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# Solutions to $y'(x) = \cos[\pi x^p y(x)^q]$

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## Abstract

The asymptotic behaviour of solutions to  $y'(x) = \cos[\pi x^p y(x)^q]$  is considered. This is a generalisation of the problem of the behaviour of  $y'(x) = \cos[\pi x y(x)]$  that was investigated by Bender, Fring and Komijani [1]. We present a derivation of the asymptotic results that follows the approach used in Kerr [2].

## 1 Introduction

In Bender, Fring and Komijani [1] a detailed asymptotic analysis of the nonlinear initial-value problem

$$y'(x) = \cos[\pi x y(x)], \quad y(0) = a \quad (1)$$

was presented which focused on the solutions for  $x \geq 0$ . They showed that for  $a > 0$  solutions could be split into classes depending on the initial conditions such that solution with  $a_{n-1} < a < a_n$  displayed an oscillatory region with  $n$  maxima before decaying monotonically to zero. They then found the result that as  $n \rightarrow \infty$ ,  $a_n \sim 2^{5/6} \sqrt{n}$ . This result was subsequently derived using a different approach by Kerr [2]. Here we use this alternative derivation to obtain equivalent results to the generalisation of the original problem where we look for asymptotic solutions to

$$y'(x) = \cos[\pi x^p y(x)^q], \quad y(0) = a \quad (2)$$

where  $p$  and  $q$  are positive integers<sup>1</sup>

Much of this derivation is essentially the same as that of Kerr [2]

## 2 Outline

The typical behaviour of solutions to (2) is essentially the same as the original problem and is shown by the solid lines in figure 1 (with  $p = q = 1$ ). There is an initial oscillatory phase where the frequency increases and the amplitude decreases as the initial value,  $y(0)$ , increases. These oscillatory solutions drift downwards until they undergo a transition to monotonic decay towards the horizontal axis.

Some of the basic behaviour of the solutions of (2) can be understood by considering the lines in the  $x$ - $y$  plane where  $x^p y^q$  is constant. The situation is shown schematically

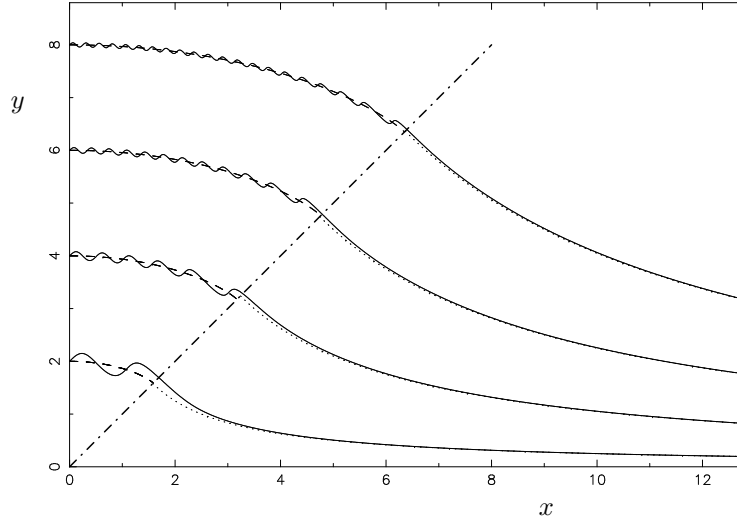


Fig. 1: Plots of solutions to (2) with  $y(0) = 2, 4, 6, 8$  and  $p = q = 1$ . The dotted lines in  $x/p > y/q$  show the curves  $x^p y^q = C$  to which these converge asymptotically as  $x \rightarrow \infty$ . The dashed lines in  $x/p < y/q$  give the estimate of the mean path of the oscillatory part of these curves.

in figure 2. The arguments here are essentially the same as those in section 3 of Kerr, except the region is now divided by the line  $x/p = y/q$  and the lines under consideration are lines of the form  $x^p y^q = c$ .

If we consider lines where  $x^p y^q = 2n$  then solutions will have gradient 1 where they intersect these lines, similarly when  $x^p y^q = 2n+1$  they will intersect with gradient  $-1$ , and when  $x^p y^q = 2n \pm 1/2$  they will intersect with gradient 0. The gradients of the solutions will have gradients with magnitude at most 1, while the lines  $x^p y^q = c$  for constants  $c$  have gradients greater than 1 in magnitude for  $x/p < y/q$ , and less than 1 for  $x/p > y/q$ . In the region  $x/p < y/q$  solutions must cross the lines  $x^p y^q = c$  from left to right, with a maximum each time it crosses a line  $x^p y^q = 2n + \frac{1}{2}$ ,  $n = 1, 2, 3, \dots$ . In the region  $x/p > y/q$  this restriction no longer holds. This results in the solutions having intrinsically different behaviour above and below the line  $x/p = y/q$ .

### 3 Solutions in the region $x/p > y/q$

This is essentially the same argument as previously in Kerr [2].

Any solution that enters a region  $2n + \frac{1}{2} \leq x^p y^q \leq 2n + 1$  is trapped in this region as  $x$  increases as the gradient of a solution on the lower boundary is 0, and on the upper boundary is  $-1$ . Indeed, in such a region any solution that is initially above a

<sup>1</sup> Strictly speaking this requirement to be positive integers can be relaxed.

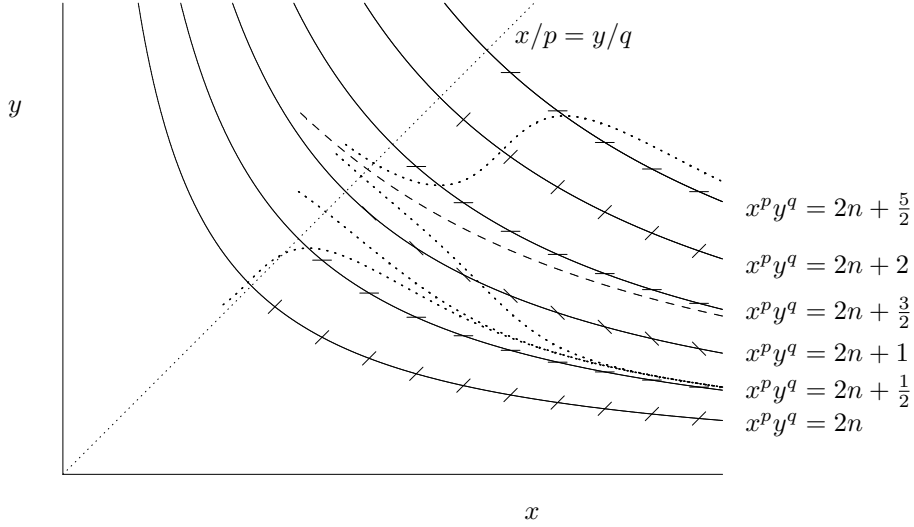


Fig. 2: Schematic plot for solutions in the region  $x/p > y/q$  and the influences of the lines of form  $x^p y^q = c$ . The dotted lines show the trajectories of various solutions, while the dashed line shows the path of the separatrix dividing solutions that converge to  $x^p y^q = 2n + 1/2$  from those that converge to  $x^p y^q = 2n + 5/2$ .

line  $x^p y^q = 2n + \frac{1}{2} + \epsilon$  will have a negative gradient of magnitude greater than  $\sin \pi \epsilon$  and so must eventually pass below  $x^p y^q = 2n + \frac{1}{2} + \epsilon$ , whose gradient tends to zero as  $x \rightarrow \infty$ . Thus all solutions in this region asymptote to the line  $x^p y^q = 2n + \frac{1}{2}$ .

All solutions in the region  $2n - \frac{1}{2} < x^p y^q < 2n + \frac{1}{2}$  will have positive gradients and so will pass into the region  $2n + \frac{1}{2} \leq x^p y^q \leq 2n + 1$  from below, and will have one maximum in the region  $x/p > y/q$ .

There is one solution in the region  $2n + 1 < x^p y^q < 2n + \frac{3}{2}$  that stays in this region. Solutions initially below this curve will pass into the region  $2n + \frac{1}{2} \leq x^p y^q \leq 2n + 1$  and remain there, while those above it will end up in the region  $2n + \frac{5}{2} \leq x^p y^q \leq 2n + 3$ . This curve is indicated by the dashed line in figure 2. By a similar argument to that given previously it can be shown that these seperatrices tend towards their asymptotes  $x^p y^q = 2n + 3/2$  from below. We will denote the point where the separatrix crosses the line  $x/p = y/q$  as  $x/p = y/q = b_n$ , and hence  $2n + 1 < p^p q^q b_n^{p+q} < 2n + 3/2$ , or  $(2n + 1)^{1/(p+q)} / (p^p q^q)^{1/(p+q)} < b_n < (2n + 3/2)^{1/(p+q)} / (p^p q^q)^{1/(p+q)}$ .

Clearly, any solution that ends up just above the curve  $x^p y^q = 2n + \frac{1}{2}$  will have crossed  $n$  lines given by  $x^p y^q = \frac{1}{2}, \frac{5}{2}, \frac{9}{2}, \dots$  and so will have  $n$  maxima. Hence any solution that crosses the line  $x/p = y/q$  with  $b_{n-1} < x/p = y/q < b_n$  will have  $n$  maxima.

## 4 Solutions in the region $x/p < y/q$

As before, for large values of  $y(0)$  the solution  $y(x)$  will tend to oscillate quickly with small amplitude. The previous arguments hold here.

If lines of constant  $x^p y^q = c$  are given locally by the lines  $x + \alpha y = C$  then the mean path of the oscillatory solution is given by

$$\frac{dy}{dx} = \frac{\sqrt{1 - \alpha^2} - 1}{\alpha}. \quad (3)$$

The value of  $\alpha$  is determined by the curves  $x^p y^q = c$ . On such curves

$$\frac{dy}{dx} = -\frac{pc^{1/q}}{qx^{(p+q)/q}} = -\frac{py}{qx}. \quad (4)$$

Since the gradient of the lines  $x + \alpha y = C$  is  $-1/\alpha$ , we find  $\alpha = qx/py$  and so the equation for the slope of the average curve is given by

$$\frac{dy}{dx} = \sqrt{\left(\frac{py}{qx}\right)^2 - 1} - \frac{py}{qx}. \quad (5)$$

This has solutions

$$\left(\sqrt{p^2 y^2 - q^2 x^2} + py\right)^p \left((p+q)y - \sqrt{p^2 y^2 - q^2 x^2}\right)^{p+q} = 2^p p^p q^{p+q} y(0)^{2p+q}. \quad (6)$$

The solution curves meet the line  $x/p = y/q$  at the point  $x/p = y/q = \beta$  when

$$y(0) = \frac{(p+q)^{\frac{p+q}{2p+q}}}{2^{\frac{p}{2p+q}}} \beta \quad (7)$$

If the  $a_n$  are the values of  $y(0)$  which correspond to the  $b_n$ , and so are solutions with  $n$  maxima then we would have here

$$a_n \approx \frac{(p+q)^{\frac{p+q}{2p+q}}}{2^{\frac{p}{2p+q}}} b_n \approx \frac{2^{\frac{1}{p+q}} (p+q)^{\frac{p+q}{2p+q}} n^{\frac{1}{p+q}}}{2^{\frac{p}{2p+q}} p^{\frac{p}{p+q}} q^{\frac{q}{p+q}}}. \quad (8)$$

## 5 Conclusions

We have used alternative derivation of Kerr [2] to analyse the generalisation the problem considered by Bender, Fring and Komijani [1]. With luck I've got the coefficients right.

## References

- [1] Carl M. Bender, Andreas Fring, and Javad Komijani. Nonlinear eigenvalue problems. Accepted for publication in *J. Phys. A*, 2014.
- [2] Oliver S. Kerr. On "nonlinear eigenvalue problem". Submitted for publication in *J. Phys. A*, 2014.