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Pollution, mortality and optimal environmental policy

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Abstract: We study an overlapping generations economy in which environmental degradation results from economic activity and affects agents' uncertain lifetimes. Life expectancy depends positively on economic activity and negatively on the stock of pollution. This can make the growth-survival relationship convex over some region and lead to two non-trivial steady states, with one a poverty trap. Uniform abatement taxes can cause the poverty trap to widen while increasing incomes at the high steady state. We also study the properties and dynamics of an optimal second-best abatement tax. It is non-homogeneous and increasing in the capital stock, and leads to a variety of dynamic possibilities, including non-existence and multiplicity of steady states, and cycles around some of the steady states, where there were none under exogenous taxes. Thus, optimal taxes can be an independent source of non-linearities.

Keywords: Overlapping generations model, poverty traps; non-convexities; multiple steady states; pollution; optimal environmental policy; optimal abatement tax.

JEL Classification: O11, O13, O23, O44.

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1 Introduction:

The process of economic growth involves externalities. Much of the literature has focused on positive externalities that are a source of sustained economic growth. In recent years, however, there has been an increasing concern that the negative externalities of pollution may have significant impact on economic well-being. The attention has been on long-run consequence of global warming due to greenhouse gas emissions. While there is consensus that global warming can have significant impact on economic welfare, there is debate as to the extent of global warming and about the right policies to tackle it. Rich and poor countries have taken very different stances on adopting measures to tackle global warming as per the Kyoto Protocol, and on controlling pollution more generally.

In this paper, we focus on the relatively short run effects of pollution on increased mortality. While there is some debate among empirical researchers on how to estimate the effect of pollution on mortality, there appears to be a consensus that increased pollution increases mortality. Some studies have suggested that upto 40% of premature mortality is related to the adverse effects of pollution (Pimentel *et. al.* [2007]) . This could be particulate matter pollution which leads to increase of respiratory diseases, water pollution that leads to water-borne diseases, CO_2 and greenhouse gases that lead to global warming which may lead to environmental disasters, carcinogens both gases and soil contaminants, etc. This effect has been extensively documented in the medical and ecology literatures. There is robust evidence that exposure to pollution leads to increased cardio-vascular disease and controlling for the different factors an increase in mortality (see Ayres [2006], Miller *et. al.* [2007], Pope *et. al.* [2004]) as well as increased incidence of chronic obstructive pulmonary disease (COPD) and increased mortality (see HEI [2010], Viegi *et. al.* [2006]). These effects are present for both developed and developing countries.

We study a two-period overlapping generations model in which the young may die prematurely and the probability of survival into old age is determined endogenously (Chakraborty [2004]). Production of the single consumption-capital good creates pollution as a by-product. Increased pollution increases the probability of premature death. The medical literature also points out that increased income can counteract some of the adverse effects of pollution via better nutrition and greater access to health care. We model this by making the survival probability increasing in per-capita income levels. Thus, there are two contrasting effects of economic growth: pollution which is life-threatening and increased income which is life-enhancing.[1] We study the impact

[1]We depart from Chakraborty [2004] in making the survival probability a function of two argu-

of this formulation on the dynamic equilibria of the economy and on the role of public policy. It is well understood that the divergence of social and private costs and benefits will lead to under-investment in pollution abatement activities. Thus, we first look at the impact of an exogenous linear income (wage) tax whose proceeds are used for pollution abatement. As such a tax need not be optimal, we also analyse the second-best tax policy. There is a well-known commitment problem in imposing taxes on future generations. Thus, following John and Pecchenino [1994] we assume that the second-best tax is set by a series of short-lived governments who decide on the tax only one period at a time, in order to maximize the expected utility of the current young generation. Since the resulting environmental improvement does not benefit the surviving old at the time in which it is enacted, we assume the government is constrained to taxing the young alone, via a tax on wage incomes.[2]

This framework generates some interesting results. The two contrary forces that affect mortality can under very intuitive conditions result in a non-convexity that gives rise to poverty traps and to sharp differences between rich and poor countries in terms of the appropriate environmental policy. Under a uniform tax, there can be a low capital steady state which resembles a poverty trap and in which there is lower per capita consumption and life expectancy and a high capital steady state, which resembles the unique steady state of a neoclassical growth model and in which per capita consumption and life expectancy are both higher. The poverty trap is a source in that any path that starts with a capital stock lower than at the steady state converges over time to the zero-consumption or trivial steady state. Furthermore, increases in the uniform tax can increase the steady state capital in the neoclassical steady state while simultaneously widening the basin of attraction of the trivial steady state.[3]

Turning to the second-best tax the results are even more striking. We show that the optimal tax is a function of the current capital stock. If this stock is below a strictly

ments, depending positively on the level of per-capita income and negatively on the level of accumulated pollution, the flow of which is a by-product of final goods production. Chakraborthy [2004] assumes that survival depends on the stock of health which in turn is an exogenous linear function of wage incomes, so in this respect our two formulations are similar, the difference arising in the further inclusion in our model of pollution as an argument.

[2]John and Pecchenino [1994] argue that the mechanism of a government setting taxes on the young to ultimately benefit the young themselves avoids the free-rider problem which would arise if the young were to enact optimal measures in a decentralised fashion and is equivalent to a Lindahl equilibrium in the amount of the public good (in this case, pollution control) chosen.

[3]The former possibility is well known: environmental degradation imposes costs that are external to each decision-maker so any policy that offsets this externality helps reduce these costs and if the balance is right, actually promotes growth (see Pautrel [2007], [2011], Economides and Philippopoulos [2008], Palivos and Varvarigos [2011] for an analysis of such effects in a variety of settings.

positive threshold, then the optimal tax is zero and there is no expenditure on pollution abatement. If the poverty trap lies within this region, then it is also an 'environmental trap' in which pollution is never abated. In the region of positive taxation, the optimal tax is weakly increasing in the capital stock. The dynamics in the case of optimal taxes are also more complex than in the case of exogenous taxes. First, optimal taxes can lead to both non-existence and multiplicity of steady state equilibria even if the underlying economy with exogenous taxes admits a unique, neoclassical steady state. Second, when the underlying economy admits two steady states, optimal taxes can convert a poverty trap into a sink and a neoclassical steady state into a source.

Third, the possibility arises that in the neighborhood of either steady state there may be endogenous fluctuations. The mechanism for this is directly related to the fact that the tax is endogenous and increasing in the capital stock. The intuition for fluctuations is as follows: starting with a high capital stock, the associated tax rate is also high. The tax has two effects. First, by reducing post-tax income it tends to decrease current savings and next period's capital stock. Second, the ensuing reduction in pollution decreases premature mortality and increases the incentive for young agents to save, thus stimulating the capital stock. Depending on the strength of the two effects resulting from this tax next period's capital stock may *decrease*. Because the capital stock is lower, the new tax rate is also lower. This has the same effects, but in the opposite direction, and may lead to an *increase* in the subsequent period's capital stock. There are already the two contrasting effects of capital on premature mortality and these alone can generate endogenous fluctuations even when the tax remains constant. However, the second-best tax can amplify these effects and lead to stronger non-linearities. It should be emphasized that the endogeneity of taxes alone can drive fluctuations since these can arise even if the underlying steady state is unique and neoclassical in behaviour.

The plan of the paper is as follows. In section 2 we review the relevant literature from both an empirical and theoretical standpoint. In section 3, the benchmark model is developed. Section 4 studies the effects of exogenous (constant) taxes, and section 5 studies the second-best optimal tax. In this section we first characterize properties of the optimal tax function, and then study the dynamics of the equilibrium trajectories. The final section concludes.

2 Relevant literature

There has been a surge of recent interest, as exemplified by this paper, on the theoretical and policy implications of a positive link between pollution and premature mortality. At the same time, empirical investigation of this link has been going on for over two decades. The most common approach regresses time series on mortality counts against short-run variations in particulate matter, PM_{10}.[4] The popularity of this approach arisee because it enables the researcher to assess how short-term changes in air pollution lead to acute health effects without the need to consider idiosyncratic risk factors. Moreover, such research studies the relationship between low levels of pollution and mortality rather than focusing on extreme episodes.

One of the earliest studies to follow this approach, Ostro [1984], used London data to estimate a simple linear regression model, finding mortality to be significantly affected by pollution even at greatly reduced concentrations. Another early study on London, Schwartz and Marcus [1990], employed autoregressive regression techniques to find pollution correlated with mortality, again at relatively low levels of PM_{10}.

Time series analysis raises some concerns. Bell *et. al* [2004] list four major concerns: (i) serial correlation of the residuals that could lead to underestimating standard errors of the estimated parameters; (ii) the choice of appropriate lag length of PM; (iii) accounting for PM measurement error; and (iv) uncertainty associated with model choice. In addition, time series analysis may be subject to confounding bias if it omits other relevant factors that vary on similar timescales, such as temperature variations and influenza epidemics which contribute to respiratory illness. Partly reflecting such concerns, the Health Effects Institute (HEI) in the United States arranged as early as in 1995 for the replication of some of the earlier studies, the results of which were published in a report by Samet *et. al.* [1995]. These authors concluded the effect of PM on mortality remained robust although the model estimates did vary depending on the different approaches used to control for confounding factors.

More recent studies have attempted to systematically compare the results from individual studies using multiple data sources and taking into account model selection and estimation problems (EPA [2002]; HEI [2003], Peng *et. al.* [2006]). Apart from issues such as confounding bias, these studies have paid attention to the limitations arising from single-city findings and from model uncertainty (see also Clyde and DeSimone-Sasinowska [1998]).

[4] PM_{10} - particulate matter - measures the mass concentration of pollutants with an aerodynamic diameter less than 10 μm (microns). PM_{10} is considered to be benchmark as particulate matter larger then this does not pass through the upper airways to reach the lungs (Ayres [2006]).

The 1996 Air Quality Criteria Document (EPA [2002]) incorporated results from 35 PM and mortality time-series studies published between 1988 and 1996. Although five of the studies on which this report was based were later found to have used a faulty software routine, the report associated a 50 μg $/m^3$ increase in 24-hour PM_{10} with a 2.5%-5% increase in premature nonaccidental mortality for the general population. Working with the National Morbidity, Mortality and Air Pollution database, Peng *et. al.* [2006], compared studies hat covered the period 1996- 2004. Differentiating between those studies that used the faulty software (70%) and those that did not (30%) they found that even the latter estimated a 1% to 8% increase in mortality risk for a 50 μg $/m^3$ 24-hour PM_{10}. From their own analysis of the database, in which they pooled the data using Bayesian hierarchical models, they reported an average (across 100 US cities) of 0.15% increase in mortality with an increase of 10 $\mu g/m^3$ in PM_{10}. Burnett *et. al.* [1998] have reported similar results using multi-city data from Canada on pollutants such as carbon monoxide, nitrogen dioxide, sulfur dioxide and ozone.

This discussion points to two relatively robust findings. First, there is a significant effect of PM pollution on mortality even at very low levels. While the estimate is sensitive to the model specification and estimation procedure, there is a robust effect of PM pollution on mortality. Second, the effect varies in the short run and across time and space so that the estimation procedures can give different results. Our model uses the first insight that pollution even at low levels affects mortality, and generates multiple steady states and non-linearities that can lead to cycles in pollution levels and the economic variables. Thus, it can give rise to the second phenomenon and thus, the care has to be taken in estimating and interpreting time series evidence. Furthermore, as the equilibria are inherently non-linear, there can be substantive problems in calibrating these models.

Turning to the theoretical literature, our paper is closely related to Pautrel [2007], [2011]; Jouvet *et. al.* [2010]; Mariani *et. al.* [2010], Varvarigos [2008], [2010] and Palivos and Varvarigos [2011]. These papers all assume a negative relationship between survival and environmental degradation.[5] Many of them also adapt Chakraborty's [2004] model with overlapping generations and uncertain lifetimes. While resembling this paper in these respects, on the key issues studied here, there are some important differences. For example, although Pautrel [2007], [2011] and Jouvet *et. al.* [2010],

[5]While environmental degradation can take many forms, most of the above papers associate it with pollution, as we do. As an exception is Mariani *et. al.* [2010] who use John and Pecchenino's [1994] formulation of environmental quality as a broad concept which could include measures such as water quality and availability of natural resources. Jouvet *et. al.* [2010] include physical space as an additional measure of environmental quality along with pollution, although the former directly enters the utility function of agents while the latter affects only survival probability.

study first- and second-best environmental policies, they do not consider the possibility of non-convexities and multiple steady states. Thus their insights differ in a qualitative way from the ones developed in this paper. Of the papers which do focus on more complex dynamics, Mariani *et. al.* [2010], Varvarigos [2010] and Palivos and Varvarigos [2011] consider multiplicity of steady states while Varvarigos [2008] and Palivos and Varvarigos [2011] consider the possibility of fluctuations around the long-run growth path.

Of the papers that hypothesise the existence of poverty traps and multiple steady states, both Mariani *et. al.* [2010] and Varvarigos [2010] assume step-wise discontinuities in the relevant state spaces of their respective models; neither of their mechanisms relies on a tradeoff between capital and pollution stocks in affecting survival probability, as in our paper. Mariani *et. al.* [2010] assume that environmental quality is the only state variable and that survival probability jumps once a threshold level of environmental quality has been crossed. Thus the interaction between capital and pollution stocks in affecting life expectancy is not studied.[6]

Varvarigos [2010] does consider capital accumulation in a model of environmental degradation via pollution and, like us, assumes that survival probability depends positively on income and negatively on pollution. However, he uses a specific functional form for this relationship, which does not generate multiplicity of steady states. To obtain the latter, he then assumes a step function in technology adoption such that at some threshold level of capital firms find it profitable to escape pollution taxes by switching to a 'clean' technology. While plausible, this explanation for multiple steady states is somewhat orthogonal to the growth-pollution-lifetime nexus which forms the core of this literature. Indeed, as our paper shows and contrary to the claim in Varvarigos [2010], the costs of pollution in terms of increased mortality are indeed sufficient to guarantee multiple equilibria. In addition, environmental policy enters only to the extent of an exogenous penalty for the use of dirty technology.

Palivos and Varvarigos [2011] comes closest to our paper in terms of the proposed rationale underlying multiple steady states. Like us, they emphasise the growth-pollution-lifetime nexus as an underlying mechanism for generating multiple steady states. In that respect the two papers are very similar. However, beyond that the issues they

[6] Mariani *et. al.* [2010] also follow John and Pecchenino [1994] in assuming that abatement activities affect environmental quality in an additively separable fashion from degradation activities, implying that 'abatement' alone can be used to improve environmental quality even if there is no degradation taking place to begin with. This possibility might be plausible for certain types of activities and certain definitions of the 'environment' but as a general formulation, it has been criticised by, among others, Economides and Philippopoulos [2008] and Varvarigos [2008].

highlight with the help of their mechanism are very different from those highlighted in this paper.

Palivos and Varvarigos [2011] focus on the possibility of endogenous fluctuations around the long-run growth path. The underlying intuition is similar in both papers: higher life expectancy induces greater savings and capital accumulation which in turn increases emissions, leading to lower life expectancy and lower savings. Although there is considerable evidence, as cited in the Introduction, of a positive short-term link between emissions and mortality rates, it is not clear whether this in itself is capable of generating the cyclical pattern of savings and capital accumulation which the above hypothesis requires to complete the circle. Indeed if the increased mortality rates associated with higher pollution mainly affect the old and infirm or the very young, the impact on savings would be negligible. Indeed, our analysis suggests that fluctuations might arise not just in the context of Palivos and Varvarigos [2011] but also as a result of second-best abatement policy.

The policy dimension of our paper is also quite different from both Varvarigos [2010] and Palivos and Varvarigos [2011]. In both cases, one goal of environmental policy is to eliminate fluctuations, something which is not conventionally considered to be a goal of protecting the environment. In any case, neither considers a second-best taxation regime. Varvarigos [2008] does not calculate any kind of optimal tax, while Palivos and Varvarigos [2011] consider an allocation of given tax revenues between pollution abatement and health-enhancing expenditures which maximises the survival probability of young agents, rather than their welfare. Our policy analysis is different not just in terms of objective functions and instruments but also in comparing the qualitative properties of the growth path with and without second-best policy.

3 Model:

Time is discrete and denoted by $t = 0, 1, \ldots$. Each period a new generation is born, indexed by its period of birth. A generation consists of a continuum of agents normalized to measure one. Agents born in period t live at most until the end of period $t+1$. There is uncertainty whether an individual will survive till old age.The probability that an agent born in period t lives until the end of period $t+1$ is denoted by π_t, while with probability $1-\pi_t$ the agent dies at the end of period t.

Each agent supplies one unit of labour inelastically when young and receives a wage w_t which is used to finance current consumption, c_t^y and savings for old age, s_t. Old agents have no labour endowment and live entirely off the proceeds of their savings. Following the literature on uncertain lifetimes, we assume that there is a perfect annuity market in which young agents buy annuities from perfectly competitive intermediaries who lend out the proceeds to firms for investment in productive capital. Each unit of time t investment results in one unit of time $t+1$ capital, k_{t+1} which becomes immediately available for production and fully depreciates in that period. Thus,

$$k_{t+1} = s_t \tag{1}$$

At time $t = 0$, k_0 is exogenously given.

3.1 Production and factor prices:

The production function is Cobb-Douglas and displays constant returns to scale. It can be expressed in intensive form:

$$y_t = Ak_t^{\alpha}$$

where y is output per worker and k is capital per worker.

The gross returns to capital and labour r_t and w_t respectively, are equal to their marginal products:

$$w_t = (1-\alpha)Ak_t^{\alpha} \tag{2}$$

$$r_t = \frac{\alpha A}{k_t^{1-\alpha}} \tag{3}$$

Because a positive fraction of savers do not live into old age, the return on period t savings for those who survive is r_{t+1}/π_t.

3.2 Pollution emission and abatement:

The production process creates a proportionate flow of pollutants:

$$\zeta_t = \gamma y_t, \quad \gamma > 0.$$

The stock of pollutants, z_t, depends both on current flows and on past stocks, according to:

$$z_t = \zeta_t + \phi z_{t-1}.$$

where ϕ represents history-dependence in the stock of pollution. $(1-\phi) \in [0,1]$ is the environment's natural capacity to regenerate itself, which we have assumed to be constant over time.[7]

Environmental policy is implemented by a succession of governments which last one period each and impose an environmental tax, τ_t on the wage incomes of the contemporaneous young.[8] The proceeds are spent on operating a carbon capture or clean-up technology that reduces the flow of pollutants. The efficiency of this technology, *i.e.* the reduction in pollution flows, is assumed to be a linear function of tax-financed expenditures. The proportionality factor is defined as $\chi \geq 0$. Thus, given the technology, the stock of pollution accumulates according to

$$z_t = \gamma y_t - \chi \tau_t w_t + \phi z_{t-1},$$

which, after substituting for w_t and redefining terms, simplifies to

$$z_t = \gamma(1 - \psi\tau_t) A k_t^{\alpha} + \phi z_{t-1}. \tag{4}$$

where $\psi = \chi(1-\alpha)/\gamma$ is assumed to lie in $[0,1]$.[9]

At time $t=0$, the existing stock of pollution, denoted by z_{-1} is exogenous. In principle z is a state variable whose initial value can be any arbitrary positive number, but as we show below, the long run dynamics of the system are driven entirely by the path of capital accumulation, so to ease exposition without losing generality we assume that $z(-1) = \zeta y(0)$, where $\zeta \geq 0$ is constant.[10]

3.3 Probability of survival and the rationale for environmental policy:

We assume that the probability of survival into old age is identical for all agents and is represented by a twice differentiable function of y_t and z_t. The level of per-capita

[7]The literature on greenhouse gases however suggests that there is a critical level of greenhouse gas build-up beyond which the natural regenerative capacity of the environment will cease to exist. This suggests that ϕ depends on the stock of greenhouse gases. See Brunekreef and Holgate [2002] for a discussion of evidence, and D'Souza and Goenka [2011] for a modeling effects of the threshold effect. Due to the threshold effect, there is the additional non-convexity from which we abstract from.

[8]The reason for restricting the incidence of environmental taxes to the young generation is explained in the section where the optimal tax policy is derived.

[9]This formulation avoids the additively separable implication that as a result of abatement, the flow of emissions can be made negative.

[10]If $\zeta = \gamma/(1-\phi)$ this would be equivalent to assuming that the inherited stock of pollution equals the value it would have taken if the capital stock had remained at k_0 since times immemorial.

income is assumed to be positively related to longevity while the stock of pollution is negatively related. In addition, we assume that, if per-capita income is zero, survival probability is at some minimal level regardless of the stock of pollution and that as the stock of pollution approaches infinity, survival probability tends to zero regardless of the level of income.

Assumption 1

$$\pi_t = \pi(k_t) = \pi(y(k_t), z(k_t)); \tag{5}$$
$$\pi \in [0,1], \quad \forall y \geq 0 \ \& \ \forall z \geq 0; \tag{6}$$
$$\frac{\partial \pi}{\partial y} \equiv \pi_y(y,z) \geq 0, \quad \forall y \geq 0; \tag{7}$$
$$\frac{\partial \pi}{\partial z} \equiv \pi_z(y,z) \leq 0, \quad \forall z \geq 0; \tag{8}$$
$$\pi(0,z) = \underline{\pi} \in [0,1] \ \forall z \geq 0; \tag{9}$$
$$\pi(y,\infty) = 0 \ \forall y \geq 0. \tag{10}$$

The only consequence of pollution in this model is that it creates a negative external effect on expected lifetimes. Given the overlapping generations framework this externality affects the young generation alone by affecting their expected lifetime utility. As only the young work, the output is not affected by pollution directly. Thus, there is a potential for welfare improvement by means of a tax on the young, the proceeds of which are spent on abating pollution. Of course, in choosing an optimal tax rate, the government has to trade off the distortionary effects of such taxes on young agent's savings decisions along with the beneficial externality arising from growing incomes.

3.4 Preferences

Each agent has a time-separable expected utility function of the form:

$$U^t = \ln c_t^y + \pi_t \ln c_{t+1}^o$$

which the agent maximises subject to the life-cycle budget constraints:

$$c_t^y \leq w_t - s_t \tag{11}$$

$$c_{t+1}^o \leq \frac{r_{t+1}}{\pi_t} s_t \tag{12}$$

where superscipt $\{y, o\}$ denotes the agent's age and subscript t the calendar time. c denotes consumption and s denotes savings. c_{t+1}^o is *ex post* consumption for an agent who survives into old-age.

The logarithmic specification has the convenience of generating an explicit solution for the dynamic path of the capital stock. Taking the first-order condition with respect to savings,

$$-\frac{1}{c_t^y} + \frac{\pi_t}{c_{t+1}^o}\frac{r_{t+1}}{\pi_t} = 0;$$

and combining with equations (11), (12) and (3), results in the following equation:

$$s_t = \frac{\pi_t}{1+\pi_t} A \cdot (1-\tau_t)(1-\alpha)k_t^\alpha$$

3.5 Equilibrium

Using the market clearing condition, *i.e.* substituting into equation (1) we have:

$$k_{t+1} = \frac{\pi_t}{1+\pi_t} A \cdot (1-\tau_t)(1-\alpha)k_t^\alpha \tag{13}$$

Given k_0, z_{-1}, the dynamic path of the economy is fully described at each point of time by recursive application of equations (4) and (13). Thus, given an abatement policy, equation (13) fully describes the dynamic equilibrium. For given k_0, the entire trajectory of the capital stock is traced out by recursive application of equation (13) while the accompanying evolution of the stock of pollution follows from recursively applying equation (4). The other variables are updated similarly.

In the following section we consider the dynamics of the economy for an exogenously given, uniform tax rate. The problem of optimal taxes is taken up in the section after that.

4 Exogenous Taxes

We first consider the case of exogenous taxes, τ, to understand the benchmark case. We examine the dynamics in the model and what are the effect of varying the tax rate.

4.1 Dynamics

A steady state is described by the following equations:

$$\pi = \pi(k) = \pi(y(k), z(k)); \tag{14}$$

$$k = \frac{\pi(k)}{1+\pi(k)} A \cdot (1-\tau)(1-\alpha)k^{\alpha}; \tag{15}$$

$$z = \frac{\gamma(1-\psi\tau)Ak^{\alpha}}{1-\phi}; \tag{16}$$

$$y = Ak_t^{\alpha}; \tag{17}$$

where π, k, z and y denote steady state values of the respective variables.

Equation (15) can be written as

$$k = \mathtt{G}(k);$$

where

$$\mathtt{G}(k) = \frac{\pi(k)}{1+\pi(k)} \Gamma k^{\alpha};$$

and $\Gamma = A \cdot (1-\tau)(1-\alpha)$ is a constant.

Under (9), at $k = 0$,

$$\mathtt{G}(0) = \frac{\underline{\pi}}{1+\underline{\pi}} \Gamma(0)^{\alpha} = 0;$$

implying that a trivial steady state exists at $k = 0$.

If π, the survival probability was constant, then $\mathtt{G}(k)$ would represent a standard concave neoclassical growth mapping, as in Diamond [1965], with $\mathtt{G}'(0) = \infty$, $\mathtt{G}''(k) < 0$, $\forall k$, so that a unique interior steady state would exist. Moreover, the dynamics would be globally stable.

However, with endogenous survival probability, other possibilities exist. Note that π is continuous and differentiable in its arguments which in turn are continuous and differentiable in k. Therefore, π is continuous and differentiable in k and $\mathtt{G}(k)$ is continuous and differentiable in k. Taking derivatives of both terms in $\mathtt{G}(k)$ and rearranging:

$$\mathtt{G}'(k) = \left[\frac{\Gamma k^{\alpha}}{1+\pi(k)}\right] \left[\alpha \frac{\pi(k)}{k} + \frac{\pi'(k)}{1+\pi(k)}\right], \tag{18}$$

it can be seen that the shape of $\mathtt{G}(k)$ can be quite different from the standard neoclassical mapping, depending on how $\pi'(k)$ varies with k. Taking the limits of the two terms inside square brackets as $k \to 0$, the first term clearly goes to zero and the limit of the second term can be expressed as:

$$\alpha \cdot \left\{ \lim_{k\to 0} \frac{\pi(k)}{k} \right\} + \left\{ \lim_{k\to 0} \frac{\pi'(k)}{1+\pi(k)} \right\}$$

where the limit of the first term is given by L'Hopital's Rule as:

$$\lim_{k\to 0} \frac{\pi(k)}{k} = \lim_{k\to 0} \pi'(k)$$

and $\lim_{k\to 0} \pi'(k) < \infty$ is a sufficient condition for the limit of $G'(k)$ to approach zero as $k \to 0$.

Thus $\lim_{k\to\infty} \pi'(k) < \infty$ is a sufficient condition for the transformation map $\mathtt{G}(k)$ to lie below the 45^o line close to the origin (see Figure 1 below). This makes it possible for multiple steady states to arise. While this condition applies to the reduced-form version of the survival probability, it is more instructive to take into account the chain of dependence of π on y and z and through these variables on k. Given the Cobb-Douglas production function assumed throughout the paper, we can express $\pi'(k)$ as:

$$\pi'(k) = \pi_y \frac{y}{k} + \pi_z \frac{z}{k}$$

In order for the sufficient condition to hold, π_y and π_z should have exponents in k which are large enough to offset the denominator. The following specialisation of Assumption 1 is sufficient to ensure this outcome, and we impose it from hereon:

Assumption 2

$$\begin{aligned} \pi_t &= \pi((y_t)^\beta, (z_t)^\delta) \\ min\{\beta, \delta\} &\geq \frac{1}{\alpha}; \end{aligned}$$

Assumption 2 implies that non-convexities exist in the relationship between survival probability and its determinants over at least some range of values of y and z. What is the justification for imposing these effects? Let us consider in turn the two determinants. While the empirical literature suggests that even low levels of pollution can result in increased mortality, it does not tell us much about the overall shape of the relationship. It is likely that at low levels of pollution, the marginal effect of pollution on mortality is low. Higher levels of capital are likely to produce an acceleration in the detrimental effects of pollution until eventually the natural bounded-ness of the the survival probability flattens out this relationship.[11]

[11] Even if the shape of the pollution-mortality relationship does not live up to the above intuition, a non-convexity could arise from the side of the income-mortality relationship. If neither effect is strong enough to satisfy Assumption 2, the steady state with exogenous taxation will be unique and neoclassical; our results on second-best taxation will then apply only to this case.

The effects of income, on the other hand, are likely to be more complex. If agents have Stone-Geary utility functions, then at very low levels of development the marginal impact of growth would be close to zero but once the survival threshold is met, further increases in income would yield positive effects. This alone would generate increasing returns in the relationship between π and y close to the trivial steady state. In addition, if investments in the technology to meet the populations' basic survival needs are subject to increasing returns then again the early relationship between income and life expectancy can display increasing returns. As for empirical evidence on this relationship, conventional understanding was based on the Preston Curve, which postulated a positive and concave relationship between per-capita incomes and life expectancy (Preston [1975]).

Recent studies, however, call into question this finding, based as it is on a simple cross-section comparison across countries. Georgiadis *et.al* [2010] have shown that if countries are disaggregated between high HDI (Human Development Index) and low HDI, the Preston Curve fits well the high HDI (mainly rich) countries but has no explanatory power for low HDI (mainly poor) ones. Moreover, Azomahou *et. al.* [2010] have shown, using historical data for a panel of 18 rich countries, that the relationship has alternating convex and concave segments.[12]

In the absence of Assumption 2, it is possible that $\mathtt{G}'(0) > 1$ and a unique steady state with globally stable dynamics would result, as in a standard neoclassical growth model. In the section of optimal taxes, we shall use the case of neoclassical dynamics as a benchmark against which to compare the dynamics which arise under Assumption 2 and multiple steady states.

While Assumption 2 implies that for low values of k: $k > \mathtt{G}(k)$, it is easy to show that the reverse is true for sufficiently large values of k. If we let $\tilde{k} = (0.5\Gamma)^{\frac{1}{1-\alpha}}$ for given Γ, α; then $\forall k \geq \tilde{k}$, $\mathtt{G}(k) \leq k$. To see this, suppose $k \geq \tilde{k}$ and that, contrary to the claim, $\mathtt{G}(k) > k$. Since $\pi \leq 1$ by definition, then $\pi/(1+\pi) \leq 0.5$ and $\mathtt{G}(k) \leq 0.5\Gamma k^{\alpha}$. By transitivity it must be the case that $0.5\Gamma k^{\alpha} > k$. But then $0.5\Gamma > k^{1-\alpha}$ and $(0.5\Gamma)^{\frac{1}{1-\alpha}} \equiv \tilde{k} > k$, leading to a contradiction.

So far we have established that either (i) there is no interior steady state or (ii) there are multiple interior steady states. To ensure (ii), note that the steady state equation

[12] Note, however, that their focus is on the causal implications of longer life expectancy on growth rather than the other way around and both their econometric specification and their theoretical model are based on this. Nonetheless their finding is indicative of the absence of smooth concavity and while our own assumed relationships place the convex segments at different levels of growth than that estimated by those authors, it should be noted that our general results do not rely on exactly where the convex portions lie.

can be rearranged as follows:

$$\Gamma = \frac{1+\pi(k)}{\pi(k)} k^{1-\alpha}$$

Given the function $\pi(k)$ and any finite and positive value of k, the right-hand side will be positive and finite. Since Γ is exogenous and positively related to A for $\tau < 1$ and $\alpha < 1$, there always exists A large enough that

$$\Gamma > \frac{1+\pi(k)}{\pi(k)} k^{1-\alpha}$$

This leads to the following result, stated without proof:

Lemma 1 *For any $\alpha \in (0,1)$ and $\tau \in (0,1)$ there exists an $\hat{A} < \infty$ and a $\hat{k} < \infty$ and asssociated $\hat{\Gamma}$: $\hat{\Gamma} = ((1+\pi(\hat{k}))/(\pi(\hat{k}))\hat{k}^{1-\alpha}$, such that $\Gamma > \hat{\Gamma}$, $\mathtt{G}(\Gamma, \hat{k}) > \hat{k}$.*

Lemma 1 implies that so long as disembodied productivity is high enough to begin with (given a function $\pi(k)$), $\mathtt{G}(k)$ will exceed k for a non-empty interval of values of k. Along with the results on the slope and level of $\mathtt{G}(k)$ derived earlier, this leads to the following proposition

Proposition 1 *If the disembodied productivity, A is large enough, and Assumption 2 holds, then there are two interior steady states, k_ℓ^* and k_h^*, such that $k_1^* < \hat{k} < k_2^*$.*

Given that two steady states exist, how do they compare with each other and what are their dynamic properties? The higher steady state, k_2^* has more capital and therefore more consumption as well as a higher stock of pollution. Despite the latter, it has greater survival probability due to the fact that higher output more than compensates for the higher stock of pollutants. To see this note that in the steady state, the survival probability must satisfy

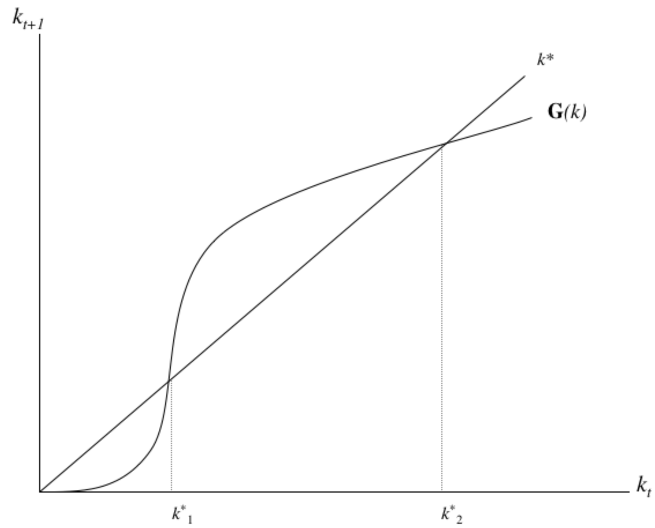
$$\pi(k) = \frac{k^{1-\alpha}}{\Gamma - k^{1-\alpha}}$$

which is increasing in k.

Figure 1 below represents the transformation map, depicting k_{t+1} function of k_t. Note that in drawing Figure 1, we hold constant the uniform tax rate, τ.

The 45^o line represents potential steady states. $\mathtt{G}(k)$ is S-shaped upwards, sharing its origin with the 45^o line and intersecting it at two other points k_ℓ^*, k_h^*. Since, for points which lie between the origin and k_1^*, $\mathtt{G}(k)$ lies below the 45^o line, any path starting off with $k_0 \in (0, k_\ell^*)$ will converge to the trivial steady state, while for points between k_ℓ^*

Figure 1: Multiple steady states



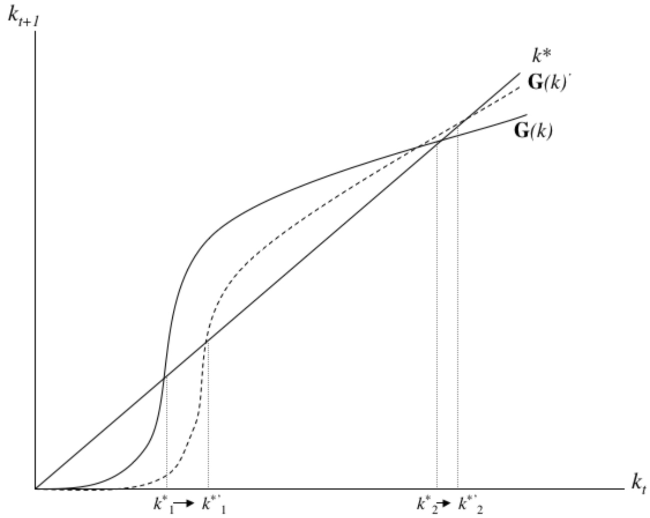
and k_h^*, $\mathtt{G}(k)$ lies above the 45^o line, any path starting off at $k_0 > k_\ell^*$ will converge to k_h^*.

k_ℓ^* represents a poverty trap not just in the sense that it is the steady state with lower levels of economic activity and pollution flows, but also in the sense that it represents a threshold starting point below which the equilibrium path of the economy converges asymptotically towards zero. We shall therefore refer to this type of steady state as a ‘poverty trap’. k_h^* represents a stable steady state, which resembles locally the unique steady state of a neoclassical growth model. We shall refer to this type of steady state as a ‘neoclassical steady state’ even when it is paired with a poverty trap.

It should finally be noted that strict concavity of $\mathtt{G}(k)$ can lead it to slope downward at some point. A necessary condition for this to happen is $\pi'(k) < 0$, which can happen at high enough values of k. To be precise, $\mathtt{G}(k)$ can slope downwards as it crosses the 45^o line from above, leading to oscillations and limit cycles in the stock of capital and the flow of emissions around the upper steady state.[13] While this possibility is of theoretical interest, to pursue it further would benefit from using a more general framework, rather than the simpler one we have chosen to characterise the effects

[13]Note that $\mathtt{G}(k)$ cannot slope downwards at the low steady state, even if $\pi'(k) < 0$.

Figure 2: A uniform increase in the tax rate.



4.2 Comparative statics:

To understand the interaction between pollution, mortality and income we carry out a comparative static exercise for an increase in the tax rate on emissions. This has the following effect on $\mathtt{G}(k)$:

$$\left.\frac{\partial \mathtt{G}(k)}{\partial \tau}\right|_k = \left[-\frac{\pi}{1+\pi} - \frac{[\pi_z \frac{\gamma\psi}{1-\phi}](1-\tau)Ak^{\alpha}}{(1+\pi)^2}\right](1-\alpha)Ak^{\alpha} \tag{19}$$

where π_z is the partial of π with respect to z alone (the effect of k on z is accounted for by the rest of the numerator in the second term). The above derivative is ambiguous in sign because $\pi_z < 0$. An increase in τ lowers net wage incomes, which at constant π tends to lower $\mathtt{G}(k)$. However, it raises π through its negative effect on z. This tends to work against the downward shift in $\mathtt{G}(k)$. However, the effects on π are weighted by k^{α}. Whatever the net effect on π, this term is likely to be dominated by the direct effect of τ on wage income for low values of k. Thus $\mathtt{G}(k)$ is likely to shift down at low levels of k even if it shifts up at higher levels. These combinations of effects are shown in Figure 2.

As can be seen, an increase in τ causes a downward shift in $\mathtt{G}(k)$ at low levels of capital stock but upwards at the high capital stock. There are two new steady states, $k_1^{*\prime}$ and $k_2^{*\prime}$, with the former being unstable and the latter being stable. Compared with their respective predecessors, *both* steady states are at higher levels of capital stock. The dynamic implications of this shift are that while the range of starting points which lead to a poverty trap has now increased from the interval $[0, k_1^*]$ to the interval $[0, k_1^{*\prime}]$ for economies that start of to the right of $k_1^{*\prime}$ will be on a path of convergence a higher steady state than before, with higher capital as well as higher expectations of longevity. In other words, with an arbitrary stationary tax, it is possible that initially poor economies become more likely to end up in a poverty trap while initially wealthy economies actually become wealthier as the curbs on pollution raise expected lifetimes and stimulate further capital accumulation.

4.3 An example of $\pi(k)$

Assuming the specific functional form:

$$\pi = \pi^A \pi^B$$

where

$$\pi^A = \frac{\underline{\pi} + y^\beta}{1 + y^\beta}$$

then it can be shown that $\pi_y^A > 0$ if $\underline{\pi} < 1$ and that $\pi_{yy}^A \leq 0$ if and only if $y \leq [(\beta - 1)/(1 + \beta)]^{1/\beta}$ so that for any $\beta > 1$, $\pi^A(y)$ is S-shaped upwards.

If similarly,

$$\pi^B = \frac{1}{1 + z^\delta}$$

then it can be shown that $\pi^B < 0$ and that $\pi_{zz}^B \leq 0$ if and only if $z \leq [(\delta - 1)/(1 + \delta)]^{1/\delta}$ so that for any $\delta > 1$, $\pi^B(z)$ is reverse S-shaped downwards.

Thus, the above function satisfies the sufficient conditions for multiple steady states. Indeed it can be shown that, after imposing the steady state relationship between y, z and k and totally differentiating, that a sufficient condition for $\pi'(k)$ to satisfy the conditions of Lemma 1 as k approaches zero is that

$$\min\{\beta, \delta\} > \frac{1}{\alpha} > 1$$

The above ensures that $\lim_{k \to 0} \pi'(k) = 0$, which is stronger than what is needed for Lemma 1.

If we consider a special case where $\underline{\pi} = 0$, then $\pi'(0) = 0$ so long as $\beta > 1/\alpha$. Actually, for that case, it can be shown that a weaker condition can suffice to generate $\mathtt{G}'(0) = 0$, *i.e.*

$$\beta > \frac{1 - \alpha}{\alpha}$$

This is because the combination of the terms

$$\mathtt{G}'(k) = \left[\frac{\Gamma k^{\alpha}}{1+\pi(k)}\right]\left[\alpha\frac{\pi(k)}{k} - \frac{\pi'(k)}{1+\pi(k)}\right].$$

can converge to zero even if each term inside the square brackets does not.

Another special case is to assume $\pi^A = \bar{\pi}$ so that growth affects survival probability only through pollution. This case could also lead to multiple steady states if $\delta > 1/\alpha$ and could also be used as a vehicle for studying optimal tax policy but because it implies a counter-factually monotonic and negative impact of growth on survival, we ignore it.

Returning to the main functional form assumed above, for the following set of parameter values,

$$\alpha = 1/3, A = 2, \gamma = 1, \underline{\pi} = 0.0, \beta = \delta = 5, \psi = 0.8, \phi = 0.1;$$

MATLAB was used to solve for steady states at different values of τ. The results were

τ	k_ℓ^*	k_h^*
0.00	0.0339	0.0965
0.15	0.0404	0.1136
0.35	0.0686	0.1026

In increasing the tax from a no-tax benchmark, the levels of capital per worker rises in *both* steady states, illustrating the possibility that an arbitrary imposition of environmental taxes can hinder growth in low-income economies, by expanding the size of the poverty trap, while simultaneously promoting it in high-income ones by increasing the size of the steady level of capital and output per worker. Increasing the tax rate even further, however, results in the conventional effect at the higher steady state, while continuing to expand the poverty trap at the lower end.

5 Optimal taxes

We now assume that in each period t, a government chooses an optimal pollution tax to maximise the lifetime welfare of the generation born in that period. In choosing the tax, it takes the inherited stocks of capital and pollution as given but takes into account the effect of its abatement policy on the savings and expected lifetimes of the contemporaneous young. Its problem is stated as:

$$\max_{\tau_t} U^t = \ln c_t^y + \pi_t \ln c_{t+1}^o$$

subject to the agents' budget constraints (11), (12), the equation of motion for capital, (13), and the equation of motion for the stock of pollution (4) and size restrictions on the tax rate: $1 \geq \tau \geq 0$.

After substituting for c_t^y, c_{t+1}^o and k_{t+1} from equations (11), (12) and (13) respectively into the objective function, the problem can be seen to be a static one:

$$\max_{\tau_t} V(k_t, \tau_t) = \ln\left(\frac{(1-\tau_t)(1-\alpha)Ak_t^\alpha}{1+\pi(k_t)}\right) + \pi(k_t)\ln\left(\frac{\alpha(1-\alpha)^\alpha A^{1+\alpha}(1-\tau)^\alpha k^{2\alpha}}{\pi(k_t)^{1-\alpha}(1+\pi(k_t))^\alpha}\right). \tag{20}$$

In other words, the optimal wage tax at time t depends only on the capital stock at time t, since capital accumulation has been endogenised and accounted for.

The first-order condition can, after rearrangement, be expressed as:

$$\frac{dV_t}{d\tau_t} = \left[\ln c_{t+1}^o - \frac{2-\alpha+\pi_t}{1+\pi_t}\right] \cdot \frac{\partial \pi_t}{\partial \tau_t} - \frac{1+\alpha\pi_t}{1-\tau_t} \leq 0; \tag{21}$$

where < 0 implies $\tau_t = 0$.

With some further restrictions, the above condition underlies a policy function, $\tau_t = h(k_t)$. Substituting the solution into equation (13) for capital accumulation yields $k_{t+1} = \mathtt{G}(h(k_t), k_t)$. The dynamic path of the economy is traced out by repeated iteration of the above. A steady state of the economy with optimal taxes is given by a pair k and $\tau = h(k)$ such that $k = \mathtt{G}(h(k), k)$.

The terms in the above expression represent the following effects: the direct effects of a tax on wage incomes, and the indirect effects working through induced changes in survival probability. The direct effects reduce both consumption and savings by the young, and are negative. These are captured by the last term in the optimality condition. The indirect effects are captured in the term inside square brackets. An environmental tax raises survival probability, leading to higher expected utility in old age. At the same time the higher survival probability reduces actual consumption at both young and old age, the first because savings are increasing in survival probability; the second because although individuals save more the return to their annuities yields less because of the higher survival ratio of the population. The last effect can be confirmed from equation (20) which is decreasing in π. The intuition is that while per-capita old-age capital increases by a factor of $[\pi/(1+\pi)]^\alpha$, the market return on a unit annuity decreases by a factor $1/\pi$. Indeed for an environment tax to be optimal, the gains from higher life expectancy have to outweigh the other effects and that in turn requires a minimum level of old-age consumption to begin with.

Proposition 2 *If k_0 is below some threshold level $\underline{k}$, then the optimal environmental tax, $\tau^* = 0$.*

Proof: From (21) we see that a necessary condition for $\tau^* > 0$ is

$$\Omega_t = \left[\ln c^o_{t+1} - \frac{2 - \alpha + \pi_t}{1 + \pi_t} \right] > 0.$$

At low levels of initial capital, k_0, this is not going to hold. This is because the negative term in Ω_t is always non-zero while the positive term approaches zero (or minus infinity given the logarithmic specification) as the capital stock approaches zero. Thus there exists some threshold level $\underline{k}$; such that for any $k_0 < \underline{k}$, $\Omega < 0$. ■

To see the potential for a positive tax at higher levels of capital, consider how Ω behaves as capital rises, abstracting for now from the equilibrium path. In principle, there will always be an arbitrarily high level of k_t such that $\Omega_t > 0$. This is because the first term in Ω_t has the potential to increase monotonically with k_t, at least after some threshold, while the second term is always bounded in the interval $[(3 - \alpha)/2, (2 + \underline{\pi} - \alpha)/(1 + \underline{\pi}]$ and within this interval, it falls with increases in π_t. c^o_{t+1} rises monotonically with k_t even when π_t rises as well. If along the dynamic path, the detrimental effects of pollution make π_t start declining in k_t, then c^o_{t+1} rises even faster with k_t. At some level of development, Ω_t will be positive and increasing in capital. The other negative term in the first-order condition is similarly bounded above at $(1 + \alpha)$, when evaluated at a zero tax rate. Thus, at a second critical level of development, an interior solution will arise for a positive optimal tax. The question is what level of development has to be reached before it arises and to what extent this level coincides with potential steady states of the economy.

To pursue these conjectures more rigorously, we first establish some general conditions for the applicability of a positive environmental tax at some threshold level of income. Let the right-hand side of equation (21) be denoted by:

$$\mathtt{H}(k_t, \tau_t) = \Omega_t \cdot \frac{\partial \pi_t}{\partial \tau_t} - \frac{1 + \alpha \pi_t}{1 - \tau_t}$$

The first condition needed for a well-behaved tax function is

$$\left. \frac{\partial \mathtt{H}}{\partial \tau_t} \right|_{\mathtt{H}=0} < 0.$$

In other words, that the second-order condition is satisfied whenever the first-order condition holds as an equality.

The second condition ensuring a well behaved tax function is:

$$\left. \frac{\partial \mathtt{H}}{\partial k_t} \right|_{\tau=0, \mathtt{H}=0} > 0.$$

In other words, evaluated at the point where the first-order condition first holds as an equality at a zero tax, it is upward sloping in k_t. Note that at very low levels of the capital stock, this may not be true but what the above condition requires is that it be true in the neighbourhood of the threshold where an optimal tax first arises.

To explore the above conditions further, differentiate $\mathtt{H}$ with respect to its arguments (time scripts will be suppressed as all variables are contemporaneous. After some manipulation, these derivatives can be written as

$$\frac{\partial \mathtt{H}}{\partial \tau} = \Omega \frac{\partial^2 \pi}{\partial \tau^2} - \frac{2\alpha}{1-\tau}\frac{\partial \pi}{\partial \tau} - \frac{1+\alpha\pi}{(1-\tau)^2} - \frac{\pi(1+\pi)+(1-\alpha)}{\pi(1+\pi)^2}\left(\frac{\partial \pi}{\partial \tau}\right)^2; \tag{22}$$

$$\frac{\partial \mathtt{H}}{\partial k} = \frac{\partial \Omega}{\partial k}\frac{\partial \pi}{\partial \tau} + \Omega \frac{\partial^2 \pi}{\partial \tau \partial k} - \frac{\alpha}{1-\tau}\frac{\partial \pi}{\partial k}; \tag{23}$$

where

$$\frac{\partial \Omega}{\partial k} = \frac{2\alpha}{k} - \frac{(1+\pi)^2 - \pi - \alpha}{(1+\pi)^2}\nu_{\pi k}$$

where $\nu_{\pi k}$ is the elasticity of survival probability with respect to capital. This is eventually decreasing in k due to the positive and eventually diminishing effects of greater income and the negative and eventually increasing effects of higher pollution. It can turn negative at some point; however, we shall restrict our analysis to cases where it remains strictly positive.

None of the above terms can be signed unambiguously but two comments are in order. First, as noted before, a positive effect of k on Ω is necessary for the first-order condition to eventually hold. What this in turn requires is that along the infra-marginal path of capital, *i.e.* before the first-order condition kicks in, there is some range of values of k where the elasticity of survival probability with respect to the capital stock (taking into account both the beneficial and detrimental effects) is sufficiently small. As noted above, this elasticity will eventually diminish with growth in the capital stock, implying the existence of a threshold value of capital after which $\partial\Omega/\partial k > 0$. From hereon we neglect consideration of values of k below this threshold, as for the purposes of deriving an environmental tax, such values of k cannot admit positive solutions for τ. Second, a sufficient condition for the second-order condition for τ to be negative is that π is concave in τ. However, this is likely to be too restrictive, given the following relationship between the second-order derivatives of π with respect to τ and z:

$$\frac{\partial^2 \pi}{\partial \tau^2} = (\psi\gamma A k^{\alpha})^2 \frac{\partial^2 \pi}{\partial z^2}$$

Thus, π will be concave in τ if and only if it is downwards concave in z. But given the likely impact of pollution levels on survival probability, this portion of the $\pi - z$

relationship applies at lower levels of pollution, when it is less likely that the first-order condition for an optimal tax will hold as an equality. At higher levels, it is unlikely that π is concave in τ. This rules out imposing concavity on the $\pi - \tau$ relationship as a sufficient condition for ensuring the validity of the second-order condition.

To proceed further, we turn to the specific example of the survival probability assumed earlier.

$$\pi = \pi^A \pi^B = \left[\frac{\underline{\pi} + y^\beta}{1 + y^\beta}\right]\left[\frac{1}{1 + z^\delta}\right]$$

In the following subsections we first analyse the sign of $\partial^2\pi/\partial\tau^2$ and then the sign of $\partial^2\pi/(\partial\tau\partial k)$

5.1 The second-order condition, $\partial \mathtt{H}/\partial\tau$

The following expressions are derived for the specific form (time scripts are again suppressed).

$$\frac{\partial\pi}{\partial\tau} = \pi^A \frac{\psi\delta\gamma A k^\alpha z^{\delta-1}}{(1 + z^\delta)^2} > 0; \tag{24}$$

$$\frac{\partial^2\pi}{\partial\tau^2} = \pi^A \frac{(\psi\gamma A k^\alpha)^2 \delta z^{\delta-2}}{(1 + z^\delta)^3}\left[(\delta+1)z^\delta - (\delta-1)\right].$$

By comparing the two expressions, the latter can be written as

$$\frac{\partial^2\pi}{\partial\tau^2} = \pi^A\left(\frac{\psi\gamma A k^\alpha\delta}{z(1 + z^\delta)} \cdot \frac{\partial\pi}{\partial\tau}\right)\left[(\delta+1)z^\delta - (\delta-1)\right]\left\{\begin{matrix} > \\ = \\ < \end{matrix}\right\} 0 \text{ as } z^\delta \left\{\begin{matrix} > \\ = \\ < \end{matrix}\right\} \frac{\delta-1}{\delta+1},$$

confirming the dependence of the sign of $\partial^2\pi/\partial\tau^2$ on that of $\partial^2\pi/\partial z^2$. To proceed further with an analysis of the second-order condition, equation (22), note from equation (4) that:

$$\gamma A k_t^\alpha = \frac{z_t - \phi z_{t-1}}{1 - \psi\tau_t}.$$

Suppressing time subscripts, let us write this as

$$\gamma A k^\alpha = \frac{z - \phi z'}{1 - \psi\tau},$$

where $z' = z_{t-1}$. The above can be further modified:

$$\frac{\partial^2\pi}{\partial\tau^2} = \pi^A\left(\frac{\psi\delta(z - \phi z')}{z(1 - \psi\tau)(1 + z^\delta)} \cdot \frac{\partial\pi}{\partial\tau}\right)\left[(\delta+1)z^\delta - (\delta-1)\right]$$

Now, from equation (21),

$$\Omega \leq \frac{1+\alpha\pi}{1-\tau}\frac{1}{\partial\pi/\partial\tau}\ \forall\, \tau$$

Thus, taking the term involving $\partial^2\pi/\partial\tau^2$ in equation (22),

$$\Omega\frac{\partial^2\pi}{\partial\tau^2} \leq \left(\frac{1+\alpha\pi}{1-\tau}\frac{\psi\delta(z-\phi z')}{z(1-\psi\tau)(1+z^\delta)}\right)\left[(\delta+1)z^\delta-(\delta-1)\right]$$

Combining with one of the other terms in equation (22)

$$\Omega\frac{\partial^2\pi}{\partial\tau^2} - \frac{1+\alpha\pi}{(1-\tau)^2} \leq \left[\frac{1+\alpha\pi}{1-\tau}\right]\left[\frac{\psi\delta(z-\phi z')\left[(\delta+1)z^\delta-(\delta-1)\right]}{z(1-\psi\tau)(1+z^\delta)} - \frac{1}{1-\tau}\right] \tag{25}$$

The sign of the above term will depend on the sign of the term inside square brackets. After some manipulation, the sign of the latter can be shown to be negative if the following holds:

$$-\frac{[1-\psi\{1+\delta(1-\tau)\}]z^\delta}{(1-\psi\tau)(1+z^\delta)(1-\tau)} < 0$$

A sufficient condition for the above term to be negative for all values of endogenous variables is $\psi < 1/(1+\delta)$.[14]

If we restrict attention to steady states, then a weaker condition suffices. Note that at a steady state, $(z-\phi z') = (1-\phi)z$ and $\partial^2\pi/\partial\tau^2$ can be written as

$$\frac{\partial^2\pi}{\partial\tau^2} = \pi^A\left(\frac{\psi\delta(1-\phi)}{(1-\psi\tau)(1+z^\delta)}\cdot\frac{\partial\pi}{\partial\tau}\right)\left[(\delta+1)z^\delta-(\delta-1)\right].$$

Repeating the steps from equation (25), we arrive at an expression whose sign, if negative, will ensure that the second-order condition is met:

$$-\frac{[1-\psi\phi\tau-\psi(1-\phi)\{1+\delta(1-\tau)\}]z^\delta}{(1-\psi\tau)(1+z^\delta)(1-\tau)}. \tag{26}$$

A sufficient condition for the sign to be negative is that

$$\psi[\phi\tau+(1-\phi)(1+\delta(1-\tau)] < 1. \tag{27}$$

In turn, the above is achieved if $\psi/[(1+\delta)(1-\phi)] < 1$, which is weaker than the general condition, since for $(1+\delta)(1-\phi) < 1$, any $\psi < 1$ will satisfy the condition, while for

[14] By extending the comparison with the sign of $\Omega\cdot\partial^2\pi/\partial\tau^2$ to other terms in the expression for $\partial^2\mathtt{H}/\partial\tau^2$ even weaker conditions can be derived. But as with the above, to ensure negativity of the second-order condition for all admissible values of endogenous variables, the above condition still applies.

$(1+\delta)(1-\phi) > 1$, the above is less restrictive than $\psi < 1/(1+\delta)$ for any $\phi > 0$.[15]

We have therefore established:

Lemma 2 *A sufficient condition for $\partial \mathtt{H}/\partial\tau$ to be negative at all values of endogenous variables and along the entire dynamic path is $\psi/(1+\delta) < 1$. In the steady state, a weaker condition suffices to ensure the validity of the second-order condition, $\psi/[(1+\delta)(1-\phi)] < 1$.*

Recall that $\psi = \dfrac{\chi(1-\alpha)}{\gamma}$, where χ is the effectiveness of the abatement technology γ is how polluting is the productive activity. As we would expect, if the first is low enough and/or the second high enough, then the second order condition holds, or in other words there is an interior solution.

5.2 The sign of $\partial \mathtt{H}/\partial k$

Note the following derivatives for the assumed function form (time indices continue to be suppressed):

$$\frac{\partial \pi^A}{\partial k} = \frac{\alpha}{k}\frac{\beta(1-\underline{\pi})y^\beta}{(1+y^\beta)^2} \tag{28}$$

$$\frac{\partial \pi^B}{\partial k} = -\frac{\alpha\gamma(1-\psi\tau)Ak^\alpha}{k}\frac{\delta z^{\delta-1}}{(1+z^\delta)^2} \tag{29}$$

$$\frac{\partial \pi}{\partial k} = \pi^B\frac{\alpha}{k}\frac{\beta(1-\underline{\pi})y^\beta}{(1+y^\beta)^2} - \pi^A\frac{\alpha\gamma(1-\psi\tau)Ak^\alpha}{k}\frac{\delta z^{\delta-1}}{(1+z^\delta)^2} \tag{30}$$

Further, using the definitions of π^A, π^B, and π, and noting that $\gamma(1-\psi\tau)Ak^\alpha = z-\phi z'$, we can express equation (30) as

$$\frac{\partial \pi}{\partial k} = \frac{\alpha\pi}{k}\left[\frac{\beta(1-\underline{\pi})y^\beta}{(1+y^\beta)(\underline{\pi}+y^\beta)} - \frac{\delta(z-\phi z')z^{\delta-1}}{(1+z^\delta)}\right]$$

which implies that

$$\nu_{\pi k} = \alpha\left[\frac{\beta(1-\underline{\pi})y^\beta}{(1+y^\beta)(\underline{\pi}+y^\beta)} - \frac{\delta(z-\phi z')z^{\delta-1}}{(1+z^\delta)}\right]$$

[15] It can be shown that if $(1+\delta)(1-\phi) > 1$, then the term inside square brackets in the numerator of equation (27) increases in τ so that it reaches a maximum at $\tau = 1$, where its value is unity; if $(1+\delta)(1-\phi) = 1$ then the term inside square brackets equals unity at all values of τ and if $(1+\delta)(1-\phi) < 1$ than the term inside the square brackets reaches a maximum at $\tau = 0$, where its value is $(1+\delta)(1-\phi)$ and is by assumption, less than unity. This is why the restriction in equation (27) applies at all values of τ.

where $\nu_{\pi k}$ has been defined as the *elasticity* of π with respect to k.[16]

Now, to derive the sign of $\partial^2 \mathtt{H}/(\partial\tau\partial k)$, we proceed in two steps. We first derive an expression for $\partial^2\pi/(\partial\tau\partial k)$ and then use it to evaluate the sign of $\partial^2 \mathtt{H}/(\partial\tau\partial k)$.

The first step is accomplished by taking the total derivative of $\partial\pi/\partial\tau$, equation (24), with respect to k. After imposing some definitions and equalities, and rearranging terms, it can be shown that:

$$\frac{k}{\partial\pi/\partial\tau}\frac{\partial^2\pi}{\partial\tau\partial k} = \nu_{\pi k} + \alpha\delta\frac{(z-\phi z')}{z(1+z^\delta)} + \alpha\phi\frac{z'}{z} > 0.$$

The full derivation is outlined in the Appendix. From here it is easy to establish the following:

Lemma 3 $H(k,\tau) = 0 \Longrightarrow \partial H/\partial k \geq 0$.

Proof: First, the expression for $\partial^2\pi/\partial\tau\partial k$ implies that

$$\frac{\partial^2\pi}{\partial\tau\partial k} \geq \frac{\partial\pi}{\partial\tau}\frac{1}{k}\nu_{\pi k}.$$

Second $F = 0$ implies that

$$\Omega = \frac{1+\alpha\pi}{1-\tau}\frac{1}{\partial\pi/\partial\tau}.$$

Therefore, referring to equation (23),

$$\Omega\frac{\partial^2\pi}{\partial\tau\partial k} = \frac{1+\alpha\pi}{1-\tau}\frac{1}{\partial\pi/\partial\tau}\frac{\partial^2\pi}{\partial\tau\partial k} \geq \frac{1+\alpha\pi}{1-\tau}\frac{1}{k}\nu_{\pi k}.$$

Now, referring to the negative term in equation (23),

$$\frac{\alpha}{(1-\tau)}\frac{\partial\pi}{\partial k} = \frac{\alpha\pi}{(1-\tau)k}\nu_{\tau k}$$

Combine the two terms in equation (23),

$$\Omega\frac{\partial^2\pi}{\partial\tau\partial k} - \frac{\alpha\pi}{(1-\tau)k}\nu_{\pi k} \geq \frac{1+\alpha\pi}{1-\tau}\frac{1}{k}\nu_{\pi k} - \frac{\alpha\pi}{(1-\tau)k}\nu_{\pi k} \geq \frac{1}{(1-\tau)k}\nu_{\pi k} \geq 0$$

■

Note that we have derived the above result for all values of τ. Thus, as an economy's capital stock grows hypothetically larger, the slack in $\mathtt{H}$ diminishes until finally an interior solution is reached.

[16]Throughout the analysis, we assume that $\nu_{\pi k}$ remains positive, although as we have noted before, a negative value is entirely possible under some conditions, and if it happens there can be oscillations around the high steady state.

5.3 Positive taxes:

From this we can now establish:

Proposition 3 *Provided that the second-order condition for optimal taxes is satisfied, and that the initial level of capital is above a threshold* $\tilde{k}$ *such that for all* $k > \tilde{k}$, $\partial\Omega/\partial k > 0$, *there (i) exists a function* $h : [\tilde{k}, \infty) \longrightarrow [0, 1]$ *such that optimal* $\tau = h(k)$*; (ii)* $h(k)$ *is (weakly) increasing in* k.

Proof: The first part follows from the strict monotonicity of $\mathtt{H}$ in both τ and k. Since $\mathtt{H}$ is strictly decreasing in τ for all k under the assumed conditions, then for any k in the relevant interval, either (i) $\mathtt{H}(0, k) \leq 0$, or (ii) $\mathtt{H}(1, k) > 0$ or (iii) $\mathtt{H}(\tau, k) = 0$ for some $\tau \in [0, 1]$. Moreover, τ uniquely solves the relevant case for $\mathtt{H}$ at given k, because for any $\tau' > \tau$, in case (i) $\tau = 0$ and $\tau' > 0$ worsens the slack in $\mathtt{H}$; in case (ii) if $\tau = 1$ then τ' lies outside the unit interval and in case (iii) since $\mathtt{H}(\tau, k) = 0$ for $\tau \in [0, 1]$, then $\mathtt{H}(\tau', k) < 0$. Similar argument rules out the possibility that $\tau' < \tau$ also solves $\mathtt{H}$ for a given k.

The second part follows from

$$\left.\frac{\partial h(k)}{\partial k}\right|_{\mathtt{H}=0} = -\frac{\mathtt{H}_k}{\mathtt{H}_\tau} \geq 0.$$

while $\forall k \in [\tilde{k}, \infty)$, $\mathtt{H}(0, k) < 0 \Rightarrow \tau = 0$ and $\mathtt{H}(1, k) > 0 \Rightarrow \tau = 1$. ■

Since $\mathtt{H}_k$ is positive, at low values of k, $\tau = 0$ so that $h(k)$ is flat at the no-tax equilibrium over this region. Note that $F(1, k) > 0$ is likely to be ruled by the fact that the negative term in $\mathtt{H}$ approaches $-\infty$ at all values of k, so the likely shape of $h(k)$ is flat at low values of k followed by an upward sloping portion which remains asymptotically bounded away from zero.

5.4 Dynamics of the optimal tax:

A steady state with optimal taxation is characterised by two simultaneous equations.

$$k = \frac{\pi(k, \tau)}{1 + \pi(k, \tau)} A \cdot (1 - \tau)(1 - \alpha)k\alpha \quad (31)$$

$$\tau = h(k) \quad (32)$$

A solution to the above equations is represented by a pair (k^*, τ^*). Define $k^* = g(\tau)$, as the value of of k^* which solves equation (31) for any admissible τ. Then $\tau^* = h(k^*)$ solves the optimal tax at this steady state. As we shall see, the interaction of optimal environmental policy with the non-convexities associated with effects of pollution create an even richer set of possible steady state equilibria than in the exogenous tax case. Before identifying these possibilities, we shall characterise the dynamic properties of different types of steady states, should each one exist. We shall then use a graphical approach to jointly charcterise the existence and dynamic properties of various steady states.

It is easy to show that

$$g'(\tau) = \frac{\frac{\partial \mathtt{G}(k^*)}{\partial \tau}}{1 - \mathtt{G}'(k^*)}. \tag{33}$$

where $\partial \mathtt{G}(k)/\partial \tau$ is given by equation (19).

The dynamics of the economy with optimal taxes are traced out by recursive application of h and $\mathtt{G}$. For any capital $k_t > \tilde{k}$, $\tau_t = h(k_t)$. Then, next period's capital stock follows:

$$k_{t+1} = \frac{\pi(h(k_t), k_t)}{1 + \pi(h(k_t), k_t)} A(1 - h(k_t))(1 - \alpha)k_t^{\alpha} = \mathtt{G}(h(k_t), k_t)$$

and so on.

This represents a first-order difference equation in k_t for any arbitrary k_0. Linearising around a steady state, the local dynamics are determined by the sign and magnitude of the expression

$$\left. \frac{dk_{t+1}}{dk_t} \right|_{k^*} = \mathtt{G}'(k^*) + \frac{\partial \mathtt{G}(k^*)}{\partial \tau} h'(k^*)$$

Using equation (33), the above can be expressed as:

$$\left. \frac{dk_{t+1}}{dk_t} \right|_{k^*} = \mathtt{G}'(k^*) + g'(\tau^*)(1 - \mathtt{G}'(k^*))h'(k^*). \tag{34}$$

where the sign of $g'(\tau^*)$ is the same as (the opposite of) the sign of $\partial \mathtt{G}(k^*)/\partial \tau$ as and when $1 - \mathtt{G}'(k^*) > 0$ (< 0).

Recall that $\partial \mathtt{G}(k)/\partial \tau$ may be positive or negative (see Figure 2) and that $\mathtt{G}'(k^*)$ may be greater than or less than one (see Figure 1). Several cases can arise depending on the respective signs and magnitudes of the above terms.

We highlight the cases of interest. In each case in the sub-section below, we restrict attention to dynamic paths which are monotonic, or equivalently, in which the right-hand side of equation (34) is non-negative.

5.4.1 Non-oscillatory paths:

1. A stable neoclassical steady state:

This case requires two conditions: (i) the transformation map cuts the 45^o line from above ($\mathtt{G}'(k^*) < 1$) and (ii) an increase in the environmental tax rate has the conventional effect of lowering the steady state capital stock ($\partial \mathtt{G}(k^*)/\partial \tau < 0$) . The combination of these suggests that $g'(\tau^*) < 0$. From equation (34),

$$\left.\frac{dk_{t+1}}{dk_t}\right|_k^* < 1,$$

so that the steady state is locally stable. Note that this is a 'well-behaved' case since it could be consistent with the possibility that $\mathtt{G}(k)$ lacks a convex portion, leading to a unique steady state. It also embodies conventional effects from higher taxes to the steady state capital stock. The resulting steady state would be stable under exogenous taxes and it remains stable with optimal taxes.[17]

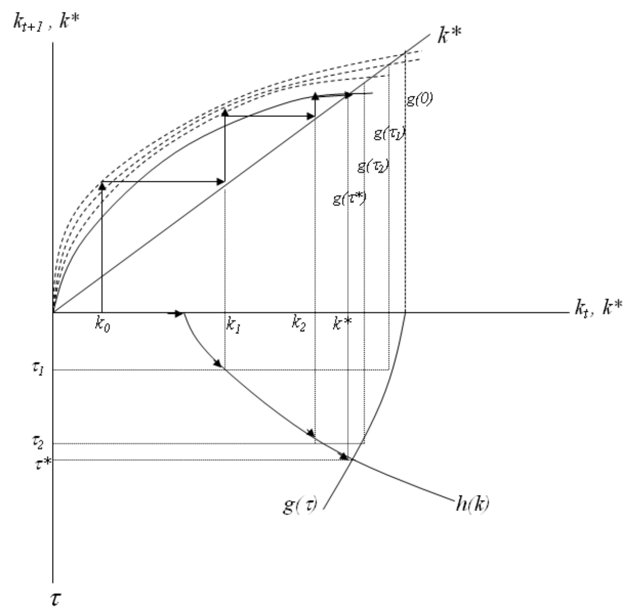
Figure 4 depicts the local dynamics around this type of steady state.

The top panel of Figure 4 shows a family of transformation maps for k_{t+1} as a function of k_t. Each map is underpinned by a specific value of the environmental tax, τ_t. The lower panel depicts the functions $g(\tau)$ and $h(k)$ in $(\tau - k)$ space. $h(k)$ is always upward sloping in this space but in keeping with case 1, $g(\tau)$ is downward sloping. Their intersection gives the combination of steady state capital and steady state taxes, $(\tau^* - k^*)$. This is the unique long-run steady state in the case depicted.

Starting at $k_0 < \tilde{k}$, the latter defined earlier as the minimum level of capital associated with active environmental policy, the optimal tax at $t = 0$ is $\tau_0 = 0$. The steady state associated with this tax is the highest dashed transformation map on the top panel, which is labeled $g(0)$. If the tax rate was held constant at this level, the capital stock would evolve monotonically towards $g(0)$ through iterative application of this map. Thus at $t = 0$, next period's capital, k_1, will be given by the vertical projection to this map from k_0. But when the economy reaches k_1, the optimal tax for that period need no longer equal zero. Indeed, as drawn, the threshold level of capital is crossed and optimal $\tau_1 > 0$, as given by the projection down from k_1 to $h(k)$. At τ_1, the horizontal projection to $g(\tau)$ gives the new steady state level of capital that would arise if the tax rate were held constant at τ_1. This means that the transformation map in the upper panel shifts downwards so it intersects the 45^o line at $g(\tau_1)$. The vertical

[17] $\mathtt{G}(k)$ would lack a convex portion if the conditions of Assumption 2 fail to be satisfied, implying relatively weak effects of pollution and income on life expectancy.

Figure 3: A stable neoclassical steady state



projection from k_1 to the new transformation map gives k_2 and so on. The dynamics are monotonically convergent with both k_t and τ_t rising in ever shorter steps towards the steady state.

2. Dynamics of optimal taxation around a poverty trap:

In this case, the steady state map cuts the 45^o line from below ($\mathtt{G}'(k^*) > 1$).

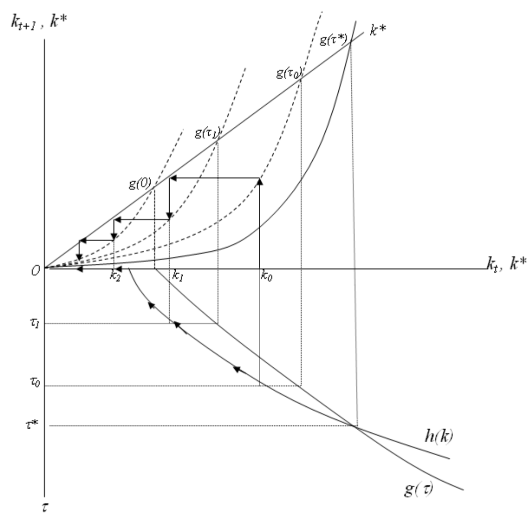
An exogenous increase in the tax rate would widen the poverty trap by increasing the steady state capital stock ($g'(\tau) > 0$). In this case it can be shown by using equation (34) that

$$\mathtt{G}'(k^*) + g'(\tau)(1 - \mathtt{G}'(k^*))h'(k^*) \left\{ \begin{array}{c} < \\ > \end{array} \right\} 1 \text{ as } g'(\tau)h'(k) \left\{ \begin{array}{c} > \\ < \end{array} \right\} 1$$

so that a poverty trap with optimal taxes could be either locally unstable (if $g'(\tau)h'(k) > 1$) or locally stable ($g'(\tau)h'(k) < 1$). Note that in the case of uniform exogenous taxes, the steady state would be unambiguously unstable. Figures 5 and 6 respectively show the local dynamics for these cases.

Figure 5 shows the unstable case. The long-run steady state is at (τ^*, k^*) where $g(\tau)$ and $h(k)$ intersect. Note that the condition under which this case arises requires that

Figure 4: A locally unstable poverty trap



$h'(k) > 1/g'(tau)$ at the steady stae. This in turn requires that $h(k)$ lies below $g(\tau)$ to the left of the steady state in the bottom half of Figure 5.

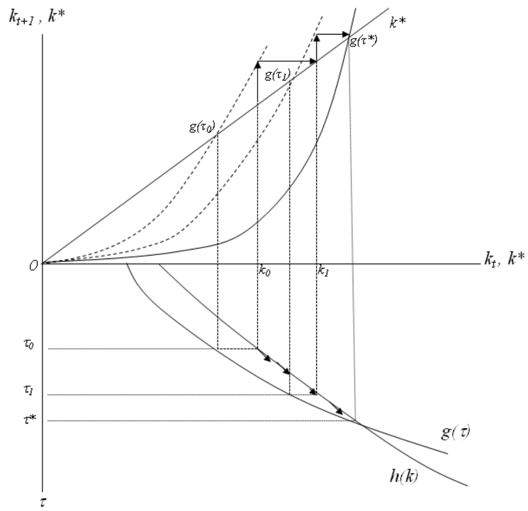
The economy starts at $k_0 < k^*$ and the initial tax rate is τ_0 as shown on the lower panel. The steady state associated with this tax rate, $g(\tau_0)$ lies above k_0 (since $g(\tau)$ lies to the right of $h(k)$ at this point). Because the steady state is locally unstable, according to the transformation map associated with it, $k_1 < k_0$. The tax rate associated with k_1 is $\tau_1 < \tau_0$ and the steady state associated with that tax rate is even lower, so that $k_2 < k_1$. Indeed at k_2, the optimal tax rate drops to zero and stays there, while the capital stock itself converges over time to the trivial steady state.

In Figure 6, we again have a poverty trap at the interior steady state (τ^*, k^*). In this case, $h'(k) < 1/g'(\tau)$, so that $h(k)$ lies above $g(\tau)$ to the left of the steady state in the bottom half of Figure 6.

When we start at a low level of capital, $k_0 < k^*$ and tax rate $\tau_0 < \tau^*$, the transformation map associated with τ_0 results in a steady state $g(\tau_0) < k_0$. Because $g(\tau_0)$ is unstable this means that $k_1 > k_0$. This sets the economy on a convergent path towards (τ^*, k^*), as can be seen by further iterations of the dynamics at $t = 1$ and $t = 2$.

Intuitively, Figure 5 represents a case in which it is optimal to impose an environmental tax at fairly low levels of steady state capital, but subsequent increases in this capital do not result in large increases in the tax rate. This is reflected in the fact that the

Figure 5: A locally stable poverty trap



$h(k)$ locus cuts the $g(\tau)$ locus from below. When the initial capital stock lies below the long-run steady state, the optimal tax rate associated with that capital stocks maps into a (transitory) steady state that lies above the initial capital stock. Since each transitory steady state, *i.e.* steady state associated with a tax rate that is held hypothetically constant, are unstable, this pushes the capital stock further below the long-run steady state, and so on.

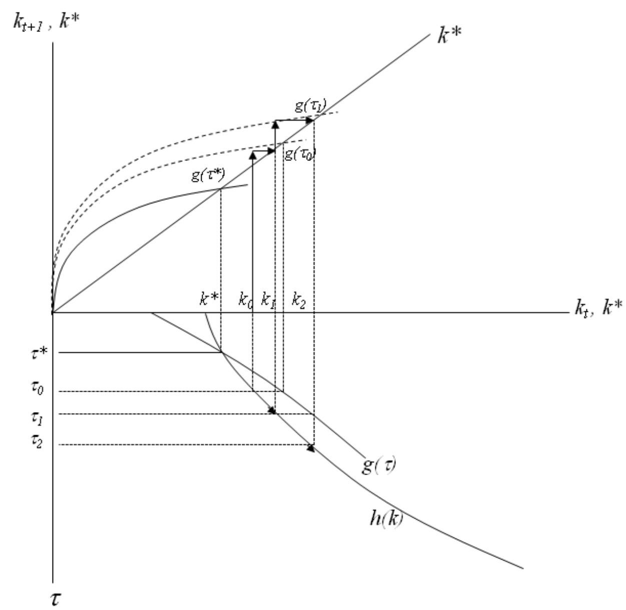
By contrast, Figure 6 reflects a case in which a positive optimal tax arises only at a relatively high level of steady state capital but is subsequently fairly sensitive to increases in capital. This results in $h(k)$ cutting $g(\tau)$ from above. When the initial capital stock is below the steady state, the optimal tax rate associated with that capital stock maps into an associated (transitory) steady state which lies below the initial capital stock. This results in next period's capital stock being higher than the initial one and closer to the long-run steady state.

<u>3: An unstable neoclassical steady state under optimal taxation:</u>

We now consider the case $g'(\tau) < 0$ and $\mathtt{G}'(k^*) < 1$.[18] This case arises when higher taxes lead to higher levels of capital in the steady state. As discussed in Section 4.2,

[18] A fourth possibility is that $g'(\tau) > 0$ and $\mathtt{G}'(k^*) > 1$, *i.e.* the steady state with exogenous taxes is unstable and an increase in the tax rate decreases the steady state capital stock. In this case, the steady state with optimal taxes is unambiguously a source.

Figure 6: A locally unstable non-poverty trap steady state.



this effect is more likely to occur at the neoclassical steady state on the exogenous-tax economy. In this case, the stability condition of case 2 is reversed, *i.e.*

$$\mathtt{G}'(k^*) + g'(\tau)(1 - \mathtt{G}'(k^*))h'(k^*) \left\{ \begin{array}{c} < \\ > \end{array} \right\} 1 \text{ as } g'(\tau)h'(k) \left\{ \begin{array}{c} < \\ > \end{array} \right\} 1$$

Figure 7 shows the local dynamics associated with the unstable case, *i.e.* when $g'(\tau)h'(k) > 1$. [19] The steady state (τ^*, k^*) would be monotonically stable if the tax rate were held constant at that value. When the tax rate is chosen optimally, however, we can see that the steady state becomes unstable for the case drawn. Since (i) both $g(\tau)$ and $h(k)$ slope upwards in $(\tau - k)$ space *and* (ii) $g(\tau)$ cuts $h(k)$ from below, then for any initial $k_0 > k^*$ (as shown in the diagram), $g(\tau_0) > k_0$. And since each potential steady state associated with a given tax rate is locally stable, $k_1 > k_0$ so that the economy moves away from k^*.

[19]The stable is analytically similar to case 1 so we do not address it separately.

5.4.2 Optimal fluctuations:

We shall now investigate the possibility that equation (34) has a negative root. Recall that in Section 3, we acknowledged the possibility that the transformation map $\mathtt{G}(k^*)$ might itself slope downwards for an exogenous and constant value of τ. This would lead to local oscillations, as in Palivos and Varvarigos [2011]. We exclude this possibility by imposing $\mathtt{G}'(k^*) > 0$. Even so, it is possible that when the effects of optimal taxes are taken into account, oscillation arise if the following condition is met:

$$\mathtt{G}'(k^*) + g'(\tau^*)(1 - \mathtt{G}'(k^*))h'(k^*) < 0$$

or, equivalently

$$\frac{\mathtt{G}'(k^*)}{1 - \mathtt{G}'(k^*)} < -g'(\tau^*)h'(k^*). \tag{35}$$

There are two situations in which this inequality can arise. Both involve steady states that would be locally stable *if* we were to impose monotonicity on the local dynamics. These are the cases depicted in Figures 4 and 6. Note that both these cases result in the root of equation (34) being less than unity. In fact, in both cases, the root need not even remain non-negative.

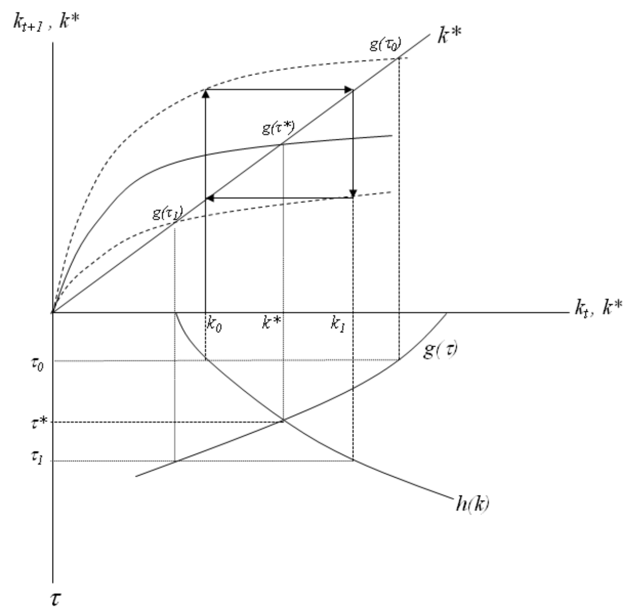
Consider Figure 8, in which all the conditions of Case 1 are met. However, as we can see, the dynamic path starting at k_0 cycles between the pair (τ_0, k_0) and (τ_1, k_1) forever. Intuitively this happens because, (i) $h(k)$ is quite 'flat', *i.e.* a large change in k induces a small increase in τ and (ii) $\mathtt{G}(k^*)$ is quite flat as it crosses the k^* line. As a consequence of these features, given that the economy starts at $k_0 < k^*$, (i) $g(\tau_0) > k^*$ and (ii) $k_1 > k^*$. But given $\tau_1 = h(k_1)$, (i) $g(\tau_1) < k^*$ and (ii) $k_2 < k^*$. Indeed, as drawn $k_2 = k_0$ so the cycle is locally stable although this is not necessarily going to be the case. The point is that oscillations can arise if these two features are present. Mathematically we can see that both these features are implied by the inequality in equation (35).

Turning to the case associated with Figure 6, it is instructive to express the inequality in equation (35) as

$$-\frac{\mathtt{G}'(k^*)}{1 - \mathtt{G}'(k^*)} > g'(\tau^*)h'(k^*).$$

In other words, for cycles to arise, it requires some combination of two factors: (i) the transformation map cuts the steady state line at a steep angle from below; (ii) a relatively high sensitivity of the optimal tax to the capital stock. Figure 9 illustrates this possibility. As drawn, the cycle that emerges when the economy starts at k_0, is explosive; however this need not be the case; the cycle could be stable or convergent.

Figure 7: Oscillations around a neoclassical steady state.



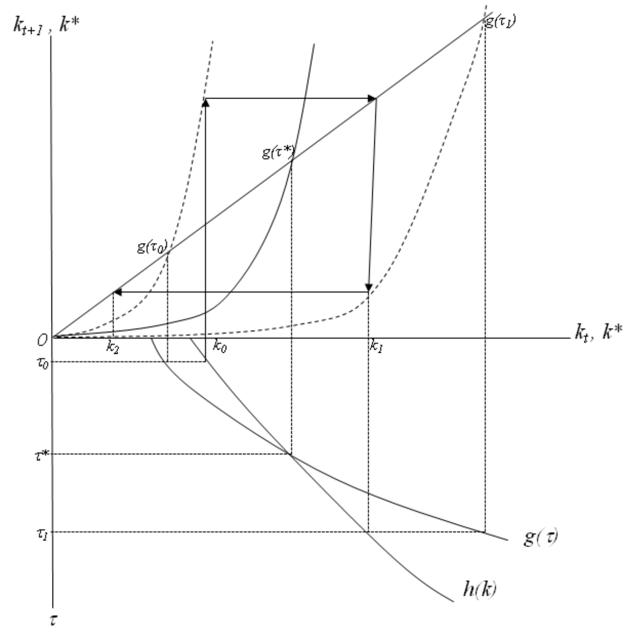
The point is that with optimal taxation, not only can a poverty trap become locally stable there can also be oscillations.

It is worth noting the difference with Palivos and Varvarigos [2011]: while they argue that environmental taxation can be used to eliminate cycles associated with the impact of pollution on uncertain lifetimes, our results suggest that second-best welfare-maximising environmental taxes can, in the same setting, be a source of oscillations.

5.4.3 Existence, uniqueness and stability of steady states:

We shall now consider the overall questions of existence, uniqueness and stability of steady state equilibrium. The approach shall make use of the local analysis carried out above. In Figure 10, we depict an economy in which multiple steady states arise at any given tax rate. Thus, the locus $g(\tau)$ is D-shaped (note that the axes have been rotated by 90^o degrees anti-clockwise in relation to Figures 4-9). The locus $h(k)$ is upward sloping throughout. Because of the shape of $g(\tau)$, the existence of a steady state with optimal taxes is not guaranteed. We have drawn three different versions of the $h(k)$ locus. With $h^1(k)$ and $h^2(k)$, there would be no interior steady state associated with optimal tax policy. It is only with $h(k)$ that interior steady states arise and in fact

Figure 8: Cycles around a poverty trap.



there are two of them: the lower steady state, S^1 has lower capital and lower taxation and because it satisfies case 2a, is locally unstable. The higher steady state, S^2 satisfies case 1 and is stable. Note, however, that as shown in the previous sub-section, local cycles are possible around this steady state. Both possibilities are illustrated in Figure 10.

In Figure 11, we show a case where $h(k)$ cuts the k-axis at a point that lies inside the D. It then cuts $g(\tau)$ at three interior points, S^1, S^2 and S^3. Both S^1 and S^2 are poverty traps and S^3 is a 'well-behaved' steady state. S^1 and S^3 are both stable, although both can give rise to cycles (the latter are shown only around S^1). Thus while poverty traps are always unstable under exogenous taxation, optimal policy can render them locally stable.

Finally, Figure 12 presents another intriguing consequence of optimal policy. The locus $g(\tau)$ is upward sloping reflecting the possibility noted in Section 4.2 that, around a stable steady state with zero taxation, initial increases in the tax rate can lead to increases in the steady state capital stock (for reasons explained in Section 4.2). If at the same time, the optimal tax does not kick in at the zero-tax steady state, then

Figure 9: Non-existence and multiplicity of steady states with optimal policy.

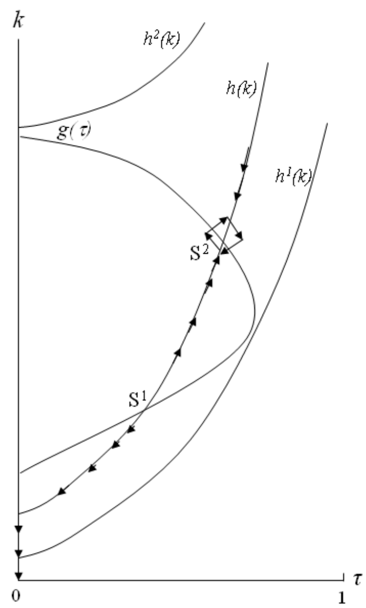


Figure 10: Multiple poverty traps.

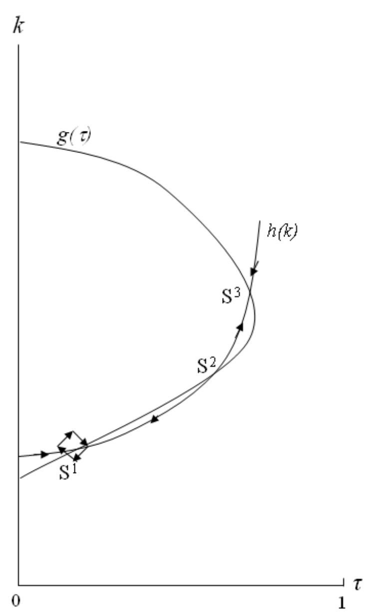
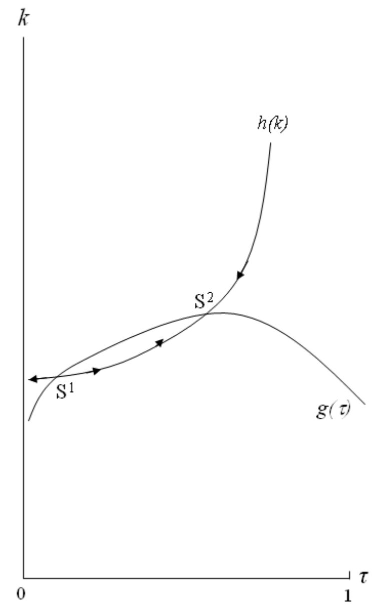


Figure 11: A locally unstable non-poverty trap



multiple steady states can arise with optimal taxation even in an economy in which exogenous taxes would result in a unique steady state. This is the case shown in Figure 12.

In this diagram, the steady state S^1 is unstable while the steady state S^2 is stable. What this suggests is that an economy that starts at a level of capital below S^1 is caught in an 'environmental trap' which results in successively lower levels of environmental controls, resulting in successively lower levels of capital. A big push in environmental protection might induce a move towards the higher steady state.

6 Conclusions:

This paper has studied an economy in which environmental degradation is a by-product of economic activity and negatively affects life expectancy. At the same time, economic growth contributes a positive effect on life expectancy. We showed that an initially bounded and subsequently convex relationship between life expectancy and its two determinants can lead to multiple interior steady states, with an unstable poverty trap and a stable, high income steady state. We examined the comparative static effects of exogenous tax abatement policy and showed that this can hurt an initially poor country while benefiting (in terms of higher levels of capital, income and life expectancy per worker) an initially rich one.

We have also demonstrated the existence of an optimal environmental tax policy and shown that it is non-homogeneous and monotonically increasing in the capital stock. From a policy point of view, this suggests that economies that are close to or just emerging from a poverty trap might impose zero or low levels of environmental protection but eventually this will rise along the growth path. This bears some resemblance to the conditions of the Kyoto Protocol which requires over the longer term to increase the set of countries that are required to take strong action.

At the same time we have shown that optimal policy might itself contribute to complex dynamics in several ways: first, a steady state with optimal taxes might not exist whereas in the underlying economy with exogenous policy, one or more interior steady states existed; second, by inducing multiple steady states iunder conditions where a unique steady state would exist with exogenous policy; third, by stabilising poverty traps which would be unstable under exogenous policy; fourth, by inducing oscillations and cycles around steady states which would otherwise be locally stable.

With respect to the last finding, we offer a word of caution. Although life expectancy is related to pollution in the data, and indeed there is evidence that short term fluctuations in air quality can lead to fluctuations in mortality rates (see Evans and Smith [2005]), it is not clear that these phenomena are in turn part of a general business cycle, *i.e.* there is a causal link between an economic boom, higher pollution, lower life expectancy and then an economic downturn. But the point of our analysis has been to identify the possibilities for complex dynamics that arise in the relationship between health, pollution and economic growth and to sound a cautionary note on the use of steady state models to study these relationships.

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APPENDIX

Derivation of $\partial^2/\partial\tau\partial k$:

Recall that

$$\frac{\partial \pi}{\partial \tau} = \frac{\pi^A \psi \gamma A k^\alpha \delta z^{\delta-1}}{(1+z^\delta)^2}$$

Note that we can also write this as

$$\frac{\partial \pi}{\partial \tau} = \frac{\pi \psi \gamma A k^\alpha \delta z^{\delta-1}}{1+z^\delta}$$

Taking the derivative of the above with respect to k (after some straightforward rearrangement):

$$\frac{\partial^2 \pi}{\partial \tau \partial k} = \frac{\alpha}{k}\frac{\partial \pi}{\partial \tau} + \frac{1}{\pi^A}\frac{\partial \pi}{\partial \tau}\frac{\partial \pi^A}{\partial k} + \frac{1}{z(1+z^\delta)}\frac{\partial \pi}{\partial \tau}[(\delta-1)-(\delta+1)z^\delta]\frac{\partial z}{\partial k}$$

where

$$\frac{\partial \pi^A}{\partial k} = \frac{\alpha}{k}\frac{\beta(1-\underline{\pi})y^\beta}{(1+y^\beta)^2} = \frac{\alpha}{k}\frac{\beta(1-\underline{\pi})y^\beta}{(1+y^\beta)}\frac{\pi(1+z^\delta)}{(\underline{\pi}+y^\beta))}$$

and

$$\frac{\partial z}{\partial k} = \frac{\alpha\gamma(1-\psi\tau)Ak^\alpha}{k} = \frac{\alpha(z-\phi z')}{k}$$

The right hand side of the main derivative can be written as

$$\frac{\partial \pi}{\partial \tau}\left[\frac{\alpha}{k} + \frac{(1+z^\delta)}{\pi^A}\frac{\alpha\pi}{k}\frac{\beta(1-\underline{\pi})y^\beta}{(1+y^\beta)(\underline{\pi}+y^\beta)} + \frac{\alpha}{k}\frac{z-\phi z'}{z(1+z^\delta)}[(\delta-1)-(\delta+1)z^\delta]\right]$$

Finally, expanding the term in square brackets involving z^δ and noting the definition of π, we get

$$\frac{\partial \pi}{\partial \tau}\left[\frac{\alpha}{k} + \frac{1}{\pi}\left\{\frac{\alpha\pi}{k}\left(\frac{\beta(1-\underline{\pi})y^\beta}{(1+y^\beta)(\underline{\pi}+y^\beta)} - \frac{(z-\phi z')\delta z^{\delta-1}}{(1+z^\delta)}\right) + \frac{\alpha\pi\delta}{k}\frac{(z-\phi z')}{(1+z^\delta)z} - \frac{\alpha\pi}{k}\frac{(z-\phi z')}{z}\right\}\right];$$

from which, noting the definition of $\partial\pi/\partial k$, it follows that

$$\frac{\partial^2 \pi}{\partial \tau \partial k} = \frac{\partial \pi}{\partial \tau}\frac{1}{k}\left[\alpha + \frac{k}{\pi}\frac{\partial \pi}{\partial k} + \frac{\alpha\delta(z-\phi z')}{(1+z^\delta)z} - \alpha + \frac{\alpha\phi z'}{z}\right];$$

leading to the desired result.