and three different strikes, i.e. ATM, ITMF ($0.85 \times$ ATMF) and OTMF ($1.15 \times$ ATMF).

For each swaption, we calculate the root mean square deviation (RMSD) of the lower and upper bounds with respect to the Monte Carlo estimation, which is used as the benchmark:

$$RMSD = \frac{1}{\sqrt{N}}\sqrt{\frac{\sum_{i=1}^{N}(B_i - MC_i)^2}{(MC_{avg})^2}}, \quad MC_{avg} = \frac{\sum_{i=1}^{N} MC_i}{N},$$

where N is the number of random trials, $B_i = LB_i$ or $B_i = UB_i$ (lower or upper bound) and MC_i is the Monte Carlo estimation of the swaption price with the i^{th} set of random parameters and MC_{avg} is the average of Monte Carlo prices over all random trials. Monte Carlo values are estimated using 10^7 simulations. Numerical results of this test are shown in Table 6.

6.2 Comments on numerical results

Numerical results are presented across a wide class of affine models for the Gaussian quadratic model and for a multiple-curve model. The tangent hyperplane lower bound and the approximation "A" of Kim (2014) produce the same prices because they are two different implementations of the same approximation. However, the new algorithm, which requires the computation of a single Fourier inversion, is faster across all models for which the characteristic function is known in its closed form. In fact, in Table 7, our implementation of the lower bound is faster than Kim's method except for the Gaussian quadratic model for which the characteristic function is available in a semi-analytical form (see Appendix G). The improvement in computational performance is more evident for swaptions with a large number of cash flows, as illustrated in Table 8. For the three factor Gaussian affine model, Nunes and Prazeres (2014) conditioning approach is more efficient than our bounds, however our aim is to find approximations that are applicable to a wider class of models and not only to Gaussian affine models. Comparing the speed of different methods is not simple because each algorithm should be optimized. However, our considerations about the efficiency of an algorithm are also justified by theoretical reasoning and confirmed by our estimations of the computational time.

Our upper bound is applicable to all affine-quadratic models, both in single- and multiple-curve frameworks, and it is particularly efficient for affine models. In the literature, upper bounds are available only for Gaussian affine models. In particular, for the three-factor affine Gaussian model, we compare our bounds with the ones proposed by Nunes and Prazeres (2014). Lower bound proposed by Nunes and Prazeres (2014) is comparable to our lower bound for all maturities and strikes. We find that our upper bound is less accurate for ATMF options but it seems to be more accurate for OTMF options (see Table 1). We observe that for the given set of parameters, price estimated using our bounds and the conditioning approach are very close. On the other hand, with reference to computational time (see Table 7), Nunes and Prazeres (2014) approach is more efficient than our bounds. However, our aim is to find approximations that are applicable to a wider class of models and not only to Gaussian affine models.

The computation of the upper bound is slower than the lower bound calculation, but it is still faster than Monte Carlo simulations for a comparable accuracy (see Table 7). In addition, the range between the lower and upper bound is always narrow so, in practice, the combined use of the two bounds provides an accurate estimate of the true price.

For the multiple-curve model, we compare our bounds with an approximate method that is widely used in the market, i.e. the freezing drift approximation (see Moreni and Pallavicini (2014)) and we find that the lower and upper bounds perform better for swaptions with long maturities (2Y and 5Y in Table 5) with comparable computational times. Moreover, the freezing technique is a generic approximation, i.e. we cannot know a priori if the approximated price underestimates or overestimates the true price.

In each table we compute the mean absolute percentage error (MAPE) of bounds with respect to Monte Carlo prices, taken as a benchmark, for fixed maturity and strike.

The RMSD computation performed for the two-factor CIR model and reported in Table 6 is an important validation for the stability of the accuracy of the bounds to changes in the parameter set. The RMSD of the lower bound for at-the-money and in-the-money options is less than 0.1% of the Monte Carlo average price, which is a good result. The relative error is larger for out-of-the-money options, in particular for the swaptions with a long swap length. Indeed, the maximum error is around 0.3% of the Monte Carlo price. The RMSDs of the upper bound are greater than the RMSDs of the lower bound, in particular for swaptions with longer swap lengths. However, the maximum RMSD of the upper bound is about 0.8% of the Monte Carlo price, which is also confirmation of the good performance of the upper bound.

Conclusions

In this paper, we propose a general lower bound formula of the swaption price based on an approximation of the exercise region. We note that previous approximations, such as the Kim (2014) and Singleton and Umantsev (2002) methods, represent a particular case of our general formula and so they can also be interpreted as lower bounds. Moreover, we provide a new algorithm to implement the lower bound that is found to be more efficient for interest rate models in which the joint characteristic function of state variables is known in analytical form. Further, this work provides a new upper bound to swaption prices that is applicable to all affine-quadratic models and that is accurate and computable in a reasonable time. Therefore, the lower bound approximation error is controlled. Finally, we extend lower and upper bounds to multiple-curve models. Numerical results confirm our hypothesis about the performance of the new algorithm in terms of computational times for the calculus of the lower bound, except for quadratic models in which the characteristic function is not analytic. Moreover, numerical tests show a very good accuracy of the new upper bound for different models across tenors, maturities and strikes.

7 Tables

Three-factor Gaussian model

Opt. Mat.	1					2					5				
Swap length	**MC**	**LB (HP)**	**UB**	**LB (CA)**	**UB (CA)**	**MC**	**LB (HP)**	**UB**	**LB (CA)**	**UB (CA)**	**MC**	**LB (HP)**	**UB**	**LB (CA)**	**UB (CA)**
1	20.817	20.817	20.818	20.817	20.817	23.555	23.554	23.555	23.554	23.555	23.207	23.207	23.208	23.207	23.207
	0.001		0.001		0.000	0.001		0.001		0.0001	0.001		0.001		0.0001
2	33.119	33.119	33.129	33.119	33.120	38.434	38.434	38.444	38.434	38.435	38.723	38.722	38.730	38.722	38.724
	0.002		0.010		0.001	0.002		0.010		0.002	0.002		0.008		0.002
5	53.312	53.312	53.396	53.312	53.323	63.688	63.686	63.764	63.686	63.702	65.684	65.683	65.741	65.683	65.700
	0.002		0.084		0.011	0.003		0.078		0.015	0.003		0.058		0.017
10	65.583	65.584	65.758	65.584	65.613	79.068	79.067	79.224	79.067	79.106	82.161	82.159	82.273	82.159	82.201
	0.003		0.174		0.029	0.004		0.157		0.039	0.004		0.114		0.042
MAPE		0.0005%	0.115%	0.0005%	0.018%		0.002%	0.086%	0.002%	0.018%		0.001%	0.061%	0.001%	0.019%
Swap length	**MC**	**LB (HP)**	**UB**	**LB (CA)**	**UB (CA)**	**MC**	**LB (HP)**	**UB**	**LB (CA)**	**UB (CA)**	**MC**	**LB (HP)**	**UB**	**LB (CA)**	**UB (CA)**
1	79.4449	79.4449	79.4451	79.4449	79.4449	78.4043	78.4039	78.4042	78.4039	78.4040	69.4421	69.4420	69.4423	69.4420	69.4421
	0.0003		0.0002		0.0001	0.0005		0.0003		0.0001	0.0005		0.0003		0.0001
2	154.5632	154.5632	154.5646	154.5632	154.5642	150.9113	150.9108	150.9131	150.9108	150.9121	131.9486	131.9485	131.9511	131.9485	131.9500
	0.0003		0.0014		0.0010	0.0005		0.0023		0.0013	0.0007		0.0026		0.0015
5	361.4695	361.4695	361.4713	361.4695	361.4793	346.2753	346.2753	346.2813	346.2753	346.2884	295.1619	295.1619	295.1717	295.1619	295.1762
	0.0001		0.0018		0.0098	0.0003		0.0061		0.0132	0.0006		0.0098		0.0143
10	636.9818	636.9818	636.9819	636.9818	637.0074	604.8099	604.8101	604.8113	604.8101	604.8443	508.8398	508.8398	508.8444	508.8398	508.8766
	0.0001		0.0001		0.0256	0.0002		0.0013		0.0342	0.0003		0.0046		0.0368
MAPE		0.00002%	0.0004%	0.00002%	0.0019%		0.0002%	0.0008%	0.0002%	0.0026%		0.0001%	0.0016%	0.0001%	0.0033%
Swap length	**MC**	**LB (HP)**	**UB**	**LB (CA)**	**UB (CA)**	**MC**	**LB (HP)**	**UB**	**LB (CA)**	**UB (CA)**	**MC**	**LB (HP)**	**UB**	**LB (CA)**	**UB (CA)**
1	1.5700	1.5700	1.5703	1.5700	1.5701	2.8242	2.8238	2.8242	2.8238	2.8239	3.7940	3.7938	3.7943	3.7938	3.7940
	0.0003		0.0003		0.0001	0.0005		0.0005		0.0002	0.0006		0.0005		0.0002
2	1.0649	1.0648	1.0671	1.0648	1.0661	2.6128	2.6123	2.6162	2.6123	2.6141	4.3224	4.3223	4.3265	4.3223	4.3242
	0.0003		0.0022		0.0013	0.0006		0.0039		0.0018	0.0008		0.0043		0.0020
5	0.1496	0.1495	0.1523	0.1495	0.1624	0.9049	0.9048	0.9141	0.9048	0.9222	2.5697	2.5696	2.5839	2.5696	2.5884
	0.0001		0.0027		0.0129	0.0004		0.0093		0.0174	0.0007		0.0144		0.0189
10	0.00268	0.00268	0.00281	0.00268	-	0.0756	0.0756	0.0776	0.0756	0.1199	0.5166	0.5166	0.5231	0.5166	0.5643
	0.00002		0.0001		-	0.0001		0.0019		0.0443	0.0003		0.0065		0.0477
MAPE		0.0327%	1.788%	0.032%	2.914%		0.017%	0.928%	0.017%	15.146%		0.003%	0.481%	0.003%	2.503%

Table 1: The three tables represent matrices of payer swaption prices for the three-factor Gaussian model at three different strikes, i.e. ATMF, ITMF ($0.85 \times$ ATMF) and OTMF ($1.15 \times$ ATMF). For each swaption, we report the price in basis points estimated with the Monte Carlo method, MC, the hyperplane approximation lower bound, LB (HP), the upper bound, UB, and the lower and upper bounds obtained with the conditional approach of Nunes and Prazeres (2014), LB (CA) and UB(CA). Monte Carlo prices are estimated using 10^9 simulations, the antithetic variates method and the exact probability distribution. Below each Monte Carlo price, the size of the confidence interval at 97.5% is reported in basis points. The distance between the lower and the upper bounds is provided below each upper bound value.

Two-factor Cox-Ingersoll-Ross model

Opt. Mat.	1					2					5				
Swap length	**MC**	**LB (HP)**	**UB**	**MC (CV)**	**Kim**	**MC**	**LB (HP)**	**UB**	**MC (CV)**	**Kim**	**MC**	**LB (HP)**	**UB**	**MC (CV)**	**Kim**
1	48.469 0.004	48.466	48.467 0.001	48.466 10^{-4}	48.466	59.359 0.005	59.361	59.362 0.001	59.361 10^{-4}	59.361	66.979 0.006	66.970	66.971 0.001	66.970 10^{-4}	66.970
2	85.876 0.008	85.871	85.883 0.012	85.871 10^{-4}	85.871	106.885 0.010	106.890	106.904 0.014	106.890 10^{-4}	106.890	123.847 0.011	123.830	123.842 0.012	123.830 10^{-4}	123.830
5	169.437 0.016	169.428	169.639 0.211	169.428 10^{-4}	169.428	216.868 0.020	216.880	217.103 0.222	216.881 10^{-4}	216.880	261.403 0.025	261.368	261.525 0.157	261.368 10^{-4}	261.368
10	265.834 0.025	265.820	266.634 0.814	265.818 0.004	265.820	344.969 0.033	344.990	345.795 0.805	344.992 0.005	344.990	422.939 0.042	422.883	423.408 0.525	422.887 0.004	422.883
MAPE		0.005%	0.108%				0.005%	0.093%				0.013%	0.044%		
Swap length	**MC**	**LB (HP)**	**UB**	**MC (CV)**	**Kim**	**MC**	**LB (HP)**	**UB**	**MC (CV)**	**Kim**	**MC**	**LB (HP)**	**UB**	**MC (CV)**	**Kim**
1	107.578 0.002	107.577	107.578 0.001	107.577 10^{-4}	107.577	116.838 0.003	116.839	116.840 0.001	116.839 10^{-4}	116.839	114.936 0.004	114.930	114.931 0.001	114.930 10^{-4}	114.930
2	208.039 0.004	208.037	208.045 0.008	208.037 10^{-4}	208.037	222.361 0.006	222.363	222.373 0.010	222.363 10^{-4}	222.363	217.219 0.008	217.208	217.217 0.009	217.208 10^{-4}	217.208
5	475.672 0.007	475.669	475.782 0.113	475.669 10^{-4}	475.669	493.297 0.011	493.301	493.454 0.152	493.301 10^{-4}	493.301	473.353 0.016	473.331	473.455 0.124	473.331 10^{-4}	473.331
10	812.488 0.009	812.482	812.917 0.435	812.482 0.002	812.482	825.213 0.017	825.218	825.772 0.554	825.219 0.003	825.218	778.608 0.028	778.572	778.993 0.420	778.573 0.002	778.572
MAPE		0.001%	0.020%				0.001%	0.027%				0.005%	0.019%		
Swap length	**MC**	**LB (HP)**	**UB**	**MC (CV)**	**Kim**	**MC**	**LB (HP)**	**UB**	**MC (CV)**	**Kim**	**MC**	**LB (HP)**	**UB**	**MC (CV)**	**Kim**
1	15.977 0.006	15.973	15.973 0.001	15.973 10^{-4}	15.973	24.445 0.007	24.446	24.447 0.001	24.446 10^{-4}	24.446	34.558 0.008	34.546	34.547 0.001	34.546 10^{-4}	34.546
2	23.730 0.011	23.724	23.733 0.009	23.724 10^{-4}	23.724	39.961 0.013	39.964	39.977 0.013	39.964 10^{-4}	39.964	61.860 0.015	61.838	61.849 0.011	61.838 10^{-4}	61.838
5	33.576 0.024	33.567	33.698 0.132	33.567 10^{-4}	33.567	68.731 0.029	68.742	68.918 0.176	68.742 0.001	68.742	124.445 0.032	124.398	124.546 0.148	124.399 0.001	124.398
10	42.469 0.039	42.458	42.873 0.414	42.459 0.005	42.458	99.027 0.048	99.049	99.626 0.577	99.045 0.005	99.049	196.344 0.056	196.265	196.742 0.476	196.266 0.005	196.265
MAPE		0.026%	0.337%				0.013%	0.232%				0.037%	0.083%		

Table 2: The three tables represent matrices of payer swaption prices for the two-factor CIR model at three different strikes, i.e. ATMF, ITMF ($0.85 \times$ ATMF) and OTMF ($1.15 \times$ ATMF). For each swaption, we report the price in basis points estimated with the Monte Carlo method, MC, the hyperplane approximation lower bound, LB (HP), the upper bound, UB, the Monte Carlo with control variable technique, MC (CV), and the approximation "A" of Kim (2014). Monte Carlo without and with control variable are estimated using 10^9 and 10^5 simulations, respectively, and the exact probability distribution. Below each Monte Carlo price, the size of the confidence interval at 97.5% is reported in basis points. The distance between the lower and the upper bounds is provided below each upper bound value.

Two-factor Gaussian model with exponential jump sizes

Opt. Mat.	1					2					5				
Swap length	**MC**	**LB (HP)**	**UB**	**MC (CV)**	**Kim**	**MC**	**LB (HP)**	**UB**	**MC (CV)**	**Kim**	**MC**	**LB (HP)**	**UB**	**MC (CV)**	**Kim**
1	403.88	403.77	403.77	403.77	403.77	434.32	434.24	434.24	434.24	434.24	473.22	473.45	473.48	473.45	473.45
	0.37		0.001	0.00001		0.39		0.005	0.00001		0.44		0.03	0.00001	
2	546.54	546.17	546.39	546.17	546.17	678.54	677.82	678.33	677.82	677.82	926.64	925.17	926.28	925.17	925.17
	0.61		0.22	0.00001		0.73		0.50	0.00004		0.91		1.11	0.0001	
5	857.58	856.93	859.96	856.93	856.93	1238.63	1236.99	1240.84	1236.99	1236.99	1986.48	1983.83	1987.31	1983.84	1983.83
	1.02		3.03	0.001		1.28		3.86	0.001		1.64		3.48	0.002	
10	963.67	962.96	964.51	962.96	962.96	1426.94	1425.19	1426.95	1425.19	1425.19	2341.44	2339.51	2340.84	2339.51	2339.51
	1.13		1.55	0.0001		1.42		1.77	0.0003		1.83		1.33	0.0003	
MAPE		0.06%	0.11%				0.10%	0.06%				0.11%	0.04%		
Swap length	**MC**	**LB (HP)**	**UB**	**MC (CV)**	**Kim**	**MC**	**LB (HP)**	**UB**	**MC (CV)**	**Kim**	**MC**	**LB (HP)**	**UB**	**MC (CV)**	**Kim**
1	428.3859	428.16	428.52	428.16	428.16	458.3401	458.37	458.37	458.37	458.37	496.7178	496.84	496.96	496.84	496.84
	0.38		0.36	0.00001		0.401507		0.00	0.00001		0.446931		0.12	0.00001	
2	640.6753	640.35	640.45	640.35	640.35	768.0223	768.01	768.30	768.01	768.01	1011.088	1010.49	1011.31	1010.49	1010.49
	0.64		0.10	0.00002		1.071384		0.29	0.0001		0.940232		0.81	0.00004	
5	1360.17	1359.44	1360.83	1359.44	1359.44	1700.289	1698.72	1700.66	1698.72	1698.72	2399.671	2396.88	2399.13	2396.88	2396.88
	1.076925		1.39	0.0005		1.335131		1.94	0.001		1.707953		2.24	0.001	
10	2500.03	2499.24	2501.74	2499.24	2499.24	2792.621	2790.91	2793.15	2790.91	2790.91	3459.408	3457.46	3458.80	3457.46	3457.46
	1.115307		2.51	0.0002		1.41797		2.24	0.0004		1.33787		1.35	0.0002	
MAPE		0.05%	0.05%				0.04%	0.02%				0.06%	0.03%		
Swap length	**MC**	**LB (HP)**	**UB**	**MC (CV)**	**Kim**	**MC**	**LB (HP)**	**UB**	**MC (CV)**	**Kim**	**MC**	**LB (HP)**	**UB**	**MC (CV)**	**Kim**
1	243.47	243.35	243.38	243.35	243.35	273.48	273.41	273.43	273.41	273.41	314.50	314.82	314.91	314.82	314.82
	0.30		0.02	0.00001		0.32		0.02	0.00001		0.36		0.09	0.00001	
2	159.23	158.92	159.82	158.92	158.92	261.82	261.00	262.25	261.00	261.00	460.58	459.27	461.01	459.27	459.27
	0.40		0.89	0.0001		0.51		1.25	0.0004		0.68		1.74	0.001	
5	136.63	136.28	137.77	136.28	136.28	265.11	264.13	267.07	264.13	264.13	591.07	589.36	591.92	589.36	589.36
	0.50		1.49	0.0007		0.68		2.94	0.001		0.99		2.56	0.002	
10	79.13	78.99	79.09	78.99	78.99	161.29	160.89	161.17	160.89	160.89	397.28	396.04	396.79	396.04	396.04
	0.41		0.10	0.00002		0.58		0.28	0.001		0.88		0.75	0.0004	
MAPE		0.17%	0.32%				0.24%	0.25%				0.25%	0.12%		

Table 3: The three tables represent matrices of payer swaption prices for the two-factor Gaussian model with jumps at three different strikes, i.e. ATMF, ITMF (0.85 × ATMF) and OTMF (1.15 × ATMF). Parameter values are calibrated to the Euribor six-month curve from January 4th, 2015. For each swaption, we report the price in basis points estimated with the Monte Carlo method, MC, the hyperplane approximation lower bound, LB (HP), the upper bound, UB, the Monte Carlo with control variable technique, MC (CV), and the approximation "A" of Kim (2014). Monte Carlo without and with control variable are estimated using 8×10^6 and 10^5 simulations, respectively, an Euler scheme with a time step equal to 0.0005 and the antithetic variates technique. Below each Monte Carlo price, the size of the confidence interval at 97.5% is reported in basis points. The distance between the lower and the upper bounds is provided below each upper bound value.

Two-factor quadratic Gaussian model

Opt. Mat.	1					2					5				
Swap length	**MC**	**LB (HP)**	**UB**	**MC (CV)**	**Kim**	**MC**	**LB (HP)**	**UB**	**MC (CV)**	**Kim**	**MC**	**LB (HP)**	**UB**	**MC (CV)**	**Kim**
2	52.35	52.32	52.33	52.32	52.32	65.68	65.69	65.69	65.69	65.69	65.87	65.87	65.87	65.87	65.87
	0.03		0.02		0.00001	0.03		0.01		0.00001	0.03		0.00		0.00001
5	106.44	106.39	106.50	106.39	106.39	129.98	130.00	130.05	130.00	130.00	130.10	130.10	130.12	130.10	130.10
	0.05		0.11		0.00001	0.06		0.06		0.00001	0.04		0.02		0.0001
10	148.35	148.32	148.54	148.31	148.32	182.14	182.17	182.25	182.17	182.17	190.54	190.53	190.64	190.53	190.54
	0.06		0.22		0.00001	0.06		0.08		0.00001	0.04		0.11		0.0004
MAPE		0.04%	0.07%				0.01%	0.05%				0.002%	0.02%		
Swap length	**MC**	**LB (HP)**	**UB**	**MC (CV)**	**Kim**	**MC**	**LB (HP)**	**UB**	**MC (CV)**	**Kim**	**MC**	**LB (HP)**	**UB**	**MC (CV)**	**Kim**
2	138.51	138.47	138.49	138.47	138.47	140.42	140.42	140.42	140.42	140.42	124.36	124.37	124.37	124.37	124.37
	0.02		0.01		0.00001	0.02		0.01		0.00001	0.02		0.00		0.00001
5	304.44	304.39	304.43	304.39	304.39	304.26	304.26	304.31	304.26	304.26	268.66	268.67	268.69	268.66	268.66
	0.02		0.05		0.00001	0.03		0.05		0.00001	0.03		0.02		0.00003
10	494.61	494.59	494.73	494.58	494.59	488.15	488.14	488.26	488.14	488.14	432.70	432.70	432.78	432.70	432.70
	0.02		0.14		0.00001	0.02		0.12		0.00001	0.04		0.08		0.0001
MAPE		0.02%	0.02%				0.001%	0.01%				0.002%	0.01%		
Swap length	**MC**	**LB (HP)**	**UB**	**MC (CV)**	**Kim**	**MC**	**LB (HP)**	**UB**	**MC (CV)**	**Kim**	**MC**	**LB (HP)**	**UB**	**MC (CV)**	**Kim**
2	11.87	11.86	11.87	11.86	11.86	23.04	23.04	23.05	23.04	23.04	27.55	27.55	27.55	27.55	27.55
	0.02		0.01		0.00001	0.03		0.00		0.00001	0.02		0.00		0.00004
5	17.90	17.89	17.95	17.89	17.89	34.52	34.54	34.56	34.54	34.54	39.84	39.82	39.86	39.82	39.84
	0.03		0.06		0.00001	0.04		0.03		0.00001	0.03		0.04		0.0004
10	12.21	12.23	12.31	12.22	12.23	27.95	27.97	28.01	27.97	27.97	36.10	36.05	36.52	36.05	36.11
	0.02		0.08		0.00001	0.03		0.04		0.00001	0.03		0.47		0.0008
MAPE		0.11%	0.37%				0.05%	0.13%				0.06%	0.41%		

Table 4: The three tables represent matrices of swaption prices for the two-factor Gaussian quadratic model at three different strikes, i.e. ATMF, ITMF (ATMF - 0.75%) and OTMF (ATMF + 0.75%). For each swaption, we report the price in basis points estimated with the Monte Carlo method, MC, the hyperplane approximation lower bound, LB (HP), the upper bound, UB, the Monte Carlo with control variable technique, MC (CV), and the approximation "A" of Kim (2014). Monte Carlo without and with control variable are estimated using 8×10^6 and 10^5 simulations, respectively, an Euler scheme with a time step equal to 0.0005 and the antithetic variates technique. Below each Monte Carlo price, the size of the confidence interval at 97.5% is reported in basis points. The distance between the lower and the upper bounds is provided below each upper bound value.

Two-factor multiple-curve Gaussian model

Opt. Mat.	1				2				5			
Swap length	**MC**	**LB (HP)**	**UB**	**Freezing**	**MC**	**LB (HP)**	**UB**	**Freezing**	**MC**	**LB (HP)**	**UB**	**Freezing**
1	37.396	37.397	38.540	37.286	56.128	56.131	56.932	55.542	77.727	77.725	77.993	72.826
	0.002		1.143	0.110	0.003		0.801	0.586	0.004		0.267	4.900
2	71.233	71.235	72.045	71.245	106.625	106.630	107.183	106.015	146.539	146.536	146.718	138.249
	0.003		0.811	0.011	0.005		0.553	0.610	0.007		0.181	8.290
5	172.640	172.644	173.660	172.748	255.820	255.832	256.526	254.594	345.864	345.859	346.086	326.836
	0.008		1.016	0.108	0.012		0.694	1.225	0.016		0.228	19.028
10	318.570	318.576	319.658	319.361	467.734	467.756	468.496	467.056	622.779	622.769	623.012	591.763
	0.015		1.083	0.792	0.022		0.740	0.678	0.029		0.243	31.016
MAPE		0.002%	1.283%	0.155%		0.005%	0.599%	0.560%		0.002%	0.141%	5.611%
Opt. Mat.	**1**				**2**				**5**			
Swap length	**MC**	**LB (HP)**	**UB**	**Freezing**	**MC**	**LB (HP)**	**UB**	**Freezing**	**MC**	**LB (HP)**	**UB**	**Freezing**
1	47.814	47.814	48.935	47.659	70.136	70.139	70.927	69.376	100.195	100.193	100.454	93.816
	0.002		1.121	0.155	0.003		0.788	0.760	0.004		0.261	6.378
2	100.760	100.762	101.539	100.693	142.769	142.774	143.312	141.805	195.296	195.293	195.470	183.977
	0.003		0.777	0.068	0.005		0.537	0.965	0.007		0.176	11.318
5	273.070	273.073	274.015	272.934	365.386	365.397	366.061	363.166	469.529	469.523	469.744	443.005
	0.007		0.942	0.135	0.011		0.664	2.220	0.015		0.221	26.524
10	557.826	557.830	558.793	558.364	706.939	706.960	707.656	704.774	861.930	861.920	862.154	817.640
	0.013		0.963	0.538	0.020		0.696	2.165	0.028		0.235	44.291
MAPE		0.001%	0.909%	0.134%		0.003%	0.448%	0.668%		0.001%	0.105%	5.737%
Opt. Mat.	**1**				**2**				**5**			
Swap length	**MC**	**LB (HP)**	**UB**	**Freezing**	**MC**	**LB (HP)**	**UB**	**Freezing**	**MC**	**LB (HP)**	**UB**	**Freezing**
1	28.556	28.556	29.677	28.481	44.051	44.054	44.841	43.610	58.779	58.778	59.039	55.111
	0.002		1.120	0.075	0.003		0.787	0.441	0.004		0.261	3.668
2	47.929	47.930	48.706	47.980	76.962	76.968	77.504	76.608	106.418	106.416	106.592	100.557
	0.003		0.776	0.051	0.005		0.536	0.354	0.007		0.176	5.862
5	99.692	99.695	100.636	99.881	170.002	170.014	170.677	169.429	245.442	245.436	245.656	232.329
	0.007		0.940	0.189	0.011		0.662	0.573	0.015		0.220	13.113
10	157.949	157.953	158.913	158.619	288.059	288.081	288.775	288.160	431.263	431.253	431.486	410.522
	0.013		0.960	0.670	0.020		0.694	0.100	0.028		0.234	20.741
MAPE		0.002%	1.776%	0.245%		0.007%	0.786%	0.458%		0.002%	0.186%	5.475%

Table 5: The three tables represent matrices of payer swaption prices for the two-factor multiple-curve Gaussian model at three different strikes, i.e. ATMF, ITMF ($0.85 \times$ ATMF) and OTMF ($1.15 \times$ ATMF). For each swaption, we report the price in basis point as estimated with the Monte Carlo method (MC), the hyperplane approximation lower bound (LB), the upper bound (UB) and the freezing technique. Monte Carlo values are estimated using 10^9 simulations, the antithetic variates method and the exact probability distribution. Below each Monte Carlo price, the size of the confidence interval at 97.5% is reported in basis points. The distance between the lower and the upper bounds is provided below each upper bound value. The error of the freezing techniques is estimated as the difference between the approximated price and the Monte Carlo price.

2-factor CIR model: RMSD calculation

RMSD - LB											
ATM	1	2	5	ITM	1	2	5	OTM	1	2	5
1	0.05%	0.07%	0.08%	1	0.01%	0.02%	0.02%	1	0.2%	0.2%	0.2%
2	0.05%	0.07%	0.08%	2	0.01%	0.01%	0.01%	2	0.2%	0.2%	0.2%
5	0.05%	0.07%	0.09%	5	0.004%	0.01%	0.01%	5	0.2%	0.3%	0.3%
10	0.05%	0.07%	0.09%	10	0.004%	0.01%	0.01%	10	0.2%	0.3%	0.3%

RMSD - UB											
ATM	1	2	5	ITM	1	2	5	OTM	1	2	5
1	0.05%	0.07%	0.08%	1	0.01%	0.02%	0.01%	1	0.2%	0.2%	0.2%
2	0.06%	0.09%	0.11%	2	0.01%	0.02%	0.02%	2	0.2%	0.3%	0.3%
5	0.14%	0.20%	0.28%	5	0.03%	0.05%	0.08%	5	0.3%	0.5%	0.6%
10	0.15%	0.22%	0.32%	10	0.04%	0.06%	0.09%	10	0.3%	0.5%	0.7%

Table 6: These tables report for each swaption the RMSD value of the bounds with respect to the Monte Carlo value obtained by randomly sampling 100 parameter sets.

3 factor Gaussian model

Overall time (sec)	MC	LB (HP)	UB	LB (CA)	UB (CA)
ATMF	32×10^2	0.084	0.140	0.024	0.024
ITMF	32×10^2	0.170	0.223	0.035	0.035
OTMF	32×10^2	0.169	0.223	0.037	0.037

2 factor CIR model

Overall time (sec)	MC	LB (HP)	UB	Kim
ATMF	23×10^2	0.146	17.054	0.391
ITMF	23×10^2	0.150	17.015	0.341
OTMF	23×10^2	0.152	17.018	0.395

2 factor Gaussian model with exponential jumps

Overall time (sec)	MC	LB (HP)	UB	Kim
ATMF	35×10^3	1.957	132.229	1.968
ITMF	35×10^3	0.868	129.218	0.977
OTMF	35×10^3	0.845	149.071	0.966

2 factor Gaussian quadratic model

Overall time (sec)	MC	LB (HP)	UB	Kim
ATMF	1.472×10^3	0.861	587.403	0.665
ITMF	1.472×10^3	1.124	635.807	0.717
OTMF	1.472×10^3	1.019	509.202	0.633

2 factor multiple-curve Gaussian model

Overall time (sec)	MC	LB (HP)	UB	Kim
ATMF	43.280	0.094	0.416	0.346
ITMF	43.3603	0.114	0.403	0.309
OTMF	42.040	0.116	0.409	0.315

Table 7: Computational times shown in the table are the overall time needed for calculating the matrices of swaption prices reported in Tables 1-5.

2-factor CIR model: comparison of the algorithms' performance

Swap length (y)	LB (HP) (sec)	Kim (sec)	LB (HP) (%)	Kim (%)
1	0.024	0.022	-	-
2	0.023	0.026	0%	20%
5	0.023	0.034	0%	55%
10	0.032	0.051	34%	132%
15	0.040	0.071	69%	225%
20	0.048	0.089	102%	305%

Table 8: For each swaption, we report in the first two columns the run time in seconds and in the last two columns the percentage variation between the run times and the first row. The maturity of the swaptions is two years and the frequency of payments is six months.

References

Ahn, D., and Dittmar, R.F., and Gallant, A.R., Quadratic term structure models: Theory and evidence. *Rev. Financ. Stud.*, 2002, **15** (1), 243-288.

Ametrano, F. M., and Bianchetti, M., Bootstrapping the illiquidity: Multiple yield curves construction for market coherent forward rates estimation *Risk Books, Incisive Media*, 2009.

Brigo, D., and Masetti, M., Risk Neutral Pricing of Counterparty Risk. In Counterparty Credit Risk Modelling: Risk Management, Pricing and Regulation, edited by Pykhtin, *Risk Books: London*, 2005.

Cheng, P., and Scaillet, O. Linear-Quadratic Jump-Diffusion Modeling. *Math. Finance*, 2007, **17**, 575-598.

Collin-Dufresne, P., and Goldstein, R.S., Pricing Swaptions Within an Affine Framework. *J. Derivatives*,2002, **10** (1), 1-18.

Duffie, D., and Kan, R., A yield-factor model of interest rates. *Math. Finance*, 1996, **6**, 379-406.

Duffie, D., Pan, P., and Singleton, K., Transform Analysis and Asset Pricing for Affine Jump-Diffusions. *Econometrica*, 2000, **68**, 1343-1376.

Duffie, D., and Singleton, K., An Econometric Model of the Term Structure of Interest-Rate Swap Yields. *J. Finance*, 1997, **52**, 1287-1321.

Fanelli, V., A Defaultable HJM Modelling of the Libor Rate for Pricing Basis Swaps after the Credit Crunch. *EJOR* , 2016, **249** (1), 238–244.

Hubalek, F., Kallsen, J., and Krawczyk, L., Variance-optimal hedging for processes with stationary independent increments. *Ann. Appl. Probab.*, 2006, **16** (2), 853-885.

Jamshidian, F., An exact bond option pricing formula. *J. Finance*, 1989, **44**, 205-209.

Kim, D.H., Spanned Stochastic Volatility in Bond markets: A Reexamination of the Relative Pricing between Bonds and Bond Options. *BIS working paper*, 2007.

Kim, D.H., Swaption pricing in affine and other models. *Math. Finance*, 2014, **24** (4), 790-820.

Leippold, M., and Wu, L., Asset Pricing under the Quadratic Class. *J. Finan. Quant. Anal.*, 2002, **2**, 271-295.

Margrabe, W., The Value of an Option to Exchange One Asset for Another. *J. Finance*, 1978, **33** (1), 177-186.

Morini, M., Solving the puzzle in the interest rate market. *SSRN eLibrary*, 2009.

Moreni, N., and Pallavicini, A., Parsimonious HJM modelling for multiple yield-curve dynamics. *Quant. Finance*, 2014 **14** (2), 199-210.

Munk, C., Stochastic Duration and Fast Coupon Bond Option Pricing in Multi-Factor Models. *Rev. Derivat.*, 1999, **3**, 157-181.

Nunes, J.P.V., and Prazeres, P.M.S., Pricing Swaptions under Multifactor Gaussian HJM Models. *Math. Finance*, 2014, **24**, 762-789.

Ritchken, P., and Sankarasubramanian, L., Volatility Structures of Forward Rates and the Dynamics of the Term Structure. *Math. Finance*, 1995, **7**, 157-176.

Schrager, D.F., and Pelsser, A.J., Pricing Swaptions and Coupon Bond Options in Affine Term Structure Models. *Math. Finance*, 2006, **16**, 673-694.

Singleton, K.J., and Umantsev, L., Pricing Coupon-Bond Options and Swaptions in Affine Term Structure Models. *Math. Finance*, 2002, **12**, 427-446.

Titchmarsh, E.C., *Introduction to the Theory of Fourier Integrals*, third edition, 1986, (Chelsea Publishing Co.: New York).

Zheng, C., Method for Swaption Pricing Under Affine Term Structure Models and Measure of Errors. *Available at SSRN: http://ssrn.com/abstract=2351162*, 2013.

A Proof Proposition 2.1

We consider the lower bound to the swaption price as in formula (4) for quadratic models:

$$LB_{\boldsymbol{\beta},\Gamma}(k;t) = P(t,T)\,\mathbb{E}_t^T\left[\left(\sum_{h=1}^n w_h\, e^{\mathbf{X}(T)^\top C_h \mathbf{X}(T)+\mathbf{b}_h^\top \mathbf{X}(T)+a_h} - 1\right) I(\mathcal{G})\right]$$

where the set $\mathcal{G} = \{\omega \in \Omega : \mathbf{X}(T)^\top \Gamma\, \mathbf{X}(T) + \boldsymbol{\beta}^\top \mathbf{X}(T) \geq k\}$.

We apply the extended Fourier transform (refer to Titchmarsh (1975) for a comprehensive treatment and to Hubalek et al. (2006) for examples of financial applications) with respect to the variable k to the T-forward expected value,

$$\psi(z) \;=\; \int_{-\infty}^{+\infty} e^{zk}\mathbb{E}_t^T\left[\left(\sum_{h=1}^n w_h\, e^{\mathbf{X}(T)^\top C_h \mathbf{X}(T)+\mathbf{b}_h^\top \mathbf{X}(T)+a_h} - 1\right) I(\mathbf{X}(T)^\top \Gamma\, \mathbf{X}(T) + \boldsymbol{\beta}^\top \mathbf{X}(T) \geq k)\right] dk.$$

Assuming that we can apply Fubini's Theorem, which is verified in concrete cases, we have

$$\begin{aligned}\psi(z) \;&=\; \mathbb{E}_t^T\left[\left(\sum_{h=1}^n w_h\, e^{\mathbf{X}(T)^\top C_h \mathbf{X}(T)+\mathbf{b}_h^\top \mathbf{X}(T)+a_h} - 1\right)\right.\\ &\qquad \left.\int_{-\infty}^{+\infty} e^{zk} I(\mathbf{X}(T)^\top \Gamma\, \mathbf{X}(T) + \boldsymbol{\beta}^\top \mathbf{X}(T) \geq k)\, dk\right].\end{aligned}$$

The function $\psi(z)$ is defined for $k \to -\infty$ if $Re(z) > 0$ and

$$\psi(z) \;=\; \mathbb{E}_t^T\left[\left(\sum_{h=1}^n w_h e^{\mathbf{X}(T)^\top C_h \mathbf{X}(T)+\mathbf{b}_h^\top \mathbf{X}(T)+a_h} - 1\right) e^{z(\mathbf{X}(T)^\top \Gamma\, \mathbf{X}(T)+\boldsymbol{\beta}^\top \mathbf{X}(T))}\right]\frac{1}{z}.$$

Using the (quadratic) characteristic function of $\mathbf{X}$, Φ, calculated under the T-forward measure, the function $\psi(z)$ can be written as

$$\psi(z) = \left(\sum_{h=1}^n w_h e^{a_h}\Phi\left(\mathbf{b}_h + z\boldsymbol{\beta}, C_h + z\Gamma\right) - \Phi\left(z\boldsymbol{\beta}, z\Gamma\right)\right)\frac{1}{z}. \tag{30}$$

Finally, the lower bound is the inverse transform of $\psi(z)$ in the sense of the Chauchy principal value integral,

$$LB_{\boldsymbol{\beta},\Gamma}(k;t) \;=\; P(t,T)\,\frac{1}{i2\pi}\lim_{\xi\to\infty}\int_{\delta-i\xi}^{\delta+i\xi} e^{-kz}\psi(z)dz,$$

where δ is a positive constant. The function $\psi(\delta + i\gamma)$ is the Fourier transform of the real function $e^{-\delta k}LB_{\boldsymbol{\beta},\Gamma}(k;t)$, then $\psi(\delta + i\gamma)$ has an even real part and an odd imaginary part. This is useful to simplify the expression above to

$$LB_{\boldsymbol{\beta},\Gamma}(k;t) \quad = \quad P(t,T)\,\frac{e^{-\delta k}}{\pi}\int_0^{+\infty} Re\left(e^{-i\gamma k}\psi(\delta+i\gamma)\right)\,d\gamma.$$

The proof for a payer swaption follows the same reasoning.

B Proof of Proposition 3.1

Here, we show the calculation of the quantity ϵ_1, defined in equation (13). The computation of the quantity ϵ_2 follows the same reasoning. Hence, we have to calculate a sum of terms that have the following form:

$$\mathbb{E}_t^T[(w_h P(T,T_h) - K_h)^+\, I(\mathcal{G}^c)].$$

Substituting into the previous expression the definition of the zero-coupon bond price $P(T,T_h)$ as in formula (5), the strike K_h as in formula (16) and the complement of the approximate exercise region $\mathcal{G}$ as defined in section 2.1, we obtain the following formulation:

$$\mathbb{E}_t^T[(w_h P(T,T_h) - K_h)^+\, I(\mathcal{G}^c)] = w_h e^{a_h}\, f(k,k_h),$$

where

$$f(k,k_h) = \mathbb{E}_t^T[(e^{\mathbf{X}(T)^\top C_h \mathbf{X}(T) + \mathbf{b}_h^\top \mathbf{X}(T)} - e^{k_h})^+\, I(\mathbf{X}(T)^\top \Gamma\, \mathbf{X}(T) + \boldsymbol{\beta}^\top \mathbf{X}(T) < k)],$$

and $k_h = \log(K_h) - \log(w_h) - a_h$. We apply the extended Fourier transform with respect to the variable k to the function $f(k,k_h)$ and by Fubini's theorem we obtain

$$\int_{-\infty}^{+\infty} e^{zk}\, f(k,k_h)\, dk = -\mathbb{E}_t^T\left[(e^{\mathbf{X}(T)^\top C_h \mathbf{X}(T) + \mathbf{b}_h^\top \mathbf{X}(T)} - e^{k_h})^+\, \frac{e^{z\,(\mathbf{X}(T)^\top \Gamma\, \mathbf{X}(T) + \boldsymbol{\beta}^\top \mathbf{X}(T))}}{z}\right].$$

The integral converges for $k \to +\infty$ if $Re(z) < 0$, then we apply a second extended Fourier transform with respect to the variable k_h,

$$\begin{aligned}
& - \int_{-\infty}^{+\infty} e^{y\,k_h}\, \frac{1}{z}\, \mathbb{E}_t^T\left[(e^{\mathbf{X}(T)^\top C_h \mathbf{X}(T) + \mathbf{b}_h^\top \mathbf{X}(T)} - e^{k_h})^+\, e^{z\,(\mathbf{X}(T)^\top \Gamma\, \mathbf{X}(T) + \boldsymbol{\beta}^\top \mathbf{X}(T))}\right] dk_h \\
= & -\frac{1}{z}\, \mathbb{E}_t^T\Bigg[\Bigg(\int_{-\infty}^{+\infty} e^{y\,k_h}\, \left(e^{\mathbf{X}(T)^\top C_h \mathbf{X}(T) + \mathbf{b}_h^\top \mathbf{X}(T)} - e^{k_h}\right) \\
& \quad I(\mathbf{X}(T)^\top C_h \mathbf{X}(T) + \mathbf{b}_h^\top \mathbf{X}(T) > k_h)\, dk_h\Bigg)\, e^{z\,(\mathbf{X}(T)^\top \Gamma\, \mathbf{X}(T) + \boldsymbol{\beta}^\top \mathbf{X}(T))}\Bigg].
\end{aligned}$$

The integral converges for $k_h \to -\infty$ if $Re(y) > 0$. Then the function $\psi(z, y)$ is in the form

$$\begin{aligned}\psi(z,y) &= \int_{-\infty}^{+\infty} dk \int_{-\infty}^{+\infty} dk_h\, e^{zk} e^{yk_h}\, f(k,k_h) \\ &= -\frac{\Phi(z\boldsymbol{\beta} + (y+1)\mathbf{b}_h, z\,\Gamma + (y+1)C_h)}{zy(y+1)}\end{aligned}$$

and it is defined for $Re(z) < 0$ and $Re(y) > 0$.

Finally, $f(k, k_h)$ is the inverse transform of $\psi(z, y)$ in the sense of a Cauchy principal value integral,

$$f(k,k_h) = \frac{1}{(i2\pi)^2} \lim_{\xi\to\infty} \lim_{\varsigma\to\infty} \int_{\delta-i\xi}^{\delta+i\xi} dz\, e^{-zk} \int_{\eta-i\varsigma}^{\eta+i\varsigma} dy\, e^{-yk_h} \psi(z,y),$$

where $\delta < 0$ and $\eta > 0$ are constants. Noting that $\psi(\delta + i\gamma, \eta + i\omega)$ is the double Fourier transform of the function $e^{\delta k} e^{\eta k_h} f(k, k_h)$, we obtain

$$f(k,k_h) = \frac{e^{-\delta k} e^{-\eta k_h}}{4\pi^2} \lim_{\xi\to\infty} \lim_{\varsigma\to\infty} \int_{-\xi}^{+\xi} d\gamma\, e^{-i\gamma k} \int_{-\varsigma}^{+\varsigma} d\omega\, e^{-i\omega k_h} \psi(\delta+i\gamma, \eta+i\omega),$$

where $\delta < 0$ and $\eta > 0$ are constants. The inner integral of the above formula is the Fourier transform of a real function, and so we can use the same symmetry properties explained in Appendix A and we obtain

$$f(k,k_h) = \frac{e^{-\delta k} e^{-\eta k_h}}{2\pi^2} \lim_{\xi\to\infty} \int_{0}^{+\xi} d\gamma\, Re\left(e^{-i\gamma k} \lim_{\varsigma\to\infty} \int_{-\varsigma}^{+\varsigma} d\omega\, e^{-i\omega k_h} \psi(\delta+i\gamma, \eta+i\omega) \right).$$

C Proof of the analytical lower bound for Gaussian affine models

Since $\mathbf{X}(T) \sim N(\boldsymbol{\mu}, V)$ in T-forward measure, then the approximate exercise region $\mathcal{G}$ becomes

$$\mathcal{G} = \{\omega \in \Omega : \boldsymbol{\beta}^\top \mathbf{X}(T) > k\} = \{\omega \in \Omega : z > d\},$$

where z is a standard normal random variable and $d = \frac{k - \boldsymbol{\beta}^\top \boldsymbol{\mu}}{\sqrt{\boldsymbol{\beta}^\top V \boldsymbol{\beta}}}$.

The lower bound expression can be written using the law of iterative expectation,

$$LB_{\boldsymbol{\beta}}(k;t) = P(t,T)\, \mathbb{E}_t^T \left[\mathbb{E}_t^T \left[\left(\sum_{h=1}^{n} w_h e^{\mathbf{b}_h{}^\top \mathbf{X}(T) + a_h} - 1 \right) |z \right] I(z > d) \right].$$

Conditionally to the random variable z, the variable $\mathbf{X}$ is distributed as a multivariate normal with mean and variance

$$\mathbb{E}_t^T[\mathbf{X}|z] = \boldsymbol{\mu} + z\,\mathbf{v} \text{ and } Var(\mathbf{X}|z) = V - \mathbf{v}\mathbf{v}^\top, \text{ with } \mathbf{v} = \frac{V\boldsymbol{\beta}}{\sqrt{\boldsymbol{\beta}^\top V\boldsymbol{\beta}}}.$$

We can now compute the inner expectation,

$$\begin{aligned} LB_{\boldsymbol{\beta}}(k;t) &= P(t,T)\left(\sum_{h=1}^{n} w_h\,\mathbb{E}_t^T\left[e^{a_h+\mathbf{b}_h^\top\boldsymbol{\mu}+z\mathbf{b}_h^\top\mathbf{v}+\frac{1}{2}V_h}I(z>d)\right] - \mathbb{E}_t^T\left[I(z>d)\right]\right) \\ &= P(t,T)\left(\sum_{h=1}^{n} w_h\,e^{a_h+\mathbf{b}_h^\top\boldsymbol{\mu}+\frac{1}{2}V_h+\frac{1}{2}d_h^2}N(d_h-d) - N(-d)\right). \end{aligned}$$

where $V_h = \mathbf{b}_h^\top(V - \mathbf{v}\mathbf{v}^\top)\mathbf{b}_h$, $d_h = \mathbf{b}_h^\top\mathbf{v}$ and $N(x)$ is the cumulative distribution function of standard normal variable. The proof for a payer swaption follows the same reasoning.

D Proof of the upper bound formula for Gaussian affine models

Since $\mathbf{X} \sim \mathcal{N}(\boldsymbol{\mu}, V)$ in T-forward measure and using the law of iterative expectations, then

$$\begin{aligned} &\mathbb{E}_t^T[(w_h e^{a_h+b_h^\top\mathbf{X}(T)} - K_h)^+ I(\beta^\top X < k)] \\ =\ &\mathbb{E}_t^T[\mathbb{E}_t^T[(w_h e^{a_h+b_h^\top\mathbf{X}(T)} - K_h)^+|Z]I(Z<d)], \\ =\ &\int_{-\infty}^{d} dz\,\frac{1}{\sqrt{2\pi}}e^{-\frac{z^2}{2}}\,\mathbb{E}_t^T[(w_h e^{a_h+\mathbf{b}_h^\top\mathbf{X}(T)} - K_h)^+|Z=z]. \end{aligned}$$

where $Z \sim \mathcal{N}(0,1)$ and $d = \frac{k-\beta^\top\mu}{\sqrt{\beta^\top V\beta}}$.

Since $\mathbf{b}_h^\top\mathbf{X}$ conditioned to the variable Z is a normal random variable with mean and variance,

$$\begin{aligned} &M_h = \mathbb{E}_t^T[\mathbf{b}_h^\top\mathbf{X}|Z=z] = \mathbf{b}_h^\top\boldsymbol{\mu} + z\mathbf{b}_h^\top\mathbf{v}, \\ &V_h = Var_t[\mathbf{b}_h^\top\mathbf{X}|Z=z] = \mathbf{b}_h^\top(V - \mathbf{v}\mathbf{v}^\top)\mathbf{b}_h \\ &\mathbf{v} = \frac{V\boldsymbol{\beta}}{\sqrt{\boldsymbol{\beta}^\top V\boldsymbol{\beta}}}, \end{aligned}$$

then the conditioned expectation can be evaluated with a Black formula,

$$\begin{aligned} &\mathbb{E}_t^T[(w_h e^{a_h+b_h^\top\mathbf{X}(T)} - K_h)^+|Z=z] \\ &= w_h e^{a_h}\left(e^{M_h+\frac{V_h}{2}}N\left(\frac{M_h - \log Y_h + V_h}{\sqrt{V_h}}\right) - Y_h N\left(\frac{M_h - \log Y_h}{\sqrt{V_h}}\right)\right), \end{aligned}$$

where $Y_h = \frac{K_h}{w_h e^{a_h}}$ and $N(x)$ is the cumulative distribution function of a standard normal variable.

E Proof of Proposition 5.1

The proof is similar to the single-curve affine Gaussian case. As in that case, $\mathbf{X}(T) \sim N(\boldsymbol{\mu}, V)$ in T-forward measure and the approximate exercise region $\mathcal{G}$ becomes

$$\mathcal{G} = \{\omega \in \Omega : \boldsymbol{\beta}^\top \mathbf{X}(T) > k\} = \{\omega \in \Omega : z > d\},$$

where z is a standard normal random variable and $d = \frac{k - \boldsymbol{\beta}^\top \boldsymbol{\mu}}{\sqrt{\boldsymbol{\beta}^\top V \boldsymbol{\beta}}}$.

The lower bound expression can be written using the law of iterative expectation,

$$LB_{\boldsymbol{\beta}}(k;t) = P(t,T)\,\mathbb{E}_t^T \left[\mathbb{E}_t^T \left[\left(\sum_{j=1}^n w_{1j} e^{\mathbf{G}_{1j}{}^\top \mathbf{X}(T) + a_{1j}} - w_{2j} e^{\mathbf{G}_{2j}{}^\top \mathbf{X}(T) + a_{2j}} \right) |z \right] I(z > d) \right].$$

Conditionally to the random variable z, the variable $\mathbf{X}$ is distributed as a multivariate normal with mean and variance

$$\mathbb{E}_t^T[\mathbf{X}|z] = \boldsymbol{\mu} + z\,\mathbf{v} \text{ and } Var(\mathbf{X}|z) = V - \mathbf{v}\mathbf{v}^\top, \text{ with } \mathbf{v} = \frac{V\boldsymbol{\beta}}{\sqrt{\boldsymbol{\beta}^\top V \boldsymbol{\beta}}}.$$

We can now compute the inner expectation,

$$\begin{aligned} LB_{\boldsymbol{\beta}}(k;t) &= P(t,T) \left(\sum_{j=1}^n w_{1j}\, \mathbb{E}_t^T \left[e^{a_{1j} + \mathbf{G}_{1j}^\top \boldsymbol{\mu} + z \mathbf{G}_{1j}^\top \mathbf{v} + \frac{1}{2} V_{1j}^G} I(z > d) \right] \right. \\ &\quad \left. - \sum_{j=1}^n w_{2j} \mathbb{E}_t^T \left[e^{a_{2j} + \mathbf{G}_{2j}^\top \boldsymbol{\mu} + z \mathbf{G}_{2j}^\top \mathbf{v} + \frac{1}{2} V_{2j}^G} I(z > d) \right] \right) \\ &= \sum_{j=1}^n w_{1j}\, e^{a_{1j} + \mathbf{G}_{1j}^\top \boldsymbol{\mu} + \frac{1}{2} V_{1j}^G + \frac{1}{2} d_{1j}^2} N(d_{1j} - d) - w_{2j}\, e^{a_{2j} + \mathbf{G}_{2j}^\top \boldsymbol{\mu} + \frac{1}{2} V_{2j}^G + \frac{1}{2} d_{2j}^2} N(d_{2j} - d). \end{aligned}$$

where $V_{ij}^G = \mathbf{G}_{ij}^\top (V - \mathbf{v}\mathbf{v}^\top) \mathbf{G}_{ij}$, $d_{ij} = \mathbf{G}_{ij}^\top \mathbf{v}$ and $N(x)$ is the cumulative distribution function of the standard normal variable. The proof for a receiver swaptions follows the same reasoning.

F Proof of Proposition 5.2

The proof is similar to the single-curve affine Gaussian case except that instead of the Black formula, we apply Margrabe's formula (Margrabe (1978)) for exchange options. Here, we show the computation of the quantity ϵ_1 defined in proposition (5.2). The evaluation of ϵ_2

follows the same steps. Since $\mathbf{X} \sim \mathcal{N}(\boldsymbol{\mu}, V)$ in T-forward measure and using the law of iterative expectations, then

$$\begin{aligned}
& \mathbb{E}_t^T[(w_{1j}\, e^{a_{1j}+G_{1j}^\top \mathbf{X}(T)} - \tilde{w}_{2j}\, e^{a_{2j}+G_{2j}^\top \mathbf{X}(T)})^+ I(\beta^\top X < k)] \\
= \; & \mathbb{E}_t^T[\mathbb{E}_t^T[(w_{1j}\, e^{a_{1j}+G_{1j}^\top \mathbf{X}(T)} - \tilde{w}_{2j}\, e^{a_{2j}+G_{2j}^\top \mathbf{X}(T)})^+ | Z] I(Z < d)], \\
& = \int_{-\infty}^{d} dz\, \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}}\, \mathbb{E}_t^T[(w_{1j}\, e^{a_{1j}+G_{1j}^\top \mathbf{X}(T)} - \tilde{w}_{2j}\, e^{a_{2j}+G_{2j}^\top \mathbf{X}(T)})^+ | Z = z].
\end{aligned}$$

where $Z \sim \mathcal{N}(0,1)$ and $d = \frac{k-\beta^\top \mu}{\sqrt{\beta^\top V \beta}}$.

$\mathbf{G}_{ij}^\top \mathbf{X}$ conditioned to the variable Z is a normal random variable with mean and variance

$$\begin{aligned}
& M_{ij} = \mathbb{E}_t^T[\mathbf{G}_{ij}^\top \mathbf{X} | Z = z] = \mathbf{G}_{ij}^\top \boldsymbol{\mu} + z \mathbf{G}_{ij}^\top \mathbf{v}, \\
& V_{ij}^G = Var_t[G_{ij}^\top \mathbf{X} | Z = z] = G_{ij}^\top (V - \mathbf{v}\mathbf{v}^\top) G_{ij} \\
& \mathbf{v} = \frac{V\boldsymbol{\beta}}{\sqrt{\boldsymbol{\beta}^\top V \boldsymbol{\beta}}}.
\end{aligned}$$

Hence, considering for each fixed j the following two underlying variables

$$\begin{aligned}
S_{1j} &= w_{1j}\, e^{a_{1j}+G_{1j}^\top \mathbf{X}(T)}, \\
S_{2j} &= \tilde{w}_{2j}\, e^{a_{2j}+G_{2j}^\top \mathbf{X}(T)},
\end{aligned}$$

the conditional expectation can be evaluated with the Margrabe formula

$$\begin{aligned}
& \mathbb{E}_t^T[(w_{1j}\, e^{a_{1j}+G_{1j}^\top \mathbf{X}(T)} - \tilde{w}_{2j}\, e^{a_{2j}+G_{2j}^\top \mathbf{X}(T)})^+ | Z = z] \\
= \; & w_{1j}\, e^{a_{1j}+M_{1j}+\frac{1}{2}V_{1j}^G}\, N(d_{1j}) - \tilde{w}_{2j}\, e^{a_{2j}+M_{2j}+\frac{1}{2}V_{2j}^G}\, N(d_{2j}), \\
d_{1j} \; = \; & \frac{\log\left(\frac{w_{1j}}{\tilde{w}_{2j}}\right) + M_{1j} + a_{1j} - M_{2j} - a_{2j} + V_{1j}^G - Cov_j}{\sqrt{V_{1j}^G + V_{2j}^G - 2Cov_j}}, \\
d_{2j} \; = \; & d_{1j} - \sqrt{V_{1j}^G + V_{2j}^G - 2Cov_j},
\end{aligned}$$

where $Cov_j = G_{1j}^\top (V - vv^\top) G_{2j}$ for $i = 1, 2$ and $j = 1, ..., d$ and $N(x)$ is the standard Gaussian cumulative distribution function.

G Models description

This section presents the considered affine and quadratic models.

G.1 Affine Gaussian models

Affine Gaussian models assign the following stochastic differential equation (SDE) to the state variable $\mathbf{X}$,

$$d\mathbf{X}(t) = K(\boldsymbol{\theta} - \mathbf{X}(t))\, dt + \Sigma\, d\mathbf{W}(t) \quad \text{and} \quad \mathbf{X}(0) = \mathbf{x}_0$$

where $\mathbf{W}_t$ is a standard d-dimensional Brownian motion, K is a $d \times d$ diagonal matrix and Σ is a $d \times d$ triangular matrix. The short rate is obtained as a linear combination of the state vector $\mathbf{X}$; it is always possible to rescale the components $X_i(t)$ and assume that $r(t) = \phi + \sum_{i=1}^{d} X_i(t)$, $\phi \in \mathbb{R}$ without loss of generality.

The ZCB formula (5) and T-forward characteristic function (6) of $\mathbf{X}$ can be obtained in closed form using the moment-generating function of a multivariate normal variable or solving the ODE system in Duffie, Pan and Singleton (2000), and the solution is given, for example, in Collin-Dufresne and Goldstein (2002).

G.2 Multi-factor CIR model

In this model, the risk-neutral dynamics of the state variates are

$$dX_i(t) \quad = \quad a_i(\,\theta_i - X_i(t))dt + \sigma_i\sqrt{X_i(t)}dW^i(t) \quad \text{and} \quad \mathbf{X}(0) = \mathbf{x}_0,$$

where $i = 1, ..., d$, $W^i(t)$ are independent standard Brownian motions, and a_i, θ_i and σ_i are positive constants. The short rate is obtained as $r(t) = \phi + \sum_{i=1}^{d} X_i(t)$, where $\phi \in \mathbb{R}$.

In multi-factor CIR models, the bond price (5) and the characteristic function (6) have closed-form expressions, which are given, for example, in Collin-Dufresne and Goldstein (2002).

G.3 Gaussian model with double exponential jumps

In this model, the risk-neutral dynamics of the state variates are

$$d\mathbf{X}(t) = K(\boldsymbol{\theta} - \mathbf{X}(t))\, dt + \Sigma\, d\mathbf{W}(t) + d\mathbf{Z}^{+}(t) - d\mathbf{Z}^{-}(t) \quad \text{and} \quad \mathbf{X}(0) = \mathbf{x}_0,$$

where $\mathbf{W}_t$ is a standard d-dimensional Brownian motion, K is a $d \times d$ diagonal matrix, Σ is a $d \times d$ triangular matrix and $\mathbf{Z}^{\pm}$ are pure jump processes whose jumps have fixed probability distribution ν on $\mathbb{R}^d$ and constant intensity $\mu^{\pm}$. The short rate is obtained as a linear combination of the state vector $\mathbf{X}$. In particular, $\mathbf{Z}^{\pm}$ are compounded Poisson processes with jump sizes that are exponentially distributed, i.e.

$$Z_l^{\pm} = \sum_{j=1}^{N^{\pm}(t)} Y_{j,l}^{\pm}$$

where $l = 1, ..., d$ is the factor index, $N^{\pm}(t)$ are Poisson processes with intensity $\frac{\mu^{\pm}}{d}$ and $Y_{j,l}^{\pm}$, for a fixed l, are independent identically distributed exponential random variables of mean parameters $m_l^{\pm}$.

Since $\mu^{\pm}$ do not depend on $\mathbf{X}$, we know that

$$\Phi(\boldsymbol{\lambda}) = \mathbb{E}_t^T \left[e^{\boldsymbol{\lambda}^\top \mathbf{X}(T)} \right] = \Phi^D(\boldsymbol{\lambda})\, e^{\tilde{A}^J(T-t,\boldsymbol{\lambda}) - A^J(T-t)} \tag{31}$$

where $\Phi^D(\boldsymbol{\lambda})$ is the T-forward characteristic function of the affine Gaussian model and the function $\tilde{A}^J(\tau, \boldsymbol{\lambda})$ is available in closed form (see Duffie, Pan and Singleton (2000) for further details).

G.4 Gaussian quadratic model

In this model, the risk-neutral dynamics of the state variates are

$$d\mathbf{X}(t) = K(\boldsymbol{\theta} - \mathbf{X}(t))\, dt + \Sigma\, d\mathbf{W}_t \quad \text{and} \mathbf{X}(0) = \mathbf{x}_0,$$

where $\mathbf{W}_t$ is a standard d-dimensional Brownian motion, $\boldsymbol{\theta}$ is a d-dimensional constant vector, K and Σ are $d \times d$ matrix. The short rate is a quadratic function of the state variates, $r(t) = a_r + b_r^\top \mathbf{X}(t) + \mathbf{X}(t)^\top C_r \mathbf{X}(t)$, $a_r \in \mathbb{R}$, $b_r \in \mathbb{R}^d$ and C_r is a $d \times d$ symmetric matrix.

We solve the system of ordinary differential equation for the functions $\tilde{A}(\tau, \boldsymbol{\lambda}, \Lambda)$, $\tilde{\mathbf{B}}(\tau, \boldsymbol{\lambda}, \Lambda)$, $\tilde{C}(\tau, \boldsymbol{\lambda}, \Lambda)$ in formula (6), using the method proposed in Cheng and Scaillet (2007). The closed-form evaluation of these functions proposed in Cheng and Scaillet (2007) requires the calculus of a matrix exponentiation and a numerical integration. However, numerical tests show that this method is much faster than numerically solving the ODE system using the Runge-Kutta or Dormand-Prince schemes.

H Multiple-curve model

We test the lower and upper bounds to the multiple-curve weighted Gaussian model presented in Moreni and Pallavicini (2014). In this model, the zero-coupon bond price process has the following dynamic:

$$P(t,T) = \frac{P(0,T)}{P(0,t)}\, e^{\int_0^t (\Sigma(s,t)-\Sigma(s,T))^\top \, dW(s) + \int_0^t (A(s,t)-A(s,T))\, ds}, \tag{32}$$

where

$\Sigma(t,T) = \int_t^T \sigma(t,u)\, du$ is a d-dimensional vector volatility function,

$W(t)$ is a d-dimensional standard Brownian motion,

$A(t,T) = \frac{1}{2}\Sigma(t,T)^\top \Sigma(t,T)$.

Moreni and Pallavicini (2014) define the risk-free forward rate F^0, which can be identified in the market using the overnight rate. It is built as the simple compounded forward rate in a classical single-curve framework. The risk-free forward rate at time t for the interval $[T-x,\, T]$ is

$$F^0(t,T,x) = \frac{1}{x}\left(\frac{P(t,T-x)}{P(t,T)} - 1\right). \tag{33}$$

Substituting equation (32) into (33), the following dynamic under the risk-neutral measure is obtained:

$$F^0(t,T,x) = \frac{1}{x}\left[(1 + x\, F^0(0,T,x)) e^{\int_0^t \Sigma^0(s,T,x)^\top \, dW(s) + \int_0^t A^0(s,T,x)\, ds} - 1\right], \tag{34}$$

where

$\Sigma^0(s,T,x) = \Sigma(s,T) - \Sigma(s,T-x) = \int_{T-x}^T \sigma(s,u)\, du$,

$A^0(s,T,x) = A(s,T) - A(s,T-x) = \frac{1}{2}\Sigma(s,T)^\top\, \Sigma(s,T) - \frac{1}{2}\Sigma(s,T-x)^\top\, \Sigma(s,T-x)$.

The Libor FRA rate $F^x(t,T,x)$ is the fair rate of a FRA contract written on the Libor rate with tenor x (usually $x =$ 1M, 3M, 6M or 12M). It is defined as

$$F^x(t,T,x) = \mathbb{E}_t^T \left[L(T-x,T)\right], \tag{35}$$

where

$L(T-x,T)$ is the spot Libor rate, fixed at time $T-x$ for the time interval $[T-x,T]$,

$\mathbb{E}_t^T[\,]$ denotes the expectation under T-forward measure, $\mathbb{P}^T$.

To model the FRA rate, these constraints have to be respected:

(i) $F^x(t,T,x)$ has to be a martingale under the T-forward measure,

(ii) $\lim_{x\to 0} F^x(t,T,x) = \lim_{x\to 0} F^0(t,T,x)$ and $F^x(t,T,x) \sim F^0(t,T,x)$ if $x \sim 0$.

Hence, under the risk-neutral $\mathbb{P}$ measure, the FRA rate is in the form

$$F^x(t,T,x) = \frac{1}{x}\left[(1 + x\, F^x(0,T,x))e^{\int_0^t \Sigma^x(s,T,x)^\top\, dW(s) + \int_0^t A^x(s,T,x)\, ds} - 1\right], \tag{36}$$

where

- $\Sigma^x(s,T,x) = \int_{T-x}^{T} \sigma(s,u;T,x)\, du$ is a d-dimensional volatility function,
- in order to satisfy condition (ii) $\sigma(s,T;T,0) = \sigma(s,T)$,
- to satisfy condition (i)

$$A^x(s,T,x) \quad = \quad -\frac{1}{2}\Sigma^x(s,T,x)^\top\, \Sigma^x(s,T,x) + \Sigma^x(s,T,x)^\top\, \Sigma(s,T). \tag{37}$$

H.1 Volatility specification

The weighted Gaussian specification of the multiple-curve model assumes a deterministic volatility in the form

$$\begin{aligned} \sigma(t,u;T,x) &= h(t)\, q(u;T,x)\, g(t,u), \\ g(t,u) &= \exp\left(-\lambda(u-t)\right), \\ h(t) &= \epsilon(t)\, h\, R, \end{aligned}$$

where λ is a deterministic array function, h is a diagonal matrix, and R is an upper triangular matrix such that $\rho = R^\top R$ is a correlation matrix. The model allows for a time-varying common volatility shape $\epsilon(t)$ of the form

$$\epsilon(t) = 1 + (\beta_0 - 1 + \beta_1\, t)e^{\beta_2\, t},$$

where β_0, β_1 and β_2 are three positive constants. Furthermore, the matrix q is given by

$$q_{i,j}(u;T,x) \quad = \quad e^{-\eta_i\, x}\, \mathrm{I}(i=j) \quad \text{for } \ i,j = 1,...,d$$

where η is a deterministic constant vector.

H.2 Markovian specification for the weighted Gaussian model

By plugging the expression for the volatility into formula (36), it is possible to work out the expression leading to the following Markovian representation of the FRA rate:

$$\log\left(\frac{1+x\,F^x(t,T,x)}{1+x\,F^x(0,T,x)}\right) = G(t,T,x)^\top\,\mathbf{X}(t) + a(t,T,x), \tag{38}$$

where $a(t,T,x)$ is a deterministic coefficient and it has the following form:

$$\begin{aligned} a(t,T,x) &= G(t,T,x)^\top\,Y(t)\left(G(t,T) - \frac{1}{2}G(t,T,x)\right) \\ (Y(t))_{ik} &= \int_0^t g_i(s,t)(h^\top(s)\,h(s))_{ik}g_k(s,t)ds \quad i,k=1,...,d, \end{aligned}$$

$G(t,T,x)$ is a deterministic vector with components

$$G_i(t,T,x) = \int_{T-x}^{T} q_{ii}(u;T,x)\,g_i(t,u)\,du,$$

$G(t,T)$ is a deterministic vector with components

$$G_i(t,T) = \int_t^T g_i(t,u)\,du,$$

and $\mathbf{X}(t)$ is a vector Markovian process with components, under the risk-neutral measure, in the form

$$X_i(t) = \sum_{j=1}^{d}\int_0^t g_i(s,t)\left(h_{i,j}^\top(s)dW_j(s) + (h^\top(s)\,h(s))_{i,j}\left(\int_s^t g_i(s,y)dy\right)\,ds\right).$$

A similar Markovian representation can be obtained for the ZCB price,

$$\log\left(P(t,T)\frac{P(0,t)}{P(0,T)}\right) = -G(t,T)^\top\,\mathbf{X}(t) + a(t,T), \tag{39}$$

where $a(t,T)$ is a deterministic coefficient and it has the following form:

$$a(t,T) \;=\; -\frac{1}{2}G(t,T)^\top Y(t)G(t,T).$$

I Parameters values

I.1 Three-factors Gaussian model and Cox–Ingersoll–Ross model

We verify the accuracy of our bounds using models and parameter values that have already been examined in the literature[3]

- Three-factor Gaussian model: $K = \begin{bmatrix} 1.0 & 0 & 0 \\ 0 & 0.2 & 0 \\ 0 & 0 & 0.5 \end{bmatrix}$, $\theta = [0, 0, 0]^\top$, $\boldsymbol{\sigma} = [0.01, 0.005, 0.002]^\top$, $\rho = \begin{bmatrix} 1 & -0.2 & -0.1 \\ -0.2 & 1 & 0.3 \\ -0.1 & 0.3 & 1 \end{bmatrix}$, $\Sigma = \mathrm{diag}(\boldsymbol{\sigma}) \quad \mathrm{chol}(\rho)$[4], $x_0 = [0.01,\ 0.005,\ -0.02]^\top$ and $\phi =$ 0.06;

- Two-factor Cox-Ingersoll–Ross model: $\mathbf{a} = [0.5080,\ -0.0010]^\top$, $\boldsymbol{\theta} = [0.4005,\ -0.7740]^\top$, $\boldsymbol{\sigma} = [0.023,\ 0.019]^\top$, $x_0 = [0.374,\ 0.258]^\top$ and $\phi =$ -0.58.

Numerical results for this model are shown in Tables 1 and 2.

Moreover, we specify the interval of parameters of the two-factor CIR model from which we extract the 100 parameters sets for the RMSD calculation: $x_0 \in [0.001, 0.5] \times [0.001, 0.5]$, $\phi \in [0.001, 1]$, $\mathbf{a} \in [0.001, 1] \times [0.001, 1]$, $\boldsymbol{\theta} \in [0.001, 1] \times [0.001, 1]$, $\boldsymbol{\sigma} \in [0.001, \sqrt{2a(1)\theta(1)}] \times [0.001, \sqrt{2a(2)\theta(2)}]$.

I.2 Two-factor Gaussian model with double exponential jumps

We test the affine Gaussian model with exponentially distributed jumps using parameter values obtained by minimization of the least square distance between the model and the market discount curve implied by bootstrapping the Euribor six-month swap curve up to 30 years. The calibration is performed on January 4^{th}, 2015, to obtain the parameters set reported below.

Parameters:

- Gaussian parameters: $K = \begin{bmatrix} 0.050926 & 0 \\ 0 & 1.3687 \end{bmatrix}$, $\theta = [0,\ 0]^\top$, $\boldsymbol{\sigma} = [0.0048887,\ 0.24025]^\top$, $\rho = \begin{bmatrix} 1 & -0.1482 \\ -0.1482 & 1 \end{bmatrix}$, $\Sigma = \mathrm{diag}(\boldsymbol{\sigma})\ \mathrm{chol}(\rho)$,

[3] Schrager and Pelsser (2006) and Duffie and Singleton (1997) for the two-factor CIR model.

[4] $\mathrm{diag}(\boldsymbol{\sigma})$ means the diagonalization of the vector $\boldsymbol{\sigma}$ and $\mathrm{chol}(\rho)$ means the Cholesky decomposition of the correlation matrix ρ, where $\boldsymbol{\sigma}$ and ρ are the volatility vector and the correlation matrix, respectively, of the original paper.

$x_0 = [0.00035256,\ 0.00035497]^\top$ and $\phi = 4.332\ \times 10^{-5}$;

- Jump parameters: $\boldsymbol{\mu}^+ = 0.4372$, $\mathbf{m}^+ = [0.027372, 0.045667]^\top$,

  $\boldsymbol{\mu}^- = 0.1101$, $\mathbf{m}^- = [0.027043, 0.012339]^\top$.

Figure 3 shows fitting of the calibration. Numerical results for this model are shown in Table 3.

[Figure 3 approximately here]

I.3 Two-factor quadratic Gaussian model

Beyond the affine framework, we test the two-factor quadratic Gaussian model using the following parameter values as proposed by Kim (2007):

$K = \begin{bmatrix} -0.0541 & 0.0361 \\ -1.2113 & 0.4376 \end{bmatrix}$,

$\theta = [0.1932,\ 0.1421]^\top$, $\Sigma = \begin{bmatrix} 0.0145 & 0 \\ 0 & 0.0236 \end{bmatrix}$, $x_0 = [0.1690,\ -0.0501]^\top$,

$a_r = 0.0444$, $b_r = [0,\ 0]^\top$ and $C_r = \begin{bmatrix} 1 & 0.4412 \\ 0.4412 & 1 \end{bmatrix}$; Numerical results for this model are shown in Table 4.

I.4 Multiple-curve two-factor Gaussian model

We verify the accuracy of our bounds using the following fixed parameters:

$\lambda = [0.0073, 4.7344]$, $\eta = [0.1581, 0.8894]$, $h = [0.0059, 0.0411]$, $\rho_{12} = -0.8577$, $\beta_0 = 1.3160$, $\beta_1 = 1.3327$ and $\beta_2 = 0.5900$. Numerical results for this model are shown in Table 5.