THE ORBIT SPACE OF A FUSION SYSTEM IS CONTRACTIBLE

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Abstract. Given a fusion system $\mathcal{F}$ on a finite p-group P, where p is a prime, we show that the partially ordered set of isomorphism classes in $\mathcal{F}$ of chains of non-trivial subgroups of P, considered as topological space, is contractible, further generalising Symonds' proof [19] of a conjecture of Webb [23, 24] and its generalisation to non-trivial Brauer pairs associated with a p-block by Barker [1].

1 Introduction

Let G be a finite group and let p be a prime divisor of the order of G. Denote by $\mathcal{P}$ the partially ordered G-set of non-trivial p-subgroups of G and by $sd(\mathcal{P})$ its barycentric subdivision; that is, $sd(\mathcal{P})$ is the G-poset of chains of non-trivial p-subgroups ordered by inclusion of chains, on which G acts by conjugation; this is a simplicial complex introduced by K. S. Brown [4] and frequently called *p-subgroup complex* or *Brown complex.* P. Symonds proved in [19] a conjecture of Webb [23, 24] which states that the orbit space $sd(\mathcal{P})/G$, viewed as topological space, is contractible, and L. Barker extended this result in [1] to the G-poset of non-trivial Brauer pairs of a p-block of G. Even though there need not be a G-action behind an abstractly given fusion system $\mathcal{F}$ on a finite p-group P, it is possible to define a space associated with $\mathcal{F}$ which coincides with the above orbit spaces in case $\mathcal{F}$ is the fusion system of a finite group or a p-block. The construction of this space involves the subdivision construction from [13]. One purpose of this paper is to show that the contractibiliy results on orbit categories carry over to the general case; the terminology is explained below:

Theorem 1.1. *Let p be a prime, let P be a finite p-group and let $\mathcal{F}$ be a fusion system on P. Let $\mathcal{C}$ be a right ideal in $\mathcal{F}$. Then the partially ordered sets $[S(\mathcal{C})]$ and $[S_{\lhd}(\mathcal{C})]$ are contractible, when viewed as topological spaces.*

Fusion systems have been introduced by Puig as an axiomatic description of fusion in finite groups and blocks; see for instance [3], [14] or the appendix below for precise definitions. A *right ideal* in a fusion system $\mathcal{F}$ on a finite p-group P is a full subcategory $\mathcal{C}$ of $\mathcal{F}$ with the property that if Q, R are subgroups of P with Q belonging to $\mathcal{C}$ and

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$\mathrm{Hom}_{\mathcal{F}}(Q,R)$ non-empty, then also R belongs to $\mathcal{C}$; see 3.2 below. Examples of right ideals in $\mathcal{F}$ include the full subcategory of all non-trivial subgroups of P and the full subcategory of all $\mathcal{F}$-centric subgroups of P (a subgroup Q of P is called *$\mathcal{F}$-centric* if $C_P(R) = Z(R)$ for any subgroup R of P which is isomorphic to Q in $\mathcal{F}$). Given a right ideal $\mathcal{C}$ in $\mathcal{F}$, the subdivision $S(\mathcal{C})$ is the category having finite chains of non-isomorphisms in $\mathcal{C}$ as objects and "obvious" commutative diagrams as morphisms; see §2 for details. The category $S_{\lhd}(\mathcal{C})$ is the full subcategory of $S(\mathcal{C})$ of chains of subgroups $Q_0 < Q_1 < \cdots < Q_m$ of P with the property that all Q_i are normal in Q_m. Chains of this form have been introduced by Knörr and Robinson [11]; we view these chains as objects of an appropriate category - see §4. All categories mentioned so far have the property that endomorphisms of objects are isomorphisms - that is, they are EI-categories in the terminology of [15]. One of the particular properties of any EI-category $\mathcal{C}$ is that the set of isomorphism classes $[\mathcal{C}]$ of $\mathcal{C}$ is in fact a partially ordered set with the order relation $[X] \leq [Y]$ if $\mathrm{Hom}_{\mathcal{C}}(X,Y) \neq \emptyset$.

The proof of 1.1 follows the pattern of Symonds' in [19]: we show in Theorem 5.12 that constant covariant functors on $[S(\mathcal{C})]$ are acyclic (that is, their cohomology vanishes in positive degree) and in Theorem 6.1 that $[S(\mathcal{C})]$ is simply connected. The contractibility follows then from a theorem of Whitehead. Instead of trying to present the shortest proof of Theorem 5.12 we take the opportunity in the Sections 4 and 5 to prove slightly more general acyclicity results - this is the second purpose of this paper. The motivation for doing so are certain open problems in block theory which admit functor cohomological interpretations, such as the question of the existence of a "classifying space" associated with any p-block - or, more generally, the question of the existence and uniqueness of finite p-local groups associated with arbitrary fusion systems in the sense of Broto, Levi, Oliver [3]. The proof of 1.1 is in any case not just a straightforward adaptation of the proofs in [19], [1], because, as mentioned above, in the more general situation of Theorem 1.1, the poset $[S(\mathcal{C})]$ can no longer be viewed as G-orbit space for some appropriate group G. Note also that, for instance, the Solomon fusion system [12] on a Sylow-2-subgroup of $\mathrm{Spin}(7,3)$ cannot be the fusion system of a finite group [17] and not even of any 2-block of a finite group by [10]. See also [5], [16] for more "exotic" fusion systems which cannot occur as fusion systems of finite groups.

2 Subdivisions and orbit spaces

We review in this Section the subdivision construction from [13] and its connection with orbit spaces of subgroup complexes. We denote as usual by Δ the category whose objects are the totally ordered set $[m] = \{0, 1, .., m\}$ and whose morphisms are the non-decreasing monotone maps $\alpha : [m] \to [n]$, where m, n are non-negative integers.

Definition 2.1. The *division category* of a category $\mathcal{C}$ is the category $D(\mathcal{C})$ defined as follows. The objects of $D(\mathcal{C})$ are the covariant functors $\sigma : [m] \to \mathcal{C}$, where m runs over the set of non-negative integers. A morphism in $D(\mathcal{C})$ from $\sigma : [m] \to \mathcal{C}$ to $\tau : [n] \to \mathcal{C}$ is a pair (α, μ) consisting of a map $\alpha : [m] \to [n]$ in Δ and an isomorphism of functors

$\mu : \sigma \cong \tau \circ \alpha$. The composition of two morphisms $(\alpha, \mu) : \sigma \to \tau$ and $(\beta, \nu) : \tau \to \rho$ in $D(\mathcal{C})$ is defined by $(\beta, \nu) \circ (\alpha, \mu) = (\beta \circ \alpha, (\nu\alpha) \circ \mu)$, where $\nu\alpha : \tau \circ \alpha \cong \rho \circ \beta \circ \alpha$ is the isomorphism of functors obtained from precomposing ν with α. The *subdivision category of a category* $\mathcal{C}$ is the full subcategory of $D(\mathcal{C})$ consisting of all faithful functors $\sigma : [m] \to \mathcal{C}$. Given an object $\sigma : [m] \to \mathcal{C}$ in $D(\mathcal{C})$, the integer m is then called the *length* of σ.

More explicitly, the objects of $D(\mathcal{C})$ can be viewed as the chains of morphisms

$$\sigma = X_0 \xrightarrow{\varphi_0} X_1 \xrightarrow{\varphi_1} \cdots \xrightarrow{\varphi_{m-1}} X_m$$

in $\mathcal{C}$, and a morphism in $D(\mathcal{C})$ from a chain of morphisms

$$\sigma = X_0 \xrightarrow{\varphi_0} X_1 \xrightarrow{\varphi_1} \cdots \xrightarrow{\varphi_{m-1}} X_m$$

to a chain of morphisms

$$\tau = Y_0 \xrightarrow{\psi_0} Y_1 \xrightarrow{\psi_1} \cdots \xrightarrow{\psi_{n-1}} Y_n$$

is a family $\mu = (\mu_i)_{0 \le i \le m}$ where for each i there is $\alpha(i) \in \{0, 1, .., n\}$ such that $\alpha(i) \le \alpha(j)$ if $i \le j$ and such that $\mu_i : X_i \to Y_{\alpha(i)}$ is an isomorphism which makes the obvious diagrams commutative; that is,

$$\mu_{i+1} \circ \varphi_i \;=\; \psi_{\alpha(i+1)-1} \circ \cdots \circ \psi_{\alpha(i)+1} \circ \psi_{\alpha(i)} \circ \mu_i$$

for any $i \in \{0, 1, .., m-1\}$ such that $\alpha(i+1) > \alpha(i)$ and $\mu_{i+1} \circ \varphi_i = \mu_i$ for any $i \in \{0, 1, .., m-1\}$ such that $\alpha(i+1) = \alpha(i)$.

Similarly, the objects of $S(\mathcal{C})$ can be viewed as the chains of morphisms

$$\sigma = X_0 \xrightarrow{\varphi_0} X_1 \xrightarrow{\varphi_1} \cdots \xrightarrow{\varphi_{m-1}} X_m$$

with the additional property that the X_i are pairwise non-isomorphic, with $0 \le i \le m$. In particular, a necessary condition for σ to be in $S(\mathcal{C})$ is that all φ_i are non-isomorphisms in $\mathcal{C}$. Note that this implies that if $(\alpha, \mu) : \sigma \to \tau$ is a morphism in $S(\mathcal{C})$, then automatically the map $\alpha : [m] \to [n]$ has to be injective. Indeed, with the above notation, the equality $\alpha(i+1) = \alpha(i)$ would imply that $\mu_{i+1} \circ \varphi_i = \mu_i$, which is impossible unless φ_i is an isomorphism.

In this paper we will consider the subdivision construction only in the context of EI-categories. Following [15], an EI-category is a small category $\mathcal{C}$ with the property that every endomorphism of an object in $\mathcal{C}$ is an isomorphism. If in addition $\mathrm{Aut}_{\mathcal{C}}(X)$ acts regularly on $\mathrm{Hom}_{\mathcal{C}}(X, Y)$ for any two objects X, Y for which the latter set is non-empty, we say that the EI-category $\mathcal{C}$ is *regular* (cf. [13, 2.1]). The set $[\mathcal{C}]$ of isomorphism classes of objects of an EI-category $\mathcal{C}$ has a structure of a partially ordered set given by $[X] \leq [Y]$ whenever $\mathrm{Hom}_{\mathcal{C}}(X,Y)$ is non-empty, where $[X]$, $[Y]$ are the isomorphism classes of objects X, Y in $\mathcal{C}$. Another important property of EI-categories is that a non-isomorphism composed with any morphism will always yield again a non-isomorphism, which in turn implies that if there is a non-isomorphism from an object X to an object Y in an EI-category $\mathcal{C}$ then X and Y cannot be isomorphic in $\mathcal{C}$. Thus, for an EI-category $\mathcal{C}$, an object in $D(\mathcal{C})$ of the form

$$\sigma = X_0 \xrightarrow{\varphi_0} X_1 \xrightarrow{\varphi_1} \cdots \xrightarrow{\varphi_{m-1}} X_m$$

belongs to $S(\mathcal{C})$ if and only if the morphisms φ_i, $0 \leq i \leq m-1$, are non-isomorphisms. In other words, $S(\mathcal{C})$ consists of all chains of non-isomorphisms in $\mathcal{C}$.

Given an object $\sigma : [m] \to \mathcal{C}$ with m positive, we define for any $i \in [m]$ the object $\sigma\backslash i$ in $S(\mathcal{C})$ by "deleting" $\sigma(i)$. Very formally speaking, $\sigma\backslash i : [m-1] \to \mathcal{C}$ is the functor defined by

$$(\sigma\backslash i)(j) = \begin{cases} \sigma(j), & 0 \leq j < i; \\ \sigma(j+1), & i \leq j \leq m-1 \end{cases}$$

and which maps a morphism $j < k$ in $[m-1]$ to either $\sigma(j < k)$ or $\sigma(j < k+1)$ or $\sigma(j+1 < k+1)$ depending on whether $j < k < i$ or $j < i \leq k$ or $i \leq j < k$, respectively. There is a canonical morphism $(\alpha, \mu) : \sigma\backslash i \to \sigma$ in $S(\mathcal{C})$ where $\alpha(j) = \begin{cases} j, & 0 \leq j < i; \\ j+1, & i \leq j \leq m-1 \end{cases}$ and where μ is the family of identity morphisms $(\sigma\backslash i)(j) = \sigma(\alpha(j))$ for $0 \leq j \leq m-1$. In particular, we have $[\sigma\backslash i] < [\sigma]$ in $[S(\mathcal{C})]$. Clearly, if there is a morphism $\sigma \to \tau$ between two objects σ, τ in $S(\mathcal{C})$ of lengths m, $m+1$, respectivley, then any such morphism factors uniquely as composition $\sigma \cong \tau\backslash i \to \tau$ of some isomorphism and the canonical morphism for a unique $i \in [m+1]$.

In general a morphism $X \to Y$ in a category $\mathcal{C}$ does not induce a group homomorphism between the automorphism groups of X and Y. One of the reasons for working with regular EI-categories is the observation that taking automorphism groups is contravariant functorial:

Proposition 2.2. *([13, 2.2]) Let $\mathcal{C}$ be a regular EI-category. There is a contravariant functor from $\mathcal{C}$ to the category of groups sending any object X in $\mathcal{C}$ to its automorphism group $\mathrm{Aut}_{\mathcal{C}}(X)$ and any morphism $\varphi : X \to Y$ in $\mathcal{C}$ to the group homomorphism $\mathrm{Aut}_{\mathcal{C}}(Y) \to \mathrm{Aut}_{\mathcal{C}}(X)$ which sends $\sigma \in \mathrm{Aut}_{\mathcal{C}}(Y)$ to the unique $\rho \in \mathrm{Aut}_{\mathcal{C}}(X)$ satisfying $\varphi \circ \rho = \sigma \circ \varphi$.*

Proof. Both φ and $\sigma \circ \varphi$ are morphisms from X to Y. Since $\mathrm{Aut}_{\mathcal{C}}(X)$ acts regularly on $\mathrm{Hom}_{\mathcal{C}}(X,Y)$ there is a unique $\rho \in \mathrm{Aut}_{\mathcal{C}}(X)$ satisfying $\varphi \circ \rho = \sigma \circ \varphi$. The result follows. $\square$

Proposition 2.3. *([13, 1.2, 1.3]) Let $\mathcal{C}$ be an EI-category and let $\sigma : [m] \to \mathcal{C}$ and $\tau : [n] \to \mathcal{C}$ be objects in $S(\mathcal{C})$. If $(\alpha, \mu), (\alpha', \mu') : \sigma \to \tau$ are two morphisms in $S(\mathcal{C})$ then $\alpha = \alpha'$, and there is a unique automorphism $(\mathrm{Id}_{[m]}, \nu)$ of σ such that $\mu' = \mu \circ \nu$. In particular, $S(\mathcal{C})$ is a regular EI-category.*

Proof. We have $\sigma \cong \tau \circ \alpha \cong \tau \circ \alpha'$, hence $\sigma(i) \cong \tau(\alpha(i)) \cong \tau(\alpha'(i))$ for any $i \in [m]$. Since τ is faithful this forces $\alpha(i) = \alpha(i')$. The natural transformations μ, μ' evaluated at $i \in [m]$ yield isomorphisms $\mu(i) : \sigma(i) \cong \tau(\alpha(i))$ and $\mu'(i) : \sigma(i) \cong \tau(\alpha(i))$. Thus $\nu(i) = (\mu(i))^{-1} \circ \mu'(i) : \sigma(i) \cong \sigma(i)$ is the unique automorphism of $\sigma(i)$ satisfying $\mu'(i) = \mu(i) \circ \nu(i)$, and hence the family $\nu = (\nu(i))_{i \in [m]}$ is the unique automorphism of the functor σ satisfying $\mu' = \mu \circ \nu$. The uniqueness property of α applied to endomorphisms of σ implies that every endomorphism of σ is of the form $(\mathrm{Id}_{[m]}, \rho)$ for some automorphism ρ of the functor σ. Thus $(\mathrm{Id}_{[m]}, \nu)$ is the unique automorphism of σ in $S(\mathcal{C})$ satisfying $(\alpha', \mu') = (\alpha, \mu) \circ (\mathrm{Id}_{[m]}, \nu)$, hence $S(\mathcal{C})$ is regular. □

Since $S(\mathcal{C})$ is an EI-category if $\mathcal{C}$ is so, the set of isomorphism classes $[S(\mathcal{C})]$ of $S(\mathcal{C})$ has a structure of partially ordered set. If $\mathcal{C}$ is itself a poset then $S(\mathcal{C})$ is just the usual barycentric subdivision of $\mathcal{C}$.

Definition 2.4. The *orbit space* of an EI-category $\mathcal{C}$ is the poset $[S(\mathcal{C})]$ viewed as topological space.

The connection with orbit spaces of subgroup complexes or Brauer pair complexes is described in the following observation [13, 4.6] (we refer to [21] for the block theoretic terminology):

Proposition 2.5. *Let G be a finite group, let k be a field of positive characteristic p and let b be a block of kG. Let $\mathcal{P}$ be a G-subposet of the G-poset of b-Brauer pairs. Choose a maximal b-Brauer pair (P, e_P) and denote, for any subgroup Q of P, by e_Q the unique block of $kC_G(Q)$ satisfying $(Q, e_Q) \leq (P, e_P)$. Let $\mathcal{F}$ be the fusion system on $\mathcal{P}$ whose morphisms are the group homomorphisms $\varphi : Q \to R$ between any two subgroups Q, R of P for which there exists $x \in G$ satisfying ${}^x(Q, e_Q) \leq (R, e_R)$ and $\varphi(u) = xux^{-1}$ for any $u \in Q$. Let $\mathcal{C}$ be the full subcategory of $\mathcal{F}$ consisting of all subgroups Q of P for which the b-Brauer pair (Q, e_Q) belongs to $\mathcal{P}$. The map sending a chain of subgroups $Q_0 < Q_1 < \cdots < Q_m$ in $\mathcal{C}$ to the chain of b-Brauer pairs $(Q_0, e_{Q_0}) < (Q_1, e_{Q_1}) < \cdots < (Q_m, e_{Q_m})$ in $\mathcal{P}$ induces an isomorphism of posets $[S(\mathcal{C})] \cong sd(\mathcal{P})/G$.*

The notation $sd(\mathcal{P})$ stands for the barycentric subdivision of $\mathcal{P}$; that is, $sd(\mathcal{P})$ is the G-poset of totally ordered sets of b-Brauer pairs in $\mathcal{P}$, ordered by inclusion. If b is the principal block of G then $\mathcal{P}$ can be identified with a G-set of p-subgroups of G. Thus Proposition 2.5 explains in what way Theorem 1.1 is indeed a generalisation to arbitrary fusion systems of the contractibility results of Symonds [19] and Barker [1].

The last result in this Section is included for completeness and not needed for the purpose of this paper; it implies that from a cohomological point of view one can always work with the subdivision category:

Proposition 2.6. *Let $\mathcal{C}$ be an EI-category. The inclusion functor $S(\mathcal{C}) \hookrightarrow D(\mathcal{C})$ has a left adjoint.*

Proof. We define a left adjoint $\Psi : D(\mathcal{C}) \to S(\mathcal{C})$ of the inclusion functor as follows. Given an object $\sigma : [m] \to \mathcal{C}$ in $D(\mathcal{C})$ let m' be the smallest non-negative integer such that there exists an object $\sigma' : [m'] \to \mathcal{C}$ in $D(\mathcal{C})$ and a morphism $(\beta, \nu) : \sigma \longrightarrow \sigma'$ in $D(\mathcal{C})$. Note that then $\beta : [m] \to [m']$ is necessarily surjective, as we always can replace $[m']$ by $\beta([m])$ and σ' by its restriction to $\beta([m])$. Also, σ' belongs to $S(\mathcal{C})$. Indeed, otherwise there would by an integer i such that $0 \leq i < m'$ and such that $\varphi_i : \sigma'_i \to \sigma'_{i+1}$ is an isomorphism, where here $\sigma'_i = \sigma'(i)$ and $\varphi_i = \sigma'(i < i+1)$. But then there would be a morphism in $D(\mathcal{C})$ from σ' to the chain obtained by deleting σ'_{i+1} given by the commutative diagram

$$\begin{tikzcd}
X'_0 \arrow[r] \arrow[d, equal] & \cdots \arrow[r] & X'_i \arrow[r, "\varphi_i"] \arrow[d, equal] & X'_{i+1} \arrow[r] \arrow[dl, "(\varphi_i)^{-1}"] & X'_{i+2} \arrow[r] \arrow[d, equal] & \cdots \arrow[r] & X'_m \arrow[d, equal] \\
X'_0 \arrow[r] & \cdots \arrow[r] & X'_i \arrow[rr] & & X'_{i+2} \arrow[r] & \cdots \arrow[r] & X'_m
\end{tikzcd}$$

contradicting the minimality of $[m']$. We want to show that the assignement $\Psi(\sigma) = \sigma'$ can be made functorial with the required adjunction property. Let $\sigma : [m] \to \mathcal{C}$ and $\tau : [n] \to \mathcal{C}$ be two objects in $D(\mathcal{C})$. Let m', n' be minimal such that there are objects $\sigma' : [m'] \to \mathcal{C}$ and $\tau' : [n'] \to \mathcal{C}$ in $D(\mathcal{C})$ for which there are morphims $(\beta, \nu) : \sigma \to \sigma'$ and $(\gamma, \tau) : \tau \to \tau'$ in $D(\mathcal{C})$. By the above, σ', τ' belong to $S(\mathcal{C})$. In order to establish the functoriality of Ψ we need to show that for any morphism $(\alpha, \mu) : sigma \ \to \tau$ in $D(\mathcal{C})$ there is a unique morphism $(\alpha', \mu') : \sigma' \to \tau'$ making the following diagram commutative:

$$\begin{tikzcd}
\sigma \arrow[r, "(\alpha,\mu)"] \arrow[d, "(\beta,\nu)"'] & \tau \arrow[d, "(\gamma,\tau)"] \\
\sigma' \arrow[r, "(\alpha',\mu')"'] & \tau'
\end{tikzcd}$$

If $0 \leq i < [m]$ such that $\beta(i) = \beta(i+1)$ then $\sigma_i \cong \sigma_{i+1}$, hence $\tau_{\alpha(i)} \cong \tau_{\alpha(i+1)}$ and so $\tau'_{\gamma(\alpha(i))} \cong \tau'_{\gamma(\alpha(i+1))}$. But since τ' is in $S(\mathcal{C})$ this forces $\gamma(\alpha(i)) = \gamma(\alpha(i+1))$. Thus there is a unique map $\alpha' : [m'] \to [n']$ such that $\alpha'(\beta(i)) = \gamma(\alpha(i))$ for all $i \in [m]$. We define μ' to be the family of isomorphisms $\mu'_{\beta(i)} : \sigma'_{\beta(i)} \cong \tau'_{\alpha'(\beta(i))} = \tau'_{\gamma(\alpha(i))}$ making the square of isomorphisms

$$\begin{array}{ccc} \sigma_i & \xrightarrow{\mu_i} & \tau_{\alpha(i)} \\ {\scriptstyle \nu_i}\downarrow & & \downarrow{\scriptstyle \tau_{\alpha(i)}} \\ \sigma'_{\beta(i)} & \xrightarrow[\mu'_{\beta(i)}]{} & \tau'_{\gamma(\alpha(i))} \end{array}$$

commutative. Clearly $(\alpha', \mu') : \sigma' \to \tau'$ is the unique morphism in $D(\mathcal{C})$ making the first diagram above commutative. Thus Ψ is a functor from $D(\mathcal{C})$ to $S(\mathcal{C})$. If τ is in $S(\mathcal{C})$, then in the above diagram, the morphism $(\gamma, \tau) : \tau \to \tau'$ is an isomorphism, and induces hence a bijection $\mathrm{Hom}_{D(\mathcal{C})}(\sigma, \tau) \cong \mathrm{Hom}_{S(\mathcal{C})}(\Psi(\sigma), \tau)$. It follows that Ψ is a left adjoint of the inclusion $S(\mathcal{C}) \hookrightarrow D(\mathcal{C})$ as claimed. □

It is well-known (see e.g. [9, 5.1(ii)]) that whenever an inclusion functor has a left adjoint, then the restriction along this inclusion functor induces an isomorphism on cohomology of contravariant functors into $\mathrm{Mod}(k)$ for some commutative ring k. Thus 2.6 implies that $H^*(D(\mathcal{C}); \mathcal{F}) \cong H^*(S(\mathcal{C}); \mathcal{F}|_{S(\mathcal{C})})$ for any contravariant functor $\mathcal{F} : D(\mathcal{C}) \to \mathrm{Mod}(k)$. See the next Section for more details on functor cohomology.

3 Cohomology of subdivisions

Let k be a commutative ring. For a small category $\mathcal{C}$ denote by $\hat{\mathcal{C}}$ the k-linear abelian category of covariant functors from $\mathcal{C}$ to $\mathrm{Mod}(k)$, with natural transformations as morphisms. Given two covariant functors $\mathcal{F}, \mathcal{G} : \mathcal{C} \to \mathrm{Mod}(k)$ we denote by $\mathrm{Hom}_{\hat{\mathcal{C}}}(\mathcal{F}, \mathcal{G})$ the k-module of natural transformations from $\mathcal{F}$ to $\mathcal{G}$. Given any object X in $\mathcal{C}$ we denote by $k\mathrm{Hom}_{\mathcal{C}}(X, -)$ the obvious functor in $\hat{\mathcal{C}}$ sending an object Y in $\mathcal{C}$ to the free k-module $k\mathrm{Hom}_{\mathcal{C}}(X, Y)$ having the morphism set $\mathrm{Hom}_{\mathcal{C}}(X, Y)$ as k-basis. By Yoneda's lemma we have a canonical isomorphism $\mathrm{Hom}_{\hat{\mathcal{C}}}(k\mathrm{Hom}_{\mathcal{C}}(X, -), \mathcal{F}) \cong \mathcal{F}(X)$ for any object X in $\mathcal{C}$ and any functor $\mathcal{F}$ in $\hat{\mathcal{C}}$, which implies in particular that $k\mathrm{Hom}_{\mathcal{C}}(X, -)$ is a projective object in the category $\hat{\mathcal{C}}$ and hence that $\hat{\mathcal{C}}$ has enough projective objects. Given any k-module A there is a unique constant functor, abusively again denoted by A, in $\hat{\mathcal{C}}$ which maps every object in $\mathcal{C}$ to A and every morphism in $\mathcal{C}$ to Id_A. The map sending A to this constant functor defines a functor $\Gamma : \mathrm{Mod}(k) \to \hat{\mathcal{C}}$. The functor Γ is obviously exact and, less obviously, has a right and left adjoint, namely the limit functor $\varprojlim_{\mathcal{C}} : \hat{\mathcal{C}} \longrightarrow \mathrm{Mod}(k)$ and the colimit functor $\varinjlim_{\mathcal{C}} : \hat{\mathcal{C}} \longrightarrow \mathrm{Mod}(k)$, respectively. In particular, $\varprojlim_{\mathcal{C}}$ preserves injectives and $\varinjlim_{\mathcal{C}}$ preserves projectives. By [6, 3.1] we have an isomorphism of functors $\varprojlim_{\mathcal{C}} \cong \mathrm{Hom}_{\hat{\mathcal{C}}}(k, -)$, where here k is understood as constant covariant functor on $\mathcal{C}$. Thus higher limits are right derived functors of $\mathrm{Hom}_{\hat{\mathcal{C}}}(k, -)$, which motivates the notation

$H^n(\mathcal{C};\mathcal{F}) = \varprojlim_{\mathcal{C}}^n(\mathcal{F})$ for any functor $\mathcal{F}$ in $\hat{\mathcal{C}}$ and any integer $n \geq 0$. More explicitly, $H^n(\mathcal{C};\mathcal{F})$ is the cohomology in degree n of the cochain complex $\mathrm{Hom}_{\hat{\mathcal{C}}}(k,\mathcal{I})$, where $\mathcal{I}$ is an injective resolution of $\mathcal{F}$ in $\hat{\mathcal{C}}$. By general abstract nonsense, this is isomorphic to the cohomology in degree n of the cochain complex $\mathrm{Hom}_{\hat{\mathcal{C}}}(\mathcal{P},\mathcal{F})$, where now $\mathcal{P}$ is a projective resolution of the constant functor k in $\hat{\mathcal{C}}$. If $\mathcal{C}$, $\mathcal{D}$ are two small categories and if $\Phi : \mathcal{D} \to \mathcal{C}$ is a covariant functor, then restriction along Φ induces a functor $\Phi^* : \hat{\mathcal{C}} \to \hat{\mathcal{D}}$ sending a functor $\mathcal{F}$ in $\hat{\mathcal{C}}$ to the functor $\mathcal{F} \circ \Phi$ in $\hat{\mathcal{D}}$. The functor Φ^* is obviously exact, and moreover, Φ^* has a left adjoint Φ_* and a right adjoint $\Phi_!$, also called the left and right Kan extension of Φ^*. It is possible to describe Φ_*, $\Phi_!$ explicitly in general (see e.g. [6, Appendix]), but we will need an explicit description only in some very particular cases of inclusions of certain subcategories. If $\mathcal{D}$ is a subcategory of $\mathcal{C}$ and $\Phi : \mathcal{D} \to \mathcal{C}$ the inclusion functor, we will write sometimes $\mathcal{F}|_{\mathcal{D}}$ instead of $\Phi^*(\mathcal{F})$.

By [13, 3.2], the cohomology of covariant functors on $[S(\mathcal{C})]$ can be computed in the same way as the Bredon cohomology of subgroup complexes (as is done in work of Grodal [8] and Symonds [19], for instance):

Proposition 3.1. *(cf. [13, 3.2]) Let $\mathcal{C}$ be an EI-category, let k be a commutative ring and let $\mathcal{A} : [S(\mathcal{C})] \to \mathrm{Mod}(k)$ be a covariant functor. There is a cochain complex of k-modules $(C(\mathcal{A}),\delta)$ with the following properties:*

(i) For any integer $n \geq 0$ component of $C(\mathcal{A})$ in degree n is equal to

$$C(\mathcal{A})^n = \bigoplus_{[\sigma]} \mathcal{A}([\sigma]) \ ,$$

where the direct sum is taken over the set of isomorphism classes $[\sigma]$ of chains σ of length n in $S(\mathcal{C})$.

(ii) The differential δ^{n-1} in degree $n-1$ of $C(\mathcal{A})$ is given by the k-linear map

$$\delta^{n-1} : C(\mathcal{A})^{n-1} \longrightarrow C(\mathcal{A})^n$$

obtained by taking the alternating sum $\delta^{n-1} = \sum_{([\sigma],i)} (-1)^i \rho_{[\sigma],i}$ over all pairs $([\sigma],i)$ consisting of an isomorphism class $[\sigma]$ of a chain σ of length n and an integer i such that $0 \leq i \leq n$, of the maps $\rho_{[\sigma],i} : \mathcal{A}([\sigma\backslash i]) \to \mathcal{A}([\sigma])$ obtained from applying the covariant functor $\mathcal{A}$ to the canonical morphism $[\sigma\backslash i] \to [\sigma]$ in $[S(\mathcal{C})]$.

(iii) We have $H^*([S(\mathcal{C})];\mathcal{A}) \cong H^*(C(\mathcal{A}))$; in other words, the complex $C(\mathcal{A})$ computes the higher limits of the functor $\mathcal{A}$.

Note that the assignment $\mathcal{A} \mapsto C(\mathcal{A})$ is exact functorial. We will need a refinement of 3.1 which computes the cohomology of covariant functors on certain types of subcategories of $[S(\mathcal{C})]$.

Definition 3.2. Let $\mathcal{C}$ be a category. A *left ideal in* $\mathcal{C}$ is a full subcategory $\mathcal{D}$ of $\mathcal{C}$ with the property that if X is an object in $\mathcal{C}$ and Y an object in $\mathcal{D}$ such that the morphism set $\mathrm{Hom}_{\mathcal{C}}(X,Y)$ is non-empty, then X belongs to the subcategory $\mathcal{D}$ as well. Dually, a *right ideal in* $\mathcal{C}$ is a full subcategory $\mathcal{D}$ of $\mathcal{C}$ with the property that if X is an object in $\mathcal{D}$ and Y an object in $\mathcal{C}$ such that the morphism set $\mathrm{Hom}_{\mathcal{C}}(X,Y)$ is non-empty, then Y belongs to the subcategory $\mathcal{D}$ as well.

If $\mathcal{F}$ is a fusion system on a finite p-group P, where p is a prime, then the full subcategory $\mathcal{F}^c$ of $\mathcal{F}$-centric subgroups of P is a right ideal in $\mathcal{F}$. If $\mathcal{D}$ is a full subcategory of an EI-category $\mathcal{C}$ then $S(\mathcal{D})$ is a left ideal in $S(\mathcal{C})$. See 4.3 for more examples of left ideals. The following collection of more or less trivial statements is essentially a pretext to introduce some notation.

Proposition 3.3. *Let $\mathcal{C}$ be a small category and let $\mathcal{D}$ be a left ideal in $\mathcal{C}$. Denote by $\Phi : \mathcal{D} \to \mathcal{C}$ the inclusion functor. Let k be a commutative ring and denote by $\hat{\mathcal{C}}$ and $\hat{\mathcal{D}}$ the k-linear abelian categories of covariant functors from $\mathcal{C}$ and $\mathcal{D}$ to $Mod(k)$, respectively.*
(i) For any covariant functor $\mathcal{G} : \mathcal{D} \to \mathrm{Mod}(k)$ there is a unique covariant functor $\mathcal{G}^{\mathcal{C}} : \mathcal{C} \to \mathrm{Mod}(k)$ such that $\mathcal{G}^{\mathcal{C}}$ vanishes on all objects outside $\mathcal{D}$ and coincides with $\mathcal{G}$ upon restriction to $\mathcal{D}$.
(ii) For any covariant functor $\mathcal{F} : \mathcal{C} \to \mathrm{Mod}(k)$ there is a unique covariant functor $\mathcal{F}_{\mathcal{D}} : \mathcal{C} \to \mathrm{Mod}(k)$ such that the restrictions to $\mathcal{D}$ of the functors $\mathcal{F}$, $\mathcal{F}_{\mathcal{D}}$ are equal and such that $\mathcal{F}_{\mathcal{D}}$ vanishes outside $\mathcal{D}$.
(iii) For any covariant functor $\mathcal{F} : \mathcal{C} \to \mathrm{Mod}(k)$ there is a unique natural transformation $\mathcal{F} \to \mathcal{F}_{\mathcal{D}}$ given by the identity maps $\mathcal{F}(X) = \mathcal{F}_{\mathcal{D}}(X)$ for all objects X in $\mathcal{D}$ and by the zero maps $\mathcal{F}(X) \to \mathcal{F}_{\mathcal{D}}(X) = 0$ for all objects X in $\mathcal{C}$ outside $\mathcal{D}$.
(iv) The restriction functor $\Phi^* : \hat{\mathcal{C}} \to \hat{\mathcal{D}}$ has an exact right adjoint $\Phi_! : \hat{\mathcal{D}} \to \hat{\mathcal{C}}$ mapping a functor $\mathcal{G}$ in $\hat{\mathcal{D}}$ to the functor $\mathcal{G}^{\mathcal{C}}$ in $\hat{\mathcal{C}}$; in particular, the restriction functor Φ^* preserves projectives.
(v) We have $\Phi^* \circ \Phi_! = \mathrm{Id}_{\hat{\mathcal{D}}}$ and for any covariant functor $\mathcal{F} : \mathcal{C} \to \mathrm{Mod}(k)$ we have $\mathcal{F}_{\mathcal{D}} = \Phi_!(\Phi^*(\mathcal{F}))$. The identity $\Phi^* \circ \Phi_! = \mathrm{Id}_{\hat{\mathcal{D}}}$ is the unit and the family of canonical natural transformations $\mathcal{F} \to \mathcal{F}_{\mathcal{D}}$ is the counit $\mathrm{Id}_{\hat{\mathcal{C}}} \to \Phi_! \circ \Phi^*$ of a right adjunction of $\Phi_!$ to Φ^*

Proof. The uniqueness of $\mathcal{G}^{\mathcal{C}}$ in (i) is trivial. The fact that $\mathcal{G}^{\mathcal{C}}$ is actually well-defined is an immediate consequence of $\mathcal{D}$ being a left ideal. Setting $\mathcal{F}_{\mathcal{D}} = (\mathcal{F}|_{\mathcal{D}})^{\mathcal{C}}$ shows (ii). Statement (iii) is yet another trivial verification, using that $\mathcal{D}$ is a left ideal. The assignment $\mathcal{G} \mapsto \mathcal{G}^{\mathcal{C}}$ defines an exact functor $\Psi : \hat{\mathcal{D}} \to \hat{\mathcal{C}}$, and one verifies that Ψ is right adjoint to Φ^*. It is a general fact that left adjoints of an exact functor preserve projectives. This proves (iv). Statement (v) is just a structural interpretation of the previous statements. □

If $\Phi : \mathcal{D} \to \mathcal{C}$ is a covariant functor between small categories, then the cohomology of functors on $\mathcal{C}$ an $\mathcal{D}$ is related via a Grothendieck spectral sequence (see e.g. [6, Appendix] for a homological version). The following well-known result is a very special case in which this spectral sequence collapses (and this is all we need for the purpose of this paper).

Proposition 3.4. *Let $\mathcal{C}$ be a small category and let $\mathcal{D}$ be a left ideal in $\mathcal{C}$. Let k be a commutative ring. Let k be a commutative ring and denote by $\hat{C}$ and $\hat{D}$ the k-linear abelian categories of covariant functors from $\mathcal{C}$ and $\mathcal{D}$ to $Mod(k)$, respectively.*

For any covariant functor $\mathcal{F} : \mathcal{C} \to \mathrm{Mod}(k)$ which vanishes on all objects outside $\mathcal{D}$ the restriction to $\mathcal{D}$ induces an isomorphism on cohomology $H^*(\mathcal{C};\mathcal{F}) \cong H^*(\mathcal{D};\mathcal{F}|_{\mathcal{D}})$.

Proof. Denote by $\Phi : \mathcal{D} \to \mathcal{C}$ the inclusion functor. Since Φ^* is exact and, by 3.3.(iv), preserves projectives, if $\mathcal{P}$ is a projective resolution of the constant functor k in $\hat{\mathcal{C}}$ then its restriction $\Phi^*(\mathcal{P})$ is a projective resolution of the constant functor k in $\hat{\mathcal{D}}$. Thus the adjunction implies an isomorphism of cochain complexes $\mathrm{Hom}_{\hat{\mathcal{C}}}(\mathcal{P};\Psi(\Phi^*(\mathcal{F}))) \cong \mathrm{Hom}_{\hat{\mathcal{D}}}(\Phi^*(\mathcal{P});\Phi^*(\mathcal{F}))$. Now if $\mathcal{F}$ vanishes outside $\mathcal{D}$ then in fact $\Psi(\Phi^*(\mathcal{F})) = \mathcal{F}$. Thus we get actually an isomorphism of cochain complexes $\mathrm{Hom}_{\hat{\mathcal{C}}}(\mathcal{P};\mathcal{F}) \cong \mathrm{Hom}_{\hat{\mathcal{D}}}(\Phi^*(\mathcal{P});\Phi^*(\mathcal{F}))$. Taking cohomology on both sides yields the result. □

Definition 3.5. Let $\mathcal{C}$ be an EI-category, let k be a commutative ring and let Σ be a left ideal in $[S(\mathcal{C})]$. For any covariant functor $\mathcal{A} : \Sigma \to \mathrm{Mod}(k)$ we define a cochain complex of k-modules $C_\Sigma(\mathcal{A})$ by setting

$$C_\Sigma(\mathcal{A}) = C(\mathcal{A}') \ ,$$

where $\mathcal{A}' = \mathcal{A}^{[S(\mathcal{C})]} : [S(\mathcal{C})] \to \mathrm{Mod}(k)$ is the unique covariant functor whose restriction to Σ is equal to $\mathcal{A}$ and which vanishes outside Σ. More explicitly,

$$C_\Sigma(\mathcal{A})^n = \bigoplus_{\substack{[\sigma]\in\Sigma \\ |\sigma|=n}} \mathcal{A}([\sigma])$$

for any integer $n \geq 0$.

Note that if $\mathcal{D}$ is a left ideal in $S(\mathcal{C})$ then $\Sigma = [\mathcal{D}]$ is a left ideal in $[S(\mathcal{C})]$, and hence 3.3 applies. In order to simplify notation, if $\mathcal{A} : [S(\mathcal{C})] \to \mathrm{Mod}(k)$ is a covariant functor, we write $C_\Sigma(\mathcal{A})$ instead of $C_\Sigma(\mathcal{A}|_\Sigma)$. Equivalently, $C_\Sigma(\mathcal{A}) = C(\mathcal{A}_\Sigma)$, where $\mathcal{A}_\Sigma$ is, as in 3.4, the unique covariant functor which coincides with $\mathcal{A}$ on Σ and vanishes outside Σ. The cochain complex $C_\Sigma(\mathcal{A})$ is obviously a direct summand of $C(\mathcal{A})$ as graded k-module, but not as complex, in general. It is though a quotient complex:

Proposition 3.6. *Let $\mathcal{C}$ be an EI-category, let k be a commutative ring and let Σ be a left ideal in $[S(\mathcal{C})]$. Let $\mathcal{A} : [S(\mathcal{C})] \to \mathrm{Mod}(k)$ be a covariant functor. The canonical projections $C(\mathcal{A})^n \to C_\Sigma(\mathcal{A})^n$, where $n \geq 0$, define an epimorphism of cochain complexes of k-modules*

$$C(\mathcal{A}) \longrightarrow C_\Sigma(\mathcal{A}) \ .$$

Proof. The complex $C(\mathcal{A})$ is functorial in $\mathcal{A}$. The canonical natural transformation $\mathcal{A} \to \mathcal{A}_\Sigma$ from 3.3.(iii) induces the required epimorphism of complexes $C(\mathcal{A}) \to C(\mathcal{A}_\Sigma) = C_\Sigma(\mathcal{A})$. □

The point of all this is that the complex $C_\Sigma(\mathcal{A})$ computes the cohomology of $\mathcal{A}$ for any covariant functor $\mathcal{A} : \Sigma \to \mathrm{Mod}(k)$, generalising 3.1 to left ideals in $[S(\mathcal{C})]$.

Proposition 3.7. *Let $\mathcal{C}$ be an EI-category, let k be a commutative ring and let Σ be a left ideal in $[S(\mathcal{C})]$. Let $\mathcal{A} : \Sigma \to \mathrm{Mod}(k)$ be a covariant functor. We have*

$$H^*(\Sigma; \mathcal{A}) \cong H^*(C_\Sigma(\mathcal{A})) \ .$$

Proof. Let $\mathcal{A}' : [S(\mathcal{C})] \to \mathrm{Mod}(k)$ be the unique covariant functor which coincides with $\mathcal{A}$ on Σ and which vanishes outside Σ. Then, by 3.4 and 3.1, we have $H^*(\Sigma; \mathcal{A}) \cong H^*([S(\mathcal{C})]; \mathcal{A}') \cong H^*(C(\mathcal{A}')) = H^*(C_\Sigma(\mathcal{A}))$. □

4 Reduction to normal chains

Throughout this Section we fix a prime p, a finite p-group P and a fusion system $\mathcal{F}$ on P. The purpose of this Section is to develop some techniques which reduce the computation of suitable covariant functors defined on $S(\mathcal{F})$ to certain subcategories.

Definition 4.1. Let $\mathcal{C}$ be a full subcategory of the fusion system $\mathcal{F}$ on P. We denote by $S_<(\mathcal{C})$ the full subcategory of $S(\mathcal{C})$ consisting of all chains of the form $\sigma = (Q_0 < Q_1 < \cdots < Q_m)$, where m is a non-negative integer and Q_i a subgroup of P belonging to $\mathcal{C}$, for $0 \leq i \leq m$.

We denote by $S_\lhd(\mathcal{C})$ the full subcategory of $S_<(\mathcal{C})$ consisting of all chains $\sigma = (Q_0 < Q_1 < \cdots < Q_m)$ in $S_<(\mathcal{C})$ with the property that Q_i is normal in Q_m for $0 \leq i \leq m$. We denote by $S_\Phi(\mathcal{C})$ the full subcategory of $S_<(\mathcal{C})$ consisting of all chains $\sigma = (Q_0 < Q_1 < \cdots < Q_m)$ in $S_<(\mathcal{C})$ with the property that the Frattini subgroup $\Phi(Q_m)$ of Q_m is contained in Q_0.

As before, if $\sigma = (Q_0 < Q_1 < \cdots < Q_m)$ is a chain of subgroups of P the integer $|\sigma| = m$ is called the *length of* σ; if in addition m is positive, we write $\sigma \backslash i = (Q_0 < \cdots < Q_{i-1} < Q_{i+1} < \cdots < Q_m)$, for any integer i such that $0 \leq i \leq m$. Note that since

$Q_m/\Phi(Q_m)$ is abelian, every chain in $S_\Phi(\mathcal{C})$ belongs in fact to $S_\lhd(\mathcal{C})$. Thus we have inclusions of full subcategories

$$S_\Phi(\mathcal{C}) \subseteq S_\lhd(\mathcal{C}) \subseteq S_<(\mathcal{C}) \subseteq S(\mathcal{C}) \ .$$

All four of the above categories are again EI-categories. We state two obvious results for future reference:

Proposition 4.2. *Let $\mathcal{C}$ be a full subcategory of the fusion system $\mathcal{F}$ on P. Suppose that $\mathcal{C}$ is closed under isomorphisms in $\mathcal{F}$. Then the inclusion $S_<(\mathcal{C}) \subseteq S(\mathcal{C})$ is an equivalence of categories. In particular, $[S_<(\mathcal{C})] \cong [S(\mathcal{C})]$ as posets.*

Proof. A chain of non-isomorphisms $Q_0 \xrightarrow{\varphi_0} Q_1 \xrightarrow{\varphi_1} \cdots \xrightarrow{\varphi_{m-1}} Q_m$ belonging to the category $S(\mathcal{C})$ is isomorphic to the chain $R_0 < R_1 < \cdots < R_m$ in $S_<(\mathcal{C})$ defined by $R_m = Q_m$ and $R_i = \varphi_{m-1} \circ \cdots \circ \varphi_{i+1} \circ \varphi_i(Q_i)$ for $0 \leq i < m$. The result follows. □

Proposition 4.3. *Let $\mathcal{C}$ be a full subcategory of the fusion system $\mathcal{F}$ on P. Suppose that $\mathcal{C}$ is closed under isomorphisms in $\mathcal{F}$. Then the categories $S_\Phi(\mathcal{C})$ and $S_\lhd(\mathcal{C})$ are left ideals in $S_<(\mathcal{C})$.*

Proof. Every subchain of a chain in $S_\lhd(\mathcal{C})$ belongs to $S_\lhd(\mathcal{C})$, and similarly, every subchain of a chain in $S_\Phi(\mathcal{C})$ belongs to $S_\Phi(\mathcal{C})$. The result follows. □

The following definition is essentially the pairing considered by Knörr and Robinson in the first part of the proof of [11, 3.3]:

Definition 4.4. Let $\mathcal{C}$ be a right ideal in the fusion system $\mathcal{F}$ on P. For any chain $\sigma = (Q_0 < Q_1 < \cdots < Q_m)$ in $S_<(\mathcal{C})$ we define a chain $z(\sigma)$ in $S_<(\mathcal{C})$ as follows:
(i) if $\Phi(Q_m) \subseteq Q_0$ we set $z(\sigma) = \sigma$;
(ii) if $\Phi(Q_m) \subseteq Q_i$ but $\Phi(Q_m) \nsubseteq Q_{i-1}$ for some positive integer i and if $\Phi(Q_m)Q_{i-1} = Q_i$ we set $z(\sigma) = \sigma\backslash i = Q_0 < \cdots < Q_{i-1} < Q_{i+1} < \cdots < Q_m$;
(iii) if $\Phi(Q_m) \subseteq Q_i$ but $\Phi(Q_m) \nsubseteq Q_{i-1}$ for some positive integer i and if $\Phi(Q_m)Q_{i-1} < Q_i$ we set $z(\sigma) = Q_0 < \cdots < Q_{i-1} < \Phi(Q_m)Q_{i-1} < Q_i < \cdots < Q_m$.

Note that in the alternative 4.4(ii) we necessarily have that $i < m$, because the equality $\Phi(Q_m)Q_{m-1} = Q_m$ is impossible by standard properties of Frattini subgroups. Thus z leaves the maximal subgroup occurring in σ unchanged. The only difference between 4.4 and the pairing defined in [11] is the requirement 4.4(i) which garantees that z leaves also the minimal subgroup occurring in σ unchanged; in particular, $z(\sigma)$ is again a chain belonging to $S_<(\mathcal{C})$. Clearly either $z(\sigma) = \sigma$ or $|z(\sigma)| = |\sigma| + 1$ or $|z(\sigma)| = |\sigma| - 1$. If $|z(\sigma)| = |\sigma| + 1$ then $z(\sigma)$ is obtained from inserting a subgroup into σ, and hence $[\sigma] < [z(\sigma)]$. Similarly, if $|z(\sigma)| = |\sigma| - 1$ then $[z(\sigma)] < [\sigma]$. The following Proposition (whose easy proof is left to the reader) collects some obvious properties of the map z, essentially stating that z defines a pairing on isomorphism classes of chains fixing those belonging to $S_\Phi(\mathcal{C})$.

Proposition 4.5. *Let $\mathcal{C}$ be a right ideal in the fusion system $\mathcal{F}$ on P. Let σ, τ be chains in $S_<(\mathcal{C})$.*
(i) We have $z(z(\sigma)) = \sigma$.
(ii) We have $z(\sigma) = \sigma$ if and only if σ belongs to $S_\Phi(\mathcal{C})$.
(iii) We have $\sigma \cong \tau$ if and only if $z(\sigma) \cong z(\tau)$.
(iv) If $[\sigma] < [z(\sigma)]$, any morphism $\sigma \to z(\sigma)$ in $S(\mathcal{C})$ induces a group isomorphism $\mathrm{Aut}_{S(\mathcal{C})}(\sigma) \cong \mathrm{Aut}_{S(\mathcal{C})}(z(\sigma))$.
(v) We have $\sigma \in S_\triangleleft(\mathcal{C})$ if and only if $z(\sigma) \in S_\triangleleft(\mathcal{C})$.

As in the previous Section , we denote by $C(\mathcal{A})$ the cochain complex of k-modules which in degree $n \geq 0$ is equal to

$$C(\mathcal{A})^n = \bigoplus_{\substack{[\sigma]\in[S(\mathcal{C})]\\ |\sigma|=n}} \mathcal{A}([\sigma])$$

with differential given by the maps $(-1)^i\mathcal{A}([\sigma] < [\tau]) : \mathcal{A}([\sigma]) \to \mathcal{A}([\tau])$ for any two chains σ, τ in $S(\mathcal{C})$ of length n, $n+1$, respectively, for which there is an integer i such that $\sigma \cong \tau\backslash i$. By 3.7 the cohomology of $C(\mathcal{A})$ is the cohomology of the functor $\mathcal{A}$. Note that in this context the complex $C(\mathcal{A})$ is bounded with non-zero components at most in the degrees $0, 1, \cdots, a$, where a is the unique integer such that $p^a = |P|$.

Proposition 4.6. *Let $\mathcal{C}$ be a right ideal in the fusion system $\mathcal{F}$ on P. Let k be a commutative ring and let $\mathcal{A} : [S(\mathcal{C})] \longrightarrow \mathrm{Mod}(k)$ be a covariant functor.*
There are unique cochain complexes of k-modules $C_\triangleleft(\mathcal{A})$ and $C_\Phi(\mathcal{A})$ such that

$$C_\triangleleft(\mathcal{A})^n = \bigoplus_{\substack{[\sigma]\in[S_\triangleleft(\mathcal{C})]\\ |\sigma|=n}} \mathcal{A}([\sigma]) \ ,$$

$$C_\Phi(\mathcal{A})^n = \bigoplus_{\substack{[\sigma]\in[S_\Phi(\mathcal{C})]\\ |\sigma|=n}} \mathcal{A}([\sigma]) \ ,$$

for any integer $n \geq 0$, and such that the canonical projections $C(\mathcal{A}) \to C_\triangleleft(\mathcal{A}) \to C_\Phi(\mathcal{A})$ define epimorphisms of cochain complexes

$$C(\mathcal{A}) \longrightarrow C_\triangleleft(\mathcal{A}) \longrightarrow C_\Phi(\mathcal{A}) \ .$$

Proof. This is Proposition 3.6 applied to the left ideals $[S_\triangleleft(\mathcal{C})]$ and $[S_\Phi(\mathcal{C})]$ in $[S_<(\mathcal{C})]$, combined with the isomorphism of posets $[S_<(\mathcal{C})] \cong [S(\mathcal{C})]$ from 4.2. □

The main result of this Section is the following Theorem which reduces the computation of the cohomology of certain covariant functors $\mathcal{A} : [S(\mathcal{C})] \longrightarrow \mathrm{Mod}(k)$ to the posets $[S_\triangleleft(\mathcal{C})]$ and $[S_\Phi(\mathcal{C})]$.

Theorem 4.7. *Let $\mathcal{C}$ be a right ideal in the fusion system $\mathcal{F}$ on P. Let k be a commutative ring and let $\mathcal{A} : [S(\mathcal{C})] \longrightarrow \mathrm{Mod}(k)$ be a covariant functor. If $\mathcal{A}$ has the property that for any σ in $S(\mathcal{C})$ such that $|\sigma| < |z(\sigma)|$ the unique morphism $[\sigma] < [z(\sigma)]$ in $[S(\mathcal{C})]$ induces an isomorphism $\mathcal{A}([\sigma]) \cong \mathcal{A}([z(\sigma)])$, then the canonical epimorphisms of cochain complexes of k-modules $C(\mathcal{A}) \to C_{\triangleleft}(\mathcal{A}) \to C_{\Phi}(\mathcal{A})$ are homotopy equivalences. In particular, we have isomorphisms on cohomology*

$$H^*([S(\mathcal{C})]; \mathcal{A}) \cong H^*([S_{\triangleleft}(\mathcal{C})]; \mathcal{A}) \cong H^*([S_{\Phi}(\mathcal{C})]; \mathcal{A}) .$$

The hypothesis in 4.7 on the functor $\mathcal{A}$ holds obviously whenever $\mathcal{A}$ is a constant functor. But there are other functors fulfilling this hypothesis: by 4.5.(iv) we have $\mathrm{Aut}_{S(\mathcal{C})}(\sigma) \cong \mathrm{Aut}_{S(\mathcal{C})}(z(\sigma))$, and hence any functor $\mathcal{A}$ whose value at $[\sigma]$ depends only on $\mathrm{Aut}_{S(\mathcal{C})}(\sigma)$ will fulfill this hypothesis. We will describe some functors with this property at the end of this Section. In order to prove 4.7 we filter the kernel of the canonical epimorphism $C(\mathcal{A}) \to C_{\Phi}(\mathcal{A})$ by subcomplexes $C_{(q)}(\mathcal{A})$ and show then that subsequent quotients of this filtration are contractible; see Appendix B, Corollary B.2. It is in this latter part that we will need the following combinatorial statement:

Lemma 4.8. *Let $\sigma = (Q_0 < Q_1 < \cdots < Q_m)$ and $\tau = (R_0 < R_1 < \cdots < R_m)$ be chains in $S_<(\mathcal{F})$ such that $|z(\sigma)| < |\sigma|$ and $|z(\tau)| < |\tau|$. Let i, j be the unique positive integers such that $z(\sigma) = \sigma\backslash i$ and $z(\tau) = \tau\backslash j$. Suppose there is a non-negative integer k such that $z(\sigma) \cong \tau\backslash k$. Then exactly one of the following statements holds:*

(i) $k = m$;

(ii) $k < i < j$;

(iii) $\sigma \cong \tau$.

Proof. Suppose that $k < m$. By the assumptions, we have an isomorphism of chains

$$Q_0 < \cdots < Q_{i-1} < Q_{i+1} < \cdots < Q_m \;\cong\; R_0 < \cdots < R_{k-1} < R_{k+1} < \cdots < R_m .$$

Since $k < m$ we have in particular $Q_m \cong R_m$. Assume first that $k > i$. Then the above isomorphism of chains implies isomorphisms $Q_{i-1} \cong R_{i-1}$ and $Q_{i+1} \cong R_i$. Since $\Phi(Q_m) \nsubseteq Q_{i-1}$ we also get $\Phi(R_m) \nsubseteq R_{i-1}$, and since $\Phi(Q_m)Q_{i-1} = Q_i < Q_{i+1}$ we get $\Phi(R_m)R_{i-1} < R_i$. This, however, would imply that $|z(\tau)| > |\tau|$, contradicting the assumptions. Thus the case $k > i$ cannot occur. Assume next that $k < i$. Then $Q_{i-1} \cong R_i$. Again, since $\Phi(Q_m) \nsubseteq Q_{i-1}$ we get $\Phi(R_m) \nsubseteq R_i$, hence $i < j$, which is alternative (ii). Assume finally that $k = i$. Then $\Phi(R_m) \nsubseteq R_{i-1}$ but $\Phi(R_m) \subseteq R_{i+1}$ and $\Phi(R_m)R_{i-1} < R_{i+1}$. Thus $k = j$, hence $\tau\backslash k = z(\tau)$. But $z(\tau) \cong z(\sigma)$ implies $\tau \cong \sigma$ by 4.5 (iii), hence alternative (iii) holds. □

Proof of Theorem 4.7. Let q be a non-negative integer. We define a subcomplex $C_{(q)}(\mathcal{A})$ of $C(\mathcal{A})$ as follows. For $n < q$ we set $C_{(q)}(\mathcal{A})^n = \{0\}$. For $n > q$ we set $C_{(q)}(\mathcal{A})^n =$

$\underset{[\sigma]}{\oplus}\ \mathcal{A}([\sigma])$ with $[\sigma]$ running over the set of isomorphism classes of chains σ in $S(\mathcal{C})$ satisfying $|\sigma| = n$ and $z(\sigma) \neq \sigma$. In other words, $C_{(q)}(\mathcal{A})^n$ is the canonical complement of $C_\Phi(\mathcal{A})^n$ in $C(\mathcal{A})^n$. In degree q we set $C_{(q)}(\mathcal{A})^q = \underset{[\sigma]}{\oplus}\ \mathcal{A}([\sigma])$ with $[\sigma]$ running over the set of isomorphism classes of chains σ in $S(\mathcal{C})$ satisfying $|\sigma| = q$ and $|z(\sigma)| = q+1$. One easily checks that the differential of $C(\mathcal{A})$ restricts to a differential on $C_{(q)}(\mathcal{A})$. Using 4.5.(ii), we have an obvious isomorphism of complexes

$$C(\mathcal{A})/C_{(0)}(\mathcal{A}) \cong C_\Phi(\mathcal{A})$$

and, for any positive integer q, we have inlusions of subcomplexes

$$C_{(q)}(\mathcal{A}) \subseteq C_{(q-1)}(\mathcal{A}) \ ;$$

by construction, all these inclusions are degreewise split. Thus, in order to show that the canonical epimorphism $C(\mathcal{A}) \to C_\Phi(\mathcal{A})$ is a homotopy equivalence, it suffices by Corollary B.2, to show that for any positive integer q the quotient complex $C_{(q-1)}(\mathcal{A})/C_{(q)}(\mathcal{A})$ is contractible. The complex $C_{(q-1)}(\mathcal{A})/C_{(q)}(\mathcal{A})$ has at most two non zero components, namely those in the degrees $q-1$ and q. More precisely, the complex $C_{(q-1)}(\mathcal{A})/C_{(q)}(\mathcal{A})$ is of the form

$$\cdots \longrightarrow 0 \longrightarrow \underset{[\rho]\in\mathcal{M}}{\oplus}\ \mathcal{A}([\tau]) \xrightarrow{\epsilon} \underset{[\sigma]\in\mathcal{N}}{\oplus}\ \mathcal{A}([\sigma]) \longrightarrow 0 \longrightarrow \cdots \ ,$$

where $\mathcal{M}$ is the set of isomorphism classes $[\rho]$ of chains ρ in $S(\mathcal{C})$ satisfying $|\rho| = q-1$ and $|z(\rho)| = q$, and where $\mathcal{N}$ is the set of isomorphism classes $[\sigma]$ of chains σ in $S(\mathcal{C})$ satisfying $|\sigma| = q$ and $|z(\sigma)| = q-1$. It follows immediately from 4.5 that the map z induces in fact bijections between the indexing sets $\mathcal{M}$ and $\mathcal{N}$. In other words, the complex $C_{(q-1)}(\mathcal{A})/C_{(q)}(\mathcal{A})$ is of the form

$$\cdots \longrightarrow 0 \longrightarrow \underset{[\sigma]\in\mathcal{N}}{\oplus}\ \mathcal{A}([z(\sigma)]) \xrightarrow{\epsilon} \underset{[\sigma]\in\mathcal{N}}{\oplus}\ \mathcal{A}([\sigma]) \longrightarrow 0 \longrightarrow \cdots \ .$$

This complex is contractible if and only if its differential ϵ is an isomorphism. We can view ϵ as a square matrix of its components $\epsilon_{[z(\sigma)],[\tau]} : \mathcal{A}([z(\sigma)]) \to \mathcal{A}([\tau])$, where $[\sigma],[\tau] \in \mathcal{N}$. The diagonal entries $\epsilon_{[z(\sigma)],[\sigma]} : \mathcal{A}([z(\sigma)] \to \mathcal{A}([\sigma])$ of this matrix are isomorphisms, by the assumptions on the functor $\mathcal{A}$. In order to show that ϵ is an isomorphism all we have to observe is that we can order the set $\mathcal{N}$ in such a way that the matrix representing ϵ is an upper triangular matrix. This is where the combinatorial Lemma 4.8 will be used. We associate with every chain σ such that $[\sigma] \in \mathcal{N}$ a pair of positive integers (m_σ, i_σ) defined as follows: if $\sigma = (Q_0 < Q_1 < \cdots < Q_m)$ we set $m_\sigma = |Q_m|$, and we denote by i_σ the unique positive integer satisfying $z(\sigma) = \sigma \backslash i_\sigma$. This makes sense as $|z(\sigma)| < |\sigma|$ for $[\sigma] \in \mathcal{N}$. Of course, the pair (m_σ, i_σ) depends only on the isomorphism class $[\sigma]$. We consider now the set of pairs of positive integers as totally ordered set with the lexicographic order; that is, for any two pairs (m,i), (n,j)

of positive integers m, n, i, j, we have $(m,i) < (n,j)$ if $m < n$ or if $m = n$ and $i < j$. In this way the assignment $[\sigma] \mapsto (m_\sigma, i_\sigma)$ is a map from $\mathcal{N}$ to the totally ordered set of pairs of positive integers. It is well-known (and easy to see) that there is a total order $\preccurlyeq$ on the set $\mathcal{N}$ such that the map $[\sigma] \mapsto (m_\sigma, i_\sigma)$ is monotone; that is such that in particular $[\sigma'] \prec [\sigma]$ if $m_{\sigma'} < m_\sigma$ or if $m_{\sigma'} = m_\sigma$ and $i_{\sigma'} < i_\sigma$. We will show that the total order $\preccurlyeq$ on $\mathcal{N}$ has the property that ϵ becomes an upper triangular matrix. Indeed, let $[\sigma]$, $[\tau]$ be different elements in $\mathcal{N}$. As pointed out before, the diagonal entry $\epsilon_{[z(\sigma)],[\sigma]}$ is an isomorphism. Suppose that the entry $\epsilon_{[z(\sigma)],[\tau]}$ is non zero. Then necessarily $z(\sigma) \cong \tau\backslash j$ for some integer $j \geq 0$. If $j = q$ then $m_\sigma < m_\tau$, hence $[\sigma] \prec [\tau]$. If $j < q$ then $j < i_\sigma < i_\tau$ by 4.8, hence again $[\sigma] \prec [\tau]$. This proves that the entry $\epsilon_{[z(\sigma)],[\tau]}$ lies above the diagonal of the matrix representing ϵ. Thus ϵ is an isomorphism, and by the above observations, this implies that the canonical epimorphism $C(\mathcal{A}) \to C_\Phi(\mathcal{A})$ is a homotopy equivalence. Since z preserves $S_\triangleleft(\mathcal{C})$ by 4.5.(v), the same arguments show that the canonical epimorphism $C_\triangleleft(\mathcal{A}) \to C_\Phi(\mathcal{A})$ is a homotopy equivalence. But then the canonical epimorphism $C(\mathcal{A}) \to C_\triangleleft(\mathcal{A})$ must be a homotopy equivalence as well. □

We conclude this Section by describing a certain class of functors fulfilling the hypotheses of 4.7. The following observation is from [13, 1.2, 1.3, 2.3] and combines 2.2 and 2.3 above:

Proposition 4.9. *Let $\mathcal{C}$ be an EI-category. There is a canonical contravariant functor from $S(\mathcal{C})$ to the category of groups sending a chain $\sigma \in S(\mathcal{C})$ to its automorphism group $\mathrm{Aut}_{S(\mathcal{C})}(\sigma)$.*

Proof. This follows from 2.2 and 2.3. □

Proposition 4.10. *Let $\mathcal{C}$ be an EI-category, let k be a commutative ring and let A be a k-module. Then, for any integer $q \geq 0$ there is a canonical covariant functor*

$$A_q : [S(\mathcal{C})] \longrightarrow \mathrm{Mod}(k)$$

such that $A_q([\sigma]) = H^q(\mathrm{Aut}_{S(\mathcal{C})}(\sigma); A)$ and such that A_q maps a morphism $[\sigma] < [\tau]$ in $[S(\mathcal{C})]$ to the map $H^q(\mathrm{Aut}_{S(\mathcal{C})}(\sigma); A) \to H^q(\mathrm{Aut}_{S(\mathcal{C})}(\tau); A)$ obtained from restriction along any group homomorphism $\mathrm{Aut}_{S(\mathcal{C})}(\tau) \to \mathrm{Aut}_{S(\mathcal{C})}(\sigma)$ given by a morphism $\sigma \to \tau$ in $S(\mathcal{C})$, for any two chains σ, τ in $S(\mathcal{C})$.

Proof. All we have to check is that the morphism $A_q([\sigma] < [\tau])$ does not depend on the choice of the morphism $\sigma \to \tau$ in $S(\mathcal{C})$. If $(\alpha, \mu), (\alpha', \mu') : \sigma \to \tau$ are two morphisms in $S(\mathcal{C})$ then, by 2.3, we have $\alpha = \alpha'$, and there is an automorphism (Id, ρ) of σ such that $(\alpha', \mu') = (\alpha, \mu) \circ (\mathrm{Id}, \rho)$. Since inner automorphisms of $\mathrm{Aut}_{S(\mathcal{C})}(\sigma)$ act trivially on $H^q(\mathrm{Aut}_{S(\mathcal{C})}(\sigma); A)$, the result follows. □

Note that with the notation of 4.10, the functor A_0 is the constant covariant functor on $[S(\mathcal{C})]$ taking the value A. By combining our previous results, we can reduce the calculation of the cohomology of functors of the form A_q to normal chains.

Theorem 4.11. *Let $\mathcal{C}$ be a right ideal in the fusion system $\mathcal{F}$. Let k be a commutative ring and let A be a k-module. For any integer $q \geq 0$ the covariant functor $A_q : [S(\mathcal{C})] \to \mathrm{Mod}(k)$ has the property that for any chain σ in $S(\mathcal{C})$ satisfying $[\sigma] < [z(\sigma)]$, the induced k-linear map $A_q([\sigma]) \to A_q([z(\sigma)])$ is an isomorphism. In particular, we have isomorphisms*

$$H^n([S(\mathcal{C})]; A_q) \cong H^n(C_{\triangleleft}(A_q))$$

for any integer $n \geq 0$.

Proof. If $[\sigma] < [z(\sigma)]$, any morphism $\sigma \to z(\sigma)$ in $S(\mathcal{C})$ induces an automorphism $\mathrm{Aut}_{S(\mathcal{C})}(z(\sigma)) \cong \mathrm{Aut}_{S(\mathcal{C})}(\sigma)$ by 4.5.(iv) and hence an isomorphism $A_q([\sigma]) \cong A_q([z(\sigma)])$. The last statement follows from 4.7. □

5 Normal chains in fusion systems

Throughout this section we fix a prime p, a finite p-group P and a fusion system $\mathcal{F}$ on P. Given a chain $\sigma = (Q_0 < Q_1 < \cdots < Q_m)$ of subgroups Q_i of P, we denote as before by $\sigma\backslash i = (Q_0 < \cdots < Q_{i-1} < Q_{i+1} < \cdots < Q_m)$ the subchain of σ obtained from deleting Q_i in σ, and we denote in addition by $\sigma_{\leq i} = (Q_0 < \cdots < Q_i)$ the subchain of σ obtained from truncating σ at Q_i, where i is any integer such that $0 \leq i \leq m$. Note that if a chain $\sigma = (Q_0 < \cdots < Q_m)$ of subgroups Q_i of P belongs to $S_{\triangleleft}(\mathcal{F})$, so do the chains $\sigma\backslash i$ and $\sigma_{\leq i}$ for all i such that $0 \leq i \leq m$; this is just a way to rephrase 4.3.

A subgroup Q of P is fully $\mathcal{F}$-normalised if $|N_P(Q)| \geq |N_P(Q')|$ for any subgroup Q' of P which is isomorphic, in the category $\mathcal{F}$, to Q. By [13, 1.6] this is equivalent to requiring that $\mathrm{Aut}_P(Q)$ is a Sylow-p-subgroup of $\mathrm{Aut}_{\mathcal{F}}(Q)$ and that Q is fully $\mathcal{F}$-centralised (that is, $|C_P(Q)| \geq |C_P(Q')|$ for any subgroup Q' of P isomorphic to Q in $\mathcal{F}$). Moreover, if Q is fully $\mathcal{F}$-normalised then by a result of Puig we have a fusion system $N_{\mathcal{F}}(Q)$ on $N_P(Q)$ (see the appendix A below, or [3, Appendix] for more details and proofs). We use this to define inductively for normal chains of subgroups of P the notion of fully $\mathcal{F}$-normalised chains.

Definition 5.1. A chain $\sigma = (Q_0 < Q_1 < \cdots < Q_m)$ in $S_{\triangleleft}(\mathcal{F})$ is called *fully $\mathcal{F}$-normalised* if Q_0 is fully $\mathcal{F}$-normalised and if either $m = 0$ or the chain $\sigma\backslash 0 = (Q_1 < \cdots < Q_m)$ is fully $N_{\mathcal{F}}(Q_0)$-normalised.

This makes sense as Q_0 is normal in all Q_i, $0 \leq i \leq m$.

Definition 5.2 Let $\sigma = (Q_0 < Q_1 < \cdots < Q_m)$ be a normal chain in $S_{\triangleleft}(\mathcal{F})$. We set $N_P(\sigma) = \bigcap_{0 \leq i \leq m} N_P(Q_i)$ and $C_P(\sigma) = C_P(Q_m)$. We denote by $\mathrm{Aut}_P(\sigma)$ the canonical image of $N_P(\sigma)$ in $\mathrm{Aut}_{S_{\triangleleft}(\mathcal{F})}(\sigma)$; that is, $\mathrm{Aut}_{S_{\triangleleft}(\mathcal{F})}(\sigma) \cong N_P(\sigma)/C_P(\sigma)$. We denote by $N_{\mathcal{F}}(\sigma)$ the category on $N_P(\sigma)$ whose morphism sets consist, for any two subgroups R, S of $N_P(\sigma)$ of all morphisms $\varphi : R \to S$ in $\mathcal{F}$ for which there exists a morphism $\psi : Q_m R \to Q_m S$ in $\mathcal{F}$ such that $\psi(Q_i) = Q_i$ for $0 \leq i \leq m$ and $\psi|_R = \varphi$.

Proposition 5.3. *Let $\sigma = (Q_0 < Q_1 < \cdots < Q_m)$ be a chain in $S_\lhd(\mathcal{F})$. If σ is fully $\mathcal{F}$-normalised then $N_\mathcal{F}(\sigma)$ is a fusion system on $N_P(\sigma)$ and $\mathrm{Aut}_P(\sigma)$ is a Sylow-p-subgroup of $\mathrm{Aut}_{S_\lhd(\mathcal{F})}(\sigma)$.*

Proof. We show first that $N_\mathcal{F}(\sigma)$ is a fusion system on $N_P(\sigma)$. If $m = 0$ this is a result of Puig (mentioned in the appendix A, or, with proofs, in [3, A.6]). If $m > 0$ then $N_\mathcal{F}(\sigma) = N_{N_\mathcal{F}(Q_0)}(\sigma\backslash 0)$ is a fusion system by induction. We show in a similar way that $\mathrm{Aut}_P(\sigma)$ is a Sylow-p-subgroup of $\mathrm{Aut}_{S_\lhd(\mathcal{F})}(\sigma)$. For $m = 0$ this is clear either by the Sylow axiom (I-BLO) as formulated in [3, 1.2] (see also the appendix A below), or by the consequence [13, 1.6] of Stancu's version in [18]. For m positive, we observe first that the inclusion of the chain $\sigma\backslash 0$ into σ induces a group isomorphism $\mathrm{Aut}_{S_\lhd(\mathcal{F})}(\sigma) \cong \mathrm{Aut}_{S_\lhd(N_\mathcal{F}(Q_0))}(\sigma\backslash 0)$. This group isomorphism restricts to an isomorphism $\mathrm{Aut}_P(\sigma) \cong \mathrm{Aut}_{N_P(Q_0)}(\sigma\backslash 0)$. By induction, $\mathrm{Aut}_{N_P(Q_0)}(\sigma\backslash 0)$ is a Sylow-p-subgroup of $\mathrm{Aut}_{S_\lhd(N_\mathcal{F}(Q_0))}(\sigma\backslash 0)$. The previous isomorphisms imply the result. □

Proposition 5.4. *Let $\sigma = (Q_0 < Q_1 < \cdots < Q_m)$ be a chain in $S_\lhd(\mathcal{F})$. Let i be an integer such that $0 \le i < m$. The following are equivalent:*

(i) The chain σ is fully $\mathcal{F}$-normalised.

(ii) The chain $\sigma_{\le i} = (Q_0 < \cdots < Q_i)$ is fully $\mathcal{F}$-normalised and the chain $(Q_{i+1} < \cdots < Q_m)$ is fully $N_\mathcal{F}(\sigma_{\le i})$-normalised.

Proof. We proceed by induction over i. For $i = 0$ this is part of the definition. Suppose $i > 0$. Assume first that (i) holds. Then the chain $\sigma\backslash 0 = (Q_1 < \cdots Q_m)$ is fully $N_\mathcal{F}(Q_0)$-normalised, and hence, by induction, the chain $(\sigma\backslash 0)_{\le i-1} = (Q_1 < \cdots < Q_i)$ is fully $N_\mathcal{F}(Q_0)$-normalised and the chain $(Q_{i+1} < \cdots < Q_m)$ is fully $N_{N_\mathcal{F}(Q_0)}((\sigma\backslash 0)_{\le i-1})$ -normalised. Since Q_0 is fully $\mathcal{F}$-normalised, it follows that $\sigma_{\le i}$ is fully $\mathcal{F}$-normalised. Since $N_{N_\mathcal{F}(Q_0)}((\sigma\backslash 0)_{\le i-1}) = N_\mathcal{F}(\sigma_{\le i})$ we also get that $(Q_{i+1} < \cdots < Q_m)$ is fully $N_\mathcal{F}(\sigma_{\le i})$-normalised. Thus (i) implies (ii). Assume now that (ii) holds. Then in particular Q_0 is fully $\mathcal{F}$-normalised and $(\sigma_{\le i})\backslash 0 = (\sigma\backslash 0)_{\le i-1} = (Q_1 < \cdots < Q_i)$ is fully $N_\mathcal{F}(Q_0)$-normalised. Since also $(Q_{i+1} < \cdots < Q_m)$ is fully $N_\mathcal{F}(\sigma_{\le i})$-normalised and $N_\mathcal{F}(\sigma_{\le i}) = N_{N_\mathcal{F}(Q_0)}(Q_1 < \cdots < Q_i)$ it follows by induction that $\sigma\backslash 0$ is fully $N_\mathcal{F}(Q_0)$-normalised. Thus σ is fully $\mathcal{F}$-normalised. □

We define now a pairing on fully normalised normal chains:

Definition 5.5. Let $\sigma = (Q_0 < Q_1 < \cdots < Q_m)$ be a fully $\mathcal{F}$-normalised chain in $S_\lhd(\mathcal{F})$. We define a chain $n(\sigma)$ in $S_\lhd(\mathcal{F})$ as follows:

(a) if $\sigma = Q_0 = P$ we set $n(\sigma) = \sigma$;

(b) if $Q_0 < P$ and $Q_m = N_P(\sigma)$ we set $n(\sigma) = \sigma\backslash m = \sigma_{\le m-1} = (Q_0 < \cdots < Q_{m-1})$;

(c) if $Q_0 < P$ and $Q_m < N_P(\sigma)$ we set $n(\sigma) = (Q_0 < \cdots < Q_m < N_P(\sigma))$.

Proposition 5.6. *For any fully $\mathcal{F}$-normalised chain σ in $S_{\lhd}(\mathcal{F})$ the chain $n(\sigma)$ is again fully $\mathcal{F}$-normalised.*

Proof. We proceed by induction over the length of σ. If $\sigma = Q_0 = P$ then $n(\sigma) = \sigma$, so there is nothing to prove. If $\sigma = Q_0$ and $Q_0 < P$ then $n(\sigma) = Q_0 < N_P(Q_0)$, and since Q_0 is fully $\mathcal{F}$-normalised, $N_P(Q_0)$ is fully $N_{\mathcal{F}}(Q_0)$-normalised as it is the unique maximal subgroup on which $N_{\mathcal{F}}(Q_0)$ is defined. Suppose that $|\sigma| \geq 1$. If $Q_0 < P$ and $Q_m = N_P(\sigma)$ then $n(\sigma) = (Q_0 < \cdots < Q_{m-1})$ is fully $\mathcal{F}$-normalised by 5.4. Assume that $Q_0 < P$ and $Q_m < N_P(\sigma)$. We have $N_P(\sigma) = N_{N_P(Q_0)}(\sigma\backslash 0) > Q_m$, hence, by induction, the chain $n(\sigma)\backslash 0 = (Q_1 < \cdots < Q_m < N_P(\sigma))$ is fully $N_{\mathcal{F}}(Q_0))$-normalised. Since σ is fully $\mathcal{F}$-normalised, in particular Q_0 is fully $\mathcal{F}$-normalised, and hence $n(\sigma)$ is fully $\mathcal{F}$-normalised. □

Proposition 5.7. *For any fully $\mathcal{F}$-normalised chain σ in $S_{\lhd}(\mathcal{F})$ we have $n(n(\sigma)) = \sigma$.*

Proof. If $\sigma = Q_0 = P$ there is nothing to prove. Assume that $Q_0 < P$. If $Q_m = N_P(\sigma)$ then $n(\sigma) = (Q_0 < \cdots < Q_{m-1})$ and $Q_{m-1} < Q_m \leq N_P(n(\sigma))$, and hence $n(n(\sigma)) = (Q_0 < \cdots < Q_{m-1} < N_P(n(\sigma)))$. In order to show that this is σ, we need to show that $Q_m = N_P(n(\sigma))$. Now if $Q_m < N_P(n(\sigma))$ then $Q_m < N_{N_P(n(\sigma))}(Q_m) = N_P(\sigma)$, contradicting the equality $Q_m = N_P(\sigma)$, so this is not possible. Finally, if $Q_m < N_P(\sigma)$ then $n(\sigma) = (Q_0 < \cdots < Q_m < N_P(\sigma))$, and then $N_P(n(\sigma)) = N_P(\sigma)$, so $n(n(\sigma)) = \sigma$. □

The next results are dedicated to showing that the above pairing on fully normalised chains passes down to isomorphism classes of chains.

Proposition 5.8. *Let $\sigma = (Q_0 < \cdots < Q_m)$ and $\tau = (R_0 < \cdots < R_m)$ be two chains in $S_{\lhd}(\mathcal{F})$ which are isomorphic in $S_{\lhd}(\mathcal{F})$. Assume that τ is fully $\mathcal{F}$-normalised. Then there is an isomorphism $\varphi : \sigma \cong \tau$ in $S_{\lhd}(\mathcal{F})$ which can be extended to $N_P(\sigma)$.*

Proof. Let $\psi : \sigma \cong \tau$ be an isomorphism in $S_{\lhd}(\mathcal{F})$; that is, ψ is a family of isomorphisms $\psi_i : Q_i \cong R_i$ such that $\psi_i = \psi_m|_{Q_i}$, where $0 \leq i \leq m$. Since R_0 is fully $\mathcal{F}$-normalised, there is an automorphism α of R_0 in $\mathcal{F}$ such that $\alpha \circ \psi_0 : Q_0 \cong R_0$ can be extended to a morphism $\gamma : N_P(\sigma) \to P$. Thus, up to replacing σ by $\gamma(\sigma)$, we may assume that $Q_0 = R_0$. Then σ and τ are in fact isomorphic in $N_{\mathcal{F}}(Q_0)$, hence so are $\sigma\backslash 0$ and $\tau\backslash 0$. By induction, there is an isomorphism $\pi : \sigma\backslash 0 \cong \tau\backslash 0$ in $N(N_{\mathcal{F}}(Q_0))$ which can be extended to $N_{N_P(Q_0)}(\sigma\backslash 0) = N_P(\sigma)$. Since $\pi(Q_0) = Q_0$, clearly π induces the required isomorphism $\sigma \cong \tau$. □

Proposition 5.9. *Let $\sigma = (Q_0 < Q_1 < \cdots < Q_m)$ be a chain in $S_{\lhd}(\mathcal{F})$ and let i be an integer such that $0 \leq i \leq m$. Let $\varphi : Q_i \to P$ be a morphism in $\mathcal{F}$ such that $\varphi(\sigma_{\leq i})$ is fully $\mathcal{F}$-normalised. Then there is a morphism $\psi : Q_m \to P$ such that $\psi(\sigma_{\leq i}) = \varphi(\sigma_{\leq i})$ and such that $\psi(\sigma)$ is fully $\mathcal{F}$-normalised.*

Proof. Induction over $m - i$. For $m - i = 0$ take $\psi = \varphi$. Suppose that $m > i$. By 5.8 we may assume that φ extends to a morphism $\tau : N_P(\sigma_{\leq i}) \to P$. Clearly $Q_m \subseteq N_P(\sigma_{\leq i})$. Set $\sigma' = \tau(\sigma)$. Then $\sigma'_{\leq i} = \varphi(\sigma_{\leq i})$ is fully $\mathcal{F}$-normalised, and by induction, $\tau(Q_{i+1}) < \cdots < \tau(Q_m)$ is isomorphic, in $N_{\mathcal{F}}(\sigma'_{\leq i})$, to a fully $N_{\mathcal{F}}(\sigma'_{\leq i})$-normalised chain. The result follows from 5.4. □

Proposition 5.10. *Let $\sigma = (Q_0 < \cdots < Q_m)$ and $\tau = (R_0 < \cdots < R_n)$ be two fully $\mathcal{F}$-normalised chains in $S_\lhd(\mathcal{F})$. We have $\sigma \cong \tau$ if and only if $n(\sigma) \cong n(\tau)$.*

Proof. Suppose there is an isomorphism $\psi : \sigma \cong \tau$. If $\sigma = Q_0 = P$ then $\sigma = \tau$ and there is nothing to prove. Assume that $N_P(\sigma) = Q_m$. Then $n(\sigma) = \sigma\backslash m = (Q_0 < \cdots < Q_{m-1})$. Clearly ψ induces an isomorphism $\sigma\backslash m \cong \tau\backslash m$, so all we have to show in this case is that $n(\tau) = \tau\backslash m$, or equivalently, $R_m = N_P(\tau)$. If $R_m < N_P(\tau)$, there is, by 5.8, an isomorphism $\tau \cong \sigma$ which can be extended to a morphism $\pi : N_P(\tau) \to P$. However, this would imply that $N_P(\sigma)$ contains $\pi(N_P(\tau)$, which is strictly bigger than R_m, contradicting the equality $Q_m = N_P(\sigma)$. This shows that $n(\tau) = \tau(m)$ and hence $n(\sigma) \cong n(\tau)$ in this case. Assume now that $Q_m < N_P(\sigma)$. Again by 5.8 the isomorphism $\psi : \sigma \cong \tau$ can be chosen in such a way that it extends to a morphism $\pi : N_P(\sigma) \to P$, and then, as before, we have $\pi(N_P(\sigma)) \subseteq N_P(\tau)$. Exchanging the roles of τ and σ yields $\pi(N_P(\sigma)) = N_P(\tau)$, and hence again, $n(\sigma) \cong n(\tau)$. Conversely, if $n(\sigma) \cong n(\tau)$ then $\sigma \cong \tau$ by the previous argument combined with 5.7. □

The following result shows that covariant functors on $[S_\lhd(\mathcal{C})]$ which are invariant under the pairing induced by n are acyclic. Clearly constant functors have that property.

Theorem 5.11. *Let $\mathcal{C}$ be a right ideal in the fusion system $\mathcal{F}$ on P and let k be a commutative ring. Let $\mathcal{A} : [S_\lhd(\mathcal{C})] \to \mathrm{Mod}(k)$ be a covariant functor. Suppose that for any fully $\mathcal{F}$-normalised chain σ in $S_\lhd(\mathcal{C})$ such that $|\sigma| < |n(\sigma)|$ the unique morphism $[\sigma] < [n(\sigma)]$ in $[S_\lhd(\mathcal{C})]$ induces an isomorphism $\mathcal{A}([\sigma]) \cong \mathcal{A}([n(\sigma)])$. Then the canonical epimorphism $C_\lhd(\mathcal{A}) \to \mathcal{A}([P])$ is a homotopy equivalence, where $\mathcal{A}([P])$ is considered as complex in degree zero. In particular, the functor $\mathcal{A}$ is acyclic.*

Proof. The pattern of the proof follows closely that of 4.7. We have $H^*([S_\lhd(\mathcal{C})]; \mathcal{A}) = H^*(C_\lhd(\mathcal{A}))$ by 3.7. For any integer $q \geq 0$ we define a subcomplex $C_\lhd^{(q)}(\mathcal{A})$ of $C_\lhd(\mathcal{A})$ as follows: for $n < q$ we set

$$C_\lhd^{(q)}(\mathcal{A})^n = 0 \; ;$$

for $n > q$ we set

$$C_\lhd^{(q)}(\mathcal{A})^n = C_\lhd(\mathcal{A})^n = \bigoplus_{[\sigma]} \mathcal{A}([\sigma])$$

with $[\sigma]$ running over the set of isomorphism classes of chains σ in $S_\lhd(\mathcal{C})$ such that $|\sigma| = n$; for $n = q$ we set

$$C_\lhd^{(q)}(\mathcal{A})^q = \bigoplus_{[\sigma]} \mathcal{A}([\sigma])$$

with $[\sigma]$ running over the set of isomorphism classes of chains σ in $S_\lhd(\mathcal{C})$ such that $|\sigma| = q$ and such that $|n(\sigma)| = q+1$. It is easy to see that the differential on $C_\lhd(\mathcal{A})$ restricts to a differential on $C_\lhd^{(q)}(\mathcal{A})$. We clearly have

$$C_\lhd(\mathcal{A})/C_\lhd^{(0)}(\mathcal{A}) \cong \mathcal{A}([P]),$$

viewed as complex concentrated in degree zero, and

$$C_\lhd^{(q-1)}(\mathcal{A}) \subseteq C_\lhd^{(q)}(\mathcal{A})$$

for any positive integer q. By construction, the inclusions $C_\lhd^{(0)}(\mathcal{A}) \subseteq C_\lhd(\mathcal{A})$ and $C_\lhd^{(q-1)}(\mathcal{A}) \subseteq C_\lhd^{(q)}(\mathcal{A})$ are degreewise split. Thus, by Corollary B.2, it suffices to show that the quotient complex $C_\lhd^{(q-1)}(\mathcal{A})/C_\lhd^{(q)}(\mathcal{A})$ is contractible, for q a positive integer. This quotient complex is of the form

$$\cdots \longrightarrow 0 \longrightarrow \bigoplus_{[\rho]\in\mathcal{M}} \mathcal{A}([\rho]) \xrightarrow{\ \epsilon\ } \bigoplus_{[\sigma]\in\mathcal{N}} \mathcal{A}([\sigma]) \longrightarrow 0 \longrightarrow \cdots$$

where $\mathcal{M}$ is the set of isomorphism classes of fully $\mathcal{F}$-normalised chains ρ in $S_\lhd(\mathcal{C})$ such that $|\rho| = q-1$, $|n(\rho)| = q$, and where $\mathcal{N}$ is the set of isomorphism classes of fully $\mathcal{F}$-normalised chains σ in $S_\lhd(\mathcal{C})$ such that $|\sigma| = q$, $|n(\sigma)| = q-1$. The map n induces a bijection between the sets $\mathcal{M}$, $\mathcal{N}$, and hence, the quotient complex $C_\lhd^{(q-1)}(\mathcal{A})/C_\lhd^{(q)}(\mathcal{A})$ is of the form

$$\cdots \longrightarrow 0 \longrightarrow \bigoplus_{[\sigma]\in\mathcal{N}} \mathcal{A}([n(\sigma)]) \xrightarrow{\ \epsilon\ } \bigoplus_{[\sigma]\in\mathcal{N}} \mathcal{A}([\sigma]) \longrightarrow 0 \longrightarrow \cdots$$

In order to see that this is contractible we only have to observe that the differential ϵ in degree $q-1$ is an isomorphism. We consider ϵ as a matrix of its components $\epsilon_{[n(\sigma)],[\tau]} : \mathcal{A}([n(\sigma)]) \to \mathcal{A}([\tau])$, where $[\sigma]$, $[\tau]$ are elements in $\mathcal{N}$. The diagonal entries $\epsilon_{[n(\sigma)],[\sigma]}$ of this matrix are isomorphisms, by the assumptions on $\mathcal{A}$. In order to show that ϵ is an isomorphism it suffices to show that $\mathcal{N}$ can be ordered in such a way that this matrix is a lower triangular matrix. Choose any total order $\preccurlyeq$ on $\mathcal{N}$ with the property that $[\tau] \prec [\sigma]$ if $m_\tau < m_\sigma$, where as before m_τ, m_σ are the orders of the maximal terms occurring in the chains τ, σ, respectively. For the component $\epsilon_{[n(\sigma)],[\tau]} : \mathcal{A}([n(\sigma)]) \to \mathcal{A}([\tau])$, to be non zero we must have $\tau\backslash j \cong n(\sigma)$ for some j. If $j = q$ then $\tau\backslash j = n(\tau)$, hence $\sigma \cong \tau$. If $j < q$ then $m_\tau = m_{n(\sigma)} < m_\sigma$. Thus this entry is below the diagonal. Consequently, ϵ is an isomorphism, which concludes the proof. $\square$

As a direct consequence of 4.7 and 5.11 we get the acyclicity result needed for the proof of Theorem 1.1.

Theorem 5.12. *Let $\mathcal{C}$ be a right ideal in the fusion system $\mathcal{F}$ on P. Then $H^n([S(\mathcal{C})];\mathbb{Z}) = H^n([S_{\lhd}(\mathcal{C})];\mathbb{Z}) = H^n([S_{\Phi}(\mathcal{C})];\mathbb{Z}) = \{0\}$ for any positive integer n, where $\mathbb{Z}$ is considered as constant covariant functor.*

Proof. The first two equalities follow from 4.7 and the rest follows from 5.11. □

We conclude this section with a characterisation of fully normalised chains in the cases where the underlying fusion system is that of a finite group or that of a block. The notation for fusion systems of finite groups and blocks is as in [14, §2]. Proposition 5.13 is the particular case of Proposition 5.14 applied to the principal block, but the proof of 5.14 in general relies on a non trivial block theoretic fact, [11, 3.1], which is not needed in 5.13, and which is why we choose to state both cases separately.

Proposition 5.13. *Let G be a finite group, let p be a prime divisor of the order of G and let P be a Sylow-p-subgroup of G. Let $\sigma = (Q_0 < Q_1 < \cdots < Q_m)$ be a chain of subgroups of P such that $Q_i \trianglelefteq Q_m$ for $0 \leq i \leq m$. Then σ is fully $\mathcal{F}_P(G)$-normalised if and only if $N_P(\sigma_{\leq i})$ is a Sylow-p-subgroup of $N_G(\sigma_{\leq i})$ for $0 \leq i \leq m$.*

Proof. Set $\mathcal{F} = \mathcal{F}_P(G)$. For $m = 0$ this is well-known; see e.g. [14, 2.2.(iii)]. For $1 \leq i \leq m$ we observe that $N_P(\sigma_{\leq i}) = N_{N_P(Q_0)}((\sigma\backslash 0)_{\leq i-1})$ and $N_G(\sigma_{\leq i}) = N_{N_G(Q_0)}((\sigma\backslash 0)_{\leq i-1})$. Thus, by induction, $\sigma\backslash 0$ is fully $N_{\mathcal{F}}(Q_0)$-normalised if and only if $N_P(\sigma_{\leq i})$ is a Sylow-p-subgroup of $N_G(\sigma_{\leq i})$ for $1 \leq i \leq m$. The result follows. □

Proposition 5.14. *Let G be a finite group, let p be a prime divisor of the order of G, let k be a field of characteristic p, let b be a block of kG and let (P, e_P) be a maximal b-Brauer pair. For any subgroup Q of P denote by e_Q the unique block of $kC_G(Q)$ satisfying $(Q, e_Q) \subseteq (P, e_P)$. Let $\sigma = (Q_0 < Q_1 < \cdots < Q_m)$ be a chain of subgroups of P such that $Q_i \trianglelefteq Q_m$ for $0 \leq i \leq m$. Then σ is fully $\mathcal{F}_{(P,e_P)}(G, b)$-normalised if and only if $N_P(\sigma_{\leq i})$ is a defect group of the block algebra $kN_G(\sigma_{\leq i}, e_{Q_i})e_{Q_i}$ for $0 \leq i \leq m$.*

Proof. Set $\mathcal{F} = \mathcal{F}_{(P,e_P)}(G, b)$. For this statement to make sense we need to invoke [11, 3.1] which implies that e_{Q_i} remains indeed a block of $kN_G(\sigma_{\leq i}, e_{Q_i})$ for $0 \leq i \leq m$. If $m = 0$ the result is again well-known; see e.g. [14, 2.4.(iii)]. For $1 \leq i \leq m$ we observe that $N_P(\sigma_{\leq i}, e_{Q_i}) = N_{N_P(Q_0)}((\sigma\backslash 0)_{\leq i-1}, e_{Q_i})$ and $N_G(\sigma_{\leq i}, e_{Q_i}) = N_{N_G(Q_0)}((\sigma\backslash 0)_{\leq i-1}, e_{Q_i})$. Thus, by induction, $\sigma\backslash 0$ is fully $N_{\mathcal{F}}(Q_0)$-normalised if and only if $N_P(\sigma_{\leq i})$ is a defect group of $kN_G(\sigma_{\leq i}, e_{Q_i})e_{Q_i}$ for $1 \leq i \leq m$. The result follows. □

6 Simple connectedness

Theorem 6.1. *Let $\mathcal{F}$ be a fusion system on a finite p-group P, where p is a prime. Let $\mathcal{C}$ be a right ideal in $\mathcal{F}$. Then the partially ordered sets $[S(\mathcal{C})]$ and $[S_{\lhd}(\mathcal{C})]$, viewed as topological spaces, are simply connected.*

Proof. We first observe that $[S(\mathcal{C})] = [S_<(\mathcal{C})]$ is path connected. Let $\sigma = (Q_0 < Q_1 < \cdots < Q_m)$ be a chain of subgroups of P belonging to $\mathcal{C}$. Set $\tau = (Q_0 < Q_1 < \cdots < Q_m < P)$ if $Q_m < P$ and $\tau = \sigma$ if $Q_m = P$. Consider P as chain of length zero in $S(\mathcal{C})$. We have morphisms in $S(\mathcal{C})$

$$\sigma \longrightarrow \tau \longleftarrow P \ ,$$

which implies that in $[S(\mathcal{C})]$ there is a path from $[\sigma]$ to $[P]$. Thus $[S(\mathcal{C})]$ is path connected. If σ is in $S_\triangleleft(\mathcal{C})$ we show by induction over $[P : Q_m]$ that there is a path from $[P]$ to $[\sigma]$. If $Q_m = P$ there is a morphism $P \to \sigma$ given by Id_P, and hence a path from $[P]$ to $[\sigma]$. If $Q_m < P$ then $Q_m < N_P(Q_m)$, and the diagram

$$\sigma = (Q_0 < \cdots < Q_m) \leftarrow Q_m \rightarrow (Q_m \triangleleft N_P(Q_m)) \leftarrow N_P(Q_m)$$

defines a path in $[S_\triangleleft(\mathcal{C})]$ from $[\sigma]$ to $[N_P(Q_m)]$. By induction, there is also a path from $[N_P(Q_m)]$ to $[P]$. Thus $S_\triangleleft(\mathcal{C})$ is path connected as well.

It remains to show that the fundamental groups of $[S(\mathcal{C})]$ and of $[S_\triangleleft(\mathcal{C})]$ are trivial. We choose $[P]$ as basepoint in $[S(\mathcal{C})]$. Let T be a loop in $[S(\mathcal{C})]$ starting and ending at $[P]$. It is well-known (and easy to see) that up to replacing T by a homotopic path, we may assume that T is contained in the 1-skeleton of $[S(\mathcal{C})]$. Then T is homotopic to the image of a path in $S_<(\mathcal{C})$ from P to P given by a diagram of morphisms in $S_<(\mathcal{C})$ of the form

$$P = \sigma_0 \rightarrow \sigma_1 \leftarrow \sigma_2 \rightarrow \cdots \leftarrow \sigma_{2n} = P$$

for some integer $n \geq 0$, where the σ_i are chains in $S_<(\mathcal{C})$ for $0 \leq i \leq 2n$. We observe first that we may always assume that the chains σ_{2k} have length zero, where $0 \leq k \leq n$, without changing the homotopy class of the path T. For $k = 0$ or $k = n$ this holds trivially. Let k be an integer such that $1 \leq k \leq n-1$. If

$$\sigma_{2k-1} = (Q_0 < \cdots < Q_m) \ ,$$

$$\sigma_{2k} = (R_0 < \cdots < R_n) \ ,$$

$$\sigma_{2k+1} = (S_0 < \cdots < S_l) \ ,$$

there is an obvious commutative diagram in $S_<(\mathcal{C})$ of the form

$$\begin{tikzcd}
\sigma_{2k-1} \arrow[d, equal] & \sigma_{2k} \arrow[l] \arrow[r] & \sigma_{2k+1} \arrow[d, equal] \\
\sigma_{2k-1} & R_n \arrow[l] \arrow[r] \arrow[u] & \sigma_{2k+1}
\end{tikzcd}$$

which shows that we may replace σ_{2k} by the chain R_n of length zero without affecting the homotopy class of the path T. Next we note that we may assume that the σ_{2k+1} have length 1, for $0 \leq k \leq n-1$, without changing the homotopy class of T. The chains σ_{2k}, σ_{2k+2} have length zero, so choose notation such that $\sigma_{2k} = Q$ and $\sigma_{2k+2} = S$ for some subgroups Q, S of P. Let $\sigma_{2k+1} = (R_0 < \cdots < R_n)$. The morphism $\sigma_{2k} \to \sigma_{2k+1}$

is given by an isomorphism $Q \cong R_i$ for a unique integer i between 0 and n. Similarly, the morphism $\sigma_{2k+1} \leftarrow \sigma_{2k+2}$ is given by an isomorphism $R_j \cong S$ for some unique integer j. If $i = j$ then $Q \cong S$ and the portion of the path of T given by $Q = \sigma_{2k} \rightarrow \sigma_{2k+1} \leftarrow \sigma_{2k+2} = S$ is homotopic to the constant path at $[Q] = [S]$, so we may eliminate this portion from the path. If $i < j$ we get an obvious commutative diagram in $S(\mathcal{C})$ of the form

$$\begin{tikzcd}
\sigma_{2k} \arrow[r] \arrow[d, equal] & \sigma_{2k+1} & \sigma_{2k+2} \arrow[l] \arrow[d, equal] \\
Q \arrow[r] & (R_i < R_j) \arrow[u] & S \arrow[l]
\end{tikzcd}$$

which shows that we may replace σ_{2k+1} by the chain of length 1 of the form $(R_i < R_j)$ without changing the homotopy class of T. A similar argument works if $i > j$. The next step is to show that we may in fact assume that the σ_{2k+1} are of the form $\sigma_{2k+1} = (Q \lhd S)$; that is, with Q normal in S. If Q is not normal in S then there is a subgroup R of S such that $Q < R < S$. By the previous arguments, we may assume that $\sigma_{2k} = Q$ and $\sigma_{2k+2} = S$, and that the morphisms $\sigma_{2k} \rightarrow \sigma_{2k+1}$ and $\sigma_{2k+1} \leftarrow \sigma_{2k+2}$ are given by the identity maps on Q and S, repsectively. Thus, we get a commutative diagram in $S_<(\mathcal{C})$ of the form

$$\begin{tikzcd}
\sigma_{2k} \arrow[rr] \arrow[d, equal] & & \sigma_{2k+1} \arrow[d, equal] & & \sigma_{2k+2} \arrow[ll] \arrow[d, equal] \\
Q \arrow[rr] \arrow[d, equal] & & (Q < S) \arrow[d] & & S \arrow[ll] \arrow[d, equal] \\
Q \arrow[rr] \arrow[d, equal] & & (Q < R < S) & & S \arrow[ll] \arrow[d, equal] \\
Q \arrow[r] & (Q < R) \arrow[ur] & R \arrow[l] \arrow[r] \arrow[u] & (R < S) \arrow[ul] & S \arrow[l]
\end{tikzcd}$$

which shows that the part of the path T from $[Q]$ to $[S]$ is homotopic to the path represented by the bottom line of the above diagram. Since the indices $[R : Q]$ and $[S : R]$ are both smaller than $[S : Q]$, we can assume that after applying the above argument a finite number of times, that the chains σ_{2k+1} are in $S_\lhd(\mathcal{C})$. This shows in particular that the path T is homotopic to a path in $[S_\lhd(\mathcal{C})]$. We keep the assumption that all σ_{2k} have length zero, $0 \leq k \leq n$, and that all σ_{2k+1} have length one and are in $S_\lhd(\mathcal{C})$. If n is 0 or 1, then clearly the path T is homotopic to the constant path at $[P]$. We assume therefore that $n \geq 2$. We choose now k such that $\sigma_{2k} = Q$ with Q having smallest possible order. If $Q = P$ we are done, so we may assume $Q < P$. Then necessarily $1 \leq k \leq n-1$, and, up to isomorphism, the part of the chain

$$\sigma_{2k-2} \rightarrow \sigma_{2k-1} \leftarrow \sigma_{2k} \rightarrow \sigma_{2k+1} \leftarrow \sigma_{2k+2}$$

is of the form

$$R \to (Q \lhd R) \leftarrow Q \to (Q \lhd S) \leftarrow S$$

for some subgroups Q, R, S of P. What we will show is that the path from $[R]$ to $[S]$ represented by this diagram is homotopic to a path represented by a diagram involving only subgroups of P which are strictly bigger than Q. If we can do this, we are done, because then after a finite number of steps we are down to a path represented by a diagram involving only P, and that yields the homotopy class of the constant path at $[P]$. First, we may assume that R and S are different. Indeed, if $R = S$ then the path represented by $(Q \lhd R) \leftarrow Q \to (Q \lhd S)$ is homotopic to the constant path at $[Q \lhd R] = [Q \lhd S]$, and so the path represented by the above sequence is homotopic to the constant path at $[R] = [S]$. If R, S are different, then at least one of them is smaller than $N_P(Q)$. Suppose that $R < N_P(Q)$. Then $R < N_{N_P(Q)}(R) = N_P(Q \lhd R)$. Set $R_1 = N_P(Q \lhd R)$. Then the chain $Q \lhd R \lhd R_1$ belongs to $S_\lhd(\mathcal{C})$. Consider the commutative diagram

$$\begin{tikzcd}
R \arrow[rr] \arrow[dd, equal] & & (Q \lhd R) \arrow[d] & & Q \arrow[ll] \arrow[dd, equal] \\
& & (Q \lhd R \lhd R_1) & & \\
R \arrow[r] & (R \lhd R_1) \arrow[ur] & R_1 \arrow[l] \arrow[r] \arrow[u] & (Q \lhd R_1) \arrow[ul] & Q \arrow[l]
\end{tikzcd}$$

This shows that we may replace R by R_1. But then, after a finite number of steps, we actually may assume that $R = N_P(Q)$. The same argument shows that we also may assume that $S = N_P(Q)$. Then in particular $R = S$, and by the previous argument, this concludes the proof. $\square$

Remark 6.2. Even though $[S_\Phi(\mathcal{C})]$ is acyclic, the above proof does not show that $[S_\Phi(\mathcal{C})]$ is contractible, because the homotopy in the last diagram above goes via the chain $Q \lhd R \lhd R_1$ which need not be in $S_\Phi(\mathcal{C})$.

7 Split exact sequences

Webb's split exact sequences [24] and their generalisations and variations in work of Bouc [2], Grodal [8], Symonds [20], Villaroel-Flores and Webb [22] can be obtained in some special cases for arbitrary fusion systems by combining 4.7 and 5.11 as follows:

Theorem 7.1. *Let p be a prime, let $\mathcal{F}$ be a fusion system on a finite p-group P and let $\mathcal{C}$ be a right ideal in $\mathcal{F}$. Let k be a commutative ring and let $\mathcal{A} : [S(\mathcal{C})] \to \mathrm{Mod}(k)$ be a covariant functor. Suppose that for any chain σ in $S_<(\mathcal{C})$ the canonical morphism between $[\sigma]$ and $[z(\sigma)]$ induces an isomorphism $\mathcal{A}([\sigma]) \cong \mathcal{A}([z(\sigma)])$, and suppose that for* *any fully $\mathcal{F}$-normalised chain τ in $S_\triangleleft(\mathcal{C})$ the canonical morphism between $[\sigma]$ and $[n(\sigma)]$ induces an isomorphism $\mathcal{A}([\sigma]) \cong \mathcal{A}([n(\sigma)])$. There is a split exact sequence*

$$0 \longrightarrow \mathcal{A}([P]) \longrightarrow \bigoplus_{\substack{[\sigma]\in[S(\mathcal{C})]\\ |\sigma|=0}} \mathcal{A}([\sigma]) \longrightarrow \bigoplus_{\substack{[\sigma]\in[S(\mathcal{C})]\\ |\sigma|=1}} \mathcal{A}([\sigma]) \longrightarrow \bigoplus_{\substack{[\sigma]\in[S(\mathcal{C})]\\ |\sigma|=2}} \mathcal{A}([\sigma]) \longrightarrow \cdots .$$

Proof. The hypotheses on $\mathcal{A}$ imply that, by 4.7 and 5.11, the canonical epimorphism $C(\mathcal{A}) \to \mathcal{A}([P])$ is a homotopy equivalence. The mapping cone of this morphism is therefore contractible - and this is precisely a sequence of the form as described in the statement. $\square$

Appendix A: fusion systems

Let p be a prime.

Definition A.1. Given a finite p-group P, a *category on P* is a category $\mathcal{F}$ whose objects are the subgroups of P and whose morphism sets $\mathrm{Hom}_{\mathcal{F}}(Q,R)$ are sets of injective group homomorphisms from Q to R, where Q, R are subgroups of P, with the following properties:

(i) the composition of morphisms in $\mathcal{F}$ is given by the usual composition of group homomorphisms;

(ii) if $Q \subseteq R$ then the inclusion map from Q to R is a morphism in $\mathcal{F}$;

(iii) if $\varphi : Q \to R$ is a morphism in $\mathcal{F}$ then so is the induced isomorphism $Q \cong \varphi(Q)$ and its inverse.

If Q, R are subgroups of P we denote by $\mathrm{Hom}_P(Q,R)$ the set of group homomorphisms from Q to R induced by conjugation with elements in P, and we write $\mathrm{Aut}_P(Q) = \mathrm{Hom}_P(Q,Q)$.

Definition A.2. Given a category $\mathcal{F}$ on a finite p-group P and a subgroup Q of P we say that

- Q is fully *$\mathcal{F}$-normalised* if $|N_P(Q)| \geq |N_P(Q')|$ for any subgroup Q' of P such that there is an isomorphism $Q' \cong Q$ in $\mathcal{F}$;
- Q is fully *$\mathcal{F}$-centralised* if $|C_P(Q)| \geq |C_P(Q')|$ for any subgroup Q' of P such that there is an isomorphism $Q' \cong Q$ in $\mathcal{F}$;
- Q is *$\mathcal{F}$-centric* if $C_P(Q') = Z(Q')$ for any subgroup Q' of P such that there is an isomorphism $Q' \cong Q$ in $\mathcal{F}$;
- Q is *$\mathcal{F}$-radical* if $O_p(\mathrm{Aut}_{\mathcal{F}}(Q)) \subseteq \mathrm{Aut}_Q(Q)$;
- Q is *weakly $\mathcal{F}$-closed* if for every morphism $\varphi : Q \to P$ we have $\varphi(Q) = Q$;
- Q is *strongly $\mathcal{F}$-closed* if for every morphism $\varphi : R \to P$ we have $\varphi(R \cap Q) = \varphi(R) \cap Q$;

Following the notation in [3], if $\varphi : Q \to P$ is a morphism in $\mathcal{F}$ we denote by N_φ the subgroup of all elements y in $N_Q(P)$ for which there exists an element $z \in N_P(\varphi(Q))$ such that $\varphi(yuy^{-1}) = z\varphi(u)z^{-1}$ for all $u \in Q$. Note that $QC_P(Q) \subseteq N_\varphi \subseteq N_P(Q)$.

Definition A.3. A category $\mathcal{F}$ on a finite p-group P is called a *fusion system* if $\mathrm{Hom}_P(Q,R) \subseteq \mathrm{Hom}_{\mathcal{F}}(Q,R)$ for all subgroups Q, R of P and if in addition the two following properties hold:

(I-S) $\mathrm{Aut}_P(P)$ is a Sylow-p-subgroup of $\mathrm{Aut}_{\mathcal{F}}(P)$;

(II-S) every morphism $\varphi : Q \to P$ such that $\varphi(Q)$ is fully $\mathcal{F}$-normalised extends to a morphism $\psi : N_\varphi \to P$ in $\mathcal{F}$ (that is, $\psi|_Q = \varphi$).

The concept of a fusion system on a finite p-group is due to L. Puig; the above axioms (I-S) and (II-S) appear in Stancu [18]. As observed in [18] - see also [14, §1] for proofs - the axioms I-S and II-S are equivalent to the a priori stronger axioms used by Broto, Levi and Oliver in [3, 1.2]

(I-BLO) if Q is a fully $\mathcal{F}$-normalised subgroup of P then Q is fully $\mathcal{F}$-centralised and $\mathrm{Aut}_P(Q)$ is a Sylow-p-subgroup of $\mathrm{Aut}_{\mathcal{F}}(Q)$;

(II-BLO) given any subgroup Q of P, every morphism $\varphi : Q \to P$ such that $\varphi(Q)$ is fully $\mathcal{F}$-centralised extends to a morphism $\psi : N_\varphi \to P$ in $\mathcal{F}$ (that is, $\psi|_Q = \varphi$).

If G is a finite group and P a Sylow-p-subgroup of G we denote by $\mathcal{F}_P(G)$ the category on P whose morphism sets are the group homomorphisms induced by conjugation with elements in G; that is, $\mathrm{Hom}_{\mathcal{F}_P(G)}(Q,R) = \mathrm{Hom}_G(Q,R)$ for any two subgroups Q, R of P. It is well-known and easy to verify that $\mathcal{F}_P(G)$ is a fusion system on P. In particular, $\mathcal{F}_P(P)$ is the fusion system on P consisting precisely of all morphisms given by conjugation with elements in P. For any fusion system $\mathcal{F}$ on P we have $\mathcal{F}_P(P) \subseteq \mathcal{F}$ by the first part of A.3.

One of the most fundamental properties of fusion systems is *Alperin's fusion theorem*, which says that any isomorphism in a fusion system $\mathcal{F}$ on a finite p-group P can be written as the composition of isomorphisms $\varphi : Q \to R$ for which there is a radical $\mathcal{F}$-centric subgroup S of P containing both Q, R and an automorphism α of S in $\mathcal{F}$ such that φ is the restriction to Q of α. See e.g. [3] or [21] for proofs and more precise statements.

Definition A.4. Given a category $\mathcal{F}$ on a finite p-group P and a subgroup Q of P we define the category $N_{\mathcal{F}}(Q)$ on $N_P(Q)$ by $\mathrm{Hom}_{N_{\mathcal{F}}(Q)}(R,R') = \{\varphi : R \to R' |\ \varphi \text{ extends to a morphism } \psi : QR \to QR' \text{ in } \mathcal{F} \text{ such that } \psi(Q) = Q\}$, for any two subgroups R, R' of $N_P(Q)$. Similarly, we define the category $C_{\mathcal{F}}(Q)$ on $C_P(Q)$ by $\mathrm{Hom}_{C_{\mathcal{F}}(Q)}(R,R') = \{\varphi : R \to R' |\ \varphi \text{ extends to a morphism } \psi : QR \to QR' \text{ in } \mathcal{F} \text{ such that } \psi|_Q = \mathrm{Id}_Q\}$.

We have clearly inclusions of categories $C_{\mathcal{F}}(Q) \subseteq N_{\mathcal{F}}(Q) \subseteq \mathcal{F}$. If $\mathcal{F} = N_{\mathcal{F}}(Q)$ for some subgroup Q of P, then clearly Q is strongly $\mathcal{F}$-closed. The converse of this statement is not true, in general.

The following result is due to Puig; see [3, Appendix, Lemma A.6] for a proof.

Proposition A.5. *Let $\mathcal{F}$ be a fusion system on a finite p-group P and let Q be a subgroup of P. If Q is fully $\mathcal{F}$-centralised, then $C_{\mathcal{F}}(Q)$ is a fusion system on $C_P(Q)$ and if Q is fully normalised, then $N_{\mathcal{F}}(Q)$ is a fusion system on $N_P(Q)$.*

Appendix B: Some homological background

Let $\mathcal{C}$ be an abelian category and let $0 \longrightarrow X \xrightarrow{f} Y \xrightarrow{g} Z \longrightarrow 0$ be a short exact sequence of chain complexes over $\mathcal{C}$. Then Z is quasi-isomorphic to the mapping cone $C(f)$ of f; in particular, Z is acyclic if and only if f is a quasi-isomorphism. It is not true, in general, that $C(f) \simeq Z$. Thus, even if Z is contractible this does not necessarily imply that f is a homotopy equivalence. One can however show that if the above exact sequence is split in each degree then $C(f) \simeq Z$. For the purpose of this paper, we need only a particular case of this fact, of which we give a direct proof for the convenience of the reader:

Theorem B.1. *Let $\mathcal{C}$ be an abelian category and let $0 \longrightarrow X \xrightarrow{f} Y \xrightarrow{g} Z \longrightarrow 0$ be a degreewise split exact sequence of chain complexes over $\mathcal{C}$. Then f is a homotopy equivalence if and only if $Z \simeq 0$.*

Proof. Let δ, ϵ, ζ be the differentials of X, Y, Z, respectively. Since the exact sequence in the statement is degreewise split, there are graded morphisms $u : Y \to X$ and $v : Z \to Y$ of degree zero such that $\mathrm{Id}_Y = fu+vg$, but u, v need not commute with the differentials of the complexes. Composing this identity with f on the right yields $f = fuf$, hence $uf = \mathrm{Id}_X$ because f is a monomorphism. Similarly, $gv = \mathrm{Id}_Z$.

Suppose first that f is a homotopy equivalence. Let $f' : Y \to X$ be a homotopy inverse of f. That is, $\mathrm{Id}_X \sim f'f$ and $\mathrm{Id}_Y \sim ff'$, or more explicitly, there are graded morphisms $a : X \to X$ and $b : Y \to Y$ of degree 1 such that

$$\mathrm{Id}_X - f'f = a\delta + \delta a \ , \quad \mathrm{Id}_Y - ff' = b\epsilon + \epsilon b \ .$$

We first show that we may assume $a = 0$, or equivalently, that $f'f = \mathrm{Id}_X$. To see this, we replace f' by $f' + au\epsilon + \delta au$. Clearly $f' \sim f' + au\epsilon + \delta au$, and we have now $(f' + au\epsilon + \delta au)f = ff' + auf\delta + \delta auf = ff' + a\delta + \delta a = \mathrm{Id}_X$, where we used the equality $uf = \mathrm{Id}_X$. Therefore we may indeed assume that $f'f = \mathrm{Id}_X$. That is, f is a split monomorphism as chain map. Consequently g is a split epimorphism as chain map. Let $g' : Z \to Y$ be a chain map satisfying $gg' = \mathrm{Id}_Z$. Set $c = gbg'$. Clearly c is a graded morphism of degree 1 from Z to Z. Composing the identity $\mathrm{Id}_Y - ff' = b\epsilon + \epsilon b$ on the left by g and on the right by g' yields $\mathrm{Id}_Z = gg' = gb\epsilon g' + g\epsilon bg' = gbg'\zeta + \zeta gbg' = c\zeta + \zeta c \sim 0$, hence $Z \simeq 0$.

Suppose conversely that $Z \simeq 0$. Then there is graded morphism $c : Z \to Z$ of degree 1 such that $\mathrm{Id}_Z = c\zeta + \zeta c$. Set now $g' = \epsilon vc + vc\zeta$. Since $gv = \mathrm{Id}_Z$ we get that $gg' = g\epsilon vc + gvc\zeta = \zeta gvc + c\zeta = \zeta c + c\zeta = \mathrm{Id}_Z$. Thus g is a split epimorphism of chain complexes. Therefore there exists a chain map $f' : Y \to X$ satisfying $\mathrm{Id}_Y = g'g + ff'$. In particular, composing this identity by f on the right yields, as above, the equality $f'f = \mathrm{Id}_X$. Moreover, $ff' = \mathrm{Id}_Y - g'g = \mathrm{Id}_Y - (\epsilon vc + vc\zeta)g = \mathrm{Id}_Y - \epsilon vcg - vcg\zeta \sim \mathrm{Id}_Y$. This shows that f' is a homotopy inverse of f which concludes the proof. □

We reformulate this in the form as used in the proofs of 4.7 and 5.11:

Corollary B.2. *Let $\mathcal{C}$ be an abelian category, let X be a chain complex over $\mathcal{C}$ and for any $q \geq 0$ let $X^{(q)}$ be a subcomplex of X such that $X^{(q)} \subseteq X^{(q-1)}$ for $q \geq 1$ and such that $X^{(q)} = 0$ for q large enough. Suppose that the inclusions $X^{(0)} \subseteq X$ and $X^{(q)} \subseteq X^{(q-1)}$ are degreewise split and that the quotients $X^{(q-1)}/X^{(q)}$ are contractible for $q \geq 1$. Then $X^{(0)}$ is contractible and the canonical epimorphism $X \to X/X^{(0)}$ is a homotopy equivalence.*

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