Quillen's stratification for fusion systems

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Abstract

The purpose of this note is to provide a reference for the fact that the proof of Quillen's stratification for finite group cohomology carries over to fusion system. As in the case of Quillen's stratification for block varieties, the proof is similar to the usual proof for group cohomology except for the use of fusion stable bisets, whose existence is due to Broto, Levi, and Oliver.

It is pointed out in work of Benson, Grodal, and Henke [3, Remark 3.7] that there seems to be no reference for Quillen's stratification for fusion systems and that such a reference would be the only missing link to generalise [3, Theorem A] to arbitrary fusion systems. See Todea [8, Theorem 1.1] for an explicit statement and proof of [3, Theorem A] for fusion systems along these lines. Quillen's stratification for arbitrary fusion systems is a straightforward adaptation of the block cohomology version in [5], which is in turn an adaptation of Benson's presentation in [2] of some of Quillen's results in [6], [7]. We mention below that this can be used to obtain alternative proofs of [4, 5.1] and of [4, 5.6]. Let p be a prime, P a finite p-group, and let $\mathcal{F}$ be a fusion system on P. Let k be an algebraically closed field of characteristic p. Denote by $H^*(P;k)^{\mathcal{F}}$ the subalgebra of $\mathcal{F}$-stable elements in $H^*(P;k)$, and by $V_{\mathcal{F}}$ the maximal ideal spectrum of $H^*(P;k)^{\mathcal{F}}$. For any subgroup Q of P denote by V_Q the maximal ideal spectrum of $H^*(Q;k)$, and set $V_Q^+ = V_Q \setminus \bigcup_{R<Q} (\mathrm{res}^Q_R)^*(V_R)$. Denote by $V_{\mathcal{F},Q}$ and $V^+_{\mathcal{F},Q}$ the images of V_Q and V_Q^+ in $V_{\mathcal{F}}$ under the map r_Q^* induced by the algebra homomorphism $r_Q : H^*(P;k)^{\mathcal{F}} \to H^*(Q;k)$ given by composing the inclusion $H^*(P;k)^{\mathcal{F}} \subseteq H^*(P;k)$ with the restriction $\mathrm{res}^P_Q : H^*(P;k) \to H^*(Q;k)$.

Theorem 1. *With the notation above, the following hold.*

(i) The variety $V_{\mathcal{F}}$ is the disjoint union of the locally closed subvarieties $V^+_{\mathcal{F},E}$, where E runs over a set of representatives of the $\mathcal{F}$-isomorphism classes of elementary abelian subgroups of P.

(ii) Let E be an elementary abelian subgroup of P. The group $W(E) = \mathrm{Aut}_{\mathcal{F}}(E)$ acts on V_E^+ and the restriction map res^P_E induces an inseparable isogeny $V_E^+/W(E) \to V^+_{\mathcal{F},E}$.

Since any subgroup of P is isomorphic, in $\mathcal{F}$, to a fully $\mathcal{F}$-centralised subgroup, the set of representatives in statement (i) of Theorem 1 may be chosen to consist of fully $\mathcal{F}$-centralised elementary abelian subgroups of P. As mentioned before, one way to prove Theorem 1 is to adapt the proof for block fusion systems; this proof follows closely Benson's presentation in [2] of parts of Quillen's original work in [6], [7], with only additional ingredient the fusion stable bisets whose existence has been shown by Broto, Levi, and Oliver in [4, 5.5].

Remark 2. Theorem 1 yields an alternative proof of [4, 5.1] stating that the product of the restriction maps $H^*(P;k) \to H^*(E;k)$, with E running over the elementary abelian subgroups of P, induces an inseparable isogeny $H^*(P;k)^{\mathcal{F}} \to \lim_{\leftarrow \mathcal{E}} H^*(E;k)$, where $\mathcal{E}$ is the full subcategory of $\mathcal{F}$ consisting of all elementary abelian subgroups of P. It also yields a proof of [4, 5.6], stating that $H^*(P;k)$ is finitely generated over $H^*(P;k)^{\mathcal{F}}$, because the varieties V_P and $V_{\mathcal{F}}$ have the same dimension (namely the rank of an elementary abelian subgroup of maximal order of P).

Following [5, §3], if Q is another finite p-group, then any finite P-Q-biset X on which P and Q act regularly on the left and right, respectively, induces a norm map $n_X : H^*(Q;k) \to H^*(P;k)$ extending Evens' norm maps in the obvious way. By [5, 3.1], if $\psi : R \to P$ is an injective group homomorphism, then $\mathrm{res}_\psi \circ n_X = n_{{}_\psi X}$, where ${}_\psi X$ is the R-Q-biset obtained from restricting the left action by P to R via ψ. If X is a transitive P-Q-biset on which P and Q act regularly, then $X = P \times_{(R,\psi)} Q$ for some subgroup R of P and some injective group homomorphism $\psi : R \to Q$; that is, X is the quotient of $P \times Q$ by the relation $(ur, v) \sim (u, \psi(r)v)$, where $u \in P$, $v \in Q$ and $r \in R$. In that case, $n_X = n_R^P \circ \mathrm{res}_\psi$, where $\mathrm{res}_\psi : H^*(Q;k) \to H^*(R;k)$ is the restriction map along ψ, and where $n_Q^P : H^*(Q;k) \to H^*(P;k)$ is the Evens norm map (see e.g. [2, 4.1]). The exact sign of n_R^P, hence of n_X, depends on a choice of coset representatives, and thus the statements below involving norm maps hold modulo keeping track of signs. Since $H^*(P;k)$ is graded-commutative (hence commutative if $p = 2$) we set $H^\cdot(P;k) = H^*(P;k)$ if $p = 2$ and $H^\cdot(P;k) = H^{\mathrm{even}}(P;k)$ if p is odd.

Proposition 3 ([4, 5.5])**.** *There is a finite P-P-biset X with the following properties.*

(i) Every transitive subbiset of X is isomorphic to $P \times_{(Q,\varphi)} P$ for some subgroup Q of P and some group homomorphism $\varphi : Q \to P$ belonging to $\mathcal{F}$.

(ii) $|X|/|P|$ is prime to p.

(iii) For any subgroup Q of P and any group homomorphism $\varphi : Q \to P$ in $\mathcal{F}$, the Q-P-bisets ${}_\varphi X$ and ${}_Q X$ are isomorphic.

Lemma 4 (cf. [5, 3.3])**.** *Let X be a finite P-P-biset fulfilling the conclusion in Proposition 3. Then, for any subgroup Q of P, there is a Q-Q-subbiset of ${}_Q X_Q$ isomorphic to Q.*

Proof. It suffices to show that X has a P-P-subbiset isomorphic to P. By Proposition 3 (i) and (ii), X has a P-P-subbiset isomorphic to ${}_\varphi P$ for some automorphism φ of P in $\mathcal{F}$. The stability condition in Proposition 3 (iii) implies the result. □

Proposition 5 (cf. [5, 3.4])**.** *Let X be a finite P-P-biset fulfilling the conclusions in Proposition 3, let Q be a subgroup of P and let Y be the Q-Q-subbiset of ${}_Q X_Q$ which is the union of all Q-Q-orbits of length $|Q|$. Set $W(Q) = \mathrm{Aut}_{\mathcal{F}}(Q)$.*

(i) The image of the norm map $n_{X_Q} : H^\cdot(Q,k) \to H^\cdot(P,k)$ is contained in $H^\cdot(P;k)^{\mathcal{F}}$.

(ii) The set Y is non empty.

(iii) For any $\zeta \in H^\cdot(Q,k)^{W(Q)}$ such that $\mathrm{res}_R^Q(\zeta) = 0$ for any proper subgroup R of Q we have $n_{{}_Q X_Q}(\zeta) = n_Y(\zeta)$.

(iv) For any $\zeta \in H^\cdot(Q,k)^{W(Q)}$ we have $n_Y(\zeta) = \zeta^{|Y|/|Q|}$.

Proof. This is identical to the proof of [5, 3.4]. □

The following translates [1, 5.6.2] and its block cohomology version [5, 3.5] to fusion systems:

Proposition 6. *Let X be a finite P-P-biset fulfilling the conclusions in Proposition 3. Let E be an elementary abelian subgroup of P and let σ_E be a homogeneous element in $H^{\cdot}(E,k)$ satisfying $\mathrm{res}^E_F(\sigma_E) = 0$ for any proper subgroup F of E. Set $W(E) = \mathrm{Aut}_{\mathcal{F}}(E)$. Let Y be the E-E-subbiset of ${}_EX_E$ which is the union of all E-E-orbits of length $|E|$. Write $|Y|/|E| = p^a m$ for some nonnegative intergers a, m, such that $(p,m) = 1$.*

(i) For any $\eta \in H^{\cdot}(E,k)^{W(E)}$ there is $\eta' \in H^*(P;k)^{\mathcal{F}}$ such that $r_E(\eta') = (\sigma_E \cdot \eta)^{p^a}$.

(ii) There is an element $\rho_E \in H^{\cdot}(P;k)^{\mathcal{F}}$ such that $r_E(\rho_E) = (\sigma_E)^{p^a}$ and such that $r_F(\rho_E) = 0$ whenever F is an elementary abelian subgroup of P such that no $\mathcal{F}$-conjugate of E is containend in F.

Proof. We may assume that η is homogeneous. Set $\zeta = n_{X_E}(1 + \sigma_E \cdot \eta)$. By Proposition 5, we have $\zeta \in H^*(P;k)^{\mathcal{F}}$. Moreover, $r_E(\zeta) = n_{{}_EX_E}(1 + \sigma_E \cdot \eta) = n_Y(1 + \sigma_E \cdot \eta) = (1 + \sigma_E \cdot \eta)^{p^a m} = (1 + (\sigma_E \cdot \eta)^{p^a})^m = 1 + m(\sigma_E \cdot \eta)^{p^a} + \tau$, where τ is a sum of elements of degree stricly bigger than $\deg((\sigma_A \cdot \eta)^{p^a}) = p^a \cdot \deg(\sigma_E \cdot \eta)$. Define η' to be the homogeneous part of ζ in degree $p^a \cdot \deg(\sigma_E \cdot \eta)$, divided by m. This shows (i). Applying (i) to $\eta = 1$ yields a homogeneous element $\rho_E \in H^{\cdot}(P;k)^{\mathcal{F}}$ such that $r_E(\rho_E) = (\sigma_E)^{p^a}$. By the construction in (i), ρ_E is a scalar multiple of the homogenous part of $n_{X_E}(1 + \sigma_E)$ in degree $p^a \cdot \deg(\sigma_E)$. Let F be another elementary abelian subgroup of P. Then $r_F(\rho_E)$ is a scalar multiple of the homogeneous part of $n_{{}_FX_E}(1 + \sigma_E)$ in degree $p^a \cdot \deg(\sigma_E)$. If E has no $\mathcal{F}$-conjugate contained in F, then the biset ${}_FX_E$ is a union of transitive bisets of the form $F \times_{(H,\psi)} E$, where H is a subgroup of F of order smaller than $|E|$, and where $\psi : H \to E$ is an injective group homomorphism. Thus $n_{{}_FX_E}(\sigma_E) = 0$, and so $r_F(\rho_E) = 0$. This completes the proof of (ii). □

Proof of Theorem 1. We follow the proof of [5, 4.2] with minor adjustments. Let E be an elementary abelian subgroup of P. By the argument in [2, 5.6] preceding [2, 5.6.2], there is a homogeneous element $\sigma_E \in H^{\cdot}(E,k)^{W(E)}$ such that V_E^+ consists of all maximal ideals in $H^{\cdot}(E,k)$ not containing σ_E, and such that $\mathrm{res}^E_F(\sigma_E) = 0$ for any proper subgroup F of E. Thus V_E^+ can be identified to the maximal ideal spectrum of the algebra $H^{\cdot}(E,k)[\sigma_E^{-1}]$, obtained from localising $H^{\cdot}(E,k)$ at σ_E. By [2, 5.4.8], the quotient $V_E^+/W(E)$ can be identified with the maximal ideal spectrum of $(H^{\cdot}(E,k)[\sigma_E^{-1}])^{W(E)}$. Let ρ_E be the element in $H^{\cdot}(P;k)^{\mathcal{F}}$ fulfilling Proposition 6 (ii). Then $V_{G,E}^+$ consists of all maximal ideals in $H^{\cdot}(P;k)^{\mathcal{F}}$ containing $\ker(r_E)$ and not containing ρ_E. Since r_E maps ρ_E to a power of σ_E, r_E induces an algebra homomorphism

$$H^{\cdot}(P;k)^{\mathcal{F}}[\rho_E^{-1}] \longrightarrow (H^{\cdot}(E,k)[\sigma_E^{-1}])^{W(E)}$$

such that, by 6, the image of this homomorphism contains a p^a-th power of every element in $(H^{\cdot}(E,k)[\sigma_E^{-1}])^{W(E)}$. Upon taking varieties, this is equivalent to saying that r_E^* induces an inseparable isogeny $V_E^+/W(E) \to V_{\mathcal{F},E}^+$. This proves (ii). If F is another fully $\mathcal{F}$-centralised elementary abelian subgroup of P such that F contains no $\mathcal{F}$-conjugate of E, then $\rho_F \in \ker(r_E)$ by Proposition 6. By the above description of $V_{\mathcal{F},E}^+$, it follows that $V_{\mathcal{F},E}^+$ and $V_{\mathcal{F},F}^+$ are disjoint. This proves (i). □

References

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