On stable equivalences and blocks with one simple module

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Abstract

Using a stable equivalence due to Rouquier, we show that Alperin's weight conjecture holds for any p-block of a finite group with defect 2 whose Brauer correspondent has a unique isomorphism class of simple modules.

1 Introduction

Throughout this paper we denote by $\mathcal{O}$ a complete discrete valuation ring with residue field $k = \mathcal{O}/J(\mathcal{O})$ of prime characteristic p and quotient field K of characteristic zero. Given a finite group G, a block algebra B of $\mathcal{O}G$ is an indecomposable direct factor of $\mathcal{O}G$ as $\mathcal{O}$-algebra; we denote by $\ell(B)$ the number of isomorphism classes of simple $k \otimes_{\mathcal{O}} B$-modules.

Theorem 1.1. *Let G be a finite group and B a block algebra of $\mathcal{O}G$ having a defect group of order at most p^2. Denote by C the Brauer correspondent of B and suppose that K, k are splitting fields for B, C. If $\ell(C) = 1$ then $\ell(B) = 1$, the inertial quotient of B is abelian, the decomposition matrices of B and C are equal and there is a p-permutation equivalence between B and C inducing an isotypy between B and C all of whose signs are positive.*

Broué's Abelian Defect Conjecture predicts more precisely that B and C are derived equivalent. If true, a result of Roggenkamp and Zimmermann would imply that B and C are actually Morita equivalent. This is known to hold if the defect groups of B are cyclic or Klein four because in that case the hypothesis of having a unique isomorphism class of simple modules implies that B and C are nilpotent, hence Morita equivalent to $\mathcal{O}P$. In order to prove Theorem 1.1 we may therefore assume that p is odd, that a defect group P of B is elementary abelian of rank 2 and that the inertial quotient of B is non trivial. We will see that this forces the inertial quotient to be abelian, and hence the Brauer correspondent C is a quantum complete intersection; see [3], [4], [13]. The main ingredient to prove Theorem 1.1 is Rouquier's stable equivalence between B and its Brauer correspondent, obtained from "gluing" together various derived equivalences at local levels. Since stable equivalences between block algebras preserve the character group $L^0(B)$ of generalised characters which vanish on p-regular elements, isometry arguments turn out to work particularly well for blocks with one simple module, because in that case the rank of $L^0(B)$ is equal to $|\mathrm{Irr}_K(B)| - 1$ and hence this subgroup will contain enough information to reconstruct the number of irreducible characters of any block stably equivalent to B.

Remark 1.2. By work of Kiyota in [14], if $p = 3$ then Alperin's weight conjecture holds for all blocks with an elementary abelian defect group of order 9 except possibly when the inertial quotient is cyclic of order 8 or quaternion of order 8 and the block is non principal (the case of non principal blocks with semi-dihedral inertial quotient of order 16 is attributed to A. Watanabe in [14, §0, Table 1]). In particular, with the notation of 1.1, Kiyota's results imply that if $p = 3$ then $\ell(B) = 1$ if and only if $\ell(C) = 1$.

2 Notation and quoted results

We review in this section some classic background material, adapted to symmetric algebras, mostly for the purpose of introducing our notation. Let A be an $\mathcal{O}$-algebra which is finitely generated free as $\mathcal{O}$-module. The algebra A is called *symmetric* if A is isomorphic, as an A-A-bimodule, to its $\mathcal{O}$-dual $A^* = \mathrm{Hom}_{\mathcal{O}}(A, \mathcal{O})$. We denote by $\mathrm{mod}(A)$ the category of finitely generated left A-modules, by $D^b(A)$ the bounded derived category of $\mathrm{mod}(A)$ and by $\underline{\mathrm{mod}}(A)$ the relatively $\mathcal{O}$-stable category of $\mathrm{mod}(A)$ (identifying to zero, in $\mathrm{mod}(A)$, all relatively $\mathcal{O}$-projective modules). We denote by $R_K(A)$ and $R_k(A)$ the Grothendieck groups of finitely generated $K \otimes_{\mathcal{O}} A$-modules and $k \otimes_{\mathcal{O}} A$-modules, respectively. For any finitely generated module U over $K \otimes_{\mathcal{O}} A$ or $k \otimes_{\mathcal{O}} A$ we denote by $[U]$ the image of U in $R_K(A)$ or $R_k(A)$, respectively. The group $R_K(A)$ is free of finite rank, having as basis the set of images, denoted $\mathrm{Irr}_K(A)$, of simple $K \otimes_{\mathcal{O}} A$-modules in $R_K(A)$. We set $\mathsf{k}(A) = |\mathrm{Irr}_K(A)|$, the number of isomorphism classes of simple $K \otimes_{\mathcal{O}} A$-modules. Similarly, the group $R_k(A)$ is free of finite rank, having as basis the set of images, denoted $\mathrm{Irr}_k(A)$, of simple $k \otimes_{\mathcal{O}} A$-modules in $R_k(A)$. We set $\ell(A) = |\mathrm{Irr}_k(A)|$, the number of isomorphism classes of simple $k \otimes_{\mathcal{O}} A$-modules. We denote by $\mathrm{Pr}_{\mathcal{O}}(A)$ the subgroup of $R_K(A)$ generated by the images of modules of the form $K \otimes_{\mathcal{O}} U$, where U is a finitely generated projective A-module. Denote by I a set of representative of the conjugacy classes of primitive idempotents in A; then $\{Ai \mid i \in I\}$ is a set of representatives of the isomorphism classes of projective indecomposable A-modules and $\{Ai/J(A)i \mid i \in I\}$ is a set of isomorphism classes of the simple $k \otimes_{\mathcal{O}} A$-modules. For $i \in I$ denote by Φ_i the image of $K \otimes_{\mathcal{O}} Ai$ in $R_K(A)$ and by φ_i the image of $Ai/J(A)i$ in $R_k(A)$. The set

$$\mathrm{IPr}_{\mathcal{O}}(A) = \{\Phi_i \mid i \in I\}$$

generates $\mathrm{Pr}_{\mathcal{O}}(A)$ and we have $\mathrm{Irr}_k(A) = \{\varphi_i \mid i \in I\}$. Similarly,we denote by $\mathrm{Pr}_k(A)$ the subgroup of $R_k(A)$ generated by the images of modules of the form $k \otimes_{\mathcal{O}} U$, where U is a finitely generated projective A-module; equivalently, $\mathrm{Pr}_k(A)$ is generated by the set $\{[k \otimes_{\mathcal{O}} Ai] \mid i \in I\}$. Assume in addition that $K \otimes_{\mathcal{O}} A$ is split semi-simple. We define a bilinear form

$$< \, , \, >_A : \; R_K(A) \times R_K(A) \longrightarrow \mathbb{Z}$$

on $R_K(A)$ by setting $< [U], [V] >_A = \dim_K(\mathrm{Hom}_{K \otimes_{\mathcal{O}} A}(U, V))$ for any two finitely generated $K \otimes_{\mathcal{O}} A$-modules U, V and extending this to $R_K(A)$ in the obvious way. Since $K \otimes_{\mathcal{O}} A$ is split semi-simple, this form is symmetric, the set $\mathrm{Irr}_K(A)$ is an orthonormal basis of $R_K(A)$, every finitely-generated $K \otimes_{\mathcal{O}} A$-module is a finite direct sum of simple $K \otimes_{\mathcal{O}} A$-modules and any simple $K \otimes_{\mathcal{O}} A$-module X is isomorphic to a direct summand of $K \otimes_{\mathcal{O}} A$. Intersecting a direct summand X of $K \otimes_{\mathcal{O}} A$ with the image $1_K \otimes A$ of A in $K \otimes_{\mathcal{O}} A$ yields an $\mathcal{O}$-free A-module Y satisfying $K \otimes_{\mathcal{O}} Y \cong X$. Thus, for any finitely generated $K \otimes_{\mathcal{O}} A$-module X there is an $\mathcal{O}$-free A-module Y satisfying $K \otimes_{\mathcal{O}} Y \cong X$. Moreover, if Y' is another $\mathcal{O}$-free A-module satisfying $K \otimes_{\mathcal{O}} Y' \cong X$ then $k \otimes Y$

and $k \otimes Y'$ have identical composition factors (with multiplicities) because the multiplicity d_S^X of a simple $k \otimes_{\mathcal{O}} A$-module S in a composition series of $k \otimes_{\mathcal{O}} Y$ is equal to

$$\dim_k(\mathrm{Hom}_A(Ai, k \otimes_{\mathcal{O}} Y)) = \mathrm{rank}_{\mathcal{O}}(\mathrm{Hom}_A(Ai, Y)) = \dim_K(\mathrm{Hom}_{K\otimes_{\mathcal{O}}A}(K \otimes_{\mathcal{O}} Ai, X)) = \dim_K(iX)$$

where i is a primitive idempotent in A satisfying $Ai/J(A)i \cong S$. We denote by

$$d_A : R_K(A) \longrightarrow R_k(A)$$

the *decomposition map* sending $[X]$ to $[k \otimes_{\mathcal{O}} Y]$, where X is a finitely-generated $K \otimes_{\mathcal{O}} A$-module and Y a finitely generated $\mathcal{O}$-free A-module Y satisfying $K \otimes_{\mathcal{O}} Y \cong X$. By the above remarks, for $\chi \in \mathrm{Irr}_K(A)$ and $i \in I$ there are unique integers d_i^χ such that

$$d_A(\chi) = \sum_{i \in I} d_i^\chi \ \varphi_i$$

and we have $d_i^\chi = \dim_K(iX)$, where X is a simple $K \otimes_{\mathcal{O}} A$-module such that $\chi = [X]$. Again since $K \otimes_{\mathcal{O}} A$ is split, d_i^χ is also the multiplicity of X in a decomposition of $K \otimes_{\mathcal{O}} Ai$, hence

$$\Phi_i = \sum_{\chi \in \mathrm{Irr}_K(A)} d_i^\chi \chi$$

The matrix $D = (d_i^\chi)$, with $\chi \in \mathrm{Irr}_K(A)$ and $i \in I$, is the *decomposition matrix of* A. The matrix $C = (d_{S_j}^{\Phi_i})$, with $i, j \in I$, is the Cartan matrix, denoted C, of A. The above considerations imply the well-known fact $C = D^t \cdot D$. We denote by $L^0(A)$ the subgroup consisting of all $Y \in R_K(A)$ such that $< X, Y >_A = 0$ for all $X \in \mathrm{Pr}_{\mathcal{O}}(A)$.

Lemma 2.1. *Let A be an $\mathcal{O}$-algebra which is finitely generated free as $\mathcal{O}$-module such that $K \otimes_{\mathcal{O}} A$ is split semi-simple. We have $L^0(A) = \ker(d_A)$. In addition, if the Cartan matrix C of $k \otimes_{\mathcal{O}} A$ is non singular then $\mathrm{Pr}_{\mathcal{O}}(A) \cap L^0(A) = \{0\}$, the decomposition map induces an isomorphism $\mathrm{Pr}_{\mathcal{O}}(A) \cong \mathrm{Pr}_k(A)$, and $R_k(A)/\mathrm{Pr}_k(A)$ is a finite abelian group of order $|\det(C)|$.*

Proof. Any element η in $R_K(A)$ can be written in the form $[K \otimes_{\mathcal{O}} Y_1] - [K \otimes_{\mathcal{O}} Y_2]$ for some finitely generated $\mathcal{O}$-free A-modules Y_1 and Y_2. Since $\mathrm{Pr}_{\mathcal{O}}(A)$ is generated by the images Φ_i of the modules $K \otimes_{\mathcal{O}} Ai$, with i running over a set of representative I of the conjugacy classes of primitive idempotents, we get that $\eta \in L^0(A)$ if and only if $< \Phi_i, \eta >_A = 0$ for all $i \in I$. This is equivalent to $\mathrm{rank}_{\mathcal{O}}(iY_1) = \mathrm{rank}_{\mathcal{O}}(iY_2)$, hence to $\dim_k(k \otimes_{\mathcal{O}} iY_1) = \dim_k(k \otimes_{\mathcal{O}} iY_2)$ for all $i \in I$. This, in turn, is just a reformulation of $\eta \in \ker(d_A)$. If the Cartan matrix of $k \otimes_{\mathcal{O}} A$ is non singular then $Pr_{\mathcal{O}}(A) \cap \ker(d_A) = \{0\}$, thus d_A induces an isomorphism $\mathrm{Pr}_{\mathcal{O}}(A) \cong \mathrm{Pr}_k(A)$. The last statement is an elementary fact. □

A fundamental result of Brauer states that for group algebras, and hence block algebras, the decomposition map is surjective. One of the numerous applications of this fact is Brauer's reciprocity, and this can be formulated for more general $\mathcal{O}$-algebras. Let A be an $\mathcal{O}$-free $\mathcal{O}$-algebra of finite rank over $\mathcal{O}$, such that $K \otimes_{\mathcal{O}} A$ is split semi-simple and such that $k \otimes_{\mathcal{O}} A$ is split. Suppose that the Cartan matrix of A is non singular and that the decompostion map $d_A : R_K(A) \to R_k(A)$ is surjective. The scalar product $< \, , \, >_A$ restricts to a bilinear form, still denoted

$$< \, , \, >_A \colon \mathrm{Pr}_{\mathcal{O}}(A) \times R_K(A) \longrightarrow \mathbb{Z}$$

This bilinear form vanishes on $\mathrm{Pr}_{\mathcal{O}}(A) \times L^0(A)$, hence induces a bilinear form

$$< \, , \, >'_A \colon \mathrm{Pr}_{\mathcal{O}}(A) \times R_k(A) \longrightarrow \mathbb{Z}$$

Note that $\mathrm{Pr}_{\mathcal{O}}(A) \cong \mathrm{Pr}_k(A)$ since the Cartan matrix of A is assumed to be non singular, so $< \, , \, >'_A$ can also be viewed as bilinear form from $\mathrm{Pr}_k(A) \times R_k(A)$ to $\mathbb{Z}$. If need arises we use the same notation $< \, , \, >'_A$ for the $\mathbb{Q}$-bilinear form from $\mathbb{Q} \otimes_{\mathbb{Z}} \mathrm{Pr}_{\mathcal{O}}(A) \times \mathbb{Q} \otimes_{\mathbb{Z}} R_k(A)$ to $\mathbb{Q}$ obtained via extension of coefficients. The following is Brauer's reciprocity:

Proposition 2.2. *Let A be an $\mathcal{O}$-free $\mathcal{O}$-algebra of finite rank over $\mathcal{O}$ such that $K \otimes_{\mathcal{O}} A$ is split semi-simple and such that $k \otimes_{\mathcal{O}} A$ is split. Suppose that the Cartan matrix of A is non singular, that the decomposition map $d_A : R_K(A) \to R_k(A)$ is surjective, and that $k \otimes_{\mathcal{O}} A$ has no projective simple module. Let I be a system of representatives of the conjugacy classes of primitive idempotents. For any $i \in I$ denote by Φ_i the image in $\mathrm{Pr}_{\mathcal{O}}(A)$ of $K \otimes_{\mathcal{O}} Ai$ and by φ_i the image of $Ai/J(A)i$ in $R_k(A)$.*
(i) We have $< \Phi_i, \varphi_j >'_A = \delta_{i,j}$ for any $i, j \in I$.
(ii) We have $L^0(A)^{\perp} = \mathrm{Pr}_{\mathcal{O}}(A)$.

Proof. Since d_A is surjective, for any $i \in I$ and any $\chi \in \mathrm{Irr}_K(A)$ there are integers m_i^{χ} satisfying

$$\varphi_i = \sum_{\chi \in \mathrm{Irr}_K(A)} m_i^{\chi} \cdot d_A(\chi) = \sum_{j \in I} \sum_{\chi \in \mathrm{Irr}_K(A)} m_i^{\chi} d_j^{\chi} \varphi_j$$

which implies that $\sum_{\chi \in \mathrm{Irr}_K(A)} m_i^{\chi} d_j^{\chi} = \delta_{i,j}$ for all $i, j \in I$. We also have $\Phi_i = \sum_{\chi \in \mathrm{Irr}_K(A)} d_i^{\chi} \chi$ and hence

$$< \Phi_i, \varphi_j >'_A = < \Phi_i, \sum_{\chi \in \mathrm{Irr}_K(A)} m_j^{\chi} \chi >_A = \sum_{\chi \in \mathrm{Irr}_K(A)} d_i^{\chi} m_j^{\chi} = \delta_{i,j}$$

proving (i). The hypotheses imply that extending coefficients yields

$$\mathbb{Q} \otimes_{\mathbb{Z}} R_K(A) = (\mathbb{Q} \otimes_{\mathbb{Z}} L^0(A)) \; \oplus \; (\mathbb{Q} \otimes_{\mathbb{Z}} \mathrm{Pr}_{\mathcal{O}}(A))$$

Let $\theta \in L^0(A)^{\perp}$. Then, in $\mathbb{Q} \otimes_{\mathbb{Z}} R_K(A)$, we have $1_{\mathbb{Q}} \otimes \theta = \sum_{i \in I} q_i \otimes \Phi_i$ for some $q_i \in \mathbb{Q}$. For any $i \in I$ we have

$$q_i = < \theta, \varphi_i >'_A = \sum_{\chi \in \mathrm{Irr}_K(A)} m_i^{\chi} < \theta, \chi >$$

which is an integer as $\theta \in R_K(A)$. This shows (ii). □

If A is a block algebra of a group algebra $\mathcal{O}G$ for some finite group G then by results of Brauer, A satisfies the hypotheses of the above Proposition. In addition, if A has a non-trivial defect group, then for any $\chi \in \mathrm{Irr}_K(A)$ there is $\lambda \in L^0(A)$ such that $< \lambda, \chi >_A \neq 0$. Indeed, if not then $\chi \in L^0(A)^{\perp} = \mathrm{Pr}_{\mathcal{O}}(A)$, but then χ would correspond to an irreducible character which vanishes on all p-singular elements, hence which belongs to a block of defect zero. The following observation is useful for explicit calculations of determinants of Cartan matrices:

Proposition 2.3. *Let G be a finite group and B a block algebra of $\mathcal{O}G$ and denote by C_B the Cartan matrix of $k \otimes_{\mathcal{O}} B$. Let $\{\psi_s \mid 1 \leq s \leq r\}$ be a basis of the abelian group $L^0(B)$ and denote by F the matrix $F = (< \psi_s, \psi_t >)_{1\leq s,t\leq r}$. We have* $\det(F) = \det(C_B)$.

Proof. Since the decomposition map $d_B : R_K(B) \to R_k(B)$ is surjective, with kernel $L^0(B)$ and since $\mathrm{Pr}_{\mathcal{O}}(B) \cong \mathrm{Pr}_k(B)$, the map d_A induces an isomorphism of abelian groups

$$R_K(B)/(L^0(B) \oplus \mathrm{Pr}_{\mathcal{O}}(B)) \cong R_k(B)/\mathrm{Pr}_k(B)$$

which is a finite abelian group of order $|\det(C_B)|$. Denote by $\{\Phi_i \mid 1 \leq i \leq \ell(B)\}$ the canonical basis of $\mathrm{Pr}_{\mathcal{O}}(B)$ of the images of the projective indecomposable B-modules in some order. The union $\{\psi_s \mid 1 \leq s \leq r\} \cup \{\Phi_i \mid 1 \leq i \leq \ell(B)\}$ is a basis of the subgroup $L^0(B) \oplus \mathrm{Pr}_{\mathcal{O}}(B)$ of $R_K(B)$. Writing this basis in terms of the canonical basis $\mathrm{Irr}_K(B)$ of $R_K(B)$ yields a square matrix of the form $(E|D)$, where D is the decomposition matrix of B. The absolute value of the determinant of this matrix is the order of the group $R_K(B)/(L^0(B) \oplus \mathrm{Pr}_{\mathcal{O}}(B))$, hence equal to $|\det(C_B)|$. Any column of E is perpendicular to any column of D, because $L^0(B)$ and $\mathrm{Pr}_{\mathcal{O}}(B)$ are orthogonal subgroups. Thus the product $(E|D)^t \cdot (E|D)$ is a block diagonal matrix whose blocks are the matrices $E^t \cdot E = F$ and $D^t \cdot D = C_B$. This implies that $\det(C_B)^2 = \det(E|D)^2 = \det(F)\det(C_B)$, whence the result. □

The following observation is a useful tool to compute $L^0(B)$ for block algebras B with normal defect:

Lemma 2.4. *Let P be a finite p-group, E a p'-subgroup of $\mathrm{Aut}(P)$ and $\alpha \in H^2(E; \mathcal{O}^\times)$. Set $A = \mathcal{O}_\alpha(P \rtimes E)$. Label the set $\mathrm{Irr}_K(A) = \{\chi_i \mid 1 \leq i \leq \mathsf{k}(A)\}$ in such a way that the subset $\mathrm{Irr}_K(A|P) = \{\chi_i \mid 1 \leq i \leq \ell(A)\}$ consists of all irreducible characters with P in their kernel. For $\ell(A) + 1 \leq j \leq \mathsf{k}(A)$ and $1 \leq i \leq \ell(A)$ denote by $a_{i,j}$ the unique integers such that $\mathrm{Res}_E(\chi_j) = \sum_{i=1}^{\ell(A)} a_{i,j}\mathrm{Res}_E(\chi_i)$, where $\mathrm{Res}_E(\chi_i)$ is the restriction to $\mathcal{O}_\alpha E$ of χ_i. Then the set*

$$\mathcal{A} = \{\chi_j - \sum_{i=1}^{\ell(A)} a_{i,j}\chi_i \mid \ell(A) + 1 \leq j \leq \mathsf{k}(A)\}$$

is a basis of the free abelian group $L^0(A)$.

Proof. Since E is a p'-group, every χ_i lifts a simple $k_\alpha E$-module S_i, for $1 \leq i \leq \ell(A)$, and hence the numbers $a_{i,j}$ are the decomposition numbers corresponding to the characters χ_j and simple module S_i. Thus the set of characters of the projective indecomposable A-modules is

$$\mathrm{IPr}_{\mathcal{O}}(A) = \{\chi_i + \sum_{j=\ell(A)+1}^{k(A)} a_{i,j}\chi_j \mid 1 \leq i \leq \ell(A)\} \ .$$

A trivial verification shows that the elements in $\mathcal{A}$ are perpendicular to this set, hence belong to $L^0(A)$. Since each element in $\mathcal{A}$ has one irreducible character appearing with multiplicity 1, the set $\mathcal{A}$ is a basis of $L^0(A)$. □

3 On stable equivalences and isometries

It is a well-known fact that a stable equivalence between two block algebras induces an isometry on the groups of generalised characters which are orthogonal to projective characters. Extending, if possible, this isometry to an isometry between the character groups is one of the standard strategies in character theory. We review this briefly and prove that an isometry obtained in this way is always perfect in the sense of Broué [6]. Let A, B be $\mathcal{O}$-algebras which are finitely generated free as $\mathcal{O}$-modules such that $K \otimes_{\mathcal{O}} A$, $K \otimes_{\mathcal{O}} B$ are split semisimple. Then any B-A-bimodule M which is finitely generated projective as right A-module induces group homomorphisms

$$\Phi_M \ : \ R_K(A) \longrightarrow R_K(B)$$

$$\varphi_M \ : \ R_k(A) \longrightarrow R_k(B)$$

satisfying $\Phi_M([K \otimes_{\mathcal{O}} U]) = [K \otimes_{\mathcal{O}} M \otimes_A U]$ and $\varphi_M([k \otimes_{\mathcal{O}} U]) = [k \otimes_{\mathcal{O}} M \otimes_A U]$ for any finitely generated $\mathcal{O}$-free A-module U. Moreover, we have $d_B \circ \Phi_M = \varphi_M \circ d_A$. If both A, B are symmetric and M is a B-A-bimodule which is finitely generated projective as left B-module and as right A-module then the functors $M \otimes_A -$ and $M^* \otimes_B -$ between the categories $\mathrm{mod}(A)$ and $\mathrm{mod}(B)$ of finitely generated modules over A and B, respectively, are both left and right adjoint to each other. In that situation we say that M *induces a stable equivalence of Morita type between A and B* if $M \otimes_A M^* \cong B \oplus W$ for some projective B-B-bimodule W and $M^* \otimes_B M \cong A \oplus V$ for some projective A-A-bimodule V. The functors $M \otimes_A -$ and $M^* \otimes_B -$ induce inverse equivalences between the relatively $\mathcal{O}$-stable categories $\underline{\mathrm{mod}}(A)$ and $\underline{\mathrm{mod}}(B)$.

Proposition 3.1. *Let A, B be symmetric $\mathcal{O}$-algebras such that $K \otimes_{\mathcal{O}} A$ and $K \otimes_{\mathcal{O}} B$ are split semi-simple and such that the Cartan matrices of $k \otimes_{\mathcal{O}} A$ and $k \otimes_{\mathcal{O}} B$ are non singular. Let M be a B-A-bimodule which is finitely generated projective as a left B-module and as a right A-module. Suppose that M induces a stable equivalence of Morita type between A and B.*

(i) Φ_M and Φ_{M^*} induce inverse isomorphisms $R_K(A)/\mathrm{Pr}_{\mathcal{O}}(A) \cong R_K(B)/\mathrm{Pr}_{\mathcal{O}}(B)$.

(ii) φ_M and φ_{M^*} induce inverse isomorphisms $R_k(A)/\mathrm{Pr}_k(A) \cong R_k(B)/\mathrm{Pr}_k(B)$; in particular, $|\det(B)| = |\det(A)|$.

(iii) Φ_M and Φ_{M^*} induce inverse isometries $L^0(A) \cong L^0(B)$.

Proof. Since the Cartan matrix of $k \otimes_{\mathcal{O}} A$ is non singular we have $\mathrm{Pr}(A) \cap L^0(A) = \{0\}$, by Lemma 2.1; similarly for B. Note that Φ_M sends $\mathrm{Pr}_{\mathcal{O}}(A)$ to $\mathrm{Pr}_{\mathcal{O}}(B)$ and $L^0(A) = \ker(d_A)$ to $L^0(B) = \ker(d_B)$, thus induces a group homomorphisms $R_K(A)/\mathrm{Pr}_{\mathcal{O}}(A) \to R_K(B)/\mathrm{Pr}_{\mathcal{O}}(B)$ and $R_k(A)/\mathrm{Pr}_k(A) \to R_k(B)/\mathrm{Pr}_k(B)$. The map $\Phi_{M^*} \circ \Phi_M$ is the map induced by tensoring with the bimodule $M^* \otimes_B M \cong A \oplus W$, where W is a projective A-A-bimodule. Thus Φ_W maps $R_K(A)$ to $\mathrm{Pr}_{\mathcal{O}}(A)$, and hence for any $\chi \in R_K(A)$ we have $\Phi_{M^*}(\Phi_M(\chi)) = \chi + \zeta$ for some $\zeta \in \mathrm{Pr}_{\mathcal{O}}(A)$. Since $\Phi_{M^*} \circ \Phi_M$ maps $L^0(A)$ to itself and since $\mathrm{Pr}(A) \cap L^0(A) = \{0\}$ this implies that if $\chi \in L^0(A)$ then $\zeta = 0$. Thus Φ_M and Φ_{M^*} induce inverse isomorphisms $R_K(A)/\mathrm{Pr}_{\mathcal{O}}(A) \cong R_K(B)/\mathrm{Pr}_{\mathcal{O}}(B)$, $R_k(A)/\mathrm{Pr}_k(A) \cong R_k(B)/\mathrm{Pr}_k(B)$ and $L^0(A) \cong L^0(B)$. Since the functors $M \otimes_A -$ and $M^* \otimes_B -$ are adjoint we get in particular for $\chi, \chi' \in L^0(A)$ that

$$< \Phi_M(\chi), \Phi_M(\chi') >_B = < \chi, \Phi_{M^*}(\Phi_M(\chi')) >_A = < \chi, \chi' >_A$$

which shows that the isomorphisms between $L^0(A)$ and $L^0(B)$ are isometries. $\square$

The surjectivity of the decomposition map is invariant under stable equivalences of Morita type:

Proposition 3.2. *Let A, B be symmetric $\mathcal{O}$-algebras such that $K \otimes_{\mathcal{O}} A$, $K \otimes_{\mathcal{O}} B$ are split semi-simple and such that $k \otimes_{\mathcal{O}} A$, $k \otimes_{\mathcal{O}} B$ are split. Suppose there is a stable equivalence of Morita type between A and B. Then the decomposition map $d_A : R_K(A) \to R_k(A)$ is surjective if and only if the decomposition map $d_B : R_K(B) \to R_k(B)$ is surjective.*

Proof. Let M be a B-A-bimodule which is finitely generated projective as left B-module and as right A-module and which induces a stable equivalence of Morita type between A and B. Suppose that d_A is surjective. Then the map $\bar{d}_A : R_K(A)/\mathrm{Pr}_{\mathcal{O}}(A) \to R_k(A)/\mathrm{Pr}_k(A)$ induced by d_A is surejective. Thus, using the fact that d_A and d_B commute with the maps Φ_M and φ_M as described above, the map $\bar{d}_B : R_K(B)/\mathrm{Pr}_{\mathcal{O}}(B) \to R_k(B)/\mathrm{Pr}_k(B)$ induced by d_B is surjective. Since $\mathrm{Im}(d_B)$ contains $\mathrm{Pr}_k(B) = d_B(\mathrm{Pr}_{\mathcal{O}}(B))$ it follows that d_B is surjective. $\square$

Let A, B be block algebras of $\mathcal{O}G$, $\mathcal{O}H$, respectively, where G, H are finite groups. Assume that K, k are splitting fields for G and H. We show next that if an isometry $L^0(A) \cong L^0(B)$ induced by a stable equivalence of Morita type extends to an isometry $R_K(A) \cong R_K(B)$, then this is a perfect isometry. We use the following notation. If M is a B-A-bimodule which is finitely generated projective as left and right module, then in particular, M is $\mathcal{O}$-free of finite rank and hence determines an element χ_M in $R_K(B \otimes_{\mathcal{O}} A^0)$. We can also regard χ_M as the character of M as $\mathcal{O}(H \times G)$-module, and then χ_M is *perfect* by a result of Broué in [6]; that is, for any $(y, x) \in H \times G$ the character value $\chi_M(y, x)$ is divisible, in $\mathcal{O}$, by the orders of $C_H(y)$ and $C_G(x)$, and $\chi_M(x, y) = 0$ if exactly one of x, y is p-regular. Any simple $K \otimes_{\mathcal{O}} A$-module X and simple $K \otimes_{\mathcal{O}} B$-module Y determines a simple $K \otimes_{\mathcal{O}} (B \otimes_{\mathcal{O}} A^0)$-module $Y \otimes_K X^*$, where $X^* = \mathrm{Hom}_K(X, K)$, and if χ is the image of X in $R_K(A)$ and η the image of Y in $R_K(B)$ we denote by $\eta \cdot \chi^*$ the image in $R_K(B \otimes_{\mathcal{O}} A^0)$ of $Y \otimes_K X^*$. If $\Phi : R_K(A) \cong R_K(B)$ is an isometry, then for any $\chi \in \mathrm{Irr}_K(A)$ we have $\Phi(\chi) = \delta_\chi \eta_\chi$ for some $\delta_\chi \in \{\pm 1\}$ and some $\eta_\chi \in \mathrm{Irr}_K(B)$. Using the above notation we can consider Φ as an element in $R_K(B \otimes_{\mathcal{O}} A^0)$, denoted χ_Φ, by setting

$$\chi_\Phi = \sum_{\chi \in \mathrm{Irr}_K(A)} \Phi(\chi) \cdot \chi^*$$

Following Broué [6], the isometry Φ is called *perfect* if χ_Φ is perfect when viewed as character of $H \times G$. One of the main features of a perfect isometry between the two blocks A and B is that it induces an isomorphism of the centers $Z(A) \cong Z(B)$; see [6] for more details.

Proposition 3.3. *Let G, H be finite groups, A a block algebra of $\mathcal{O}G$ and B a block algebra of $\mathcal{O}H$. Suppose that K, k are splitting fields for G and H. Let M be a B-A-bimodule which is finitely generated projective as left B-module and as right A-module such that M induces a stable equivalence of Morita type between A and B. Assume that the isometry $L^0(A) \cong L^0(B)$ induced by Φ_M extends to an isometry $\Phi : R_K(A) \cong R_K(B)$. Then $\chi_\Phi - \chi_M \in \mathrm{Pr}_{\mathcal{O}}(A \otimes_{\mathcal{O}} B^0)$; in particular, Φ is a perfect isometry.*

Proof. For $x \in G$ denote by $c(x)$ the conjugacy class of x in G and by τ_x the restriction to A of the $\mathcal{O}$-linear map $\mathcal{O}G \to \mathcal{O}$ sending $x' \in c(x)$ to 1 and every other group element to 0. If x is p-singular then $\tau_x \in K \otimes_{\mathbb{Z}} L^0(A)$, and if x runs over the p-singular elements of G then τ_x runs over a spanning set of the K-space $K \otimes_{\mathbb{Z}} L^0(A)$ of K-valued central functions on $K \otimes_{\mathcal{O}} A$ which are orthogonal to $\mathrm{Pr}_{\mathcal{O}}(A)$. Extend Φ, Φ_M in the obvious way to $K \otimes_{\mathbb{Z}} R_K(A)$. Since Φ and Φ_M coincide on $L^0(A)$ we have $\Phi_M(\tau_x) = \Phi(\tau_x)$. Since Φ_M is induced by tensoring with M we also have

$$\Phi_M(\tau_x)(y) = \frac{1}{|G|} \sum_{s \in G} \chi_M(y, s)\tau_x(s) = \frac{|c(x)|}{|G|} \chi_M(y, x)$$

and a similar reasoning yields

$$\Phi(\tau_x)(y) = \frac{|c(x)|}{|G|} \chi_\Phi(y, x)$$

This shows that $\chi_\Phi(y, x) = \chi_M(y, x)$ for any p-singular $x \in G$ and any $y \in H$. Exchanging the roles of A, B shows that also $\chi_\Phi(y, x) = \chi_M(y, x)$ for any p-singular $y \in H$ and any $x \in G$. Thus $\chi_\Phi - \chi_M$ vanishes outside the p-regular elements of $H \times G$, hence belongs to $\mathrm{Pr}_{\mathcal{O}}(A \otimes_{\mathcal{O}} B^0)$. Since χ_M is perfect by the assumptions on M this implies that χ_Φ is perfect. $\square$

In what follows, A, B are symmetric $\mathcal{O}$-algebras such that $K \otimes_{\mathcal{O}} A$, $K \otimes_{\mathcal{O}} B$ are split semisimple. Given a B-A-bimodule M which is finitely generated projective as right A-module, the group homomorphism $\Phi_M : R_K(A) \to R_K(B)$ induced by the functor $M \otimes_A -$ depends only on the image $[K \otimes_{\mathcal{O}} M]$ of M in $R_K(B \otimes_{\mathcal{O}} A^{op})$ because we have an obvious isomorphism $K \otimes_{\mathcal{O}} (M \otimes_A U) \cong (K \otimes_{\mathcal{O}} M) \otimes_{(K \otimes_{\mathcal{O}} A)} (K \otimes_{\mathcal{O}} U)$ for any finitely generated $\mathcal{O}$-free A-module U. Rouquier's stable equivalence for blocks of defect 2 is in fact induced by a complex rather than a bimodule; we describe briefly how these are linked (what follows are well-known formalities, included for the convenience of the reader). A bounded complex X of B-A-bimodules is said to *induce a stable equivalence*, if the components of X are finitely generated projective as left B-modules, right A-modules, and if there are isomorphisms of complexes of bimodules $X \otimes_A X^* \cong B \oplus Z$ and $X^* \otimes_B X \cong A \oplus Y$ with Y and Z homotopy equivalent to bounded complexes of projective B-B-bimodules and A-A-bimodules, respectively. By a result of Rickard in [20], there is a canonical functor $D^b(B \otimes_{\mathcal{O}} A^{op}) \to \underline{\mathrm{mod}}(B \otimes_{\mathcal{O}} A^{op})$. The following well-known lemma shows that if M is the image in $\underline{\mathrm{mod}}(B \otimes_{\mathcal{O}} A^{op})$ of a complex X inducing a stable equivalence, then M induces a stable equivalence of Morita type between A and B.

Lemma 3.4. *Let A, B be symmetric $\mathcal{O}$-algebras and X a bounded complex of B-A-bimodules inducing a stable equivalence. Then the image M of X in $\underline{\mathrm{mod}}(B \otimes_{\mathcal{O}} A^{op})$ induces a stable equivalence of Morita type.*

Proof. A formal verification shows that the image of X^* in $\underline{\mathrm{mod}}(A \otimes_{\mathcal{O}} B^{op})$ is isomorphic to M^* and that the image of $X \otimes_A X^*$ in $\underline{\mathrm{mod}}(B \otimes_{\mathcal{O}} B^{op})$ is isomorphic to $M \otimes_A M^*$. By the assumptions on X, the image of $X \otimes_A X^*$ in $\underline{\mathrm{mod}}(B \otimes_{\mathcal{O}} B^{op})$ is isomorphic to B. Similary, the image of $X^* \otimes_B X$ in $\underline{\mathrm{mod}}(A \otimes_{\mathcal{O}} A^{op})$ is isomorphic to A, whence the result. $\square$

The purpose of the next proposition is to show that if X is a splendid complex inducing a stable equivalence between block source algebras A, B and if the isometry $L^0(A) \cong L^0(B)$ induced by the image M of X in $\underline{\mathrm{mod}}(B \otimes_{\mathcal{O}} A^{op})$ extends to an isometry $R_K(A) \cong R_K(B)$, then X can be modified by a projective B-A-bimodule in such a way that the resulting image in the Grothendieck group of B-A-bimodules is a p-permutation equivalence - and hence Φ is in fact part of an isotypy. In order to state this properly, we use the following notation. For U a bounded complex of finitely generated $\mathcal{O}$-free A-modules set $[K \otimes_{\mathcal{O}} U] = \sum_{i \in \mathbb{Z}} (-1)^i [K \otimes_{\mathcal{O}} U_i] \in R_K(A)$, where U_i is the component of U in degree i for $i \in \mathbb{Z}$. If U, U' are quasi-isomorphic bounded complexes of finitely generated $\mathcal{O}$-free A-modules then $K \otimes_{\mathcal{O}} U$, $K \otimes_{\mathcal{O}} U'$ are homotopy equivalent complexes of $K \otimes_{\mathcal{O}} A$-modules (because both are split and quasi-isomorphic as $K \otimes_{\mathcal{O}} A$ is split semi-simple and the functor $K \otimes_{\mathcal{O}} -$ is exact); in particular, $[K \otimes_{\mathcal{O}} U] = [K \otimes_{\mathcal{O}} U']$. For X a bounded complex of B-A-bimodules which are finitely generated projective as right A-modules, the functor $X \otimes_A -$ induces a group homomorphism $\Phi_X : R_K(A) \to R_K(B)$ by setting $\Phi_X([K \otimes_{\mathcal{O}} U]) = [K \otimes_{\mathcal{O}} (X \otimes_A U)]$; one checks easily that $\Phi_X = \sum_{i \in \mathbb{Z}} (-1)^i \Phi_{X_i}$, where X_i is the component of X in degree i. By the above remarks the map Φ_X depends only on the image $[K \otimes_{\mathcal{O}} X]$ of X in $R_K(B \otimes_{\mathcal{O}} A^{op})$, and hence if $X \to X'$ is a quasi-isomorphism of bounded complexes of B-A-bimodules which are finitely generated projective as right A-modules then $\Phi_X = \Phi_{X'}$. Furthermore, if Y is a bounded complex of A-B-bimodules which are finitely generated projective as right B-modules then $\Phi_Y \circ \Phi_X = \Phi_{Y \otimes_B X}$. If A, B are block source algebras with a common defect group P, a complex of B-A-modules X is called *splendid* if its components are finite direct sums of summands of the B-A-bimodules $B \otimes_{\mathcal{O}Q} A$, with Q running over the subgroups of P. A p-permutation equivalence between A and B is essentially a splendid generalised B-A-bimodule inducing an isomorphism $R_K(A) \cong R_K(B)$; this concept is due to Boltje and Xu [5]. We refer to [16] for more details regarding the terminology involving splendid complexes and p-permutation equivalences between source algebras.

Proposition 3.5. *Let A, B be block source algebras with a common defect group such that $K \otimes_{\mathcal{O}} A$, $K \otimes_{\mathcal{O}} B$ are split semi-simple and such that $k \otimes_{\mathcal{O}} A$, $k \otimes_{\mathcal{O}} B$ are split. Let X be a bounded complex of B-A-bimodules inducing a stable equivalence, and let M be a B-A-bimodule such that M is isomorphic, in $\underline{\mathrm{mod}}(B \otimes_{\mathcal{O}} A^{op})$, to the canonical image of X under Rickard's functor $D^b(B \otimes_{\mathcal{O}} A^{op}) \to \underline{\mathrm{mod}}(B \otimes_{\mathcal{O}} A^{op})$. If the isometry $L^0(A) \cong L^0(B)$ induced by M extends to an isometry $\Phi : R_K(A) \cong R_K(B)$ then there is a split bounded complex of projective B-A-bimodules W such that $\Phi = \Phi_{X \oplus W}$ and $\Phi^{-1} = \Phi_{(X \oplus W)^*}$. If moreover X is splendid then the image of $X \oplus W$ in the Grothendieck group of B-A-bimodules is a p-permutation equivalence between A and B.*

Proof. By construction of M, the characters $[K \otimes_{\mathcal{O}} X]$ and $[K \otimes_{\mathcal{O}} M]$ differ by the character of a generalised projective $B \otimes_{\mathcal{O}} A$-module, and the character $[K \otimes_{\mathcal{O}} M]$ differs from the character determined by Φ by the character of a generalised projective $B \otimes_{\mathcal{O}} A$-module. Thus there is a split bounded complex of projective B-A-bimodules W such that $[K \otimes_{\mathcal{O}} (X \oplus W)]$ is the character of Φ, hence $\Phi = \Phi_{X \oplus W}$. Since tensoring by the dual of $X \oplus W$ is left and right adjoint to the functor given by tensoring with $X \oplus W$, the statement $\Phi^{-1} = \Phi_{(X \oplus W)^*}$ follows immediately. Thus the composition of the maps $\Phi_{X \oplus W}$ and $\Phi_{(X \oplus W)^*}$ is the identity on $R_K(B)$, hence

$$[K \otimes_{\mathcal{O}} ((X \oplus W) \otimes_A (X \oplus W)^*)] = [K \otimes_{\mathcal{O}} B]$$

Since also

$$(X \oplus W) \otimes_A (X \oplus W)^* \simeq B \oplus Z'$$

for some bounded complex Z' of projective B-B-bimodules this forces that $[K \otimes_{\mathcal{O}} Z'] = 0$. But if the character of a generalised projective B-B-bimodule is zero, then actually the generalised bimodule is zero itself because the Cartan matrix of B is non-singular. Thus the image of Z' in the Grothendieck group of B-B-bimodules (with respect to split exact sequences) is zero. Equivalently, the images of $(X \oplus W) \otimes_A (X \oplus W)^*$ and B in the Grothendieck group of B-B-bimodules are equal. Since X is splendid and W consists of projective B-B-bimodules, the complex $X \oplus W$ is still splendid, and hence its image in the Grothendieck group of B-A-bimodules is a p-permutation equivalence. □

4 Stable equivalences and one simple module

Theorem 4.1. *Let A, B be symmetric $\mathcal{O}$-algebras such that $K \otimes_{\mathcal{O}} A$, $K \otimes_{\mathcal{O}} B$ are split semi-simple and such that $k \otimes_{\mathcal{O}} A$, $k \otimes_{\mathcal{O}} B$ are split and have non singular Cartan matrices. Suppose that $\ell(A) = 1$ and that for any $\eta \in \mathrm{Irr}_K(B)$ there is $\lambda \in L^0(B)$ such that $< \eta, \lambda >_B \neq 0$. Denote by $d_1, d_2, .., d_r$ the different dimensions of the simple $K \otimes_{\mathcal{O}} A$-modules and by m_i the number of isomorphism classes of simple $K \otimes_{\mathcal{O}} A$-modules of dimension d_i, where $1 \leq i \leq r$. Assume that d_i divides d_{i+1} and that $m_i > 2\frac{d_{i+1}}{d_i}$ for $1 \leq i \leq r-1$. Suppose that there is a stable equivalence of Morita type between A and B. Then there is an isomorphism $R_K(A) \cong R_K(B)$ which maps $\mathrm{Irr}_K(A)$ onto $\mathrm{Irr}_K(B)$ and induces an isomorphism $\mathrm{Pr}_{\mathcal{O}}(A) \cong \mathrm{Pr}_{\mathcal{O}}(B)$. In particular, $\ell(B) = 1$ and the decomposition matrices of A and B are equal.*

Proof. Since $k \otimes_{\mathcal{O}} A$ has a unique isomorphism class of simple modules, the rank of $\mathrm{Pr}_{\mathcal{O}}(A)$ is one, and hence the rank of $L^0(A)$ is $|\mathrm{Irr}_K(A)| - 1$. Since the stable equivalence of Morita type between A and B induces an isometry $L^0(B) \cong L^0(A)$ we get $|\mathrm{Irr}_K(A)| \leq |\mathrm{Irr}_K(B)|$. Note that $|\mathrm{Irr}_K(A)| = \sum_{i=1}^r m_i$. Since $k \otimes_{\mathcal{O}} A$ is also split it is a matrix algebra over its basic algebra. We therefore may assume that A is basic. Similarly, we may assume that B is basic. Then A, as left A-module, is projective indecomposable, and we have

$$[K \otimes_{\mathcal{O}} A] = \sum_{\chi \in \mathrm{Irr}_K(A)} d^\chi \cdot \chi$$

where $d^\chi = \dim_K(X)$ if $\chi = [X]$ for some simple $K \otimes_{\mathcal{O}} A$-module X. For $1 \leq i \leq r$ denote by Λ_i the subset of $\mathrm{Irr}_K(A)$ of isomorphism classes of simple $K \otimes_{\mathcal{O}} A$-modules of dimension d_i. In particular, $|\Lambda_i| = m_i$. By regrouping simple modules of the same degree we may rewrite this in the form

$$[K \otimes_{\mathcal{O}} A] = \sum_{i=1}^r d_i (\sum_{\chi \in \Lambda_i} \chi)$$

Thus the elements $\chi - \chi'$, with $\chi, \chi' \in \Lambda_i$, where $1 \leq i \leq r$, and $\chi - \frac{d_{i+1}}{d_i}\chi'$ with $\chi \in \Lambda_{i+1}$, $\chi' \in \Lambda_i$, where $1 \leq i \leq r-1$ are all in $L^0(A)$. More precisely, if we choose an element $\chi_i \in \Lambda_i$ and set $\Lambda'_i = \Lambda_i - \{\chi_i\}$ then the following set is a basis of the free abelian group $L^0(A)$:

$$\mathcal{B} \ = \ \cup_{i=1}^r \ \{\chi_i - \chi'_i \mid \chi'_i \in \Lambda'_i\} \ \cup \ \{\chi_{i+1} - \frac{d_{i+1}}{d_i}\chi_i \mid 1 \leq i \leq r-1\}$$

Indeed, $\mathcal{B}$ is clearly linearly independent and has $\sum_{i=1}^r (m_i - 1) + r - 1 = |\mathrm{Irr}_K(A)| - 1$ elements. One checks that if a $\mathbb{Z}$-linear combination of elements in $\mathcal{B}$ is divisible, in $R_K(A)$, by a positive integer q then all coefficients of that $\mathbb{Z}$-linear combination are divisible by q, which shows that

$\mathcal{B}$ is indeed a basis of $L^0(A)$. Note that $|\Lambda_i| > 2\frac{d_{i+1}}{d_i} \geq 4$ for $1 \leq i \leq r-1$. Denote by M a B-A-bimodule which is finitely generated projective as left B-module and as right A-module such that M and M^* induce a stable equivalence of Morita type between A and B. In particular, by Proposition 3.1, the map Φ_M restricts to an isometry $L^0(A) \cong L^0(B)$. For any i such that $1 \leq i \leq r$, the elements $\chi - \chi'$, with $\chi, \chi' \in \Lambda_i$, have norm 2, and thus, are mapped by Φ_M to elements of norm 2, hence an element of the form $\eta - \eta'$ for some $\eta, \eta' \in \mathrm{Irr}_K(B)$. Thus if $m_i = 2$ then Λ_i determines a unique subset $\Delta_i = \{\eta, \eta'\}$ having exactly two elements, but this does not determine a canonical bijection between Λ_i and Δ_i. If $m_i \geq 3$ then Λ_i contains a third element χ'', and then $< \chi - \chi', \chi - \chi" >= 1$, and hence $\chi - \chi''$ is mapped to either $\eta - \eta''$ or $\eta'' - \eta'$ for some $\eta'' \in \mathrm{Irr}_K(B)$ different from both η, η'. Thus, if $m_i \geq 3$ then there is a subset Δ_i of $\mathrm{Irr}_K(B)$ and a sign $\delta_i \in \{\pm\}$ such that Φ_M induces a uniquely determined bijection $\chi \mapsto \eta_\chi$ between Λ_i and Δ_i with the property

$$\Phi_M(\chi - \chi') = \delta_i(\eta_\chi - \eta_{\chi'})$$

for all $\chi, \chi' \in \Lambda_i$. This case applies to all indices i such that $1 \leq i \leq r-1$ because of the assumption $m_i > 2\frac{d_{i+1}}{d_i}$. We denote by η_i the image, in Δ_i, of χ_i under this bijection. If $m_r = 1$ we set $\Delta_r = \mathrm{Irr}_K(B) - \cup_{i+1}^{r-1}\Delta_i$. Since $|\mathrm{Irr}_K(A)| \leq |\mathrm{Irr}_K(B)|$, the set Δ_r is non empty. After possibly replacing M by $\Omega_{B\otimes_{\mathcal{O}}A^0}(M)$ we may assume that $\delta_1 = 1$. We will show inductively that the signs δ_i are all 1. Let i be an integer such that $\delta_i = 1$ and such that $1 \leq i \leq r-1$. Consider the element

$$\mu_i = \chi_{i+1} - \frac{d_{i+1}}{d_i}\chi_i$$

in $L^0(A)$. We have $< \mu_i, \chi_i - \chi' >_A = -\frac{d_{i+1}}{d_i}$ for all $\chi' \in \Lambda_i'$. Thus

$$\Phi_M(\mu_i) = (a - \frac{d_{i+1}}{d_i})\eta_i + a\sum_{\eta\in\Delta_i'}\eta + \Psi_i$$

for some integer a and some element Ψ_i in $R_K(B)$ not involving any of the elements in Δ_i. If $m_{i+1} \geq 2$ then $\Psi_i \neq 0$ because for any $\chi' \in \Lambda_{i+1}'$ we have $< \mu_i, \chi_{i+1} - \chi' >= 1$ and hence Ψ_i must involve one of the two characters occurring in $\Phi_i(\chi_{i+1} - \chi')$. Taking norms on both sides in the above equality yields

$$1 + \frac{d_{i+1}^2}{d_i^2} \geq (a - \frac{d_{i+1}}{d_i})^2 + (m_i - 1)a^2 = m_i a^2 - 2a\frac{d_{i+1}}{d_i} + \frac{d_{i+1}^2}{d_i^2}$$

which is equivalent to

$$1 \geq m_i a^2 - 2a\frac{d_{i+1}}{d_i}$$

Since $m_i > 2\frac{d_{i+1}}{d_i}$ this implies

$$1 \geq m_i a^2 - 2a\frac{d_{i+1}}{d_i} \geq m_i(a^2 - a) \geq 0$$

and all numbers involved in these inequalities are integers, hence equal to 1 or 0. Thus a is either 1 or 0. Note that if $\Psi_i \neq 0$ (which happens in particular for $1 \leq i \leq r-2$) then the left most inequalities are proper inequalities which forces $a = 0$, hence

$$\Phi_M(\mu_i) = -\frac{d_{i+1}}{d_i}\eta_i + \Psi_i$$

Comparing norms again forces that the norm of Ψ_i is 1. If $m_{i+1} \geq 3$ then for any $\chi', \chi'' \in \Lambda'_{i+1}$ we have

$$< \Phi(\mu_i), \delta_{i+1}(\eta_{i+1} - \eta_{\chi'}) >=< \mu_i, \chi_{i+1} - \chi' >_A = 1$$
$$< \Phi(\mu_i), \delta_{i+1}(\eta_{\chi'} - \eta_{\chi''}) >_B =< \mu_i, \chi' - \chi'' >_A = 0$$

which shows that $\Phi_M(\mu_i)$ involves η_{i+1}. Since μ_i cannot come from a module, the signs force

$$\Phi_M(\mu_i) = \eta_{i+1} - \frac{d_{i+1}}{d_i}\eta_i$$

hence $\delta_{i+1} = 1$. The only thing that remains to be seen is that we can exclude the pathological case $a = 1$ and $\Psi_i = 0$. This can occur only for $i = r-1$. What happens in that case is that then $m_r = 1$. By the assumptions, every element of $\mathrm{Irr}_K(B)$ occurs in at least one element of $L^0(B)$. Moreover, $|\mathrm{Irr}_K(B)| \geq |\mathrm{Irr}_K(A)|$, so all elements $\eta \in \Delta_r$ occur in $\Phi_M(\mu_{r-1})$, contradicting $\Psi_{r-1} = 0$. The bijections $\Lambda_i \cong \Delta_i$ constructed above yield therefore an isometry $\Phi : R_K(A) \cong R_K(B)$ mapping $\mathrm{Irr}_K(A)$ to $\mathrm{Irr}_K(B)$. This implies that $\mathrm{Irr}_k(B)$ has a unique element because $|\mathrm{Irr}_K(B)| - |\mathrm{Irr}_k(B)|$ is equal to the rank of $L^0(B)$, hence to the rank of $L^0(A)$ which in turn is equal to $|\mathrm{Irr}_K(A)| - |\mathrm{Irr}_k(A)| = |\mathrm{Irr}_K(A)| - 1$. Since Φ extends the isometry $L^0(A) \cong L^0(B)$ induced by Φ_M it must send $\mathrm{Pr}_\mathcal{O}(A)$ to $\mathrm{Pr}_\mathcal{O}(B)$. For the same reason the inverse of Φ sends $\mathrm{Pr}_\mathcal{O}(B)$ to $\mathrm{Pr}_\mathcal{O}(A)$. Thus Φ induces an isomorphism $\mathrm{Pr}_\mathcal{O}(A) \cong \mathrm{Pr}_\mathcal{O}(B)$, and both groups are isomorphic to $\mathbb{Z}$. In particular, Φ maps the image in $R_K(A)$ of a projective indecomposable A-module to the image in $R_K(B)$ of a projective indecomposable B-module. The statement on the decomposition matrices follows. □

5 Proof of Theorem 1.1

Definition 5.1 (cf. [12]). A finite group H is said to be of *central type* if H has an ordinary irreducible character χ satisfying $\chi(1)^2 = |H : Z(H)|$.

Proposition 5.2. *Let P be an elementary abelian p-group of order p^2, and let $E \leq \mathrm{Aut}(P)$ be a p'-group. Suppose that there exists a central extension*

$$1 \to Z \to \tilde{E} \to E \to 1$$

with Z a cyclic p'-group such that there is a K-valued linear character of Z which is covered by a unique irreducible character of $\tilde{E}$. Then E is abelian.

Proof. The hypothesis implies that $Z = Z(\tilde{E})$ and that $\tilde{E}$ is of central type. We first show that E has abelian Sylow 2-subgroups. Let $\tilde{U}$ be a Sylow 2-subgroup of $\tilde{E}$, and let $U := \tilde{U}Z/Z \cong \tilde{U}/\tilde{U}\cap Z$, a Sylow 2-subgroup of E. By [10, Lemma 2.2] or [8, Theorem 2], $\tilde{U}$ is also of central type and $\tilde{U} \cap Z = Z(\tilde{U})$. Now E and hence U is a subgroup of $GL_2(p)$. Let R be a Sylow 2-subgroup of $GL_2(p)$ containing U. First suppose that $p \equiv 3(\text{mod } 4)$. Then R has a cyclic subgroup R_0 of index 2. So, $U_0 := R_0 \cap U$ is a cyclic subgroup of index at most 2 in U. On the other hand, by [10, Corollary 3.2], either $U = 1$ or U_0 is not self-centralising in U. Since U_0 has index at most 2 in U, it follows that U is abelian.

Let us suppose now that $p \equiv 1(\text{mod } 4)$. In this case, $R \cong (R_1 \times R_2) \rtimes \langle t\rangle$ where $R_1 = \langle g\rangle$ and $R_2 = \langle h\rangle$ are cyclic groups of equal order, t has order 2 and $tgt^{-1} = g$, $tht^{-1} = gh^{-1}$. In particular, $R_1 = Z(R)$. Set $R_0 = R_1 \times R_2$ and for $0 \le i \le 2$ set $U_i = U \cap R_i$. For any subgroup V of R, denote by $\tilde{V}$ the inverse image of V in $\tilde{U}$ and denote by X_V the set of ordinary irreducible characters of $\tilde{V}$ covering θ. Note that if $V_1 \le V_2$ are subgroups of U, with V_1 normal in V_2, then X_{V_2} is the set of irreducible characters of $\tilde{V}_2$ covering some character in X_{V_1}.

If $U_0 = U$, then U is abelian and we are done. We assume therefore that U_0 has index 2 in U. If U_0 is cyclic, then by the same argument as above, U is abelian. Thus, we may assume that U_0 is of rank 2, say U_0 is a direct product of a cyclic group of order 2^α and a cyclic group of order 2^β with $\alpha \ge \beta$. We claim that $\beta = \alpha - 1$. Indeed, let V be a cyclic subgroup of U_0 of order 2^α. Since U/U_0 has order 2, X_{U_0} has two elements which are transitively permuted by $\tilde{U}$ (otherwise there would be at least two characters of $\tilde{U}$ which cover characters of X_U and hence cover θ). Thus there are at most two orbits in the action of $\tilde{U}_0$ on X_V. Since V is cyclic of order 2^α, $|X_V| = 2^\alpha$ and since $|U_0 : V| = 2^\beta$ each orbit of $\tilde{U}_0$ on X_V has at most 2^β elements. Thus, $\alpha \ge \beta \ge \alpha - 1$. If $\beta = \alpha$, then $|\tilde{U}/Z(\tilde{U})| = |U| = 2^{2\alpha+1}$ is not a square, a contradiction to $\tilde{U}$ being of central type. Thus, $\beta = \alpha - 1$, proving the claim.

We note here also that if V is a cyclic subgroup of U_0 of order 2^α which is normal in U, then the fact that there is only one orbit of $\tilde{U}$ on X_V along with the fact that $|U : V| = 2^\alpha = |X_V|$ means that $\tilde{U}$ acts faithfully, freely and transitively on X_V. In particular, $C_{\tilde{U}}(\tilde{V}) = \tilde{V}$.

Next, we claim that $U_1 = U \cap R_1$ has order 2^α. For this, first we note that since U_0/U_1 is a subgroup of R_0/R_1, U_0/U_1 is cyclic. So, U_1 has order at least $2^{\alpha-1}$. Suppose if possible that U_1 has order $2^{\alpha-1}$. Then U_1 has a complement in U_0 (any $\langle v\rangle$ such that the coset of v generates U_0/U_1 is a complement to U_1 in U_0). Write $U_1 = \langle g^a\rangle$ and let $W = \langle g^i h^j\rangle$ be a complement to U_1 in U_0. The order of W being 2^α, it follows that the order of h^j is greater than or equal to the order of g^i for otherwise, the unique involution of $\langle g\rangle$ (which is also the unique involution of U_1) would be a power of $g^i h^j$. So, denoting by n_2 the 2-part of a natural number n, we have that $i_2 \ge j_2$. In particular, $(j + 2i)_2 = j_2$. Let s be an element of $U - U_0$. Since $sg^i h^j s^{-1} = g^{j+2i} g^{-i} h^{-j} \in U_0$ we get that $g^{j+2i} \in U_1$, that is, $(j + 2i)_2 \ge a_2$. So, $i_2 \ge j_2 = (j + 2i)_2 \ge a_2$ which means that the order of $g^i h^j$ is less than or equal to the order of g^a. This proves the claim.

So, U_1 is a cyclic normal subgroup of U of order 2^α contained in U_0. Hence, by the above remarks, $C_{\tilde{U}}(\tilde{U}_1) = \tilde{U}_1$. Write $\tilde{U} \cap Z = \langle z\rangle$ and let $\tilde{u} \in \tilde{U}_0$ be a generator of $\tilde{U}_1/\tilde{U} \cap Z$. Since $U_1 \le Z(U)$, the rule $\tilde{v} \to [\tilde{u}, \tilde{v}]$ defines a homomorphism from $\tilde{U} \to \tilde{U}\cap Z$, with kernel $C_{\tilde{U}}(\tilde{U}_1) = \tilde{U}_1$. Since Z is cyclic, this means in particular that $\tilde{U}/\tilde{U}_1$ and hence U/U_1 is cyclic. Let $v \in U$ be such that vU_1 generates U/U_1. Then, $v \notin U_0$, so $v = g^i h^j t$ for some i and j. Then $v^2 = g^{2i} h^j t h^j t = g^{2i} h^j g^j h^{-j} = g^{2i+j} \in R_1 \cap U = U_1$. This means that $\alpha = 1$, hence U has order 4 and is abelian.

Suppose, if possible that E is not abelian. Since E is a p'-group, the faithful two-dimensional representation of E on the field of p elements represented by the inclusion $E \leq \mathrm{Aut}(P)$ is absolutely irreducible and lifts to a faithful 2-dimensional irreducible representation of E over $\mathbb{C}$. By the classification of the non-abelian finite subgroups of $GL_2(\mathbb{C})$ (see for instance [9, Theorem 26.1]), either E has a normal abelian subgroup of index 2 or $E/Z(E)$ is isomorphic to one of A_4, S_4 or A_5 and $Z(E)$ consists of scalar matrices (considering E as subgroup of $GL_2(\mathbb{C})$). In particular, $Z(E)$ is cyclic. But A_4, S_4 and A_5 all have non-abelian Sylow 2-subgroups, whereas by the above E has abelian Sylow 2-subgroups. Thus, E has a normal abelian subgroup, say N of index 2. So the Sylow l-subgroups of E for odd primes l are abelian, normal in E and centralise each other. Furthermore, we have shown above that the Sylow 2-subgroups of E are abelian. Thus it suffices to show that any 2-element of E centralises the Sylow l-subgroup of E for any odd prime l.

Let l be an odd prime, Q the Sylow l-subgroup of E and u a 2-element of E. Since Q is abelian and normal in E, $Q = C_Q(u) \times [\langle u \rangle, Q]$. Let $\tilde{Q}$ be the Sylow l-subgroup of $\tilde{E}$, so that $Q := \tilde{Q}Z/Z$ and let $\tilde{u}$ be a 2-element in $\tilde{U}$ lifting u. By [10, Lemma 2.2], $\tilde{Q}$ is of central type with $Z(\tilde{Q}) = Z \cap \tilde{Q}$. So Q is non-cyclic, hence of l-rank 2. By conjugating Q into the subgroup of diagonal matrices in $GL_2(\mathbb{C})$, we see that $Q/Z(GL_2(\mathbb{C})) \cap Q$ is cyclic. Thus, $C_Q(u)$ is non-trivial. Also, we may assume that $[\langle u \rangle, Q]$ is non-trivial as otherwise u centralises Q and we are done. Thus, both $C_Q(u)$ and $[\langle u \rangle, Q]$ are non-trivial cyclic groups. Let $\tilde{Q}_1$ be the inverse image of $C_Q(u)$ in $\tilde{Q}$ and let $\tilde{Q}_2$ be the inverse image of $[\langle u \rangle, Q]$ in $\tilde{Q}$. Both $\tilde{Q}_1$ and $\tilde{Q}_2$ are abelian groups; along with $Z \cap \tilde{Q}$, they generate $\tilde{Q}$ and $Z(\tilde{Q}) = Z \cap \tilde{Q}$. Hence $C_{\tilde{Q}_2}(\tilde{Q}_1) \leq Z \cap \tilde{Q}$. Now , $\tilde{u}$ stablises the normal series $1 \leq Z \cap \tilde{Q} \leq \tilde{Q}_1$, $\tilde{Q}_1$ is an l-group and u is an l' -element. Hence, $\tilde{u}$ centralises $\tilde{Q}_1$. Also, $\tilde{u}$ centralises $Z \cap \tilde{Q} = C_{\tilde{Q}_2}(\tilde{Q}_1)$. Thus by the $A \times B$-lemma (applied with $A = \langle u \rangle$, $B = \tilde{Q}_1$ and $P = \tilde{Q}_2$), it follows that $\tilde{u}$ centralises $\tilde{Q}_2$ and hence $\tilde{Q}$. □

Proposition 5.3. *Let P be an elementary abelian p-group of order p^2, and let $E \leq \mathrm{Aut}(P)$ be an abelian p'-group. Suppose that there exists a central extension*

$$1 \to Z \to \tilde{E} \to E \to 1$$

with Z a cyclic p'-group such that there is a K-valued linear character θ of Z which is covered by a unique irreducible character of $\tilde{E}$. Then $|E| = l^2$ for some natural number l dividing $(p-1)$. Let $e_\theta \in \mathcal{O}Z$ be the central idempotent corresponding to θ and set $A := \mathcal{O}(P \rtimes \tilde{E})e_\theta$, $m := \frac{p-1}{l}$. The degree of an element of $\mathrm{Irr}_K(A)$ is either l or l^2 and $\mathrm{Irr}_K(A)$ has $2p-1$ elements of degree l and m^2 elements of degree l^2.

Proof. By [8, Lemma 2], E is a direct product of two isomorphic groups. In particular, $|E| = l^2$ for some l. If E acts irreducibly on P, (where P is viewed as a vector space over the field of p elements), then $E = C_E(E)$ is cyclic, a contradiction. Thus, P is a direct product $P_1 \times P_2$ of two 1-dimensional E-invariant spaces. Hence E is conjugate to a subgroup of the group H of diagonal matrices in $GL_2(p)$, l divides $(p-1)$ and E is a direct product of cyclic groups of order l. But H has a unique subgroup isomorphic to the direct product of two cyclic groups of order l, hence E is conjugate to that unique subgroup. In other words, $E = E_1 \times E_2$, where E_1 acts faithfully and regularly on P_1 and centralises P_2 and E_2 acts faithfully and regularly on P_2 and centralises P_1.

Set $N = P \rtimes \tilde{E}$ and consider an irrreducible K-valued character $\xi = \xi_1 \times \xi_2$ of P, where ξ_i is an irreducible character of P_i, $i = 1, 2$. If ξ_1 and ξ_2 are both trivial, then there is exactly one character of N covering $\xi \times \theta$ and that has dimension l. Supose that ξ_1 is trivial and ξ_2 is non-trivial. The inertial subgroup of $\xi \times \theta$ is the inverse image $\tilde{E}_1$ of E_1 in $\tilde{E}$, $\tilde{E}/\tilde{E}_1$ is cyclic of order l and $|\tilde{E} : \tilde{E}_1| = l$. Thus, there are l irreducible characters of $\tilde{E}$ covering $\xi \times \theta$, each of dimension l. Also, the $\tilde{E}$-orbit of ξ has l-elements. Thus, there are $2(p-1)$ irreducible characters covering characters of the form $\xi_1 \times \xi_2 \times \theta$ where exactly one of ξ_1 and ξ_2 is trivial, and they all have dimension l. Finally, if neither ξ_1 nor ξ_2 is trivial, then the inertial subgroup of ξ in $\tilde{E}$ is Z, hence there is exactly one character of N covering $\xi \times \theta$ and it has dimension l^2. The $\tilde{E}$-orbit of ξ has size l^2. Thus, there are m^2 characters of N covering characters of the form $\xi_1 \times \xi_2 \times \theta$ where none of ξ_1 and ξ_2 is trivial, and each of these has dimension l^2. □

Proof of Theorem 1.1. Since the theorem is well known when P is cyclic, we may assume that P is elementary abelian of order p^2. By Rouquier's work [21, 6.3] (see also [16, Theorem A.2]) there is a stable equivalence of Morita type between B and C given by a splendid complex (in the slightly more restrictive sense of the notion "splendid" as used in [16]). By the structure theory of blocks with normal defect groups [15], [17, 14.6] and Proposition 5.2 the Brauer correpondent C of B has as source algebra a twisted group algebra of the form $A = \mathcal{O}_\alpha(P \rtimes E)$ for some abelian p'-subgroup E of $\mathrm{Aut}(P)$ and some $\alpha \in H^2(E; \mathcal{O}^\times)$. Then for a suitable central extension

$$1 \to Z \to \tilde{E} \to E \to 1$$

of E by a cyclic p'-group Z, determined by α, the algebra A is isomorphic to a block algebra $\mathcal{O}(P \rtimes \tilde{E})e_\theta$ for some block e_θ of $P \rtimes \tilde{E}$. By Proposition 5.3, A and a source algebra of B satisfy the hypotheses of Theorem 4.1, with a stable equivalence which is induced by a splendid complex. Thus, by 3.5, this stable equivalence induces a p-permutation equivalence. Since the blocks B, C have the same local structure it follows from [16, Theorem 1.4] that this p-permutation equivalence induces an isotypy. □

6 Block cohomology need not be an Ext-algebra

One of the technical difficulties with non principal p-blocks is that they do not have, in general, a canonical module which would play the role of the trivial module. If there were such a module, one reasonable expectation would be that its Ext-algebra should be isomorphic to the block cohomology because this is what happens in the principal block case. In the context of a block algebra B which has, up to isomorphism, a unique simple module S, it may be tempting to think that S would play that role. However, even the smallest known and well understood example in [3], [4] of a non nilpotent block with one isomorphism class of simple modules has the property that its block cohomology is not isomorphic to the Ext-algebra of a module over that block, and this is what we are going to describe in this section. Let k be an algebraically field of odd prime characteristic p and let $P = C_p \times C_p$ be an elementary abelian group of order p^2. Set $G = P \rtimes Q_8$, where Q_8 is a quaterion group of order 8 with $Z(Q_8)$ acting trivially on P and with $Q_8/Z(Q_8) \cong C_2 \times C_2$ acting in such a way that each factor C_2 inverts a generator of the corresponding factor C_p of P. Then $C_G(P) = P \times Z(Q_8)$. Thus kG has two blocks, namely the principal block $b_0 = \frac{1}{2}(1+z)$ and the block $b_1 = \frac{1}{2}(1-z)$, where z is the generator of $Z(Q_8)$. The block algebra $B_1 = kGb_1$ has, up to isomorphism, a unique simple module S. We have $\dim_k(S) = 2$; more precisely, $\mathrm{Res}^G_{Q_8}(S)$ is the unique 2-dimensional simple kQ_8-module. Both blocks have the same local structure, namely the fusion system of the group G, and hence the block cohomology of B_1 is $H^*(B_1) \cong H^*(B_0) = H^*(G,k)$.

Proposition 6.1. *With the notation above, there is no finitely generated B_1-module V such that $\mathrm{Ext}^*_{B_1}(V,V) \cong H^*(B_1)$.*

Proof. Suppose that there is a finitely generated B_1 module V such that $\mathrm{Ext}^*_{B_1}(V,V) \cong H^*(B_1)$. Then in particular $\mathrm{Ext}^0_{B_1}(V,V) \cong \mathrm{End}_{B_1}(V) \cong H^0(B_1) \cong k$. But since all composition factors of V are isomorphic to S this implies $V \cong S$. The generator z of $Z(Q_8)$ acts as -1 on S, hence also on its dual S^*, and thus as identity on $S \otimes_k S^*$. This shows that $S \otimes_k S^*$ is a module over the principal block algebra B_0. we have $B_0 \cong k(P \rtimes (C_2 \times C_2))$, and hence B_0 has four pairwise non isomorphic simple modules all of which have dimension one. Since P acts as identity on S it follows that $S \otimes_k S^*$ is a semi-simple four-dimensional B_0-module. By a result of Benson and Carlson [2, Theorem 2.1] or [1, Theorem 3.1.9], the trivial B_0-module occurs exactly with multiplicity 1 in $S \otimes_k S^*$. The remaining three summands are non-trivial one-dimensional modules. If one regards S and $S \otimes_k S^*$ as kQ_8-modules, then S is stable under any automorphism of Q_8, hence so is $S \otimes_k S^*$. The one-dimensional non trivial kQ_8-modules are permuted transitively by an automorphism of order three of Q_8. Thus, as B_0-module, $S \otimes_k S^* = k \oplus T_1 \oplus T_2 \oplus T_3$ is a direct sum of four pairwise non-isomorphic simple B_0-modules of dimension one, with k the trivial module. We have

$$\mathrm{Ext}^*_{B_1}(S,S) \cong \mathrm{Ext}^*_{kG}(k, S \otimes S^*) \cong H^*(G;k) \oplus \mathrm{Ext}^*_{kG}(k, T_1 \oplus T_2 \oplus T_3)$$

Since the Ext-quiver of B_0 is connected, the summand $\mathrm{Ext}^*_{kG}(k, T_1 \oplus T_2 \oplus T_3)$ is not zero. But then $\mathrm{Ext}^*_{B_1}(S,S)$ cannot be isomorphic to $H^*(B_1)$. $\square$

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