Systemic Risk: An Asymptotic Evaluation

Alexandru V. Asimit[a] and Jinzhu Li[b],*

[a] Cass Business School, City University London,

London EC1Y 8TZ, United Kingdom.

E-mail: asimit@city.ac.uk

[b] School of Mathematical Science and LPMC,

Nankai University, Tianjin 300071, P.R. China.

E-mail: lijinzhu@nankai.edu.cn

February 5, 2017

Abstract

The *systemic risk (SR)* has been shown to play an important role in explaining the financial turmoils in the last several decades and understanding this source of risk has been a particular interest amongst academics, practitioners, and regulators. The precise mathematical formulation of the SR is still scrutinised, but the main purpose is to evaluate the financial distress of a system as a result of the failure of one component of the financial system in question. Many of the mathematical definitions of the SR are based on evaluating expectations in extreme regions and therefore, *Extreme Value Theory (EVT)* represents the key ingredient in producing valuable estimates of the SR and even its decomposition per individual components of the entire system. Without doubt, the prescribed dependence model amongst the system components has a major impact over our asymptotic approximations. Thus, this paper considers various well-known dependence models in the EVT literature that allow us to generate SR estimates. Interestingly, our findings reveal sensible results. That is, the SR has a significant impact under asymptotic dependence, while weak tail dependence, known as asymptotic independence, produces an insignificant loss over the regulatory capital.

Keywords: asymptotics; dependence; max-domain of attraction; regular variation; rapid variation; systemic risk

Mathematics Subject Classification: Primary 62P05; Secondary 62H20, 60E05

1 Introduction

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and denote by $L_+(\mathbb{P})$ the set of non-negative random variables with ultimate right tails. Consider $X \in L_+(\mathbb{P})$ and $Y \in L_+(\mathbb{P})$ as two random insurance risks possessing distribution functions F and G, respectively. The corresponding survival functions are $\overline{F} := 1 - F$ and $\overline{G} := 1 - G$.

*Corresponding author: Phone: +86-22-23501233.

The *systemic risk (SR)* has played an important role in explaining the recent financial turmoils from the banking and insurance industries and understanding this source of risk has been a particular interest amongst academics, practitioners, and regulators. The precise mathematical formulation of the SR is still debated, but the main purpose is to evaluate the financial distress of a system as a result of the failure of one component of the financial system in question. We follow in this paper the precise methodology from Acharya *et al.* (2012), where the SR represents the expected capital shortfall of the system when one component of the system is in financial distress. The system could be viewed as the entire industry or a conglomerate/group of firms, while individual components could be a single firm within the industry, a legal entity/subsidiary of the group, or even a line of business. In a nutshell, this paper evaluates conditional expectations of the under-capitalisation of the system when one component is under-capitalised. This could be viewed as an insurance definition, rather than a corporate finance definition, but it is sufficiently general to be acceptable for applications within banking and insurance industries.

Alternative SR definitions to that provided in Acharya *et al.* (2012) have appeared in various forms in the existing literature. For example, a quite similar approach is given in Adrian and Brunnermeier (2009), while specific SR definitions to a generic banking system are investigated in Acharya (2009) and Rogers and Veraart (2013). A more comprehensive work is given in Chen *et al.* (2013), where an axiomatic approach to the SR is developed. Finally, Feinstein *et al.* (2016) investigates a more general mathematical representation for the SR, which incorporates many of the previously-mentioned approaches.

We choose the more simple SR definition from Acharya *et al.* (2012), since it is more transparent than all other definitions from the pedagogical point of view. That is, we consider the following expected shortfall:

$$\rho_{X,Y}(q) := \mathbb{E}\Big[\big(X - t_1(q)\big)_+ | Y > t_2(q)\Big], \tag{1.1}$$

where $t_1(q)$ and $t_2(q)$ are two positive functions for $q \in (0,1)$ such that $\lim_{q\uparrow 1} t_2(q) = \infty$, and by definition, $b_+ := \max\{b, 0\}$ for any real number b. The definition from (1.1) is very flexible and it is very sensitive to the chosen extreme region, which naturally could be related to the level of available capital. In this paper, we aim to find asymptotic evaluations for $\rho_{X,Y}(q)$ as $q \uparrow 1$.

The rest of this paper consists of four sections. Section 2 introduces some necessary preliminaries of various concepts and notations, Section 3 shows our main asymptotic results for $\rho_{X,Y}(q)$, Section 4 provides some SR applications of our main results, and all proofs are relegated in Section 5.

2 Preliminaries

Let $\big\{Z, Z_i; i \geq 1\big\}$ be a sequence of independent and identically distributed random variables with common distribution function V possessing an ultimate right tail. *Extreme Value Theory (EVT)* assumes that there are constants $a_n > 0$ and $b_n \in (-\infty, \infty)$ such that

$$\lim_{n\to\infty} \mathbb{P}\left(a_n\left(\max_{1\le i\le n} Z_i - b_n\right) \le x\right) = H(x), \quad x \in (-\infty, \infty).$$

In this case, H is called an *Extreme Value Distribution* and V is said to belong to the *max-domain of attraction of* H, denoted by $V \in \mathrm{MDA}(H)$. By the Fisher-Tippett Theorem (see Fisher and Tippett, 1928), if the limit distribution H is non-degenerate, then it is of one of the following two types: $\Phi_\alpha(x) = \exp\{-x^{-\alpha}\}$ for all $x > 0$ with $\alpha > 0$, or $\Lambda(x) = \exp\left\{-\mathrm{e}^{-x}\right\}$ for all $x \in (-\infty, \infty)$. In the first case, Z has a *Fréchet* tail, which implies that its survival function is *regularly varying* at ∞ with index $-\alpha$ for some $\alpha > 0$, i.e.

$$\lim_{t\to\infty} \frac{\overline{V}(tx)}{\overline{V}(t)} = x^{-\alpha}, \quad x > 0.$$

We signify the above relation by $V \in \mathcal{R}_{-\alpha}$. In other words, $V \in \mathrm{MDA}(\Phi_\alpha)$ if and only if $V \in \mathcal{R}_{-\alpha}$. In the second case, Z has a Gumbel tail and it is well-known that there exists a positive auxiliary function $a(\cdot)$ such that $a(t) = o(t)$ as $t \to \infty$ and

$$\lim_{t\to\infty} \frac{\overline{V}(t + xa(t))}{\overline{V}(t)} = \mathrm{e}^{-x}, \quad x \in (-\infty, \infty). \tag{2.1}$$

As usual, we write $V \in \mathrm{MDA}(\Lambda)$ if (2.1) holds. Additionally, by Theorem 3.3.26 of Embrechts *et al.* (1997), the auxiliary function a can be chosen such that

$$a(t) = \frac{1}{\overline{V}(t)} \int_t^\infty \overline{V}(s)\mathrm{d}s, \quad t \in (-\infty, \infty). \tag{2.2}$$

Moreover, relation (2.1) implies that Z has a rapidly varying tail, written as $V \in \mathcal{R}_{-\infty}$, which by definition means that

$$\lim_{t\to\infty} \frac{\overline{V}(tx)}{\overline{V}(t)} = 0, \quad x > 1.$$

We refer the reader to Bingham *et al.* (1987) or Embrechts *et al.* (1997) for further details of the above statements.

For a distribution function V with an ultimate right tail, we define its lower Matuszewska index as

$$\alpha_V^* := \sup\left\{-\frac{\log \overline{V}^*(x)}{\log x} : x > 1\right\} \in [0, \infty],$$

where $\overline{V}^*(x) := \limsup_{t\to\infty} \overline{V}(tx)/\overline{V}(t)$. It is clear that $0 < \alpha_V^* \le \infty$ if and only if $\overline{V}^*(x) < 1$ for some $x > 1$. In this case, Proposition 2.2.1 of Bingham *et al.* (1987) tells us that, for every $0 < \alpha' < \alpha_V^*$, there are some $K > 1$ and $t_0 > 0$ such that the relation

$$\frac{\overline{V}(tx)}{\overline{V}(t)} \le Kx^{-\alpha'} \tag{2.3}$$

holds for all $tx > t > t_0$. It is not difficult to see that if $V \in \mathcal{R}_{-\alpha}$ for some $0 < \alpha \le \infty$ then $\alpha_V^* = \alpha$.

We now introduce the concept of *copula*, which is a commonly-used tool for measuring dependence amongst random variables. Let Z_1 and Z_2 be two random variables with distribution functions V_1 and V_2, respectively. It is well-known that the dependence structure associated with the distribution of a random vector can be characterized in terms of its copula, whenever it exists. A bivariate copula is a two-dimensional distribution function defined on $[0,1]^2$ with uniformly distributed marginals. Due to Sklar's Theorem (see Sklar, 1959), if V_1 and V_2 are continuous, then there exists a unique copula, $C(\cdot,\cdot)$, such that

$$\mathbb{P}\big(Z_1 \le x, Z_2 \le y\big) = C\big(V_1(x), V_2(y)\big).$$

Similarly, the *survival copula*, $\hat{C}(\cdot,\cdot)$, is defined as the copula corresponding to the joint survival function satisfying

$$\mathbb{P}\big(Z_1 > x, Z_2 > y\big) = \hat{C}\big(\overline{V}_1(x), \overline{V}_2(y)\big).$$

Clearly, $C(\cdot,\cdot)$ and $\hat{C}(\cdot,\cdot)$ are connected by the following relation:

$$\hat{C}(u,v) = u + v - 1 + C(1-u, 1-v), \quad (u,v) \in [0,1]^2.$$

We refer the reader to Nelsen (2006) for a comprehensive discussion on copulas.

For a non-decreasing function $f(\cdot)$, define its *generalized inverse function* by

$$f^{\leftarrow}(y) = \inf\big\{x : f(x) \ge y\big\},$$

where by convention, $\inf \emptyset = \infty$. Two random variables Z_1 and Z_2 with distribution functions V_1 and V_2 are said to be *asymptotically independent* if

$$\lim_{q \uparrow 1} \mathbb{P}\big(\, Z_2 > V_2^{\leftarrow}(q) \,|\, Z_1 > V_1^{\leftarrow}(q)\big) = 0. \tag{2.4}$$

Moreover, Z_1 and Z_2 are said to be *asymptotically dependent* if

$$\liminf_{q \uparrow 1} \mathbb{P}\big(\, Z_2 > V_2^{\leftarrow}(q) \,|\, Z_1 > V_1^{\leftarrow}(q)\big) > 0. \tag{2.5}$$

Recall that the concept of asymptotic independence stems from Definition 5.30 of McNeil *et al.* (2005) and not only, while the asymptotic dependence is related to equation (1.2) of Asimit *et al.* (2011). It is not difficult to find that, if Z_1 and Z_2 are continuous random variables with copula $C(\cdot,\cdot)$, then (2.4) and (2.5) can be respectively rewritten as

$$\lim_{u \downarrow 0} \frac{\hat{C}(u,u)}{u} = 0 \quad \text{and} \quad \liminf_{u \downarrow 0} \frac{\hat{C}(u,u)}{u} > 0. \tag{2.6}$$

An important notion for detailing our examples is the *vague convergence*. Let $\{\mu_n; n \ge 1\}$ be a sequence of measures on a locally compact Hausdorff space $\mathbb{B}$ with countable base. Then, μ_n converges vaguely to some measure μ, written as $\mu_n \xrightarrow{v} \mu$, if for all continuous functions f with compact support we have

$$\lim_{n \to \infty} \int_{\mathbb{B}} f \,\mathrm{d}\mu_n = \int_{\mathbb{B}} f \,\mathrm{d}\mu.$$

Note that we deal only with Radon measures, i.e. measures that are finite for every compact set in $\mathbb{B}$. A thorough background on vague convergence is given by Kallenberg (1983) and Resnick (1987).

We end this section with a summary of notations used in this paper. Unless otherwise stated, all limit relationships hold as $q \uparrow 1$ or $t \to \infty$, which will be further specified whenever a relation appears. For two positive functions $f_1(\cdot)$ and $f_2(\cdot)$, we write $f_1(\cdot) \sim f_2(\cdot)$ if $\lim f_1(\cdot)/f_2(\cdot) = 1$, write $f_1(\cdot) = O\big(f_2(\cdot)\big)$ if $\limsup f_1(\cdot)/f_2(\cdot) < \infty$, write $f_1(\cdot) = o\big(f_2(\cdot)\big)$ if $\lim f_1(\cdot)/f_2(\cdot) = 0$, and write $f_1(\cdot) \asymp f_2(\cdot)$ if both $f_1(\cdot) = O\big(f_2(\cdot)\big)$ and $f_2(\cdot) = O\big(f_1(\cdot)\big)$. Finally, $\mathbf{1}_{\{\cdot\}}$ represents the indicator function.

3 Main Results of Expected Shortfall

This section investigates asymptotic approximations for the expected shortfall $\rho_{X,Y}(q)$ defined in (1.1). Our main results require a general assumption stated as Assumption 3.1, in which (3.1) describes a general dependence structure including both asymptotic independence and asymptotic dependence cases. Recall that the distribution functions of X and Y are F and G, respectively.

Assumption 3.1. *Let $\overline{F}(t) = O\big(\overline{G}(t)\big)$ and let the limit*

$$\lim_{t\to\infty} \mathbb{P}\big(X > tx | Y > t\big) := h(x) \in [0,1] \tag{3.1}$$

exist almost everywhere for $x > 0$.

Remark 3.1. *Clearly, Assumption 3.1 may not be symmetric with respect to X and Y, i.e. given that (X,Y) satisfies Assumption 3.1 we can not conclude that the same assumption holds for (Y,X). However, if the limit*

$$\lim_{t\to\infty} \overline{G}(tx)/\overline{F}(t) := \kappa(x) \in (0,\infty) \tag{3.2}$$

exists at $x = 1$ and almost everywhere for other $x > 0$, then (X,Y) satisfies Assumption 3.1 if and only if the same assumption holds for (Y,X). In fact, if (3.1) holds for (X,Y) then we have for almost all $x > 0$ that

$$\lim_{t\to\infty} \mathbb{P}\left(Y > tx | X > t\right) = \lim_{t\to\infty} \frac{\mathbb{P}\left(Y > tx, X > tx\frac{1}{x}\right)}{\overline{G}(tx)} \frac{\overline{G}(tx)}{\overline{F}(t)} = h\big(1/x\big)\kappa(x),$$

provided that condition (3.2) holds. This indicates that (3.1) also holds for (Y,X) with $h(\cdot)$ replaced by $h(1/\cdot)\kappa(\cdot)$. Similarly, one may verify that if relation (3.1) holds for (Y,X) with a function $h(\cdot)$, then (3.1) also holds for (X,Y) with a function $h(1/\cdot)/\kappa(1/\cdot)$.

Remark 3.2. *It is not difficult to check that if $F \in \mathcal{R}_{-\alpha}$ with $0 < \alpha \le \infty$, then (X,X) satisfies Assumption 3.1 with*

$$h(x) = \begin{cases} 1, & 0 < x \le 1, \\ x^{-\alpha}\mathbf{1}_{\{\alpha<\infty\}} + 0 \cdot \mathbf{1}_{\{\alpha=\infty\}}, & x > 1. \end{cases}$$

For properties of the function h in (3.1), we refer the reader to Lemma 3.1 of Asimit and Li (2016). The following Proposition 3.1 gives some sufficient conditions to verify the asymptotic independence and asymptotic dependence between X and Y that satisfy Assumption 3.1 within the scope of regular variation and rapid variation.

Proposition 3.1. *Let X and Y satisfy Assumption 3.1.*

(i) Assume that $F \in \mathcal{R}_{-\alpha}$, $G \in \mathcal{R}_{-\alpha}$ for some $\alpha > 0$, and $\lim_{t\to\infty} \overline{F}(t)/\overline{G}(t) = r$ for some $r \geq 0$. Then, X and Y are asymptotically independent if there is some $0 < r_1 < r^{1/\alpha}$ (hence, $r > 0$ must hold) such that $h(r_1) = 0$. Moreover, X and Y are asymptotically dependent if there is some $r_2 > r^{1/\alpha}$ such that $h(r_2) > 0$.

(ii) Assume that $F \in \mathcal{R}_{-\infty}$, $G \in \mathcal{R}_{-\infty}$, and $\overline{F}(tr) \asymp \overline{G}(t)$ as $t \to \infty$ for some $r > 0$. Then, X and Y are asymptotically independent if there is some $0 < r_1 < r$ such that $h(r_1) = 0$. Moreover, X and Y are asymptotically dependent if there is some $r_2 > r$ such that $h(r_2) > 0$.

We are now ready to state our first asymptotic result for $\rho_{X,Y}(q)$, which is provided in the next theorem. Recall that α_F^* represents the lower Matuszewska index of F as defined in Section 2.

Theorem 3.1. *Consider the expected shortfall $\rho_{X,Y}(q)$ defined as (1.1). Assume that $1 < \alpha_F^* \leq \infty$ and Assumption 3.1 holds. If $\lim_{q\uparrow 1} t_1(q)/t_2(q) = c$ for some $c \geq 0$, then*

$$\lim_{q\uparrow 1} \frac{\rho_{X,Y}(q)}{t_2(q)} = \int_c^\infty h(x)\mathrm{d}x. \tag{3.3}$$

Observing (3.3), one may find that this relation could fail to provide a precise approximation for $\rho_{X,Y}(q)$ under various asymptotic independence cases, for which the function h from (3.1) is often 0 on (c, ∞). To overcome this drawback, we next introduce between X and Y a general asymptotic independence assumption, under which a precise approximation for $\rho_{X,Y}(q)$ can be obtained within the scope of regular variation and rapid variation.

Assumption 3.2. *There is some $\sigma > 0$ such that*

$$\mathbb{P}\big(X > t_1, Y > t_2\big) \sim \sigma \overline{F}\big(t_1\big)\overline{G}\big(t_2\big), \quad \big(t_1, t_2\big) \to (\infty, \infty).$$

Clearly, if (X, Y) satisfies Assumption 3.2, then it satisfies Assumption 3.1 with $h(x) \equiv 0$. Hence, Assumption 3.2 is actually a refinement to Assumption 3.1 in the asymptotic independence case. It should be clear how to translate Assumption 3.2 into a copula form. That is, if (X, Y) has copula $C(\cdot, \cdot)$, then there is some $\sigma > 0$ such that

$$\hat{C}(u, v) \sim \sigma uv, \quad (u, v) \to (0+, 0+).$$

The above relation clearly represents the asymptotic independence in view of (2.6). It is interesting to note that Li (2016) proposes Assumption 3.2 in the context of ruin theory and provides many copula examples to illustrate the high degree of generality of this assumption. Some commonly-used copulas satisfying Assumption 3.2 are listed in Examples 3.1–3.3, but for further details one could find in Section 2 of Li (2016).

Example 3.1. *The Farlie-Gumbel-Morgenstern (FGM) copula*

$$C(u, v) = uv + \theta uv(1 - u)(1 - v), \qquad \theta \in (-1, 1],$$

satisfies Assumption 3.2 with $\sigma = 1 + \theta$.

Example 3.2. *The Ali-Mikhail-Haq copula*

$$C(u,v) = \frac{uv}{1-\theta(1-u)(1-v)}, \qquad \theta \in (-1,1],$$

satisfies Assumption 3.2 with $\sigma = 1+\theta$*.*

Example 3.3. *The Frank copula*

$$C(u,v) = -\frac{1}{\theta} \ln \left(1 + \frac{\left(\mathrm{e}^{-\theta u} - 1\right)\left(\mathrm{e}^{-\theta v} - 1\right)}{\mathrm{e}^{-\theta} - 1} \right), \qquad \theta \neq 0,$$

satisfies Assumption 3.2 with $\sigma = \theta \mathrm{e}^{\theta} / \left(\mathrm{e}^{\theta} - 1\right)$*.*

Our second main asymptotic result for $\rho_{X,Y}(q)$ is now given in the next theorem.

Theorem 3.2. *Consider the expected shortfall* $\rho_{X,Y}(q)$ *defined as (1.1) with* $\lim_{q\uparrow 1} t_1(q) = \infty$*. Assume that Assumption 3.2 holds.*

(i) If $F \in \mathcal{R}_{-\alpha}$ *for some* $\alpha > 1$*, then*

$$\rho_{X,Y}(q) \sim \frac{\sigma}{\alpha - 1} t_1(q) \overline{F}\big(t_1(q)\big), \quad q \uparrow 1.$$

(ii) If $F \in \mathrm{MDA}(\Lambda)$ *with an auxiliary function* a*, then*

$$\rho_{X,Y}(q) \sim \sigma a\left(t_1(q)\right) \overline{F}\big(t_1(q)\big), \quad q \uparrow 1.$$

4 Applications to Systemic Risk

This section is devoted to apply our main findings from Section 3 to provide SR evaluations within a system consisting of a finite number of components. Assume that there are d lines of businesses or legal entities with random liabilities $X_1, \ldots, X_d$ and that the regulator sets a total capital in amount of $\sum_{i=1}^{d} C_i$, where C_i is the capital allocated to each entity. In what follows, we write $S_d := \sum_{i=1}^{d} X_i$. Without loss of generality, we assume that the first line of business or legal entity is in financial distress. Therefore, the aggregate SR becomes

$$SR := \mathbb{E}\left[\left(S_d - \sum_{i=1}^{d} C_i \right)_+ \middle| X_1 > C_1 \right].$$

The individual SR contribution to the k^{th} component is defined as follows:

$$SR_k := \mathbb{E}\Big[\big(X_k - C_k\big)_+ | X_1 > C_1 \Big], \quad k \in \{1, \ldots, d\}.$$

Clearly, the above definitions heavily depend on the way the regulatory capital is defined. Now, a $CVaR_q$-based regulatory environment requires a total capital of $CVaR_q\big(S_d\big)$ and the most common and practical capital decomposition rule is the Euler one, where C_k is replaced by the following:

$$C_{k,CVaR}(q) := \mathbb{E}\Big[X_k|S_d > VaR_q\big(S_d\big)\Big], \quad k \in \{1, \ldots, d\}.$$

Thus, the SR and SR_k's under the $CVaR_q$-based allocations can be expressed as

$$SR_{CVaR}(q) := \mathbb{E}\left[\left(S_d - \sum_{i=1}^{d} C_{i,CVaR}(q)\right)_+ \middle| X_1 > C_{1,CVaR}(q)\right] \tag{4.1}$$

and

$$SR_{k,CVaR}(q) := \mathbb{E}\Big[\big(X_k - C_{k,CVaR}(q)\big)_+ |X_1 > C_{1,CVaR}(q)\Big], \quad k \in \{1, \ldots, d\}. \tag{4.2}$$

If the entire system is VaR_q regulated, then the total capital is $VaR_q\big(S_d\big)$ and it is decomposed at the individual component (via Euler decomposition) in the following fashion:

$$\mathbb{E}\Big[X_k|S_d = VaR_q\big(S_d\big)\Big], \quad k \in \{1, \ldots, d\}.$$

However, this capital allocation raises many issues and it is difficult to evaluate or estimate and hence, a more practical solution (for example, see Overbeck, 2000, Kalkbrener, 2005, or Bluhm *et al.*, 2006) is to use a surrogate $CVaR_{\beta(q)}$-type allocation rule in the following fashion:

$$C_{k,VaR}(q) := \mathbb{E}\Big[X_k|S_d > VaR_{\beta(q)}\big(S_d\big)\Big], \quad k \in \{1, \ldots, d\},$$

where

$$\beta(q) := \inf_{u\in(0,1]} \Big\{VaR_q\big(S_d\big) \le CVaR_u\big(S_d\big)\Big\}.$$

As before, we write the SR and SR_k's under the VaR_q-based allocations as

$$SR_{VaR}(q) := \mathbb{E}\left[\left(S_d - \sum_{i=1}^{d} C_{i,VaR}(q)\right)_+ \middle| X_1 > C_{1,VaR}(q)\right] \tag{4.3}$$

and

$$SR_{k,VaR}(q) := \mathbb{E}\Big[\big(X_k - C_{k,VaR}(q)\big)_+ |X_1 > C_{1,VaR}(q)\Big], \quad k \in \{1, \ldots, d\}. \tag{4.4}$$

The next corollary of Theorem 3.1 enables us to obtain asymptotic approximations for $SR_{CVaR}(q)$, $SR_{k,CVaR}(q)$, $SR_{VaR}(q)$, and $SR_{k,VaR}(q)$ defined in (4.1)–(4.4) as $q \uparrow 1$.

Corollary 4.1. *Consider the expected shortfall $\rho_{X,Y}(q)$ defined as (1.1) with*

$$t_1(q) = \mathbb{E}\left[X\,|Z > t(q)\right] \;\; \textit{and} \;\; t_2(q) = \mathbb{E}\left[Y\,|Z > t(q)\right],$$

where $Z \in L_+(\mathbb{P})$ is another random variable and $t(q)$ is a positive function for $q \in (0,1)$ such that $\lim_{q\uparrow 1} t(q) = \infty$. Assume that (X,Y), (X,Z), and (Y,Z) satisfy Assumption

3.1 with limiting functions h, h_1, and h_2, respectively, such that $\int_0^\infty h_2(x)\mathrm{d}x > 0$. Assume further that $1 < \alpha_F^* \le \infty$ and $1 < \alpha_G^* \le \infty$. Then, we have

$$\lim_{q\uparrow 1} \frac{\rho_{X,Y}(q)}{t(q)} = \int_{\widetilde{h}_1}^{\infty} h\left(\frac{x}{\widetilde{h}_2}\right)\mathrm{d}x, \tag{4.5}$$

where

$$\widetilde{h}_i = \int_0^\infty h_i(x)\mathrm{d}x, \quad i \in \{1, 2\}.$$

Corollary 4.1 is applicable in multiple situations. We next give four general examples where $(X_1, \ldots, X_d)$ is a non-negative random vector with marginal distribution functions $F_1, \ldots, F_d$.

Example 4.1. *Assume that there is some function $H_\Psi(\cdot)$ such that the relation*

$$\lim_{t\to\infty} \frac{\mathbb{P}(X_1 > tx_1, \ldots, X_d > tx_d)}{\overline{F}_1(t)} = H_\Psi(\mathbf{x}) \tag{4.6}$$

holds for every $\mathbf{x} := (x_1, \ldots, x_d) \in [0,\infty]^d \backslash \{\mathbf{0}\}$.

By Lemma 2.2 of Asimit et al. (2011), in this case $X_1, \ldots, X_d$ are pairwise asymptotically dependent, which can also be verified by our Proposition 3.1(i). Relation (4.6) implies that the relation

$$\frac{\mathbb{P}((X_1/t, \ldots, X_d/t) \in \cdot)}{\overline{F}_1(t)} \xrightarrow{v} \mu_\Psi(\cdot), \quad t \to \infty, \tag{4.7}$$

holds on $[0,\infty]^d \backslash \{\mathbf{0}\}$ with a measure μ_Ψ such that

$$\mu_\Psi\big(\mathbf{y} : y_i > x_i, \text{ for all } i \in \{1, \ldots, d\}\big) = H_\Psi(\mathbf{x}).$$

Example 4.1 of Asimit and Li (2016) tells us that $F_1 \in \mathcal{R}_{-\alpha}$ for some $\alpha \in [0,\infty)$ with $\alpha_{F_1}^* = \alpha$,

$$\lim_{t\to\infty} \frac{\overline{F}_k(t)}{\overline{F}_1(t)} = \mu_\Psi(\mathbf{y} : y_k > 1) > 0, \quad k \in \{1, \ldots, d\}, \tag{4.8}$$

$$\lim_{t\to\infty} \frac{\mathbb{P}(S_d > tx)}{\overline{F}_1(t)} = \mu_\Psi\left(\mathbf{y} : \sum_{i=1}^d y_i > x\right) > 0, \quad x > 0, \tag{4.9}$$

and (X_k, S_d) satisfies Assumption 3.1 with

$$h_{k,S}(x) = \frac{\mu_\Psi\left(\mathbf{y} : y_k > x, \sum_{i=1}^d y_i > 1\right)}{\mu_\Psi\left(\mathbf{y} : \sum_{i=1}^d y_i > 1\right)}\mathbf{1}_{\{0<x<1\}} + \frac{\mu_\Psi(\mathbf{y} : y_k > 1)}{\mu_\Psi\left(\mathbf{y} : \sum_{i=1}^d y_i > 1\right)} x^{-\alpha}\mathbf{1}_{\{x\ge 1\}} \tag{4.10}$$

for each $k \in \{1, \ldots, d\}$. Note that (4.8) and (4.9) indicate that the distribution functions of X_k's and S_d belong to $\mathcal{R}_{-\alpha}$. Assuming $\alpha > 1$, we may now derive the asymptotic approximations for $SR_{CVaR}(q)$, $SR_{k,CVaR}(q)$, $SR_{VaR}(q)$, and $SR_{k,VaR}(q)$ defined in (4.1)–(4.4) via Corollary 4.1.

We first deal with $SR_{CVaR}(q)$ and $SR_{VaR}(q)$, which quantify the aggregate SR evaluations. Recalling (4.1), (4.3), and Corollary 4.1, it is quite transparent to evaluate $SR_{CVaR}(q)$ and $SR_{VaR}(q)$ via Corollary 4.1 by setting

$$X = Z = S_d, \quad Y = X_1, \tag{4.11}$$

and

$$t(q) = \begin{cases} VaR_q\left(S_d\right), & \text{for the } CVaR_q\text{-based SR}, \\ VaR_{\beta(q)}(S_d), & \text{for the } VaR_q\text{-based SR}. \end{cases} \tag{4.12}$$

In view of (4.10), $(Y,Z) = \big(X_1, S_d\big)$ satisfies Assumption 3.1 with $h_{1,S}$, which corresponds to the function h_2 in Corollary 4.1. It follows from (4.7) that $(X,Y) = \big(S_d, X_1\big)$ satisfies Assumption 3.1 with

$$h_{S,1}(x) = \mathbf{1}_{\{0<x\le 1\}} + \mu_\Psi\left(\mathbf{y} : y_1 > 1, \sum_{i=1}^d y_i > x\right)\mathbf{1}_{\{x>1\}},$$

which corresponds to the function h in Corollary 4.1. In fact, the above relation may be also obtained by Remark 3.1 and (4.9) with noting the fact that μ_Ψ is homogeneous, i.e. $\mu_\Psi\left(x\mathbf{A}\right) = x^{-\alpha}\mu_\Psi\left(\mathbf{A}\right)$ for every $x > 0$ and $\mathbf{A} \subset [0,\infty]^d \backslash \{\mathbf{0}\}$, but utilising (4.7) is more transparent for this example. Moreover, recalling Remark 3.2, we know that $(X,Z) = (S_d, S_d)$ satisfies Assumption 3.1 with

$$h_{S,S}(x) = \mathbf{1}_{\{0<x\le 1\}} + x^{-\alpha}\mathbf{1}_{\{x>1\}}, \tag{4.13}$$

which corresponds to the function h_1 in Corollary 4.1. Now, plugging $h = h_{S,1}$, $h_1 = h_{S,S}$, and $h_2 = h_{1,S}$ into (4.5), we have that

$$\begin{aligned} \lim_{q\uparrow 1}\frac{SR_{CVaR}(q)}{VaR_q\left(S_d\right)} &= \lim_{q\uparrow 1}\frac{SR_{VaR}(q)}{VaR_{\beta(q)}(S_d)} \\ &= \int_{\alpha/(\alpha-1)}^{\infty} h_{S,1}\left(\frac{x}{\widetilde{h}_{1,S}}\right)\mathrm{d}x \\ &= \int_{\alpha/(\alpha-1)}^{\infty} \mu_\Psi\left(\mathbf{y} : y_1 > 1, \sum_{i=1}^d y_i > \frac{x}{\widetilde{h}_{1,S}}\right)\mathrm{d}x, \end{aligned}$$

where $\widetilde{h}_{1,S} = \int_0^\infty h_{1,S}(x)\mathrm{d}x$. Note that the last step of the above derivations follows from the fact that

$$\widetilde{h}_{1,S} \le \int_0^\infty h_{S,S}(x)\mathrm{d}x = \alpha/\left(\alpha-1\right),$$

which is a consequence of the fact that $h_{1,S}(x) \le h_{S,S}(x)$ for all $x > 0$.

We next focus on the individual SR contributions, namely $SR_{k,CVaR}(q)$ and $SR_{k,VaR}(q)$. Recalling (4.2), (4.4), and Corollary 4.1, we evaluate $SR_{k,CVaR}(q)$ and $SR_{k,VaR}(q)$ via Corollary 4.1 by setting (4.12) and

$$X = X_k, \quad Y = X_1, \quad Z = S_d. \tag{4.14}$$

By keeping (4.7) in mind, it is not difficult to check that $(X, Y) = (X_k, X_1)$ *satisfies Assumption 3.1 with*

$$h_{k,1}(x) = \mu_\Psi\left(\mathbf{y} : y_1 > 1, y_k > x\right).$$

Also, $(X, Z) = \big(X_k, S_d\big)$ *and* $(Y, Z) = \big(X_1, S_d\big)$ *satisfy Assumption 3.1 with* $h_{k,S}$ *and* $h_{1,S}$ *given by (4.10). Plugging* $h = h_{k,1}$*,* $h_1 = h_{k,S}$*, and* $h_2 = h_{1,S}$ *into (4.5) gives that, for each* $k \in \{1, \ldots, d\}$*,*

$$\lim_{q\uparrow 1} \frac{SR_{k,CVaR}(q)}{VaR_q\big(S_d\big)} = \lim_{q\uparrow 1} \frac{SR_{k,VaR}(q)}{VaR_{\beta(q)}(S_d)} = \int_{\widetilde{h}_{k,S}}^{\infty} \mu_\Psi\left(\mathbf{y} : y_1 > 1, y_k > \frac{x}{\widetilde{h}_{1,S}}\right) \mathrm{d}x,$$

where $\widetilde{h}_{k,S} = \int_0^\infty h_{k,S}(x)\mathrm{d}x$*.*

Example 4.2. *Let* $F_1 \in \mathcal{R}_{-\alpha}$ *for some* $\alpha > 1$*. Assume that* $\overline{F}_k(t) \sim b_k \overline{F}_1(t)$ *holds as* $t \to \infty$ *with* $b_k > 0$ *for each* $k \in \{1, \ldots, d\}$*. Moreover, for any* $1 \le i \ne j \le d$*, it holds that*

$$\lim_{t\to\infty} \frac{\mathbb{P}\big(X_i > tx, X_j > t\big)}{\overline{F}_1(t)} = 0, \quad x > 0. \tag{4.15}$$

One may check via Proposition 3.1(i) that $X_1, \ldots, X_d$ *are pairwise asymptotically independent. In addition,* $F_k \in \mathcal{R}_{-\alpha}$ *with* $\alpha^*_{F_k} = \alpha > 1$ *for each* $k \in \{1, \ldots, d\}$*. Further, by Lemma 2.1 of Davis and Resnick (1996) or Theorem 3.1 of Chen and Yuen (2009), we have*

$$\lim_{t\to\infty} \frac{\mathbb{P}\big(S_d > t\big)}{\overline{F}_1(t)} = \sum_{i=1}^d b_i > 0, \tag{4.16}$$

which implies that the distribution function of S_d *also belongs to* $\mathcal{R}_{-\alpha}$*.*

Consider first $SR_{CVaR}(q)$ *and* $SR_{VaR}(q)$ *that are further evaluated by setting (4.11) and (4.12). Similar derivations to those in Example 4.3 of Asimit and Li (2016) imply that, for every* $0 < x \le 1$*,*

$$\lim_{t\to\infty} \mathbb{P}\big(\, X_k > tx |\, S_d > t\big) = \frac{b_k}{\sum_{i=1}^d b_i}, \quad k \in \{1, \ldots, d\}.$$

Now, for every $x > 1$*, it holds that*

$$\lim_{t\to\infty} \mathbb{P}\big(\, X_k > tx |\, S_d > t\big) = \lim_{t\to\infty} \frac{\overline{F}_k(tx)}{\overline{F}_1(tx)} \frac{\overline{F}_1(tx)}{\overline{F}_1(t)} \frac{\overline{F}_1(t)}{\mathbb{P}\big(S_d > t\big)} = \frac{b_k x^{-\alpha}}{\sum_{i=1}^d b_i}, \quad k \in \{1, \ldots, d\},$$

where in the last step we used (4.16), $F_1 \in \mathcal{R}_{-\alpha}$*, and* $\overline{F}_k(t) \sim b_k \overline{F}_1(t)$ *as* $t \to \infty$*. Hence,* $\big(X_k, S_d\big)$ *satisfies Assumption 3.1 with*

$$h_{k,S}(x) = \frac{b_k\big(\mathbf{1}_{\{0<x\le 1\}} + x^{-\alpha} \cdot \mathbf{1}_{\{x>1\}}\big)}{\sum_{i=1}^d b_i}, \quad k \in \{1, \ldots, d\}, \tag{4.17}$$

which together with $b_1 = 1$ *indicate that* $(Y, Z) = \big(X_1, S_d\big)$ *satisfies Assumption 3.1 with*

$$h_{1,S}(x) = \frac{\mathbf{1}_{\{0<x\le 1\}} + x^{-\alpha} \cdot \mathbf{1}_{\{x>1\}}}{\sum_{i=1}^d b_i}.$$

Thus, by Remark 3.1, (4.16), and $F_1 \in \mathcal{R}_{-\alpha}$*, we get that* $(X,Y) = \big(S_d, X_1\big)$ *satisfies Assumption 3.1 with*

$$h_{S,1}(x) = h_{1,S}(1/x) \lim_{t\to\infty} \frac{\mathbb{P}\big(S_d > tx\big)}{\overline{F}_1(tx)} \frac{\overline{F}_1(tx)}{\overline{F}_1(t)} = \mathbf{1}_{\{0<x\le 1\}} + x^{-\alpha}\mathbf{1}_{\{x>1\}}.$$

Moreover, by Remark 3.2, $(X,Z) = \big(S_d, S_d\big)$ *satisfies Assumption 3.1 with* $h_{S,S}$ *given by (4.13). Plugging* $h = h_{S,1}$*,* $h_1 = h_{S,S}$*, and* $h_2 = h_{1,S}$ *into (4.5) leads to*

$$\begin{aligned} \lim_{q\uparrow 1} \frac{SR_{CVaR}(q)}{VaR_q\big(S_d\big)} &= \lim_{q\uparrow 1} \frac{SR_{VaR}(q)}{VaR_{\beta(q)}\big(S_d\big)} \\ &= \int_{\alpha/(\alpha-1)}^{\infty} \left(\frac{x}{\alpha/(\alpha-1)} \sum_{i=1}^{d} b_i \right)^{-\alpha} \mathrm{d}x \\ &= \frac{\alpha}{(\alpha-1)^2 \left(\sum_{i=1}^{d} b_i\right)^{\alpha}}. \end{aligned} \tag{4.18}$$

Next, we turn our attention to $SR_{k,CVaR}(q)$ *and* $SR_{k,VaR}(q)$ *that are evaluated by keeping in mind the settings specified in (4.12) and (4.14). Clearly, (4.15) and Remark 3.2 imply that* $(X,Y) = \big(X_k, X_1\big)$ *satisfies Assumption 3.1 with* $h_{k,1}(x) \equiv 0$ *for* $k \neq 1$ *and*

$$h_{1,1}(x) = \mathbf{1}_{\{0<x\le 1\}} + x^{-\alpha}\mathbf{1}_{\{x>1\}}.$$

Furthermore, $(X,Z) = \big(X_k, S_d\big)$ *and* $(Y,Z) = \big(X_1, S_d\big)$ *satisfy Assumption 3.1 with* $h_{k,S}$ *and* $h_{1,S}$ *as given by (4.17). Plugging* $h = h_{k,1}$*,* $h_1 = h_{k,S}$*, and* $h_2 = h_{1,S}$ *into (4.5) leads to*

$$\lim_{q\uparrow 1} \frac{SR_{1,CVaR}(q)}{VaR_q\big(S_d\big)} = \lim_{q\uparrow 1} \frac{SR_{1,VaR}(q)}{VaR_{\beta(q)}\big(S_d\big)} = \frac{\alpha}{(\alpha-1)^2 \sum_{i=1}^{d} b_i} \tag{4.19}$$

and

$$\lim_{q\uparrow 1} \frac{SR_{k,CVaR}(q)}{VaR_q\big(S_d\big)} = \lim_{q\uparrow 1} \frac{SR_{k,VaR}(q)}{VaR_{\beta(q)}\big(S_d\big)} = 0, \quad k \in \{2,\ldots,d\}. \tag{4.20}$$

As mentioned immediately after Theorem 3.1, although Corollary 4.1 is applicable to Example 4.2, it fails to provide precise approximations for $SR_{k,CVaR}(q)$ and $SR_{k,VaR}(q)$ with $k \in \{2,\ldots,d\}$ in (4.20) under such a framework of pairwise asymptotic independence. To remedy this drawback, we further discuss the next example, namely Example 4.3, which is a special case of Example 4.2 and allows us to obtain precise approximations for all $SR_{k,CVaR}(q)$ and $SR_{k,VaR}(q)$ with $k \in \{1,\ldots,d\}$ by the help of Theorem 3.2.

Example 4.3. *Consider the same set of conditions as in Example 4.2 with the additional condition that Assumption 3.2 holds for any* $\big(X_i, X_j\big)$ *with* $1 \le i < j \le d$*, i.e. there are some* $\sigma_{ij} = \sigma_{ji} > 0$ *such that*

$$\mathbb{P}\big(X_i > t_1, X_j > t_2\big) \sim \sigma_{ij}\overline{F}_i\big(t_1\big)\overline{F}_j\big(t_2\big), \quad \big(t_1, t_2\big) \to (\infty,\infty). \tag{4.21}$$

Clearly, (4.21) implies (4.15) due to $\overline{F}_k(t) \sim b_k\overline{F}_1(t)$ *as* $t \to \infty$ *and hence, this example is a special case of Example 4.2. Therefore, (4.18) and (4.19) still hold for* $SR_{CVaR}(q)$*,*

$SR_{VaR}(q)$, $SR_{1,CVaR}(q)$, and $SR_{1,VaR}(q)$. For each $k \neq 1$, by recalling (1.1), (4.2), and (4.4), we may find that the quantity $t_1(q)$ in (1.1) for $SR_{k,CVaR}(q)$ and $SR_{k,VaR}(q)$ corresponds to $\mathbb{E}\big[X_k | S_d > t(q)\big]$, where $t(q)$ is given by (4.12). As analysed in Example 4.2, $\big(X_k, S_d\big)$ satisfies Assumption 3.1 with $h_{k,S}$ shown in (4.17). Thus, applying Theorem 3.1 to $t_1(q)$, we have

$$t_1(q) \sim \left(\int_0^\infty h_{k,S}(x)\mathrm{d}x\right) t(q) = \frac{\alpha b_k}{(\alpha-1)\sum_{i=1}^d b_i} t(q), \quad q \uparrow 1, \tag{4.22}$$

which implies that $t_1(q) \to \infty$ as $q \uparrow 1$. Then, applying Theorem 3.2(i) to $SR_{k,CVaR}(q)$ with $t(q) = VaR_q\big(S_d\big)$ leads to

$$\begin{aligned} SR_{k,CVaR}(q) &\sim \frac{\sigma_{k1}}{\alpha-1} t_1(q) \overline{F}_k\big(t_1(q)\big) \\ &\sim \frac{\alpha^{1-\alpha}\sigma_{k1} b_k^{2-\alpha}}{(\alpha-1)^{2-\alpha}\left(\sum_{i=1}^d b_i\right)^{1-\alpha}} t(q)\overline{F}_1\big(t(q)\big), \quad q \uparrow 1, \end{aligned} \tag{4.23}$$

where the last equivalence is due to (4.22), $F_1 \in \mathcal{R}_{-\alpha}$, and $\overline{F}_k(t) \sim b_k \overline{F}_1(t)$ as $t \to \infty$. By using the same arguments, we could find that (4.23) also holds for $SR_{k,VaR}(q)$ with $t(q) = VaR_{\beta(q)}\big(S_d\big)$. Hence, we may conclude that the following holds for each $k \in \{2, \ldots, d\}$:

$$\begin{aligned} \lim_{q\uparrow 1} \frac{SR_{k,CVaR}(q)}{VaR_q\big(S_d\big)\overline{F}_1\Big(VaR_q\big(S_d\big)\Big)} &= \lim_{q\uparrow 1} \frac{SR_{k,VaR}(q)}{VaR_{\beta(q)}\big(S_d\big)\overline{F}_1\Big(VaR_{\beta(q)}\big(S_d\big)\Big)} \\ &= \frac{\alpha^{1-\alpha}\sigma_{k1} b_k^{2-\alpha}}{(\alpha-1)^{2-\alpha}\left(\sum_{i=1}^d b_i\right)^{1-\alpha}}. \end{aligned}$$

We next show an example concerning the rapid variation case, in which the dependence structure is motivated by Mitra and Resnick (2009) (see also Asimit *et al.*, 2011, Hashorva and Li, 2015, and Asimit and Li, 2016).

Example 4.4. *Let $F_1 \in \mathrm{MDA}(\Lambda)$ with an auxiliary function a. Assume that $\overline{F}_k(t) \sim b_k \overline{F}_1(t)$ holds as $t \to \infty$ with $b_k > 0$ for each $k \in \{1, \ldots, d\}$. Moreover, for any $1 \leq i \neq j \leq d$, Assumption 3.2 holds for $\big(X_i, X_j\big)$, i.e. (4.21) holds, and we also have*

$$\lim_{t\to\infty} \frac{\mathbb{P}\left(X_i > a(t)x, X_j > t\right)}{\overline{F}_1(t)} = 0, \quad \textit{for every } x > 0, \tag{4.24}$$

and

$$\lim_{t\to\infty} \frac{\mathbb{P}\left(X_i > L_{ij}a(t), X_j > L_{ij}a(t)\right)}{\overline{F}_1(t)} = 0, \quad \textit{for some } L_{ij} > 0. \tag{4.25}$$

Note that, if $a(t) \to \infty$ as $t \to \infty$, then (4.24) is implied by (4.21) and $\overline{F}_k(t) \sim b_k \overline{F}_1(t)$ as $t \to \infty$. This example is a special case of Example 4.3 of Asimit and Li (2016), by which we know that (4.16) also holds and $\big(X_k, S_d\big)$ satisfies Assumption 3.1 with

$$h_{k,S}(x) = \frac{b_k}{\sum_{i=1}^d b_i} \mathbf{1}_{\{0<x\leq 1\}} + 0 \cdot \mathbf{1}_{\{x>1\}}. \tag{4.26}$$

Relation (4.16) implies that the distribution function of S_d also belongs to $\mathrm{MDA}(\Lambda) \subset \mathcal{R}_{-\infty}$ *with the auxiliary function a. Then, by the similar arguments used in Example 4.2, we can check that Corollary 4.1 is also applicable to this example but fails to provide precise approximations for all of $SR_{CVaR}(q)$, $SR_{VaR}(q)$, $SR_{k,CVaR}(q)$, and $SR_{k,VaR}(q)$ with $k \in \{1, \ldots, d\}$. Thus, we have to seek precise approximations by other methods.*

As before, consider first $SR_{CVaR}(q)$ and $SR_{VaR}(q)$. Recalling (1.1), (4.1) and (4.3), we find that the quantities $t_1(q)$ and $t_2(q)$ in (1.1) for $SR_{CVaR}(q)$ and $SR_{VaR}(q)$ correspond respectively to $\mathbb{E}\big[S_d|S_d > t(q)\big]$ and $\mathbb{E}\big[X_1|S_d > t(q)\big]$, where $t(q)$ is given by (4.12). Recalling Remark 3.2, we know that (S_d, S_d) satisfies Assumption 3.1 with

$$h_{S,S}(x) = \mathbf{1}_{\{0<x\le 1\}} + 0 \cdot \mathbf{1}_{\{x>1\}}.$$

Thus, applying Theorem 3.1 to $t_1(q)$ and $t_2(q)$, we have

$$t_1(q) \sim \left(\int_0^\infty h_{S,S}(x) \mathrm{d}x \right) t(q) = t(q), \quad q \uparrow 1, \tag{4.27}$$

and

$$t_2(q) \sim \left(\int_0^\infty h_{1,S}(x) \mathrm{d}x \right) t(q) = \frac{1}{\sum_{i=1}^d b_i} t(q), \quad q \uparrow 1, \tag{4.28}$$

where we used (4.26) and $b_1 = 1$. Clearly, (4.27) and (4.28) means that both $t_1(q) \to \infty$ and $t_2(q) \to \infty$ as $q \uparrow 1$. Now, for any two numbers t_1 and t_2, we have

$$\begin{aligned}
\mathbb{P}(S_d > t_1, X_1 > t_2) &= \mathbb{P}(S_d > t_1) - \mathbb{P}(S_d > t_1, X_1 \le t_2) \\
&\le \mathbb{P}(S_d > t_1) - \mathbb{P}\left(\bigcup_{i=2}^d \{X_i > t_1\}, X_1 \le t_2 \right) \\
&= \mathbb{P}(S_d > t_1) - \mathbb{P}\left(\bigcup_{i=2}^d \{X_i > t_1\} \right) + \mathbb{P}\left(\bigcup_{i=2}^d \{X_i > t_1\}, X_1 > t_2 \right) \\
&\le \mathbb{P}(S_d > t_1) - \sum_{i=2}^d \mathbb{P}(X_i > t_1) + \sum_{2 \le i < j \le d} \mathbb{P}(X_i > t_1, X_j > t_2) \\
&\quad + \sum_{i=2}^d \mathbb{P}(X_i > t_1, X_1 > t_2).
\end{aligned}$$

Combining the above estimate with (4.16), (4.21), and $\overline{F}_k(t) \sim b_k \overline{F}_1(t)$ as $t \to \infty$, we obtain that

$$\limsup_{(t_1,t_2)\to(\infty,\infty)} \frac{\mathbb{P}\big(S_d > t_1, X_1 > t_2\big)}{\overline{F}_1\big(t_1\big)} \le \sum_{i=1}^d b_i - \sum_{i=2}^d b_i + 0 + 0 = b_1 = 1.$$

On the other hand, if we restrict the point (t_1, t_2) within the range $\{(x_1, x_2) : x_1 \ge x_2\}$, then it is trivial that

$$\liminf_{\substack{(t_1,t_2)\to(\infty,\infty) \\ t_1 \ge t_2}} \frac{\mathbb{P}\big(S_d > t_1, X_1 > t_2\big)}{\overline{F}_1\big(t_1\big)} \ge \liminf_{\substack{(t_1,t_2)\to(\infty,\infty) \\ t_1 \ge t_2}} \frac{\overline{F}_1\big(t_1\big)}{\overline{F}_1\big(t_1\big)} = 1.$$

Thus, it holds that

$$\lim_{\substack{(t_1,t_2)\to(\infty,\infty)\\ t_1\geq t_2}} \frac{\mathbb{P}\big(S_d > t_1, X_1 > t_2\big)}{\overline{F}_1\big(t_1\big)} = 1. \tag{4.29}$$

Therefore, with $t(q) = VaR_q\big(S_d\big)$, we have

$$\begin{aligned} SR_{CVaR}(q) &= \int_{t_1(q)}^{\infty} \mathbb{P}\big(S_d > x | X_1 > t_2(q)\big)\ \mathrm{d}x \\ &= \frac{1}{\overline{F}_1\big(t_2(q)\big)} \int_{t_1(q)}^{\infty} \mathbb{P}\big(S_d > x, X_1 > t_2(q)\big)\ \mathrm{d}x \\ &\sim \frac{1}{\overline{F}_1\big(t_2(q)\big)} \int_{t_1(q)}^{\infty} \overline{F}_1\,(x)\ \mathrm{d}x \\ &= \frac{a\big(t_1(q)\big)\overline{F}_1\big(t_1(q)\big)}{\overline{F}_1\big(t_2(q)\big)}, \quad q\uparrow 1, \end{aligned} \tag{4.30}$$

where the third step is due to (4.29) and $t_1(q) \geq t_2(q)$ and the last step follows from (2.2). Obviously, (4.30) also holds for $SR_{VaR}(q)$ with $t(q) = VaR_{\beta(q)}\big(S_d\big)$. Note that, although we have (4.27) and (4.28), we can not further refine (4.30) to a form with respect to $t(q)$, because in the rapid variation case $\overline{F}_1$ is much more sensitive to its variable than that in the regular variation case shown in Example 4.3.

Finally, we deal with $SR_{k,CVaR}(q)$ and $SR_{k,VaR}(q)$ for each $k \in \{1,\ldots,d\}$. As analysed in Example 4.3, the quantity $t_1(q)$ in (1.1) for $SR_{k,CVaR}(q)$ and $SR_{k,VaR}(q)$ corresponds to

$$\mathbb{E}\big[X_k|S_d > t(q)\big] =: t_{1,k}(q),$$

where $t(q)$ is given by (4.12). For $k = 1$, with $t(q) = VaR_q\big(S_d\big)$, it is easy to see from (2.2) that

$$\begin{aligned} SR_{1,CVaR}(q) &= \int_{t_{1,1}(q)}^{\infty} \mathbb{P}\big(X_1 > x | X_1 > t_{1,1}(q)\big)\ \mathrm{d}x \\ &= \frac{1}{\overline{F}_1\big(t_{1,1}(q)\big)} \int_{t_{1,1}(q)}^{\infty} \overline{F}_1\,(x)\ \mathrm{d}x \\ &= a\big(t_{1,1}(q)\big). \end{aligned} \tag{4.31}$$

For $k \neq 1$, since $\big(X_k, S_d\big)$ satisfies Assumption 3.1 with $h_{k,S}$ shown in (4.26), Theorem 3.1 tells us that

$$t_{1,k}(q) \sim \left(\int_0^{\infty} h_{k,S}(x)\mathrm{d}x\right) t(q) = \frac{b_k}{\sum_{i=1}^d b_i} t(q), \quad q \uparrow 1,$$

which indicates that $t_{1,k}(q) \to \infty$ as $q \uparrow 1$. Thus, applying Theorem 3.2(ii) to $SR_{k,CVaR}(q)$ with $t(q) = VaR_q\big(S_d\big)$ leads to

$$SR_{k,CVaR}(q) \sim \sigma_{k1} b_k a\big(t_{1,k}(q)\big)\overline{F}_1\big(t_{1,k}(q)\big), \quad q \uparrow 1, \tag{4.32}$$

where we used $\overline{F}_k(t) \sim b_k\overline{F}_1(t)$ as $t \to \infty$. Similarly, we can check that (4.31) and (4.32) also hold for $SR_{k,VaR}(q)$ with $t(q) = VaR_{\beta(q)}\big(S_d\big)$.

We end this section with four interesting remarks, in which Remarks 4.1–4.3 explain in great details some specific scenarios of Examples 4.2–4.4 while Remark 4.4 further discusses the limit of the right-hand side of (4.30) in Example 4.4 as $q \uparrow 1$.

Remark 4.1. *The dependence structure given by (4.21) in Example 4.3 is satisfied if the random vector $(X_1, \ldots, X_d)$ follows a multivariate FGM copula, which means that*

$$\mathbb{P}\big(X_1 \le x_1, \ldots, X_d \le x_d\big) = \left(\prod_{i=1}^{d} F_i(x_i)\right)\left(1+\sum_{k=2}^{d} \sum_{1\le j_1<\cdots<j_k\le d} \theta_{j_1\cdots j_k}\overline{F}_{j_1}(x_{j_1})\cdots\overline{F}_{j_k}(x_{j_k})\right),$$

where $|\theta_{j_1\cdots j_k}| \le 1$ with $2 \le k \le d$ and $1 \le j_1 < \cdots < j_k \le d$ are some real numbers such that the right-hand side of the above relation is a proper multivariate distribution function (for details, see Hashorva and Hüsler, 1999). In this setting, for any $1 \le i < j \le d$,

$$\mathbb{P}\big(X_i > x_i, X_j > x_j\big) = \overline{F}_i(x_i)\overline{F}_j(x_j)\Big(1 + \theta_{ij}F_i(x_i)F_j(x_j)\Big),$$

which implies that (4.21) holds with $\sigma_{ij} = \sigma_{ji} = 1 + \theta_{ij}$ given that $\theta_{ij} > -1$.

Remark 4.2. *Consider another well-known dependence structure for the random vector $\big(X_1, \ldots, X_d\big)$. That is, for any $1 \le i \ne j \le d$, there is some bounded function $g_{ij} : (0,\infty) \mapsto (0,\infty)$ such that the relation*

$$\mathbb{P}\big(X_i > t|X_j = x\big) \sim g_{ij}(x)\overline{F}_i(t), \quad t \to \infty, \tag{4.33}$$

holds uniformly for $x \in (0,\infty)$ (for details, see Asimit and Badescu, 2010, Li et al., 2010, Asimit et al., 2011, Chen and Yuen, 2012, Yang et al., 2016, and the references therein). If $\overline{F}_k(t) \sim b_k\overline{F}_1(t)$ holds as $t \to \infty$ with $b_k > 0$ for each $k \in \{1, \ldots, d\}$, as we assumed in Example 4.2, then it is not difficult to check that (4.33) implies (4.15). In fact, noting the uniformity of (4.33), we have for every $x > 0$ that

$$\begin{aligned}\mathbb{P}\big(X_i > tx, X_j > t\big) &= \int_t^\infty \mathbb{P}\big(\, X_i > tx|\, X_j = y\big)\mathbb{P}\big(X_j \in \mathrm{d}y\big)\\ &\sim \overline{F}_i(tx)\int_t^\infty g_{ij}(y)\mathbb{P}\big(X_j \in \mathrm{d}y\big)\\ &= \overline{F}_i(tx)\cdot O\big(\overline{F}_1(t)\big)\\ &= o\big(\overline{F}_1(t)\big), \quad t \to \infty,\end{aligned} \tag{4.34}$$

where the third step is due to $\overline{F}_k(t) \sim b_k\overline{F}_1(t)$ as $t \to \infty$ and the boundedness of g_{ij}. Assuming that $F_1 \in \mathcal{R}_{-\alpha}$ for some $\alpha > 1$, relation (3.13) of Asimit et al. (2011) gives an asymptotic approximation for $\mathbb{E}\big[X_k|S_d > t\big]$ as $t \to \infty$, which in our tail equivalence case is reduced to

$$\mathbb{E}\big[X_k|S_d > t\big] \sim \frac{\alpha b_k}{(\alpha-1)\sum_{i=1}^d b_i}t, \quad t \to \infty.$$

The above relation coincides with (4.22) derived via Theorem 3.1 and our approach also implies that the extra condition (3.8) of Asimit et al. (2011) is redundant for the tail equivalence case. Moreover, if we further assume that the limit

$$\lim_{x\to\infty} g_{ij}(x) := g^*_{ij} > 0 \tag{4.35}$$

exists for any $1 \leq i \neq j \leq d$, then it follows from the similar derivations as in justifying (4.34) (see also Proposition 2.1 of Li, 2016) that Assumption 3.2 holds for $\left(X_i, X_j\right)$ with $\sigma_{ij} = g_{ij}^* = g_{ji}^*$. Therefore, if (4.33) holds together with (4.35), which is true for all examples about (4.33) appeared in the aforementioned literatures, then this setting is a special case of the dependence structure given in Example 4.3.

Remark 4.3. *Although condition (4.25) in Example 4.4 seems especially strong, it is also satisfied by many interesting scenarios. For instance, under the other conditions of Example 4.4, if the relation*

$$\overline{F}_1^2(La(t)) = o\left(\overline{F}_1(t)\right), \quad t \to \infty, \tag{4.36}$$

holds for some $L > 0$ (hence, $\lim_{t\to\infty} a(t) = \infty$ must hold), then (4.21) obviously implies (4.25). Two well-known examples of F_1 satisfying (4.36) include the LogNormal case in which

$$\overline{F}_1(t) = \overline{\Phi}\left(\frac{\log t - \mu}{\delta}\right), \quad t > 0,$$

with $a(t) = \delta^2 t / (\log t - \mu)$ and the case in which

$$\overline{F}_1(t) = \mathrm{e}^{-(\log t)^\gamma}, \quad t > 1, \tag{4.37}$$

with $a(t) = t / \left(\gamma (\log t)^{\gamma - 1}\right)$, where $\Phi(\cdot)$ is the standard normal distribution function, $\mu \in (-\infty, \infty)$, $\delta > 0$, and $\gamma > 1$ (for details, see Proposition 4.1 of Asimit et al., 2011 and the discussions after it).

Remark 4.4. *Denote by $I(q)$ the right-hand side of (4.30) obtained in Example 4.4, i.e.*

$$I(q) := \frac{a\big(t_1(q)\big)\overline{F}_1\big(t_1(q)\big)}{\overline{F}_1\big(t_2(q)\big)}.$$

In general, we can not conclude that $I(q)$ tends to 0 or ∞ as $q \uparrow 1$. However, if the auxiliary function a is regularly varying with some index $\nu \in (-\infty, 1)$, written here as $a \in \mathcal{R}_\nu$, then we can verify that $I(q) \to 0$ as $q \uparrow 1$. In fact, recalling (4.27) and (4.28) in Example 4.4, we have $t_2(q) \sim w t_1(q)$ as $q \uparrow 1$ with $w = 1/\sum_{i=1}^d b_i \in (0,1)$. Hence, for any $0 < w_1 < w < w_2 < 1$, the relation

$$\frac{a\big(t_1(q)\big)\overline{F}_1\big(t_1(q)\big)}{\overline{F}_1\big(w_1 t_1(q)\big)} \leq I(q) \leq \frac{a\big(t_1(q)\big)\overline{F}_1\big(t_1(q)\big)}{\overline{F}_1\big(w_2 t_1(q)\big)} \tag{4.38}$$

holds for q in the left neighborhood of 1. Then, we can obtain that $I(q) \to 0$ as $q \uparrow 1$ if we can prove that

$$J_2(t) := \frac{a(t)\,\overline{F}_1(t)}{\overline{F}_1(w_2 t)} \to 0, \quad t \to \infty.$$

By the representation of Von Mises functions (for example, see Proposition 1.4 of Resnick, 1987), we have, as $t \to \infty$,

$$J_2(t) \sim a(t) \exp\left\{-\int_{w_2 t}^{t} \frac{1}{a(z)} \mathrm{d}z\right\} = \exp\left\{-\left(\int_{w_2}^{1} \frac{t}{a(tz)} \mathrm{d}z - \log a(t)\right)\right\}. \tag{4.39}$$

Since $a \in \mathcal{R}_\nu$ with $\nu \in (-\infty, 1)$, for some $\nu < \nu^* < 1$, we have $a(t) \le t^{\nu^*}$ for all large t. Thus, for t large enough,

$$\int_{w_2}^{1} \frac{t}{a(tz)} \mathrm{d}z - \log a(t) \ge \left(\int_{w_2}^{1} \frac{1}{z^{\nu^*}} \mathrm{d}z \right) t^{1-\nu^*} - \nu^* \log t,$$

which clearly tends to ∞ as $t \to \infty$. Combining this fact with (4.39) gives $J_2(t) \to 0$ as $t \to \infty$, which implies $I(q) \to 0$ as $q \uparrow 1$ by (4.38). Note that, since $a(t) = o(t)$ as $t \to \infty$, if $a \in \mathcal{R}_\nu$ for some index ν, then ν must be in the range $(-\infty, 1]$. Nevertheless, for the critical case of $a \in \mathcal{R}_1$, both $I(q) \to 0$ and $I(q) \to \infty$ are possible as $q \uparrow 1$. It depends on the specific form of the function a and even the value of w. Actually, the survival function given as (4.37) in Remark 4.3 is just an example. Concretely speaking, assume that (4.37) holds for F_1 with $a(t) = t/\left(\gamma (\log t)^{\gamma-1}\right) \in \mathcal{R}_1$, where $\gamma > 1$. For $i \in \{1, 2\}$, we consider the quantities

$$J_i(t) := \frac{a(t)\,\overline{F}_1(t)}{\overline{F}_1(w_i t)} = \frac{1}{\gamma} \exp\left\{ -(\log t)^\gamma + (\log t + \log w_i)^\gamma + \log t - (\gamma - 1) \log\log t \right\}.$$

It is clear that $(\log t + \log w_i)^\gamma - (\log t)^\gamma \sim \gamma \log w_i (\log t)^{\gamma-1}$ as $t \to \infty$. By this fact and $\log w_i < 0$ for $i \in \{1, 2\}$, we know that $\lim_{t\to\infty} J_2(t) = 0$ ($\lim_{q\uparrow 1} I(q) = 0$) if $\gamma > 2$, $\lim_{t\to\infty} J_1(t) = \infty$ ($\lim_{q\uparrow 1} I(q) = \infty$) if $1 < \gamma < 2$, while the limits also depend on the value of w if $\gamma = 2$.

We conclude this section with a summary of our findings that have shown to be quite sensible. Examples 4.1–4.4 consider the case where the system components have set individual levels of capital that are allocated from the aggregate regulatory capital set for the entire system. Specifically, Example 4.1 deals with an asymptotic dependent portfolio of risks and all the SRs, both at aggregate and individual levels, become very large in the limit, suggesting a significant impact of the SR when the asymptotic dependence is present. The results change in Examples 4.2–4.4, when the asymptotic independence arises. Examples 4.2 and 4.3 show an infinite limit for the aggregate SR, while conclusive results for the individual SR are obtained only for Example 4.3, which is a special case of Examples 4.2. The individual SR in Example 4.3 becomes negligible in the limit and the same pattern is observed in Example 4.4. In a nutshell, provided that one component of the system is under-capitalised, all other components are individually under-capitalised only if the asymptotic dependence is present; otherwise, the individual under-capitalisation may be insignificant. On the other hand, provided one component of the system is under-capitalised, the entire system is overall under-capitalised whenever the individual risk distributions are regularly varying, which shows how vulnerable the entire system is. Thus, a chain reaction in the system is likely to happen, as a result of financial distress observed in one single system component, if the asymptotic dependence arises, otherwise all other components are solvent even though the whole system is under financial distress.

5 Proofs

Proof of Proposition 3.1. We first deal with case (i), in which $F \in \mathcal{R}_{-\alpha}$, $G \in \mathcal{R}_{-\alpha}$ for some $\alpha > 0$, and $\lim_{t\to\infty} \overline{F}(t)/\overline{G}(t) = r$ for some $r \geq 0$. By Proposition 0.8(vi) of Resnick (1987) or Lemma 2.1 of Asimit *et al.* (2011), we have

$$\lim_{q\uparrow 1} \frac{F^{\leftarrow}(q)}{G^{\leftarrow}(q)} = r^{1/\alpha}.$$

The above relation implies that, for $0 < r_1 < r^{1/\alpha} < r_2$ and q in the left neighborhood of 1, we have

$$r_1 G^{\leftarrow}(q) \leq F^{\leftarrow}(q) \leq r_2 G^{\leftarrow}(q). \tag{5.1}$$

Hence, if Assumption 3.1 holds with $h(r_1) = 0$ then

$$\lim_{q\uparrow 1} \mathbb{P}\big(X > F^{\leftarrow}(q) | Y > G^{\leftarrow}(q)\big) \leq \lim_{q\uparrow 1} \mathbb{P}\big(X > r_1 G^{\leftarrow}(q) | Y > G^{\leftarrow}(q)\big) = h(r_1) = 0,$$

which together with (2.4) indicate that X and Y are asymptotically independent. On the other hand, if Assumption 3.1 holds with $h(r_2) > 0$ then

$$\liminf_{q\uparrow 1} \mathbb{P}\big(X > F^{\leftarrow}(q) | Y > G^{\leftarrow}(q)\big) \geq \liminf_{q\uparrow 1} \mathbb{P}\big(X > r_2 G^{\leftarrow}(q) | Y > G^{\leftarrow}(q)\big) = h(r_2) > 0,$$

which together with (2.5) indicate that X and Y are asymptotically dependent.

We next focus on case (ii), in which $F \in \mathcal{R}_{-\infty}$, $G \in \mathcal{R}_{-\infty}$, and $\overline{F}(tr) \asymp \overline{G}(t)$ as $t \to \infty$ for some $r > 0$. By Lemma 2.4 of Asimit *et al.* (2011), we have

$$F^{\leftarrow}(q) \sim r G^{\leftarrow}(q), \quad q \uparrow 1,$$

which implies that (5.1) holds for $0 < r_1 < r < r_2$ and q in the left neighborhood of 1. Thus, case (ii) can be verified in a similar manner to case (i) above. □

Proof of Theorem 3.1. By (2.3), for every $1 < \alpha' < \alpha_F^*$ there are some $K_1 > 1$ and $t_0 > 0$ such that the relation

$$\frac{\overline{F}(tx)}{\overline{F}(t)} \leq K_1 x^{-\alpha'}$$

holds for all $tx > t > t_0$. Since $\overline{F}(t) = O\big(\overline{G}(t)\big)$, there is some $K_2 > 1$ such that the relation

$$\frac{\overline{F}(t)}{\overline{G}(t)} \leq K_2$$

holds for all $t \geq 0$. Thus, letting $K = K_1 K_2$, we have for all $tx > t > t_0$ that

$$\mathbb{P}\big(X > tx | Y > t\big) = \frac{\mathbb{P}\big(X > tx, Y > t\big)}{\overline{F}(t)} \frac{\overline{F}(t)}{\overline{G}(t)} \leq \frac{\overline{F}(tx)}{\overline{F}(t)} \frac{\overline{F}(t)}{\overline{G}(t)} \leq K x^{-\alpha'}. \tag{5.2}$$

It is clear that

$$\rho_{X,Y}(q) = \int_{t_1(q)}^{\infty} \mathbb{P}\big(X > x | Y > t_2(q)\big)\, \mathrm{d}x = t_2(q) \int_{t_1(q)/t_2(q)}^{\infty} \mathbb{P}\big(X > t_2(q) x | Y > t_2(q)\big)\, \mathrm{d}x,$$

which is equivalent to

$$\frac{\rho_{X,Y}(q)}{t_2(q)} = \int_0^\infty \mathbb{P}\big(X > t_2(q)x|Y > t_2(q)\big)\mathbf{1}_{\{x>t_1(q)/t_2(q)\}}\,\mathrm{d}x. \tag{5.3}$$

Keeping in mind (5.2) and $t_2(q) \to \infty$ as $q \uparrow 1$, we find that the relation

$$\mathbb{P}\big(X > t_2(q)x|Y > t_2(q)\big) \le Kx^{-\alpha'}$$

holds for q in the left neighborhood of 1 and $x > 1$. Hence, for q in the left neighborhood of 1, the integrand of (5.3) is not greater than

$$\mathbb{P}\big(X > t_2(q)x|Y > t_2(q)\big) \le \mathbf{1}_{\{0<x\le 1\}} + Kx^{-\alpha'}\mathbf{1}_{\{x>1\}},$$

which is obviously integrable on $(0,\infty)$. This fact together with (3.1) and $\lim_{q\uparrow 1} t_1(q)/t_2(q) = c$ allow us to apply the Dominated Convergence Theorem to (5.3) to obtain (3.3). This completes the proof. □

Proof of Corollary 4.1. Applying Theorem 3.1 to the given $t_1(q)$ and $t_2(q)$ leads to

$$\lim_{q\uparrow 1}\frac{t_i(q)}{t(q)} = \int_0^\infty h_i(x)\,\mathrm{d}x = \widetilde{h}_i, \quad i \in \{1,2\}. \tag{5.4}$$

Hence, we have

$$\lim_{q\uparrow 1}\frac{t_1(q)}{t_2(q)} = \frac{\widetilde{h}_1}{\widetilde{h}_2}.$$

Applying Theorem 3.1 once again to $\rho_{X,Y}(q)$ and noting (5.4), one may get that

$$\lim_{q\uparrow 1}\frac{\rho_{X,Y}(q)}{t(q)} = \widetilde{h}_2\int_{\widetilde{h}_1/\widetilde{h}_2}^\infty h(x)\,\mathrm{d}x = \int_{\widetilde{h}_1}^\infty h\left(\frac{x}{\widetilde{h}_2}\right)\,\mathrm{d}x.$$

This completes the proof. □

Proof of Theorem 3.2. Noting Assumption 3.2 and $\lim_{q\uparrow 1} t_1(q) = \infty$, we have, as $q \uparrow 1$,

$$\begin{aligned}
\rho_{X,Y}(q) &= \int_{t_1(q)}^\infty \mathbb{P}\big(X > x|Y > t_2(q)\big)\,\mathrm{d}x \\
&= \frac{1}{\overline{G}\big(t_2(q)\big)}\int_{t_1(q)}^\infty \mathbb{P}\big(X > x, Y > t_2(q)\big)\,\mathrm{d}x \\
&\sim \sigma\int_{t_1(q)}^\infty \overline{F}(x)\mathrm{d}x \\
&\sim \begin{cases} \dfrac{\sigma}{\alpha-1}t_1(q)\overline{F}\big(t_1(q)\big), & \text{if } F \in \mathcal{R}_{-\alpha}, \\[2ex] \sigma a\big(t_1(q)\big)\overline{F}\big(t_1(q)\big), & \text{if } F \in \mathrm{MDA}(\Lambda), \end{cases}
\end{aligned}$$

where the last step follows from Karamata's Theorem (see Theorem 1.5.11(ii) of Bingham *et al.*, 1987) when $F \in \mathcal{R}_{-\alpha}$ while from (2.2) when $F \in \mathrm{MDA}(\Lambda)$. This completes the proof. □

Acknowledgment. The research of Jinzhu Li was supported by the National Natural Science Foundation of China (Grant No. 11371020).

References

[1] Acharya, V. A theory of systemic risk and design of prudential bank regulation. *Journal of financial stability* 5 (2009), no. 3, 224–255.

[2] Acharya, V.; Engle, R.; Richardson, M. Capital shortfall: A new approach to ranking and regulating systemic risks. *The American Economic Review* 102 (2012), no. 3, 59–64.

[3] Adrian, T.; Brunnermeier, M. K. CoVaR. *Technical report, Federal Reserve Bank of New York Staff Reports* (2009), no. 348.

[4] Asimit, A. V.; Badescu, A. L. Extremes on the discounted aggregate claims in a time dependent risk model. *Scandinavian Actuarial Journal* 2010 (2010), no. 2, 93–104.

[5] Asimit, A. V.; Furman, E.; Tang, Q.; Vernic, R. Asymptotics for risk capital allocations based on conditional tail expectation. *Insurance: Mathematics and Economics* 49 (2011), no. 3, 310–324.

[6] Asimit, A. V.; Li, J. Extremes for coherent risk measures. *Insurance: Mathematics and Economics* 71 (2016), 332–341.

[7] Bingham, N. H.; Goldie, C. M.; Teugels, J. L. *Regular Variation*. Cambridge University Press, Cambridge, 1987.

[8] Bluhm, C.; Overbeck, L.; Wagner, C. *An Introduction to Credit Risk Modeling*. Boca Raton, FL: CRC Press/Chapman & Hall., 2006.

[9] Chen, C.; Iyengar, G.; Moallemi, C. C. An axiomatic approach to systemic risk. *Management Science* 59 (2013), no 6, 1373–1388.

[10] Chen, Y.; Yuen, K. C. Sums of pairwise quasi-asymptotically independent random variables with consistent variation. *Stochastic Models* 25 (2009), no. 1, 76–89.

[11] Chen, Y.; Yuen, K. C. Precise large deviations of aggregate claims in a size-dependent renewal risk model. *Insurance: Mathematics and Economics* 51 (2012), no. 2, 457–461.

[12] Davis, R. A.; Resnick, S. I. Limit theory for bilinear processes with heavy-tailed noise. *The Annals of Applied Probability* 6 (1996), no. 4, 1191–1210.

[13] Embrechts, P.; Klüppelberg, C.; Mikosch, T. *Modelling Extremal Events for Insurance and Finance*. Springer-Verlag, Berlin, 1997.

[14] Feinstein, Z.; Rudloff, B.; Weber, S. Measures of Systemic Risk. *Working paper* (2016), arXiv:1502.07961.

[15] Fisher, R. A.; Tippett, L. H. C. Limiting forms of the frequency distribution of the largest or smallest member of a sample. *Mathematical Proceedings of the Cambridge Philosophical Society* 24 (1928), no. 2, 180–190.

[16] Hashorva, E.; Hüsler, J. Extreme values in FGM random sequences. *Journal of Multivariate Analysis* 68 (1999), no. 2, 212–225.

[17] Hashorva, E.; Li, J. Tail behavior of weighted sums of order statistics of dependent risks. *Stochastic Models* 31 (2015), no. 1, 1–19.

[18] Kalkbrener, M. An axiomatic approach to capital allocation. *Mathematical Finance* 15 (2005), no. 3, 425–437.

[19] Kallenberg, O. *Random Measures.* Third Edition. Akademie-Verlag, Berlin, 1983.

[20] Li, J. A revisit to asymptotic ruin probabilities for a bidimensional renewal risk model. *Working paper* (2016), https://ssrn.com/abstract=2910357.

[21] Li, J.; Tang, Q.; Wu, R. Subexponential tails of discounted aggregate claims in a time-dependent renewal risk model. *Advances in Applied Probability* 42 (2010), no. 4, 1126–1146.

[22] McNeil, A. J.; Frey, R.; Embrechts, P. *Quantitative Risk Management. Concepts, Techniques and Tools.* Princeton University Press, Princeton, 2005.

[23] Mitra, A.; Resnick, S. I. Aggregation of rapidly varying risks and asymptotic independence. *Advances in Applied Probability* 41 (2009), no. 3, 797–828.

[24] Nelsen, R. B. *An Introduction to Copulas.* Second edition. Springer, New York, 2006.

[25] Overbeck, L. *Allocation of Economic Capital in Loan Portfolios* in Proceedings "Measuring Risk in Complex Stochastic Systems", W. Härdle and G. Stahl, eds., Lecture Notes in Statistics, Vol. 147. Berlin: Springer, 2000.

[26] Resnick, S. I. *Extreme Values, Regular Variation, and Point Processes.* Springer-Verlag, New York, 1987.

[27] Rogers, L. C. G.; Veraart, L. A. M. Failure and rescue in an interbank network. *Management Science* 59 (2013), no 4, 882–898.

[28] Sklar, A. Fonctions de répartion à n dimensions et leurs marges. (French) *Publications de l'Institut de Statistique de l'Université de Paris* 8 (1959), 229–231.

[29] Yang, H.; Gao, W.; Li, J. Asymptotic ruin probabilities for a discrete-time risk model with dependent insurance and financial risks. *Scandinavian Actuarial Journal* 2016 (2016), no. 1, 1–17.