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TRIVIAL SOURCE BIMODULE RINGS FOR BLOCKS AND p-PERMUTATION EQUIVALENCES

Markus Linckelmann

Abstract. We associate with any p-block of a finite group a Grothendieck ring of certain p-permutation bimodules. We extend the notion of p-permutation equivalences introduced by Boltje and Xu [4] to source algebras of p-blocks of finite groups. We show that a p-permutation equivalence between two source algebras A, B of blocks with a common defect group and same local structure induces an isotypy.

1 Introduction

Let p be a prime, let $\mathcal{O}$ be a complete discrete valuation ring having a quotient field K of characteristic zero and a algebraically closed residue field k of prime characteristic p. For any finite p-group P and any interior P-algebra A we denote by $\mathcal{T}(A)$ the Grothendieck group, with respect to split exact sequences, of finite direct sums of summands of the A-A-bimodules $A \underset{\mathcal{O}Q}{\otimes} A$, with Q running over the subgroups of P. If A is a source algebra of a block of a finite group algebra $\mathcal{O}G$ then the group $\mathcal{T}(A)$ has a ring structure:

Theorem 1.1. *Let A be a source algebra of a block of a finite group with defect group P. The tensorproduct over A induces an associative bilinear multiplication on $\mathcal{T}(A)$ through which $\mathcal{T}(A)$ becomes a unitary ring having an augmentation and an antipode.*

We will restate and prove the various parts of Theorem 1.1 in greater generality in 8.2, 8.6 and 12.3. We show in 8.4 that the ring $\mathcal{T}(A)$ does not depend on the choice of a source algebra of the considered block $b \in Z(\mathcal{O}G)$ of a finite group G, and is hence an invariant of the block b. One may think of $\mathcal{T}(A)$ as a bimodule version for source algebras of the trivial source ring of a finite group and call $\mathcal{T}(A)$ the *trivial source bimodule ring of the block b*. If $A = \mathcal{O}P$ then $\mathcal{T}(A)$ is isomorphic to the Burnside ring of P (cf. 12.1), which is also the trivial source ring in that case. Results of Alperin-Broué [1] and Broué-Puig [9] imply that a source algebra A of a block with defect group P determines a *fusion system* on P. For P a finite p-group, A an interior P-algebra and Q a subgroup of P, the Brauer construction with respect to the conjugation action of Q on A yields an interior $C_P(Q)$-algebra $A(\Delta Q)$. In certain cases, this is compatible with the ring structure on $\mathcal{T}(A)$:

Theorem 1.2. *Let A be a source algebra of a block with defect group P and fusion system $\mathcal{F}$. For any fully $\mathcal{F}$-centralised subgroup Q of P the tensorproduct over $A(\Delta Q)$ induces an associative unitary ring structure on $\mathcal{T}(A(\Delta Q))$ and the Brauer construction induces a unitary ring homomorphism $\mathcal{T}(A) \longrightarrow \mathcal{T}(A(\Delta Q))$.*

This is proved in 9.1 and 9.2. The difficult part is to show that the Brauer construction does actually induce a map from $\mathcal{T}(A)$ to $\mathcal{T}(A(\Delta Q))$; the fact that this map is multiplicative follows then

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from a result in [15]. This Theorem does not hold, in general, for the block algebra $\mathcal{O}Gb$ instead of A. For A, B two source algebras of blocks of finite groups having a common defect group P we denote more generally by $\mathcal{T}(A,B)$ the Grothendieck group, with respect to split exact sequences, of finite direct sums of summands of the A-B-bimodules $A \underset{\mathcal{O}Q}{\otimes} B$, with Q running over the subgroups of P. Since A, B are symmetric $\mathcal{O}$-algebras, the $\mathcal{O}$-dual of an A-B-bimodule of the form $A \underset{\mathcal{O}Q}{\otimes} B$ is isomorphic to the B-A-module $B \underset{\mathcal{O}Q}{\otimes} A$. In other words, $\mathcal{O}$-duality induces a group isomorphism $\mathcal{T}(A,B) \cong \mathcal{T}(B,A)$. For X an element in $\mathcal{T}(A,B)$ we denote by X^* its image in $\mathcal{T}(B,A)$ under this isomorphism. The map sending a pair (M,N) consisting of an A-B-bimodule M and a B-A-bimodule N to the A-A-bimodule $M \underset{B}{\otimes} N$ induces a bilinear map $- \underset{B}{\cdot} -$ from $\mathcal{T}(A,B) \times \mathcal{T}(B,A)$ to the Grothendieck group of A-A-bimodules, again taken with respect to split exact sequences. The following definition is the source algebra version of the concept of a p-permutation equivalence between blocks of finite groups, as introduced by Boltje and Xu [4].

Definition 1.3. Let A, B be source algebras of blocks of finite groups with a common defect group P. A *p-permutation equivalence between A and B* is an element $X \in \mathcal{T}(A,B)$ such that $X \underset{B}{\cdot} X^* = [A]$ in the Grothendieck group of A-A-bimodules and $X^* \underset{A}{\cdot} X = [B]$ in the Grothendieck group of B-B-bimodules.

A p-permutation equivalence between source algebras A, B induces a p-permutation equivalence in the sense of Boltje and Xu [4]. The converse is true for principal blocks, but not for arbitrary blocks in general (cf. 7.5, 7.6). What one gains by adopting the slightly more restrictive definition 1.3 is a generalisation of [4, 1.11] to arbitrary blocks:

Theorem 1.4. *Let A, B be source algebras of blocks b, c of finite groups G, H, respectively, having a common defect group P and the same fusion system on P. If there is a p-permutation equivalence between the source algebras A and B then the blocks b and c are isotypic.*

This is proved in 10.1; as in [4], we will show that the isotypy between b and c is obtained from applying the Brauer construction to the p-permutation equivalence. Following [15, 1.10] or [17, 1.1] a splendid derived equivalence between source algebras A, B of blocks of finite groups with a common defect group P is a derived equivalence given by a bounded two-sided tilting complex X of A-B-bimodules whose components in each degree are direct sums of summands of the A-B-bimodules $A \underset{\mathcal{O}Q}{\otimes} B$, where Q runs over the subgroups of P. A two-sided tilting complex X of this form is then called a *splendid tilting complex* of A-B-bimodules. This is a variation of Rickard's version [25] of a splendid equivalence - our definition is again slightly more restrictive in that it is relative to a choice of source idempotents. The motivation then as here is to make the notion "splendid" work for arbitrary blocks. If X is a bounded complex of A-B-bimodules, we set $[X] = \sum_{i \in \mathbb{Z}} (-1)^i [X_i]$, where $[X_i]$ is the isomorphism class of the degree i component of X viewed as element in the Grothendieck group of A-B-bimodules. We obtain the result corresponding to [4, 1.5] in this setting:

Theorem 1.5. *Let A, B be source algebras of blocks of finite groups having a common defect group P and let X be a splendid tilting complex of A-B-bimodules. The image $[X]$ of X in $\mathcal{T}(A,B)$ is a p-permutation equivalence.*

This is an immediate consequence of [14, 11.4.2]. Combining the present material with a result of Puig [24] yields the following invariance statement:

Theorem 1.6. *Let A, B be source algebras of blocks b, c of finite groups G, H, respectively, having a common defect group P. Suppose that $\mathcal{O}Gb$ and $\mathcal{O}Hc$ are Morita equivalent via a bimodule with endo-permutation source. Then $\mathcal{T}(A) \cong \mathcal{T}(B)$.*

The structure theory of nilpotent blocks [23] in conjunction with 1.6 implies:

Corollary 1.7. *Let A be a source algebra of a nilpotent block with defect group P. Then $\mathcal{T}(A) \cong \mathcal{T}(\mathcal{O}P)$.*

Further remarks in §13 include the observation that the Hochschild cohomology and block cohomology of a block are modules over the trivial source bimodule ring of the block (cf. 13.2), as well as considerations on how far off the trivial source bimodule ring is from being commutative (cf. 13.3).

Convention: all $\mathcal{O}$-algebras are assumed to be finitely generated as $\mathcal{O}$-modules, and all modules are finitely generated left modules, unless stated otherwise. Given two $\mathcal{O}$-algebras A, B, we assume that for any A-B-bimodule M the left and right $\mathcal{O}$-module structure on M concident; in other words, M can be viewed as $A \underset{\mathcal{O}}{\otimes} B^{op}$-module. Given finite groups G, H, an $\mathcal{O}G$-$\mathcal{O}H$-bimodule M can be viewed as $\mathcal{O}(G \times H)$-module, with $(x, y) \in G \times H$ acting on M by sending $m \in M$ to xmy^{-1}. In particular, if M is an indecomposable $\mathcal{O}G$-$\mathcal{O}H$-bimodule, it makes sense to consider its vertices and sources as $\mathcal{O}(G \times H)$-module.

§2 Fusion systems

As a consequence of results in [1], [9], any block of a finite group gives rise to a *fusion system* on any of its defect groups (see 4.4 below). When dealing with fusion systems - a concept introduced by Puig in the early 1990's - we follow the terminology in [18, §2]. Given a fusion system $\mathcal{F}$ on a finite p-group P, a subgroup Q is called *fully $\mathcal{F}$-centralised* if $|C_P(Q)| \geq |C_P(\varphi(Q))|$ for any $\varphi \in \mathrm{Hom}_{\mathcal{F}}(Q, P)$. By a result of Puig, any fully $\mathcal{F}$-centralised subgroup Q of P gives rise to a fusion system, denoted by $C_{\mathcal{F}}(Q)$, on $C_P(Q)$. See e.g. [19] for a more detailed introduction to fusion systems. We collect in this section some elementary facts on centralisers in fusion systems.

Proposition 2.1. *Let $\mathcal{F}$ be a fusion system on a finite p-group P, let Q be a fully $\mathcal{F}$-centralised subgroup of P and let S be a subgroup of $C_P(Q)$. Then S is fully $C_{\mathcal{F}}(Q)$-centralised if and only if QS is fully $\mathcal{F}$-centralised.*

Proof. Suppose that S is fully $C_{\mathcal{F}}(Q)$-centralised. Let $\psi : QS \to P$ be a morphism in $\mathcal{F}$. Denote by $\tau : \psi(Q) \to Q$ the inverse of the isomorphism $\psi|_Q : Q \cong \psi(Q)$ induced by ψ. Since Q is fully $\mathcal{F}$-centralised, τ extends to a morphism $\sigma : \psi(Q)C_P(\psi(Q)) \to QC_P(Q)$ in $\mathcal{F}$ (cf. [18]). Note that $\psi(S) \subseteq C_P(\psi(Q))$. By construction, we have $\sigma \circ \psi|_Q = \mathrm{Id}_Q$. Thus $\sigma \circ \psi|_S$ is a morphism in $C_{\mathcal{F}}(Q)$. Since S is fully $C_{\mathcal{F}}(Q)$-centralised, we have

$$|C_P(QS)| = |C_{C_P(Q)}(S)| \geq |C_{C_P(Q)}(\sigma(\psi(S)))| = |C_P(Q\sigma(\psi(S))|$$

Now σ sends $C_P(\psi(QS))$ to $C_P(Q\sigma(\psi(S)))$, and hence $|C_P(Q\sigma(\psi(S)))| \geq |C_P(\psi(QS)|$, which proves that QS is fully $\mathcal{F}$-centralised. Conversely, suppose that QS is fully $\mathcal{F}$-centralised. Let $\tau : S \to C_P(Q)$ be a morphism in $C_{\mathcal{F}}(Q)$. That is, τ extends to a morphism $\sigma : QS \to P$ in $\mathcal{F}$ such that $\sigma|_Q = \mathrm{Id}_Q$. Since QS is fully $\mathcal{F}$-centralised, we have

$$|C_{C_P(Q)}(S)| = |C_P(QS)| \geq |C_P(\sigma(QS))| = |C_P(Q\tau(S))| = |C_{C_P(Q)}(\tau(S))|$$

which proves that S is fully $C_{\mathcal{F}}(Q)$-centralised. □

Let P, Q be finite p-groups, let $\mathcal{F}$ be a fusion system on P and let $\mathcal{G}$ be a fusion system on Q. By [6], the category on $P \times Q$ generated by $\mathcal{F}$ and $\mathcal{G}$ is a fusion system on $P \times Q$, denoted by $\mathcal{F} \times \mathcal{G}$. In particular, $\mathcal{F}$ determines a fusion system $\mathcal{F} \times \mathcal{F}$ on $P \times P$.

Proposition 2.2. *Let $\mathcal{F}$ be a fusion system on a finite p-group P, let Q be a subgroup of P, let $\tau \in \mathrm{Hom}_{\mathcal{F}}(Q, P)$ and set $\Delta_\tau Q = \{(u, \tau(u)) \mid u \in Q\}$. The subgroup $\Delta_\tau Q$ of $P \times P$ is fully $\mathcal{F} \times \mathcal{F}$-centralised if and only if Q and $\tau(Q)$ are fully $\mathcal{F}$-centralised.*

Proof. Let $\sigma : \Delta_\tau Q \to P$ be a morphism in $\mathcal{F} \times \mathcal{F}$. By the definition of $\mathcal{F} \times \mathcal{F}$, this means there are morphisms $\varphi \in \mathrm{Hom}_{\mathcal{F}}(Q, P)$ and $\psi \in \mathrm{Hom}_{\mathcal{F}}(\tau(Q), P)$ such that $\sigma(u, \tau(u)) = (\varphi(u), \psi(\tau(u)))$ for all $u \in Q$. Clearly $C_{P\times P}(\Delta_\tau Q) = C_P(Q) \times C_P(\tau(Q))$ and $C_{P\times P}(\sigma(\Delta_\tau Q)) = C_P(\varphi(Q)) \times C_P(\psi(\tau(Q)))$. Thus the inequality $|C_{P\times P}(\sigma(\Delta_\tau Q))| \leq |C_{P\times P}(\Delta_\tau Q)|$ holds for all morphisms $\sigma : \Delta_\tau Q \to P \times P$ in $\mathcal{F} \times \mathcal{F}$ if and only if the inequalities $|C_P(\varphi(Q))| \leq |C_P(Q)|$ and $|C_P(\psi(\tau(Q)))| \leq |C_P(\tau(Q))|$ hold for all $\varphi \in \mathrm{Hom}_{\mathcal{F}}(Q, P)$ and all $\psi \in \mathrm{Hom}_{\mathcal{F}}(\tau(Q), P)$. The result follows. □

Proposition 2.3. *Let $\mathcal{F}$ be a fusion system on a finite p-group P and let Q be a fully $\mathcal{F}$-centralised subgroup of P. Then the fusion systems $C_{\mathcal{F}\times\mathcal{F}}(\Delta Q)$ and $C_{\mathcal{F}}(Q) \times C_{\mathcal{F}}(Q)$ on $C_{P\times P}(\Delta Q) = C_P(Q) \times C_P(Q)$ are equal.*

Proof. Since Q is fully $\mathcal{F}$-centralised it follows from 2.2 that ΔQ is fully $\mathcal{F} \times \mathcal{F}$-centralised. Thus both $C_{\mathcal{F}\times\mathcal{F}}(\Delta Q)$ and $C_{\mathcal{F}}(Q) \times C_{\mathcal{F}}(Q)$ are fusion systems on $C_{P\times P}(\Delta Q) = C_P(Q) \times C_P(Q)$. Let R be a subgroup of $C_P(Q) \times C_P(Q)$. Denote by R_1, R_2 the images of R in $C_P(Q)$ under the two projections of the direct product $C_P(Q) \times C_P(Q)$ onto its first and second factor, respectively. Let $\varphi : R \to C_P(Q) \times C_P(Q)$ be a morphism in $C_{\mathcal{F}\times\mathcal{F}}(\Delta Q)$. That is, φ extends to a morphism $\psi : R\Delta Q \to P \times P$ in $\mathcal{F} \times \mathcal{F}$ such that $\psi|_{\Delta Q} = \mathrm{Id}_{\Delta Q}$. The images in P of $R\Delta Q$ under the two projections of $P \times P$ onto its factors are equal to $R_1 Q$ and $R_2 Q$. Thus ψ being a morphism in $\mathcal{F} \times \mathcal{F}$ is equivalent to the existence of morphisms $\psi_1 : R_1 Q \to P$ and $\psi_2 : R_2 Q \to P$ which are the identity on Q and which induce ψ. The result follows. □

§3 The generalised Brauer construction

3.1. The Brauer construction has its origins in Brauer's observation that the canonical linear projection $kG \to kC_G(P)$ induces an algebra homomorphism $Z(kG) \to Z(kC_G(P))$, for any finite group G and any p-subgroup P of G. Broué [7] observed that this extends to a surjective algebra homomorphism $(kG)^{\Delta P} \to kC_G(P)$. This in turn has been extended to G-algebras in [9], to modules in [11], and generalised further by Boltje and Külshammer in [3]. We briefly recall the definition and some of the basic properties of this construction, following [3, §2]. Let P be a finite p-group and let $\theta : P \to \mathcal{O}^\times$ be a group homomorphism. For any $\mathcal{O}P$-module M we set

$$M^{(P,\theta)} = \{m \in M \mid ym = \theta(y)m \ (\forall y \in P)\} .$$

For $\theta = 1$ (the trivial group homomorphism) this is the usual $\mathcal{O}$-submodule of P-fixpoints in M; we write M^P instead of $M^{(P,1)}$. For Q a subgroup of P and $\tau = \theta|_Q : Q \to \mathcal{O}^\times$ we denote by $[P/Q]$ a system of representatives in P of the cosets P/Q, and for $m \in M^{(Q,\tau)}$ we set $\mathrm{Tr}^{(P,\theta)}_{(Q,\tau)}(m) = \sum_{y\in[P/Q]} \theta(y)^{-1} ym$. This expression does not depend on the choice of $[P/Q]$ and hence defines an $\mathcal{O}$-linear map $\mathrm{Tr}^{(P,\theta)}_{(Q,\tau)} : M^{(Q,\tau)} \to M^{(P,\theta)}$. We set $M^{(P,\theta)}_{(Q,\tau)} = \mathrm{Im}(\mathrm{Tr}^{(P,\theta)}_{(Q,\tau)})$. If $\theta = 1$ we write Tr^P_Q

and M_Q^P instead of $\mathrm{Tr}_{(Q,\tau)}^{(P,\theta)}$ and $M_{(Q,\tau)}^{(P,\theta)}$, respectively. The generalised Brauer construction [3, 2.2] is defined by

$$M(P,\theta) = M^{(P,\theta)}/(\sum_Q M_{(Q,\theta|_Q)}^{(P,\theta)} + J(\mathcal{O})M^{(P,\theta)})$$

where Q runs over the set of all proper subgroups of P. The Brauer homomorphism is defined to be the canonical surjection

$$\mathrm{Br}_{(P,\theta)}^M : M^{(P,\theta)} \longrightarrow M(P,\theta)$$

If no confusion arises we will write $\mathrm{Br}_{(P,\theta)}$ instead of $\mathrm{Br}_{(P,\theta)}^M$. If moreover θ is the trivial homomorphism we write $M(P)$ instead of $M(P,\theta)$ and Br_P instead of $\mathrm{Br}_{(P,\theta)}$. One can play back this construction to the usual Brauer construction as follows: for $\theta : P \to \mathcal{O}^\times$ a group homomorphism, denote by $\bar{\theta}$ the $\mathcal{O}$-algebra automorphism of $\mathcal{O}P$ sending $u \in P$ to $u\theta(u^{-1})$. For M an $\mathcal{O}P$-module denote by ${}_{\bar{\theta}}M$ the $\mathcal{O}P$-module obtained from restricting the action of $\mathcal{O}P$ along $\bar{\theta}$; that is, ${}_{\bar{\theta}}M = M$ as $\mathcal{O}$-module, and $y \in P$ acts on $m \in M$ as multiplication by $y\theta(y)^{-1}$. Then $M^{(P,\theta)} = ({}_{\bar{\theta}}M)^P$ and we have

$$M(P,\theta) = ({}_{\bar{\theta}}M)(P) \ .$$

Note that if M is a permutation $\mathcal{O}P$-module and if $\theta : P \to \mathcal{O}^\times$ is a non-trivial group homomorphism then $M(P,\theta) = \{0\}$.

3.2. If G is a finite group having P as subgroup, then $\mathcal{O}G$ is an $\mathcal{O}(P \times P)$-module with $(u,v) \in P \times P$ acting on $a \in \mathcal{O}G$ as uav^{-1}. For Q a subgroup of P we set $\Delta Q = \{(u,u) \mid u \in Q\}$. This is a subgroup of $P \times P$, hence acts on $\mathcal{O}G$ (this is the conjugation action of Q on $\mathcal{O}G$). The corresponding Brauer construction $(\mathcal{O}G)(\Delta Q)$ is well-known to be canonically isomorphic to $kC_G(Q)$, and under the canonical identification $(\mathcal{O}G)(\Delta Q) = kC_G(Q)$ the Brauer homomorphism

$$\mathrm{Br}_{\Delta Q} : (\mathcal{O}G)^{\Delta Q} \longrightarrow kC_G(Q)$$

is a surjective algebra homomorphism induced by the linear map sending $x \in G$ to its image in $kC_G(Q)$ if $x \in C_G(Q)$ and to zero otherwise.

3.3. Let A be an interior P-algebra; that is, A is an $\mathcal{O}$-algebra endowed with a group homomorphism $\sigma : P \to A^\times$ (or equivalently, with a unitary algebra homomorphism $\mathcal{O}P \to A$). Then A becomes an $\mathcal{O}(P \times P)$-module with $(u,v) \in P \times P$ acting on $a \in A$ by $\sigma(u)a\sigma(v^{-1})$. For Q a subgroup of P, the Brauer construction $A(\Delta Q)$ becomes then an interior $C_P(Q)$-algebra, with multiplication induced by that in $A^{\Delta Q}$ via the Brauer homomorphism $\mathrm{Br}_{\Delta Q}^A$, and with structural homomorphism

$$C_P(Q) \overset{\sigma|_{C_P(Q)}}{\longrightarrow} (A^\times)^{\Delta Q} = (A^{\Delta Q})^\times \overset{\mathrm{Br}_{\Delta Q}}{\longrightarrow} A(\Delta Q)^\times$$

If M is an A-module, then M can be viewed as $\mathcal{O}P$-module via $\sigma : P \to A^\times$. The A-module structure on M induces then an $A(\Delta Q)$-module structure on the corresponding Brauer construction $M(Q)$, for Q a subgroup of P. If A and B are two interior P-algebras with structural homomorphisms $\sigma : P \to A^\times$ and $\tau : P \to B^\times$, and if M is an A-B-bimodule, then M can be viewed as $\mathcal{O}(P \times P)$-module with $(u,v) \in P \times P$ acting on $m \in M$ as $\sigma(u)m\tau(v^{-1})$. For Q a subgroup of P, the A-B-bimodule structure on M restricts to an $A^{\Delta Q}$-$B^{\Delta Q}$-bimodule structure on $M^{\Delta Q}$ which in turn induces an $A(\Delta Q)$-$B(\Delta Q)$-bimodule structure on $M(\Delta Q)$. In particular, if G, H are two finite groups having P as common p-subgroup and if M is an $\mathcal{O}G$-$\mathcal{O}H$-bimodule, we regard $M(\Delta Q)$ as $kC_G(Q)$-$kC_H(Q)$-bimodule through the canonical identifications.

The Brauer construction is functorial, sends p-permutation modules to p-permutation modules, commutes with duality, and in some cases, commutes with tensor products. See [28, §11, §27] for more details. Calculating the Brauer construction for p-permutation modules is particularly easy:

Lemma 3.4. *([28, (27.6)]) Let P be a finite p-group and let M be an $\mathcal{O}P$-module having a P-stable $\mathcal{O}$-basis X. Then the image in $M(P)$ of the fixpoint set X^P under Br_P^M is a k-basis of $M(P)$.*

Lemma 3.5. *([15, 6.1.(iii)]) Let P be a finite p-group and let M be an indecomposable $\mathcal{O}P$-$\mathcal{O}P$-bimodule having a $P \times P$-stable $\mathcal{O}$-basis such that M is projective as left and right $\mathcal{O}P$-module. Then $M(\Delta P)$ is non zero if and only if $M \cong \mathcal{O}P$ as $\mathcal{O}P$-$\mathcal{O}P$-bimodule.*

3.6 Let P be a finite p-group, let $\theta : P \to \mathcal{O}^\times$ be a group homomorphism, let A be an interior P-algebra with structural homomorphism $\sigma : P \to A^\times$ and let M be an A-module. Denote by ${}_{\bar\theta}A$ the interior P-algebra which is equal to A as $\mathcal{O}$-algebra, with structural homomorphism sending $y \in P$ to $\sigma(y)\theta(y^{-1})$. Note that $\sigma(y)$ and $\sigma(y)\theta(y^{-1})$ act in the same way on A. Thus $({}_{\bar\theta}A)(\Delta P) = A(\Delta P)$ as k-algebras. More precisely, for Q a subgroup of P we get that

$$({}_{\bar\theta}A)(\Delta Q) = A(\Delta Q)$$

as interior $C_P(Q)$-algebras. The A-module M is still a ${}_{\bar\theta}A$-module, and its restriction to $\mathcal{O}P$ via the structural homomorphism is now ${}_{\bar\theta}M$. Thus, using 3.3, we get that $M(P,\theta)$ is an $A(\Delta P)$-module with a module structure which is induced by the $A^{\Delta P}$-module structure on $M^{(P,\theta)}$. In particular, if $A = \mathcal{O}Gb$ for some finite group G containing P and some idempotent $b \in Z(\mathcal{O}G)$ it follows that $M(P,\theta)$ is a $kC_G(P)\mathrm{Br}_{\Delta P}(b)$-module.

3.7 By [4, Theorem 2.4], the generalised Brauer construction can be used to "lift" generalised decomposition maps. We describe this briefly. Let G be a finite group and let b be an idempotent in $Z(\mathcal{O}G)$. Denote by $\mathbb{Z}\mathrm{Irr}_K(G,b)$ the group of generalised characters of G over K associated with b and by $\mathbb{Z}\mathrm{IBr}_K(G,b)$ the corresponding group of generalised Brauer characters. If $b = 1$ we write $\mathbb{Z}\mathrm{Irr}_K(G)$ instead of $\mathbb{Z}\mathrm{Irr}_K(G,b)$. Denote by $\bar b$ the image of b in kG. For a $kG\bar b$-module N denote by $[[N]]$ its Brauer character viewed as element in the group $\mathbb{Z}\mathrm{IBr}_K(G,b)$ associated with b (which can be identified with the Grothendieck group of $kG\bar b$-modules taken with respect to all short exact sequences). For u a p-element in G and e an idempotent in $Z(kC_G(u))$ the generalised decomposition map

$$d_{(G,b)}^{(u,e)} : \mathbb{Z}\mathrm{Irr}_K(G,b) \longrightarrow K \underset{\mathbb{Z}}{\otimes} \mathbb{Z}\mathrm{IBr}_K(C_G(u),e)$$

is defined by $d_{(G,b)}^{(u,e)}(\chi)(s) = \chi(eus)$ for all p'-elements s in $C_G(u)$. An $\mathcal{O}G$-module M is called a *linear source module* if M is a finite direct sum of summands of $\mathcal{O}G$-modules of the form $\mathcal{O}G \underset{\mathcal{O}Q}{\otimes} {}_\theta\mathcal{O}$, where Q runs over the p-subgroups of G and θ over the group homomorphisms from Q to $\mathcal{O}^\times$ and where ${}_\theta\mathcal{O}$ is the rank one module determined by θ.

Theorem 3.8. *([4, Theorem 2.4]) Let G be a finite group, let u be a p-element in G, let b be an idempotent in $Z(\mathcal{O}G)$ and let e be an idempotent in $Z(kC_G(u))$. Let M be a linear source $\mathcal{O}Gb$-module and let χ be the character of M. We have*

$$d_{(G,b)}^{(u,e)}(\chi) \;=\; \sum_{\theta \in \mathrm{Hom}(\langle u\rangle, \mathcal{O}^\times)} \theta(u) \otimes [[M(\langle u\rangle, \theta)]] \;.$$

The generalised Brauer construction "commutes" with "cutting by idempotents":

Lemma 3.9. *Let G be a finite group, let P be a p-subgroup of G, let $\theta : P \to \mathcal{O}^\times$ be a group homomorphism and let i be an idempotent in $(\mathcal{O}G)^{\Delta P}$ such that $\mathrm{Br}_{\Delta P}(i) \neq 0$. For any $\mathcal{O}G$-module M there is a natural isomorphism of $\mathrm{Br}_{\Delta P}(i)kC_G(P)\mathrm{Br}_{\Delta P}(i)$-modules*

$$(iM)(P,\theta) \cong \mathrm{Br}_{\Delta P}(i)M(P,\theta) \ .$$

Proof. The $kC_G(P)$-module structure on $M(P,\theta)$ is induced by the $(\mathcal{O}G)^{\Delta P}$-module structure on $M^{(P,\theta)}$. That is, for $a \in (\mathcal{O}G)^{\Delta P}$ and $m \in M^{(P,\theta)}$ we have

$$am + \ker(\mathrm{Br}^M_{(P,\theta)}) = \mathrm{Br}_{\Delta P}(a)(m + \ker(\mathrm{Br}^M_{(P,\theta)})) \ .$$

Since i commutes with P we have $(iM)^{(P,\theta)} = i(M^{(P,\theta)})$. It follows that the map sending $im + \ker(\mathrm{Br}^M_{(P,\theta)})$ to $\mathrm{Br}_{\Delta P}(i)(m+\ker(\mathrm{Br}^M_{(P,\theta)})$ is in fact an equality $(iM)(P,\theta) = \mathrm{Br}_{\Delta P}(i)M(P,\theta)$ of subsets of $M(P,\theta)$. □

3.10. Let P be a finite p-group and let A be an interior P-algebra. Following [21] a *local point of a subgroup Q of P on A* is an $(A^{\Delta Q})^\times$-conjugacy class δ of primitive idempotents in $A^{\Delta Q}$ such that $\mathrm{Br}_{\Delta Q}(\delta) \neq 0$. Since $\mathrm{Br}_{\Delta Q} : A^{\Delta Q} \to A(\Delta Q)$ is a surjective algebra homomorphism, the standard lifting theorems for idempotents imply that then $\mathrm{Br}_{\Delta Q}(\delta)$ is a conjugacy class of primitive idempotents in $A(\Delta Q)$. If Q, R are subgroups of P, δ a local point of Q on A and ϵ a local point of R on A, we write $Q_\delta \subseteq R_\epsilon$ if $Q \subseteq R$ and if there are $i \in \delta$, $j \in \epsilon$ satisfying $ij = ji = i$. We refer to [28] for a detailed account on local pointed groups.

§4 Almost source algebras

Let G be a finite group and let b be a block of $\mathcal{O}G$; that is b is a primitive idempotent in $Z(\mathcal{O}G)$. Let P be a defect group of b; that is, P is a maximal p-subgroup of G with the property that $\mathrm{Br}_{\Delta P}(b) \neq 0$, where $\mathrm{Br}_{\Delta P} : (\mathcal{O}G)^{\Delta P} \to kC_G(P)$ is the Brauer homomorphism as described in the previous section. Equivalently, P is a minimal subgroup of G such that $b \in (\mathcal{O}G)^G_P$. Since $\mathrm{Br}_{\Delta P}(b) \neq 0$ there must be a primitive idempotent $i \in (\mathcal{O}Gb)^{\Delta P}$ satisfying $\mathrm{Br}_{\Delta P}(i) \neq 0$. Such an idempotent i is called a *source idempotent* and the algebra $A = i\mathcal{O}Gi$ is called a *source algebra* of b. This concept is due to Puig [21]. A source algebra is always considered as *interior P-algebra*; that is, together with the group homomorphism $P \to A^\times$ sending $u \in P$ to $ui = iu = iui \in A^\times$. If P' is another defect group of b and $i' \in (\mathcal{O}Gb)^{\Delta P'}$ a source idempotent then there is an $\mathcal{O}$-algebra isomorphism $\alpha : i\mathcal{O}Gi \cong i'\mathcal{O}Gi'$ and an element $x \in G$ such that ${}^xP = P'$ and such that $\alpha(ui) = {}^xui'$ for all $u \in P$. Source algebras of a block b are Morita equivalent to the block algebra $\mathcal{O}Gb$; more generally:

Proposition 4.1. *([21, 3.5]) Let G be a finite group, let b be a block of $\mathcal{O}G$, let P be a defect group of b and let i be an idempotent in $(\mathcal{O}Gb)^{\Delta P}$ such that $\mathrm{Br}_{\Delta P}(i) \neq 0$. The block algebra $\mathcal{O}Gb$ and the algebra $i\mathcal{O}Gi$ are Morita equivalent via the $\mathcal{O}Gb$-$i\mathcal{O}Gi$-bimodule $\mathcal{O}Gi$ and its $\mathcal{O}$-dual $i\mathcal{O}G$.*

Proof. We sketch a proof for the convenience of the reader. The obvious map $iOG \underset{\mathcal{O}G}{\otimes} \mathcal{O}Gi \to i\mathcal{O}Gi$ given by multiplication in $\mathcal{O}G$ is clearly an isomorphism. For any idempotent $j \in (\mathcal{O}Gb)^{\Delta P}$, $(\mathcal{O}Gj)^{\Delta G}_{\Delta P}$ is a left ideal, hence an ideal in the commutative algebra $Z(\mathcal{O}Gb)$. Since $b \in (\mathcal{O}Gb)^{\Delta G}_{\Delta P}$, Rosenberg's Lemma [28, (4.9)] implies that $b \in (\mathcal{O}Gj)^{\Delta G}_{\Delta P}$ for some primitive idempotent $j \in (\mathcal{O}Gb)^{\Delta P}$. Since $\mathrm{Br}_{\Delta P}(b) \neq 0$ this implies $\mathrm{Br}_{\Delta P}(j) \neq 0$ and hence that j belongs to a local point of P on $\mathcal{O}Gb$. Since all local points of P on $\mathcal{O}Gb$ are $N_G(P)$-conjugate it follws that $b \in (\mathcal{O}Gi)^{\Delta G}_{\Delta P}$. Thus there is $c \in (\mathcal{O}G)^{\Delta P}$ such that $\mathrm{Tr}^{\Delta G}_{\Delta P}(ci) = b$. Now $\mathrm{Tr}^{\Delta G}_{\Delta P}(ci)$ belongs to the two-sided ideal $\mathcal{O}Gi\mathcal{O}G$ generated

by i, and hence b is in the image of the the map $\mathcal{O}Gi \underset{i\mathcal{O}Gi}{\otimes} i\mathcal{O}G \to \mathcal{O}G$ given by multiplication in $\mathcal{O}G$. Therefore this map is surjective and the result follows. □

Source algebras are known to be "relatively separable" with respect to their defect groups. As before, this is true in slightly greater generality:

Proposition 4.2. *Let G be a finite group, let b be a block of $\mathcal{O}G$, let P be a defect group of b and let i be an idempotent in $(\mathcal{O}Gb)^{\Delta P}$ such that $\mathrm{Br}_{\Delta P}(i) \neq 0$. Set $A = i\mathcal{O}Gi$. The canonical map $A \underset{\mathcal{O}P}{\otimes} A \to A$ induced by multiplication in A splits as homomorphism of A-A-bimodules; in particular, A is isomorphic to a direct summand of $A \underset{\mathcal{O}P}{\otimes} A$ as A-A-bimodule.*

Proof. As in the proof of 4.1, write $b = \mathrm{Tr}_{\Delta P}^{\Delta G}(ci)$ for some $c \in (\mathcal{O}G)^{\Delta P}$. One checks that the map sending $a \in \mathcal{O}G$ to $\sum_{x \in [G/P]} axci \otimes ix^{-1}$ in $\mathcal{O}Gi \underset{\mathcal{O}P}{\otimes} i\mathcal{O}G$ is a section for the canonical $\mathcal{O}G$-$\mathcal{O}G$-bimodule homomorphism $\mathcal{O}Gi \underset{\mathcal{O}G}{\otimes} i\mathcal{O}G \to \mathcal{O}Gb$ given by multiplication in $\mathcal{O}G$. Multiplying this homomorphism by i on the left and right yields the required section as stated. □

Note that the idempotent $i \in (\mathcal{O}Gb)^{\Delta P}$ in 4.1 and 4.2 need not be primitive (if it is, then A is a source algebra of b). A crucial property of a source idempotent $i \in (\mathcal{O}Gb)^{\Delta P}$ is that, by [9, 1.8], for any subgroup Q of P the idempotent $\mathrm{Br}_{\Delta Q}(i)$ in $kC_G(Q)$ belongs in fact to a unique block algebra of $kC_G(Q)$. In the following definition we keep this key property but require no longer that i is primitive:

Definition 4.3 Let G be a finite group, let b be a block of $\mathcal{O}G$, let P be a defect group of b. An idempotent i in $(\mathcal{O}Gb)^{\Delta P}$ is called an *almost source idempotent* if $\mathrm{Br}_{\Delta P}(i) \neq 0$ and for every subgroup Q of P there is a unique block e_Q of $kC_G(Q)$ such that $\mathrm{Br}_{\Delta Q}(i) \in kC_G(Q)e_Q$. The interior P-algebra $i\mathcal{O}Gi$ is then called an *almost source algebra of the block b*.

Remark 4.4. As pointed out before, any source idempotent is an almost source idempotent, by [9, 1.8]. The choice of an almost source idempotent $i \in (\mathcal{O}Gb)^{\Delta P}$ determines a fusion system $\mathcal{F}$ as follows. For any subgroup Q of P denote by e_Q the unique block of $kC_G(Q)$ such that $\mathrm{Br}_{\Delta Q}(i)e_Q \neq 0$. For any two subgroups Q, R of P, the set $\mathrm{Hom}_{\mathcal{F}}(Q, R)$ is the set of all group homomorphisms $\varphi : Q \to R$ for which there is an element $x \in G$ satisfying $\varphi(u) = xux^{-1}$ for all $u \in Q$ and satisfying $xe_Qx^{-1} = e_{xQx^{-1}}$, which is equivalent to the inclusion of Brauer pairs ${}^x(Q, e_Q) \subseteq (R, e_R)$ as defined in [1]. It follows from the results in [1] that $\mathcal{F}$ is indeed a fusion system; see e.g. [18, §2] or [19] for some more details. Moreover, a subgroup Q of P is fully $\mathcal{F}$-centralised if and only if $C_P(Q)$ is a defect group of the block e_Q of $kC_G(Q)$. Given a subgroup Q of P it is always possible to find a subgroup R of P such that $Q \cong R$ in $\mathcal{F}$ and such that R is fully $\mathcal{F}$-centralised. The following observation explains why we will need to work with fully centralised subgroups and almost source idempotents rather than source idempotents:

Proposition 4.5. *Let G be a finite group, let b be a block of $\mathcal{O}G$, let P be a defect group of b and let i be an almost source idempotent in $(\mathcal{O}Gb)^{\Delta P}$. Let $\mathcal{F}$ be the fusion system determined by i on P, let Q be a fully $\mathcal{F}$-centralised subgroup of P and let e be the unique block of $kC_G(Q)$ such that $\mathrm{Br}_{\Delta Q}(i)e \neq 0$. The idempotent $\mathrm{Br}_{\Delta Q}(i)$ in $(kC_G(Q)e)^{\Delta C_P(Q)}$ is an almost source idempotent of the block e of $kC_G(Q)$.*

Proof. Since Q is fully $\mathcal{F}$-centralised, the group $C_P(Q)$ is a defect group of e (cf. [18, §2]). The idempotent $\mathrm{Br}_{\Delta Q}(i)$ belongs to $kC_G(Q)^{C_P(Q)}$ and satisfies $\mathrm{Br}_{\Delta C_P(Q)}(\mathrm{Br}_{\Delta Q}(i)) = \mathrm{Br}_{\Delta QC_P(Q)}(i)$ which

is non zero because even $\mathrm{Br}_{\Delta P}(i)$ is non zero. Moreover, for every subgroup S of $C_P(Q)$ we have $C_{C_P(Q)}(S) = C_P(QS)$ and $C_{C_G(Q)}(S) = C_G(QS)$. Thus the unique block f of $C_G(QS)$ satisfying $\mathrm{Br}_{\Delta QS}(i)f \neq 0$ is also the unique block of $kC_{C_G(Q)}(S)$ satisfying $\mathrm{Br}_{\Delta S}(\mathrm{Br}_{\Delta Q}(i))f \neq 0$, and so $\mathrm{Br}_{\Delta Q}(i)$ is an almost source idempotent. □

Proposition 4.6. *Let G be a finite group, let b be a block of $\mathcal{O}G$, let P be a defect group of b and let i be an almost source idempotent in $(\mathcal{O}Gb)^{\Delta P}$. Set $A = i\mathcal{O}Gi$ and let $\mathcal{F}$ be the fusion system of A on P. Let Q be a fully $\mathcal{F}$-centralised subgroup of P and let e be the unique block of $kC_G(Q)$ such that $\mathrm{Br}_{\Delta Q}(i)e \neq 0$.*

(i) The block algebra $kC_G(Q)e_Q$ is Morita equivalent to the algebra $A(\Delta Q)$ via the bimodule $kC_G(Q)\mathrm{Br}_{\Delta Q}(i)$ and its dual.

(ii) For any local point δ of Q on $\mathcal{O}Gb$ satisfying $\mathrm{Br}_{\Delta Q}(\delta)e_Q \neq \{0\}$ we have $\delta \cap A \neq \emptyset$.

(iii) The canonical map $A(\Delta Q) \underset{kC_P(Q)}{\otimes} A(\Delta Q) \to A(\Delta Q)$ induced by multiplication in $A(\Delta Q)$ splits as homomorphism of $A(\Delta Q)$-$A(\Delta Q)$-bimodules.

Proof. By 4.5, $\mathrm{Br}_{\Delta Q}(i)$ is an almost source idempotent, and thus 4.1.(i) applies to $kC_G(Q)e$ and $\mathrm{Br}_{\Delta Q}(i)$ which implies statement (i). If δ is a local point of Q on $\mathcal{O}Gb$ satisfying $\mathrm{Br}_{\Delta Q}(e_Q) \neq \{0\}$ then $\mathrm{Br}_{\Delta Q}(\delta)$ is a conjugacy class of primitive idempotents in $kC_G(Q)e$. Since $kC_G(Q)e$ is Morita equivalent to $A(\Delta Q) = \mathrm{Br}_{\Delta Q}(i)kC_G(Q)\mathrm{Br}_{\Delta Q}(i)$ there is $j \in \delta$ such that $\mathrm{Br}_{\Delta Q}(j) \in \mathrm{Br}_{\Delta Q}(i)kC_G(Q)\mathrm{Br}_{\Delta Q}(i)$. The usual lifting theorems for idempotents imply that j can be chosen in $A^{\Delta Q} = i(kG)^{\Delta Q}i$, whence statement (ii). Statement (iii) follows from 4.1.(ii) applied to $kC_G(Q)$, e_Q, $\mathrm{Br}_{\Delta Q}(i)$ and $C_P(Q)$ instead of $\mathcal{O}G$, b, i and P, respectively. □

This explains why it is technically easier to work at the source algebra level: while $(\mathcal{O}Gb)(\Delta Q) = kC_G(Q)\mathrm{Br}_{\Delta Q}(b)$ may decompose as product of more than one block algebra, the algebra $A(\Delta Q)$ remains indecomposable so long as one chooses Q to be fully centralised with respect to the fusion system of A on P. The following result is a tool to switch back and forth between block and source algebras:

Proposition 4.7. *Let G be a finite group, let b be a block of $\mathcal{O}G$, let P be a defect group of b and let i be an almost source idempotent in $(\mathcal{O}Gb)^{\Delta P}$. Set $A = i\mathcal{O}Gi$ and let $\mathcal{F}$ be the fusion system of A on P. Let Q be a fully $\mathcal{F}$-centralised subgroup of P and let e be the unique block of $kC_G(Q)$ such that $\mathrm{Br}_{\Delta Q}(i)e \neq 0$. Let $\theta : Q \to \mathcal{O}^\times$ be a group homomorphism. For any A-module M there is a natural isomorphism of $kC_G(Q)e$-modules*

$$e((\mathcal{O}Gi \underset{A}{\otimes} M)(Q,\theta)) \cong kC_G(Q)\mathrm{Br}_{\Delta Q}(i) \underset{A(\Delta Q)}{\otimes} M(Q,\theta)\ .$$

Proof. Since $\mathrm{Br}_{\Delta Q}(i) \in kC_G(Q)e$, both sides in the statement are indeed $kC_G(Q)e$-modules. By 4.6.(i), multiplication by $\mathrm{Br}_{\Delta Q}(i)$ induces a Morita equivalence between $kC_G(Q)e$ and $A(\Delta Q)$. Thus, in order to show that there is a natural isomorphism as stated, it suffices to show this after multiplying both sides with $\mathrm{Br}_{\Delta Q}(i)$. Using 3.9, the left side becomes

$$\mathrm{Br}_{\Delta Q}(i)(kGi \underset{A}{\otimes} M)(Q,\theta)) = (ikGi \underset{A}{\otimes} M)(Q,\theta) = (A \underset{A}{\otimes} M)(Q,\theta) \cong M(Q,\theta)$$

and the right side becomes

$$\mathrm{Br}_{\Delta Q}(i)kC_G(Q)\mathrm{Br}_{\Delta Q}(i) \underset{A(\Delta Q)}{\otimes} M(Q,\theta) \cong M(Q,\theta)\ ;$$

both sides are naturally isomorphic as $A(\Delta Q)$-modules. □

Proposition 4.8. *Let G be a finite group, b a block of $\mathcal{O}G$, P a defect group of b and $i \in (\mathcal{O}Gb)^{\Delta P}$ an almost source idempotent. Let Q be a subgroup of P and let e, e' be two different blocks of $kC_G(Q)$. For any subgroup R of P and any direct summand Y of $\mathcal{O}Gi \underset{\mathcal{O}R}{\otimes} i\mathcal{O}G$ we have $eY(\Delta Q)e' = \{0\}$.*

Proof. We may assume $\mathcal{O} = k$. Since i is an almost source idempotent, for any subgroup S of P there is a unique block e_S of $kC_G(S)$ satisfying $\mathrm{Br}_{\Delta S}(i)e_S \neq 0$. The image of the set $G \times G$ in $kG \underset{kR}{\otimes} kG$ is a k-basis which is stable under the action of ΔQ. In order to compute $Y(\Delta Q)$ we have to determine the ΔQ-fixpoints in this basis. For $x, y \in G$ the image $x \otimes y$ in $kG \underset{kR}{\otimes} kG$ is fixed under the action of ΔQ if for every $u \in Q$ there is $r_u \in R$ such that $ux = xr_u$ and $yu^{-1} = (r_u)^{-1}y$, or equivalently, if and only if $xy \in C_G(Q)$ and $Q^x = {}^yQ \subseteq R$. Thus we have

$$(kG \underset{kR}{\otimes} kG)^{\Delta Q} = \sum_{(x,y)} k(x \otimes y) \ \oplus \ \ker(\mathrm{Br}_{\Delta Q})$$

where (x, y) runs over the set of pairs in $G \times G$ satisfying $xy \in C_G(Q)$ and $Q^x \subseteq R$. Consequently, we get that

$$(ekGi \underset{kR}{\otimes} ikGe')^{\Delta Q} = \sum_{(x,y)} k(exi \otimes iye') \ + \ \ker(\mathrm{Br}_{\Delta Q})$$

with (x, y) as before. Write $exi = e({}^xi)x$. Since i is an almost source idempotent we have $e({}^xi) \in \ker(\mathrm{Br}_{\Delta Q})$ unless $e = {}^x(e_{Q^x})$. Similarly, we have $({}^yi)e' \in \ker(\mathrm{Br}_{\Delta Q})$ unless $e' = (e_{{}^yQ})^y$. But these two equalities would imply the contradiction $e = e'$ since $xy \in C_G(Q)$. Thus at least one of $e({}^xi)$, $({}^yi)e'$ is contained in $\ker(\mathrm{Br}_{\Delta Q})$, whence the result. □

Proposition 4.9. *Let G, H be finite groups, let b, c be blocks of $\mathcal{O}G, \mathcal{O}H$ with defect group P, Q, respectively. Let $i \in (\mathcal{O}Gb)^{\Delta P}$ and $j \in (\mathcal{O}Hc)^{\Delta Q}$ be almost source idempotents, let $\mathcal{F}$ be the fusion system on P determined by i and let $\mathcal{G}$ be the fusion system on Q determined by j. Identify $\mathcal{O}G \underset{\mathcal{O}}{\otimes} \mathcal{O}H$ and $\mathcal{O}(G \times H)$ through the canonical isomorphism.*

(i) $b \otimes c$ is a block of $\mathcal{O}(G \times H)$ and $P \times Q$ is a defect group of $b \otimes c$.

(ii) $i \otimes j$ is an almost source idempotent of $b \otimes c$.

(iii) The fusion system determined by $i \otimes j$ on $P \times Q$ is equal to $\mathcal{F} \times \mathcal{G}$.

Proof. Statement (i) is well-known (and easy to prove). For R a subgroup of $P \times Q$, denote by R_1 the image of R under the projection $P \times Q \to P$ and by R_2 the image of R under the projection $P \times Q \to Q$. Then $R \subseteq R_1 \times R_2$ and $C_{G \times H}(R) = C_G(R_1) \times C_H(R_2)$. Thus $\mathrm{Br}_{\Delta R}(i \otimes j) = \mathrm{Br}_{\Delta R_1}(i) \otimes \mathrm{Br}_{\Delta R_2}(j)$. Therefore, if e_1, f_2 are the unique blocks of $kC_G(R_1)$, $kC_H(R_2)$, respectively, satisfying $\mathrm{Br}_{\Delta R_1}(i)e_1 \neq 0$ and $\mathrm{Br}_{\Delta R_2}(j)f_2 \neq 0$ then $e = e_1 \otimes f_2$ is the unique block of $C_{G \times H}(R)$ satisfying $\mathrm{Br}_{\Delta R}(i \otimes j)e \neq 0$. This shows (ii), and (iii) is clear. □

Proposition 4.10. *Let G be a finite group, let b be a block of $\mathcal{O}G$, let P be a defect group of b and let $i \in (\mathcal{O}Gb)^{\Delta P}$ be an almost source idempotent. Let α be the anti-automorphism of $\mathcal{O}G$ sending $x \in G$ to x^{-1}. Set $b^0 = \alpha(b)$ and $i^0 = \alpha(i)$. Then b^0 is a block of $\mathcal{O}G$ having P as defect group and $i^0 \in (\mathcal{O}Gb^0)^{\Delta P}$ is an almost source idempotent of b^0. Moreover, the fusion systems on P determined by i and by i^0 are equal.*

Proof. Straightforward verification. □

Combining 4.7 and 4.9, 4.10 yields another statement about switching between block algebras and source algebras for bimodules which will be needed in the proof of 1.4:

Proposition 4.11. *Let G, H be finite groups, let b, c be blocks of $\mathcal{O}G$, $\mathcal{O}H$, respectively, having a common defect group P. Let $i \in (\mathcal{O}Gb)^{\Delta P}$ and $j \in (\mathcal{O}Hc)^{\Delta P}$ be almost source idempotents. Suppose that i and j determine the same fusion system $\mathcal{F}$ on P. Let Q be a fully $\mathcal{F}$-centralised subgroup of P and let e and f be the unique blocks of $kC_G(Q)$ and $kC_H(Q)$, respectively, satisfying $\mathrm{Br}_{\Delta Q}(i)e \neq 0$ and $\mathrm{Br}_{\Delta Q}(j)f \neq 0$. Let X be an A-B-bimodule and set $Y = e((\mathcal{O}Gi \underset{A}{\otimes} X \underset{B}{\otimes} j\mathcal{O}H)(\Delta Q))f$. Then*

$$Y \cong kC_G(Q)\mathrm{Br}_{\Delta Q}(i) \underset{A(\Delta Q)}{\otimes} X(\Delta Q) \underset{B(\Delta Q)}{\otimes} \mathrm{Br}_{\Delta Q}(j)kC_H(Q)$$

as $kC_G(Q)e$-$kC_H(Q)f$-bimodules and

$$\mathrm{Br}_{\Delta Q}(i)Y\mathrm{Br}_{\Delta Q}(j) \cong X(\Delta Q)$$

as $A(\Delta Q)$-$B(\Delta Q)$-bimodules.

Proof. By 2.2, ΔQ is fully $\mathcal{F} \times \mathcal{F}$-centralised. By 4.7 applied to $G \times H$, the almost source idempotent $i \otimes j^0$, the subgroup ΔQ of $P \times P$ and the block $e \otimes f^0$ of $C_{G\times H}(\Delta Q)$ we get the first of the two isomorphisms, and multiplying this on the left and on the right by $\mathrm{Br}_{\Delta Q}(i)$ and $\mathrm{Br}_{\Delta Q}(j)$, respectively, yields the second isomorphism. □

§5 Fusion in almost source algebras

By [22, 3.1], the fusion system $\mathcal{F}$ of a block b of $\mathcal{O}G$ can be read off the $\mathcal{O}P$-$\mathcal{O}P$-bimodule structure of the source algebra $A = i\mathcal{O}Gi$ of b. The following Propositions collect some technical details around this theme, similar to material in [15, 17], extended here to almost source idempotents.

Proposition 5.1. *Let G be a finite group, let b be a block of $\mathcal{O}G$ with defect group P, let $i \in (\mathcal{O}Gb)^{\Delta P}$ be an almost source idempotent and set $A = i\mathcal{O}Gi$. Denote by $\mathcal{F}$ the fusion system of A on P. Let Q be a fully $\mathcal{F}$-centralised subgroup of P and let $\varphi : Q \to P$ be a morphism in $\mathcal{F}$. Set $R = \varphi(Q)$. Denote by e_Q, e_R the unique blocks of $kC_G(Q)$, $kC_G(R)$ satisfying $\mathrm{Br}_{\Delta Q}(i)e_Q \neq 0$ and $\mathrm{Br}_{\Delta R}(i)e_R \neq 0$.*

(i) For any primitive idempotent n in $(\mathcal{O}Gb)^{\Delta R}$ satisfying $\mathrm{Br}_{\Delta R}(n)e_R \neq 0$ there is a primitive idempotent m in $A^{\Delta Q}$ satisfying $\mathrm{Br}_{\Delta Q}(m) \neq 0$ such that $m\mathcal{O}G \cong {}_\varphi(n\mathcal{O}G)$ as $\mathcal{O}Q$-$\mathcal{O}Gb$-bimodules and such that $\mathcal{O}Gm \cong (\mathcal{O}Gn)_\varphi$ as $\mathcal{O}Gb$-$\mathcal{O}Q$-bimodules.

(ii) For any primitive idempotent n in $A^{\Delta R}$ satisfying $\mathrm{Br}_{\Delta R}(n) \neq 0$ there is a primitive idempotent m in $A^{\Delta Q}$ satisfying $\mathrm{Br}_{\Delta Q}(m) \neq 0$ such that $mA \cong {}_\varphi(nA)$ as $\mathcal{O}Q$-A-bimodules and such that $Am \cong (An)_\varphi$ as A-$\mathcal{O}Q$-bimodules.

Proof. Since φ belongs to the fusion system $\mathcal{F}$ there is an element $x \in G$ such that $\varphi(u) = xux^{-1}$ for all $u \in Q$ and such that ${}^x(e_Q) = e_R$. Let n be a primitive idempotent in $(\mathcal{O}Gb)^{\Delta R}$ satisfying $\mathrm{Br}_{\Delta R}(n)e_R \neq 0$. Denote by ν be the local point of R on $\mathcal{O}Gb$ containing n. Since ${}^xQ = R$ there is a local point μ of Q on $\mathcal{O}Gb$ such that $\nu = {}^x\mu$. Since $\mathrm{Br}_{\Delta R}(\nu)e_R \neq 0$ we have $\mathrm{Br}_{\Delta Q}(\mu)e_Q \neq 0$. This implies that $\mathrm{Br}_{\Delta Q}(\mu)$ is in fact a conjugacy class of primitive idempotents in $kC_G(Q)e_Q$. Now Q is fully $\mathcal{F}$-centralised, and hence by 4.6.(ii), μ must contain an element m such that $m \in A$. Moreover, n^x and m belong both to the same point μ of Q on $\mathcal{O}Gb$, and hence there is an element $c \in ((\mathcal{O}Gb)^{\Delta Q})^\times$ such that ${}^{xc}m = n$. The map sending $ma \in m\mathcal{O}G$ to $xcma = nxca$ is the required isomorphism $m\mathcal{O}G \cong {}_\varphi(n\mathcal{O}G)$. The map sending am to $amc^{-1}x^{-1} = ac^{-1}x^{-1}n$ is the required isomorphism $\mathcal{O}Gm \cong (\mathcal{O}Gn)_\varphi$. This proves (i). Statement (ii) is played back to (i) via the Morita equivalence between $\mathcal{O}Gb$ and A. Let n be a primitive idempotent in $A^{\Delta R}$ satisfying $\mathrm{Br}_{\Delta R}(n) \neq 0$. The unit element i of A satisfies $\mathrm{Br}_{\Delta R}(i)e_R = \mathrm{Br}_{\Delta R}(i)$ by the uniqueness of the inclusion of Brauer pairs. Thus $\mathrm{Br}_{\Delta R}(n)e_R \neq 0$. Therefore statement (i) applies and yields a primitive idempotent $m \in A^{\Delta Q}$

satisfying $\mathrm{Br}_{\Delta Q}(m) \neq 0$ such that there are isomorphisms $m\mathcal{O}G \cong {}_\varphi(n\mathcal{O}G)$ and $\mathcal{O}Gm \cong (\mathcal{O}Gn)_\varphi$. Multiplying these isomorphisms by i on the right and on the left, respectively, yields the isomorphisms as stated in (ii). □

The following is [17, 7.7, 7.8] generalised to almost source idempotents.

Proposition 5.2. *Let G be a finite group, let b be a block of $\mathcal{O}G$ with defect group P, let i be an almost source idempotent in $(\mathcal{O}Gb)^{\Delta P}$ and set $A = i\mathcal{O}Gi$. Denote by $\mathcal{F}$ be the fusion system of A on P. Let Q, R be subgroups of P.*

(i) Every indecomposable direct summand of A as $\mathcal{O}Q$-$\mathcal{O}R$-bimodule is isomorphic to $\mathcal{O}Q \underset{\mathcal{O}S}{\otimes} {}_\varphi\mathcal{O}R$ for some subgroup S of Q and some morphism $\varphi : S \to R$ belonging to $\mathcal{F}$.

(ii) If $\varphi : Q \to R$ is an isomorphism in $\mathcal{F}$ such that R is fully $\mathcal{F}$-centralised then ${}_\varphi\mathcal{O}R$ is isomorphic to a direct summand of A as $\mathcal{O}Q$-$\mathcal{O}R$-bimodule.

Proof. (i) Let Y be an indecomposable direct summand of A as $\mathcal{O}Q$-$\mathcal{O}R$-bimodule. Then Y has a $Q \times R$-stable $\mathcal{O}$-basis on which Q and R act freely on the left and on the right, respectively. Thus

$$Y \cong \mathcal{O}Q \underset{\mathcal{O}S}{\otimes} {}_\varphi\mathcal{O}R$$

for some subgroup S of Q and some injective group homomorphism $\varphi : S \to R$. Set $T = \varphi(S)$. Restricting Y to $S \times T$ shows that ${}_\varphi\mathcal{O}T$ is isomorphic to a direct summand of Y, hence of A, as $\mathcal{O}S$-$\mathcal{O}T$-bimodule. Now A is a direct summand of $\mathcal{O}G$ as $\mathcal{O}P$-$\mathcal{O}P$-bimodule. In particular, ${}_\varphi\mathcal{O}T$ is isomorphic to a direct summand of $\mathcal{O}G$ as $\mathcal{O}S$-$\mathcal{O}T$-bimodule, hence isomorphic to $\mathcal{O}Sy^{-1} = y^{-1}\mathcal{O}T$ for some element $y \in G$ such that ${}^yS = T$ and such that ${}^ys = \varphi(s)$ for all $s \in S$. Then $\mathcal{O}S$ is isomorphic to a direct summand of $i\mathcal{O}Giy = i\mathcal{O}Gy^{-1}iy$ as $\mathcal{O}S$-$\mathcal{O}S$-bimodule. Thus $\mathrm{Br}_{\Delta S}(i\mathcal{O}Gy^{-1}iy) \neq 0$ by 3.5. Since $\mathrm{Br}_{\Delta S}(i) \in kC_G(S)e_S$ this forces also that $\mathrm{Br}_{\Delta S}(y^{-1}iy)e_S \neq 0$. Conjugating by y yields that $\mathrm{Br}_{\Delta T}(i)^y e_S \neq 0$. But then necessarily ${}^ye_S = e_T$ because e_T is the unique block of $kC_G(T)$ with the property $\mathrm{Br}_{\Delta T}(i)e_T \neq 0$. This shows that φ is a morphism in the fusion system $\mathcal{F}$, whence (i).

(ii) Since φ belongs to $\mathcal{F}$ there is an element $x \in G$ such that $\varphi(u) = {}^xu$ for all $u \in Q$ and such that ${}^xe_Q = e_R$. Let μ be a local point of Q on $\mathcal{O}Gb$ such that $\mu \cap A \neq \emptyset$. Set $\nu = {}^x\mu$; that is, ν is the local point of R on $\mathcal{O}Gb$ such that ${}^x(Q_\delta) = R_\nu$. Since $\mu \cap A \neq \emptyset$ we have $\mathrm{Br}_{\Delta Q}(\mu)e_Q \neq 0$. Conjugating by x implies that $\mathrm{Br}_{\Delta R}(\nu)e_R \neq 0$. Since R is fully $\mathcal{F}$-centralised, it follows from 4.6.(ii) that also $\nu \cap A \neq \emptyset$. Let $m \in \mu \cap A$ and let $n \in \nu \cap A$. Note that n and ${}^xm$ belong both to ν, hence are conjugate in $(A^R)^\times$. Since $\mathrm{Br}_{\Delta Q}(m) \neq 0$ we get $(m\mathcal{O}Gm)(\Delta Q) \neq \{0\}$, hence $m\mathcal{O}Gm$ has a direct summand isomorphic to $\mathcal{O}Q$ as $\mathcal{O}Q$-$\mathcal{O}Q$-bimodule. Therefore, $m\mathcal{O}Gmx^{-1} = m\mathcal{O}Gxmx^{-1} \cong m\mathcal{O}Gn = mAn$ has a direct summand isomorphic to $\mathcal{O}Q_{\varphi^{-1}} \cong {}_\varphi\mathcal{O}R$ as $\mathcal{O}Q$-$\mathcal{O}R$-bimodule. □

Corollary 5.3. *With the notation of 5.2, the fusion system $\mathcal{F}$ is generated by the set of inclusions between subgroups of P and automorphisms φ of any subgroup Q of P for which ${}_\varphi\mathcal{O}Q$ is isomorphic to a direct summand of A as $\mathcal{O}Q$-$\mathcal{O}Q$-bimodule.*

Proof. This follows from 5.2 and Alperin's fusion theorem. □

6 Vertices of p-permutation modules for almost source algebras

The Brauer construction sends p-permutation modules to p-permutation modules. Combining material of the previous sections we can be more precise regarding vertices:

Proposition 6.1. *Let A be an almost source algebra of a block with defect group P and fusion system $\mathcal{F}$. Let R be a subgroup of P and let X be an indecomposable direct summand of the A-module $A \underset{\mathcal{O}R}{\otimes} \mathcal{O}$. Then X is isomorphic to a direct summand of $Am \underset{\mathcal{O}Q}{\otimes} \mathcal{O}$ for some fully $\mathcal{F}$-centralised subgroup Q of P which is isomorphic, in $\mathcal{F}$, to a subgroup of R, and some primitive idempotent $m \in A^{\Delta Q}$ satisfying $\mathrm{Br}_{\Delta Q}(m) \neq 0$.*

Proof. Since X is indecomposable, X is isomorphic to a direct summand of $An \underset{\mathcal{O}R}{\otimes} \mathcal{O}$ for some primitive idempotent $n \in A^{\Delta R}$. Up to replacing R by a proper subgroup we may assume by [15, 2.6] that $\mathrm{Br}_{\Delta R}(n) \neq 0$. Let Q be a fully $\mathcal{F}$-entralised subgroup of P such that there is an isomorphism $\varphi : Q \cong R$ in $\mathcal{F}$. By 5.1.(ii) there is a primitive idempotent $m \in A^{\Delta Q}$ such that $\mathrm{Br}_{\Delta Q}(m) \neq 0$ and such that $Am \cong (An)_\varphi$ as A-$\mathcal{O}Q$-bimodules. Then $An \underset{\mathcal{O}R}{\otimes} \mathcal{O} \cong Am \underset{\mathcal{O}Q}{\otimes} \mathcal{O}$, and hence in particular, X is isomorphic to a direct summand of $A \underset{\mathcal{O}Q}{\otimes} \mathcal{O}$ as stated. □

We use 6.1 to track to vertices of indecomposable summands of p-permutation modules after applying the Brauer construction:

Theorem 6.2. *Let A be an almost source algebra of a block with defect group P and fusion system $\mathcal{F}$. Let Q, R be subgroups of P such that Q is fully $\mathcal{F}$-centralised. Let Y be an indecomposable direct summand of the $A(\Delta Q)$-module $(A \underset{\mathcal{O}R}{\otimes} \mathcal{O})(Q)$. Then Y is isomorphic to a direct summand of $A(\Delta Q)m \underset{kS}{\otimes} k$ for some subgroup S of $C_P(Q)$ containing $Z(Q)$ and some primitive idempotent $m \in A(\Delta Q)^{\Delta S}$ satisfying $\mathrm{Br}_{\Delta S}(m) \neq 0$ such that S is fully $C_{\mathcal{F}}(Q)$-centralised and QS is isomorphic in $\mathcal{F}$ to a subgroup of R.*

Proof. By 4.6, $A(\Delta Q)$ is isomorphic to a direct summand of $A(\Delta Q) \underset{kC_P(Q)}{\otimes} A(\Delta Q)$. Tensoring with $- \underset{A(\Delta Q)}{\otimes} Y$ implies that Y is isomorphic to a direct summand of $A(\Delta Q) \underset{kC_P(Q)}{\otimes} Y$. Since Y is indecomposable, Y is in fact isomorphic to a direct summand of $A(\Delta Q) \underset{kC_P(Q)}{\otimes} W$ for some indecomposable direct summand of Y as $kC_P(Q)$-module. Thus we need to determine the $kC_P(Q)$-module structure of Y. Let X be a P-P-stable $\mathcal{O}$-basis of A. We use the notation $X \otimes 1$ for the image of the set X in $A \underset{\mathcal{O}R}{\otimes} \mathcal{O}$. The set $X \otimes 1$ is a P-stable $\mathcal{O}$-basis of $A \underset{\mathcal{O}R}{\otimes} \mathcal{O}$. Thus, in order to compute $(A \underset{\mathcal{O}R}{\otimes} \mathcal{O})(Q)$ we need to determine the Q-fixpoints in the set $X \otimes 1$. For $x, y \in X$ the images $x \otimes 1$ and $y \otimes 1$ in $X \otimes 1$ are equal if and only if there is an element $r \in R$ such that $y = xr$. Therefore, $x \otimes 1 \in (X \otimes 1)^Q$ if and only if for every $u \in Q$ there is $r_u \in R$ such that $ux = xr_u$. In that case, r_u is then uniquely determined by u because P acts freely on the right of X. Since P acts also freely on the left of X, the map sending $u \in Q$ to $r_u \in R$ is an injective group homomorphism $\varphi : Q \to R$. Set

$$U_\varphi = \{x \in X \mid ux = x\varphi(u)\ (\forall\ u \in Q)\}$$

Note that by 5.2.(i) any φ arising in this way belongs to the fusion system $\mathcal{F}$ because $\mathcal{O}Qx = x\mathcal{O}\varphi(Q) \cong {}_\varphi\mathcal{O}\varphi(Q)$ is a direct summand of A as $\mathcal{O}Q$-$\mathcal{O}\varphi(Q)$-bimodule. It follows from the above that

$$(X \otimes 1)^Q = \bigcup_\varphi U_\varphi \otimes 1$$

where φ runs over the set $\mathrm{Hom}_{\mathcal{F}}(Q, R)$. Let $\varphi, \psi \in \mathrm{Hom}_{\mathcal{F}}(Q, R)$ and denote by $\tilde{\varphi}$, $\tilde{\psi}$ their images in the orbit space $\mathrm{Inn}(R)\backslash\mathrm{Hom}_{\mathcal{F}}(Q, R)$, where the group of inner automorphisms $\mathrm{Inn}(R)$ of R acts by composition of group homomorphisms on the set $\mathrm{Hom}_{\mathcal{F}}(Q, R)$. Suppose there is an element $x \in U_\varphi$

and $y \in U_\psi$ such that $x \otimes 1 = y \otimes 1$. Then there is an element $r \in R$ such that $y = xr$, and for all $u \in U$, we have $ux = x\varphi(u)$ and $uy = y\psi(u)$. Thus $uxr = x\varphi(u)r = xr\psi(u)$, hence $\varphi(u)r = \psi(u)r$, or equivalently, $\varphi(u) = {}^r\psi(u)$ for all $u \in Q$. This means that $\tilde{\varphi} = \tilde{\psi}$. It follows that $U_\varphi \otimes 1 = U_\psi \otimes 1$ if $\tilde{\varphi} = \tilde{\psi}$ and that $U_\varphi \otimes 1 \cap U_\psi \otimes 1 = \emptyset$ if $\tilde{\varphi} \neq \tilde{\psi}$. In other words, $(X \otimes 1)^Q$ is in fact the disjoint union

$$(X \otimes 1)^Q = \bigcup_\varphi U_\varphi \otimes 1$$

with φ running over a set of representatives in $\mathrm{Hom}_{\mathcal{F}}(Q,R)$ of $\mathrm{Inn}(R)\backslash\mathrm{Hom}_{\mathcal{F}}(Q,R)$. Therefore, by 3.4, we have

$$(A \underset{\mathcal{O}R}{\otimes} \mathcal{O})(Q) \cong k((X \otimes 1)^Q) = \underset{\varphi}{\oplus} k(U_\varphi \otimes 1)$$

with φ running as before over a set of representatives in $\mathrm{Hom}_{\mathcal{F}}(Q,R)$ of $\mathrm{Inn}(R)\backslash\mathrm{Hom}_{\mathcal{F}}(Q,R)$. This is a decomposition of $(A \underset{\mathcal{O}R}{\otimes} \mathcal{O})(Q)$ as $kC_P(Q)$-module because the subsets U_φ of X are invariant under the left action of $C_P(Q)$. Thus the direct summand W of Y as $kC_P(Q)$-module is isomorphic to a direct summand of $k(U_\varphi \otimes 1)$ for some $\varphi \in \mathrm{Hom}_{\mathcal{F}}(Q,R)$. Now W is a permutation $kC_P(Q)$-module, and hence, in order to determine its structure, it suffices to determine the stabilisers in $C_P(Q)$ of basis elements $x \otimes 1 \in U_\varphi \otimes 1$. Let $x, y \in U_\varphi$ such that $x \otimes 1 = y \otimes 1$. Then, on one hand, $y = xr$ for some $r \in R$, and on the other hand, $ux = x\varphi(u)$ and $uy = y\varphi(u)$ for all $u \in Q$. Thus $uxr = x\varphi(u)r = xr\varphi(u)$, which implies $r \in C_P(\varphi(Q))$. It follows that the stabiliser S in $C_P(Q)$ of an element $x \otimes 1 \in U_\varphi \otimes 1$ consists of all elements $z \in C_P(Q)$ such that $zx = xr_z$ for some $r_z \in C_P(\varphi(Q))$. In particular $Z(Q) \subseteq S$ because $zx = x\varphi(z)$ for $z \in Z(Q)$. It follows that there is a well-defined group homomorphism $\psi : QS \longrightarrow R$ given by $\psi(uz) = \varphi(u)r_z$ for all $u \in Q$ and $z \in S$. Note that ψ has the property $uzx = x\psi(uz)$, so $\mathcal{O}(QS)x = x\mathcal{O}(\psi(QS)) \cong {}_\psi\mathcal{O}\psi(QS)$ is a direct summand of A as $\mathcal{O}(QS)$-$\mathcal{O}\psi(QS)$-bimodule. Hence ψ belongs to the fusion system $\mathcal{F}$ by 5.2.(i). This shows that Y is isomorphic to a direct summand of $A(\Delta Q) \underset{kS}{\otimes} k$ for some subgroup S of $C_P(Q)$ containing $Z(Q)$ for which there exists a morphism $\psi : QS \to R$ in $\mathcal{F}$. We need to show that S can furthermore be chosen to be fully $C_{\mathcal{F}}(Q)$-centralised. Applying 6.1 to $A(\Delta Q)$ and its fusion system $C_{\mathcal{F}}(Q)$ shows that Y is isomorphic to a direct summand of $A(\Delta Q)m \underset{kT}{\otimes} k$ for some fully $C_{\mathcal{F}}(Q)$-centralised subgroup T of $C_P(Q)$ for which there exists a morphism $\tau : T \to S$ in $C_{\mathcal{F}}(Q)$ and some primitive idempotent $m \in A(\Delta Q)^{\Delta T}$ satisfying $\mathrm{Br}_{\Delta T}(m) \neq 0$. By the definition of $C_{\mathcal{F}}(Q)$, the morphism τ extends to a morphism $\sigma : QT \to QS$ in $\mathcal{F}$ satisfying $\sigma|_Q = \mathrm{Id}_Q$. Thus replacing S by T and ψ by $\psi \circ \sigma$ concludes the proof. □

Corollary 6.3. *Let A be an almost source algebra of a block with defect group P and fusion system $\mathcal{F}$. Let Q, R be subgroups of P such that Q is fully $\mathcal{F}$-centralised. If $\mathrm{Hom}_{\mathcal{F}}(Q,R) = \emptyset$ then $(A \underset{\mathcal{O}R}{\otimes} \mathcal{O})(Q) = \{0\}$.*

Corollay 6.4. *Let A, B be almost source algebras of blocks of finite groups having a common defect group P and the same fusion system $\mathcal{F}$. Let Q, R be subgroups of P such that Q is fully $\mathcal{F}$-centralised. If $\mathrm{Hom}_{\mathcal{F}}(Q,R) = \emptyset$ then $(A \underset{\mathcal{O}R}{\otimes} B)(\Delta Q) = \{0\}$.*

Proof. This follows from 6.3 in conjunction with 4.9 and 2.2. □

Corollary 6.5. *Let A be an almost source algebra of a block with defect group P and fusion system $\mathcal{F}$. Let Q be a fully $\mathcal{F}$-centralised subgroup of P. Every indecomposable direct summand of $(A \underset{\mathcal{O}Q}{\otimes} \mathcal{O})(Q)$ is isomorphic to $A(\Delta Q)m \underset{kZ(Q)}{\otimes} k$ for some primitive idempotent $m \in A(\Delta Q)$.*

Proof. By 6.2, an indecomposable direct summand Y of $(A \underset{\mathcal{O}Q}{\otimes} \mathcal{O})(Q)$ is isomorphic to a direct summand of $A(\Delta Q)m \underset{kS}{\otimes} k$ for some subgroup S of $C_P(Q)$ containing $Z(Q)$ and some primitive idempotent $m \in A(\Delta Q)^{\Delta S}$ satisfying $\mathrm{Br}_{\Delta S}(m) \neq 0$ such that QS is isomorphic to a subgroup of Q in $\mathcal{F}$. This forces $S \subseteq Q$, hence $S = Z(Q)$. Since the image of $Z(Q)$ in $A(\Delta Q)^\times$ belongs to $Z(A(\Delta Q))$, a primitive idempotent in $A(\Delta Q)^{Z(Q)}$ is primitive in $A(\Delta Q)$ and hence the module $A(\Delta Q)m \underset{kZ(Q)}{\otimes} k$ is indecomposable since it is an indecomposable projective module for the algebra $A(\Delta Q) \underset{kZ(Q)}{\otimes} k$. □

7 Grothendieck groups of trivial and linear source modules for almost source algebras

We translate in this section some of the concepts and terminology related to trivial source and linear source modules for block algebras to the source algebra level.

Definition 7.1. Let A, B be $\mathcal{O}$-algebras. We denote by $\mathcal{R}(A, B)$ the Grothendieck group of A-B-bimodules which are finitely generated as left A-modules and as right B-modules, with respect to split exact sequences. We denote by $\mathcal{P}(A, B)$ the subgroup of $\mathcal{R}(A, B)$ generated by the A-B-bimodules which are finitely generated projective as left A-modules and as right B-modules. For A, B, C three $\mathcal{O}$-algebras we denote by

$$- \underset{B}{\cdot} - \ : \ \mathcal{R}(A, B) \times \mathcal{R}(B, C) \longrightarrow \mathcal{R}(A, C)$$

the group homomorphism induced by the tensorproduct $M \underset{B}{\otimes} N$ over B of an A-B-bimodule M and a B-C-bimodule N. We consider $\mathcal{R}(A) = \mathcal{R}(A, A)$ as unitary associative ring with multiplication $- \underset{A}{\cdot} -$ and unit element $[A]$, the isomorphism class of A as A-A-bimodule in $\mathcal{R}(A)$, and we consider $\mathcal{R}(A, B)$ as $\mathcal{R}(A)$-$\mathcal{R}(B)$-bimodule via $- \underset{A}{\cdot} -$ and $- \underset{B}{\cdot} -$. Clearly the bilinear map $- \underset{B}{\cdot} -$ induces a bilinear map

$$- \underset{B}{\cdot} - \ : \ \mathcal{P}(A, B) \times \mathcal{P}(B, C) \longrightarrow \mathcal{P}(A, C) \ ;$$

in particular, $\mathcal{P}(A) = \mathcal{P}(A, A)$ is a unitary subring of $\mathcal{R}(A)$.

Following [21], given a finite group P, an *interior P-algebra* is an $\mathcal{O}$-algebra A endowed with a group homomorphism $P \to A^\times$, or equivalently, endowed with a unitary algebra homomorphism $\mathcal{O}P \to A$.

Definition 7.2. Let P be a finite group and let A, B be interior P-algebras. We denote by $\mathcal{T}(A, B)$ the subgroup of $\mathcal{R}(A, B)$ generated by the finite direct sums of summands of the A-B-bimodules $A \underset{\mathcal{O}Q}{\otimes} B$, where Q runs over the subgroups of P.

We will also need a source algebra version of linear source modules. Let Q be a finite p-group. Any group homomorphism $\theta : Q \to \mathcal{O}^\times$ induces an $\mathcal{O}$-algebra automorphism sending $u \in Q$ to $\theta(u)u$.

Given an $\mathcal{O}Q$-module U we denote by ${}_\theta U$ the $\mathcal{O}Q$-module which is equal to U as $\mathcal{O}$-module, with $u \in Q$ acting as $\theta(u)u$ on U. We use a similar notation for right modules and bimodules.

Definition 7.3 Let P be a finite p-group and let A, B be interior P-algebras. We denote by $\mathcal{L}(A,B)$ the Grothendieck group of finite direct sums of summands of A-B-bimodules of the form $A \underset{\mathcal{O}Q}{\otimes} {}_\theta B$, with Q running over the subgroups of P and θ over the set $\mathrm{Hom}(Q, \mathcal{O}^\times)$ of group homomorphisms from Q to $\mathcal{O}^\times$.

Clearly $\mathcal{T}(A,B)$ is a subgroup of $\mathcal{L}(A,B)$. If A, B are finitely generated projective over $\mathcal{O}P$ then $\mathcal{T}(A,B)$ and $\mathcal{L}(A,B)$ are subgroups of $\mathcal{P}(A,B)$. If A is isomorphic to a direct summand of $A \underset{\mathcal{O}P}{\otimes} A$ as A-A-bimodule then $[A]$ belongs to $\mathcal{T}(A) = \mathcal{T}(A,A)$. By 4.2 this case occurs whenever A is an almost source algebra of a block with P as defect group. The groups $\mathcal{T}(A,B)$ and $\mathcal{L}(A,B)$ are independent of the choice of almost source idempotents in the following sense:

Theorem 7.4. *Let G, H be finite groups and let b, c be blocks of $\mathcal{O}G$, $\mathcal{O}H$ with a common defect group P, respectively. Let $i \in (\mathcal{O}Gb)^{\Delta P}$, $j \in (\mathcal{O}Hc)^{\Delta P}$ be almost source idempotents. Set $A = i\mathcal{O}Gi$ and $B = j\mathcal{O}Hj$. Let $i' \in (\mathcal{O}Gb)^{\Delta P}$, $j' \in (\mathcal{O}Hc)^{\Delta P}$ be idempotents satisfying $i = ii' = i'i$ and $j = jj' = j'j$. Set $A' = i'\mathcal{O}Gi'$ and $B' = j'\mathcal{O}Hj'$. The map sending an A-B-bimodule M to the A'-B'-bimodule $A'i \underset{A}{\otimes} M \underset{B}{\otimes} jB'$ induces injective group homomorphisms*

$$\mathcal{T}(A,B) \longrightarrow \mathcal{T}(A',B'),$$

$$\mathcal{L}(A,B) \longrightarrow \mathcal{L}(A',B') \ .$$

If the idempotents i', j' are also almost source idempotents and if i, j determine the same fusion system on P then these maps are isomorphisms whose inverses are induced by the map sending an A'-B'-bimodule N to the A-B-bimodule iNj.

Proof. The condition $i = ii' = i'i$ is equivalent to $i \in A'$, so the statement makes sense. Let M be a direct summand of $A \underset{\mathcal{O}R}{\otimes} {}_\theta B$ for some subgroup R of P and some group homomorphism $\theta : R \to \mathcal{O}$. Then $A'i \underset{A}{\otimes} M \underset{B}{\otimes} jB'$ is a direct summand of $A'i \underset{A}{\otimes} A \underset{\mathcal{O}R}{\otimes} {}_\theta B \underset{B}{\otimes} jB' \cong A'i \underset{\mathcal{O}R}{\otimes} {}_\theta jB'$, which is a direct summand of $A' \underset{\mathcal{O}R}{\otimes} {}_\theta B'$, Thus the map sending M to $A'i \underset{A}{\otimes} M \underset{B}{\otimes} jB'$ induces group homomorphisms $\mathcal{T}(A,B) \to \mathcal{T}(A',B')$ and $\mathcal{L}(A,B) \to \mathcal{L}(A',B')$. These group homomorphisms are injective because A, A' and B, B' are Morita equivalent via the bimodules $A'i$, $B'j$ and their duals, respectively, as a consequence of 4.1. Suppose now that i' and j' are almost source idempotents. In order to show the surjectivity of the above maps we may assume that i, j are source idempotents. Let N be an indecomposable direct summand of $A' \underset{\mathcal{O}R}{\otimes} {}_\theta B'$ for some group homomorphism $\theta : Q \to \mathcal{O}^\times$. Thus N is an indecomposable direct summand of $A'n \underset{\mathcal{O}R}{\otimes} {}_\theta sB'$ for some primitive idempotents $n \in (A')^{\Delta R}$ and $s \in (B')^{\Delta R}$. By choosing R of minimal order we may assume that $\mathrm{Br}_{\Delta R}(n) \neq 0$ and $\mathrm{Br}_{\Delta R}(s) \neq 0$. In other words, n belongs to a local point ν of Q on A', and s belongs to a local point σ of R on B'. Since i', j' are almost source idempotents, the defect group P has unique local points γ, δ on A', B', respectively, and so $i \in \gamma$ and $j \in \delta$. Denote by $\mathcal{F}$ the common fusion system of G and H on P. Let $\varphi : Q \cong R$ be an isomorphism in $\mathcal{F}$ such that Q is fully $\mathcal{F}$-centralised. By 5.1 there is a primitive idempotent $m \in A^{\Delta Q}$ satisfying $\mathrm{Br}_{\Delta Q}(m) \neq 0$ such that $\mathcal{O}Gm \cong (\mathcal{O}Gn)_\varphi$ as $\mathcal{O}Gb$-$\mathcal{O}Q$-bimodules. Multiplying by i' yields an isomorphism $A'm \cong (A'n)_\varphi$ as A'-$\mathcal{O}Q$-bimodules. Similarly, there is a primitive idempotent $r \in B^{\Delta Q}$ satisfying $\mathrm{Br}_{\Delta Q}(r) \neq 0$ such that $rB' \cong {}_\varphi sB'$ as $\mathcal{O}Q$-B'- bimodules. Then ${}_{\theta\circ\varphi} rB' \cong {}_{\varphi\theta} sB'$ as $\mathcal{O}Q$-B'- bimodules. Thus N is isomorphic to a direct summand of $A'n \underset{\mathcal{O}R}{\otimes} {}_\theta sB' \cong A'm \underset{\mathcal{O}Q}{\otimes} {}_\tau sB'$, where $\tau = \theta \circ \varphi : Q \to \mathcal{O}^\times$. Since $m \in A, r \in B$ the

latter is isomorphic to a direct summand of $A'i \underset{\mathcal{O}Q}{\otimes} {}_\tau jB'$. Thus N is in fact a direct summand of $A'i \underset{\mathcal{O}Q}{\otimes} {}_\tau jB'$. Multiplication by i on the left and by j on the right is a Morita equivalence between the category of A'-B'-bimodules and the category of A-B-bimodules. Thus, setting $M = iNj$ we get that $N \cong A'i \underset{A}{\otimes} M \underset{B}{\otimes} jB'$. Since by construction M is a direct summand of $A \underset{\mathcal{O}Q}{\otimes} {}_\tau B$ the surjectivity of the given maps follows. □

Following the notation in [4, 1.2], if G, H are finite groups, b a block of $\mathcal{O}G$ and c a block of $\mathcal{O}H$ with a common defect group P, then $T^{\Delta P}(\mathcal{O}Gb, \mathcal{O}Hc)$ denotes the Grothendieck group of $\mathcal{O}Gb$-$\mathcal{O}Hc$-bimodules (with respect to split exact sequences) whose indecomposable direct summands have trivial source and a vertex contained in ΔP, when viewed as $\mathcal{O}(G \times H)$-modules. Similarly, $L^{\Delta P}(\mathcal{O}Gb, \mathcal{O}Hc)$ denotes the Grothendieck group of linear source $\mathcal{O}Gb$-$\mathcal{O}Hc$-bimodules (with respect to split exact sequences) whose indecomposable direct summands have a linear source and a vertex contained in ΔP. The following consequence of Theorem 7.4 implies in particular that the notion of p-permutation equivalence as introduced by Boltje and Xu is equivalent to the source algebra version 1.3 above in case the involved blocks are of principal type. Recall that b is said to be of *principal type* if $\mathrm{Br}_{\Delta Q}(b)$ is a block of $kC_G(Q)$ for every subgroup Q of a defect group P of b. Equivalently, b is of principal type if and only if b itself is an almost source idempotent. If b is the principal block of $\mathcal{O}G$ then by Brauer's Third Main Theorem, $\mathrm{Br}_{\Delta Q}(b)$ is the principal block of $kC_G(Q)$ for any subgroup Q of a Sylow-p-subgroup P of G, and hence in particular and not surprisingly, the principal block b is of principal type.

Corollary 7.5. *Let G, H be finite groups and let b, c be blocks of $\mathcal{O}G$, $\mathcal{O}H$ with a common defect group P, respectively. Let $i \in (\mathcal{O}Gb)^{\Delta P}$, $j \in (\mathcal{O}Hc)^{\Delta P}$ be almost source idempotents. Set $A = i\mathcal{O}Gi$ and $B = j\mathcal{O}Hj$. The map sending an A-B-bimodule M to the $\mathcal{O}Gb$-$\mathcal{O}Hc$-bimodule $\mathcal{O}Gi \underset{A}{\otimes} M \underset{B}{\otimes} j\mathcal{O}H$ induces injective group homomorphisms*

$$\mathcal{T}(A, B) \longrightarrow T^{\Delta P}(\mathcal{O}Gb, \mathcal{O}Hc) \ ,$$

$$\mathcal{L}(A, B) \longrightarrow L^{\Delta P}(\mathcal{O}Gb, \mathcal{O}Hc) \ .$$

If the blocks b, c are of principal type and have the same fusion system these maps are isomorphisms.

Proof. For any subgroup R of P and any group homomorphism $\theta : R \to \mathcal{O}^\times$ we have an isomorphism of $\mathcal{O}(G \times H)$-modules

$$\mathcal{O}G \underset{\mathcal{O}R}{\otimes} {}_\theta\mathcal{O}H \cong \mathrm{Ind}_{\Delta R}^{G \times H}({}_\theta\mathcal{O})$$

sending $x \otimes y^{-1}$ to $(x, y) \otimes 1$ for $x \in G$, $y \in H$. Thus $\mathcal{T}(\mathcal{O}Gb, \mathcal{O}Hc) \cong T^{\Delta P}(\mathcal{O}Gb, \mathcal{O}Hc)$ and $\mathcal{L}(\mathcal{O}Gb, \mathcal{O}Hc) \cong L^{\Delta P}(\mathcal{O}Gb, \mathcal{O}Hc)$. Thus 7.4 applied to $i' = b$ and $j' = c$ yields the injective group homomorphisms as stated. If b and c are of principal type then b and c are almost source idempotents, and hence, if they determine the same fusion system it follows again from 7.4 that the maps in the statement are isomorphisms. □

Remark 7.6. For non principal blocks the maps in 7.5 need not be isomorphisms, in general. If $\mathcal{O}Gb$ is a block algebra whose defect group P has two different local points γ, γ' on $\mathcal{O}Gb$ then $\mathcal{O}Gi \underset{\mathcal{O}P}{\otimes} i'\mathcal{O}G$ is a p-permutation $\mathcal{O}(G \times G)$-module, where $i \in \gamma$, $i' \in \gamma'$, but not all indecomposable summands of $i\mathcal{O}Gi \underset{\mathcal{O}P}{\otimes} i'\mathcal{O}Gi$ as $i\mathcal{O}Gi$-$i\mathcal{O}Gi$-bimodules will be summands of bimodules of the form $i\mathcal{O}Gi \underset{\mathcal{O}Q}{\otimes} i\mathcal{O}Gi$. This is because $\gamma' = {}^x\gamma$ for some $x \in N_G(P)$ not contained in $N_G(P_\gamma)$, so conjugation by x is an automorphism of P which does not belong to the A-fusion on P.

For the sake of completeness we include some further observations relating the notation of the present paper to that in [4]. Let A be an almost source algebra of a block b of a finite group G with defect group P. Consider $\mathcal{O}$ as trivial interior P-algebra with structural homomorphism $P \to \mathcal{O}^\times$ sending any element $u \in P$ to $1_\mathcal{O}$. Then $\mathcal{T}(A,\mathcal{O})$ is the Grothendieck group of sums of summands of the left A-modules $A \underset{\mathcal{O}Q}{\otimes} \mathcal{O}$, where Q runs over the subgroups of P. Similarly, $\mathcal{L}(A,\mathcal{O})$ is the Grothendieck group of sums of summands of the left A-modules $A \underset{\mathcal{O}Q}{\otimes} {}_\theta\mathcal{O}$, where $Q \subseteq P$ and $\theta \in \mathrm{Hom}(Q,\mathcal{O}^\times)$. Following the notation of Boltje and Xu [4], we denote by $T(\mathcal{O}Gb)$ the Grothendieck group of p-permutation $\mathcal{O}Gb$-modules; that is, $T(\mathcal{O}Gb)$ is generated by the isomorphism classes of summands of $\mathcal{O}Gb \underset{\mathcal{O}Q}{\otimes} \mathcal{O}$, with $Q \subseteq P$. Similarly, we denote by $L(\mathcal{O}Gb)$ the Grothendieck group generated by the linear source $\mathcal{O}Gb$-modules; that is, $L(\mathcal{O}Gb)$ is generated by the isomorphism classes of summands of $\mathcal{O}Gb \underset{\mathcal{O}Q}{\otimes} {}_\theta\mathcal{O}$, with $Q \subseteq P$ and $\theta \in \mathrm{Hom}(Q,\mathcal{O}^\times)$.

Proposition 7.7. *Let G be a finite group, let b be a block of $\mathcal{O}G$, let P be a defect group of b and let $i \in (\mathcal{O}Gb)^{\Delta P}$ be an idempotent satisfying $\mathrm{Br}_{\Delta P}(i) \neq 0$ and set $A = i\mathcal{O}Gi$. The functor $\mathcal{O}Gi \underset{A}{\otimes} - : \mathrm{mod}(A) \to \mathrm{mod}(\mathcal{O}Gb)$ induces group isomorphisms $\mathcal{T}(A,\mathcal{O}) \cong T(\mathcal{O}Gb)$ and $\mathcal{L}(A,\mathcal{O}) \cong L(\mathcal{O}Gb)$.*

Proof. Let Q be a subgroup of P, let $\theta : Q \to \mathcal{O}^\times$ be a group homomorphism and let Y be a direct summand of $A \underset{\mathcal{O}Q}{\otimes} {}_\theta\mathcal{O}$. Then $\mathcal{O}Gi \underset{A}{\otimes} Y$ is a direct summand of $\mathcal{O}Gi \underset{\mathcal{O}Q}{\otimes} {}_\theta\mathcal{O}$, hence of $\mathcal{O}Gb \underset{\mathcal{O}Q}{\otimes} {}_\theta\mathcal{O}$. Thus the functor $\mathcal{O}Gi \underset{A}{\otimes} -$ induces group homomorphisms $\mathcal{T}(A,\mathcal{O}) \to T(\mathcal{O}Gb)$ and $\mathcal{L}(A,\mathcal{O}) \to L(\mathcal{O}Gb)$. These group homomorphisms are injective because the functor $\mathcal{O}Gi \underset{A}{\otimes} -$ is an equivalence, by 4.1. To see that they are also surjective, let X be an indecomposable linear source $\mathcal{O}Gb$-module. It follows from [13, 6.3] that one can choose a vertex R of X such that X is isomorphic to a direct summand of $\mathcal{O}Gi \underset{\mathcal{O}R}{\otimes} {}_\rho\mathcal{O}$ for some group homomorphism $\rho : R \to \mathcal{O}^\times$. Thus the functor $\mathcal{O}Gi \underset{A}{\otimes} -$ induces the isomorphisms as stated. □

With the notation of 7.5, the map given by $- \underset{\mathcal{O}Hc}{\cdot} -$ sends $T^{\Delta P}(\mathcal{O}Gb,\mathcal{O}Hc) \times L(\mathcal{O}Hc)$ to $L(\mathcal{O}Gb)$ (cf. [2, 2.3]). Clearly the map $- \underset{B}{\cdot} -$ sends $\mathcal{T}(A,B) \times \mathcal{L}(B,\mathcal{O})$ to $\mathcal{L}(A,\mathcal{O})$. These two maps correspond to each other via the maps from 7.5 and 7.7:

Proposition 7.8. *Let G, H be finite groups and let b, c be blocks of $\mathcal{O}G$, $\mathcal{O}H$ with a common defect group P, respectively. Let $i \in (\mathcal{O}Gb)^{\Delta P}$, $j \in (\mathcal{O}Hc)^{\Delta P}$ be almost source idempotents. Set $A = i\mathcal{O}Gi$ and $B = j\mathcal{O}Hj$. We have a commutative diagram*

$$\begin{array}{ccc} \mathcal{T}(A,B) \times \mathcal{L}(B,\mathcal{O}) & \xrightarrow{\quad - \underset{B}{\cdot} - \quad} & \mathcal{L}(A,\mathcal{O}) \\ \downarrow & & \downarrow \cong \\ T^{\Delta P}(\mathcal{O}Gb,\mathcal{O}Hc) \times L(\mathcal{O}Hc) & \xrightarrow[\quad - \underset{\mathcal{O}Hc}{\cdot} - \quad]{} & L(\mathcal{O}Gb) \end{array}$$

where the vertical arrows are induced by those in 7.5 and 7.7.

Proof. Let M be a direct summand of an A-B-bimodule of the form $A \underset{\mathcal{O}Q}{\otimes} B$ for some subgroup Q of P and let Y be a direct summand of a left B-module $B \underset{\mathcal{O}R}{\otimes} {}_\theta\mathcal{O}$ for some subgroup R of P and some group homomorphism $\theta : R \to \mathcal{O}^\times$. Then $M \underset{B}{\otimes} Y$ is a direct summand of $A \underset{\mathcal{O}Q}{\otimes} B \underset{\mathcal{O}R}{\otimes} {}_\theta\mathcal{O}$. Since B is

a permutation $\mathcal{O}(Q \times R)$-module it follows that $M \underset{B}{\otimes} Y$ is a direct sum of linear source A-modules. The composition of maps $\mathcal{T}(A,B) \times \mathcal{L}(B,\mathcal{O}) \to T^{\Delta P}(\mathcal{O}Gb, \mathcal{O}Hc) \times L(\mathcal{O}Hc) \to L(\mathcal{O}Gb)$ is induced by the map

$$(M,Y) \mapsto (\mathcal{O}Gi \underset{A}{\otimes} M \underset{B}{\otimes} j\mathcal{O}H, \mathcal{O}Hj \underset{B}{\otimes} Y) \mapsto \mathcal{O}Gi \underset{A}{\otimes} M \underset{B}{\otimes} j\mathcal{O}H \underset{\mathcal{O}Hc}{\otimes} \mathcal{O}Hj \underset{B}{\otimes} Y \cong \mathcal{O}Gi \underset{A}{\otimes} M \underset{B}{\otimes} Y$$

and the composition of maps $\mathcal{T}(A,B) \times \mathcal{L}(B,\mathcal{O}) \to \mathcal{L}(B,\mathcal{O}) \cong L(\mathcal{O}Gb)$ is induced by the map

$$(M,Y) \mapsto M \underset{B}{\otimes} Y \mapsto \mathcal{O}Gi \underset{A}{\otimes} M \underset{B}{\otimes} Y$$

and hence both maps coincide. □

§8 Ring and module structures of $\mathcal{T}(A,B)$ and $\mathcal{L}(A,B)$

As a consequence of [17, §2] we get some extra structure on $\mathcal{T}(A,B)$ in case A, B are source algebras. The following theorem extends [17, 2.3.(i)] to linear source modules over almost source algebras.

Theorem 8.1. *Let A, B, C be almost source algebras of p-blocks of finite groups with a common defect group P all having the same fusion system $\mathcal{F}$ on P. The tensorproduct over B induces bilinear maps*

$$-\underset{B}{\cdot}- : \mathcal{L}(A,B) \times \mathcal{L}(B,C) \longrightarrow \mathcal{L}(A,C) \ ,$$
$$-\underset{B}{\cdot}- : \mathcal{T}(A,B) \times \mathcal{T}(B,C) \longrightarrow \mathcal{T}(A,C) \ .$$

Proof. Let M be an indecomposable direct summand of $A \underset{\mathcal{O}Q}{\otimes} {}_\theta B$ and let N be an indecomposable direct summand of $B \underset{\mathcal{O}R}{\otimes} {}_\eta C$ for some subgroups Q, R of P and some group homomorphisms $\theta : Q \to \mathcal{O}^\times$ and $\eta : R \to \mathcal{O}^\times$. Let X be an indecomposable direct summand of $M \underset{B}{\otimes} N$. We have to show that X is isomorphic to a direct summand of $A \underset{\mathcal{O}S}{\otimes} {}_\tau C$ for some subgroup S of P and some group homomorphism $\tau : S \to \mathcal{O}^\times$, such that τ is trivial whenever both θ, η are trivial. The bimodule X is isomorphic to a direct summand of

$$A \underset{\mathcal{O}Q}{\otimes} {}_\theta B \underset{\mathcal{O}R}{\otimes} {}_\eta C \ .$$

Since X is indecomposable, there is an indecomposable direct summand W of B as $\mathcal{O}Q$-$\mathcal{O}R$-bimodule such that X is isomorphic to a direct summand of

$$A \underset{\mathcal{O}Q}{\otimes} {}_\theta W \underset{\mathcal{O}R}{\otimes} {}_\eta C \ .$$

By 5.2.(i), the $\mathcal{O}Q$-$\mathcal{O}R$-bimodule W is isomorphic to $\mathcal{O}Q \underset{\mathcal{O}S}{\otimes} {}_\varphi \mathcal{O}R$ for some subgroup S of Q and some morphism $\varphi : S \to R$ belonging to the fusion system $\mathcal{F}$. Thus X is isomorphic to a direct summand of

$$A \underset{\mathcal{O}S}{\otimes} {}_\varphi({}_\sigma C) \ ,$$

where $\sigma : \varphi(S) \to \mathcal{O}^\times$ is the group homomorphism sending $\varphi(y)$ to $\theta(y)\eta(\varphi(y))$, for every $y \in S$. Since X is indecomposable, X is in fact isomorphic to a direct summand of $An \underset{\mathcal{O}S}{\otimes} {}_\varphi({}_\sigma sC)$ for some primitive

idempotent $n \in A^{\Delta S}$ and some primitive idempotent $s \in C^{\Delta\varphi(S)}$. By [15, 2.6] we may assume that n, s belong to local points of S, $\varphi(S)$ on A, C, respectively. Let T be a fully $\mathcal{F}$-centralised subgroup of P such that there is an isomorphism $\psi : T \to S$ in $\mathcal{F}$. By 5.1 there are primitive idempotents $m \in A^{\Delta T}$ and $r \in C^{\Delta T}$ such that $Am \cong An_\psi$ as A-$\mathcal{O}T$-bimodules, and such that $rC \cong {}_{\varphi\psi}sC$ as $\mathcal{O}T$-C-bimodules. Thus X is a direct summand of

$$Am \underset{\mathcal{O}T}{\otimes} {}_\tau rC$$

where $\tau = \sigma \circ \psi : T \to \mathcal{O}^\times$. This shows that the isomorphism class of X belongs to $\mathcal{L}(A, C)$. By construction, if θ, η are trivial, so is τ, and hence in that case, the isomorphism class of X belongs to $\mathcal{T}(A, C)$. □

Corollary 8.2. *Let A be an almost source algebra of a p-block of a finite group with defect group P. The abelian groups $\mathcal{T}(A)$ and $\mathcal{L}(A)$ are unitary subrings of $\mathcal{P}(A)$.*

Proof. Applying 8.1 to $A = B = C$ shows that $\mathcal{T}(A)$ and $\mathcal{L}(A)$ are rings; the fact that they are unitary follows from 4.2. □

Corollary 8.3. *Let A, B be almost source algebras of p-blocks of finite groups with a common defect group P and the same fusion system. Then $\mathcal{L}(A, B)$ is an $\mathcal{L}(A)$-$\mathcal{L}(B)$-bimodule and $\mathcal{T}(A, B)$ is an $\mathcal{T}(A)$-$\mathcal{T}(B)$-bimodule, with left and right module structure induced by the tensorproducts over A and B, respectively.*

Proof. This follows from 8.1 applied to the cases $A = B$ and $B = C$. □

Corollary 8.4. *Let A, B be almost source algebras of p-blocks of finite groups with a common defect group P and the same fusion system. Let X be a p-permutation equivalence between A and B. The map sending $Y \in \mathcal{T}(A)$ to $X^* \underset{A}{\cdot} Y \underset{A}{\cdot} X$ is a ring isomorphism*

$$\mathcal{T}(A) \cong \mathcal{T}(B) \ .$$

Proof. For Y, Z in $\mathcal{T}(A)$ we have $(X^* \underset{A}{\cdot} Y \underset{A}{\cdot} X) \underset{B}{\cdot} (X^* \underset{A}{\cdot} Z \underset{A}{\cdot} X) = X^* \underset{A}{\cdot} Y \underset{A}{\cdot} (X \underset{B}{\cdot} X^*) \underset{A}{\cdot} Z \underset{A}{\cdot} X = X^* \underset{A}{\cdot} Y \underset{A}{\cdot} [A] \underset{A}{\cdot} Z \underset{A}{\cdot} X = X^* \underset{A}{\cdot} (Y \underset{A}{\cdot} Z) \underset{A}{\cdot} X$, which shows that the given map is a ring homomorphism. Exchanging the roles of A, B and X, X^* yields the inverse. □

The ring structure of $\mathcal{T}(A)$ does not depend on the choice of the almost source idempotents:

Proposition 8.5. *Let G be a finite group, let b be a block of $\mathcal{O}G$, let P, P' be defect groups of b, let $i \in (\mathcal{O}Gb)^{\Delta P}$ and $i' \in (\mathcal{O}Gb)^{\Delta P'}$ be almost source idempotents. Set $A = i\mathcal{O}Gi$ and $A' = i'\mathcal{O}Gi'$. There is a canonical ring isomorphism $\mathcal{T}(A) \cong \mathcal{T}(A')$.*

Proof. As a consequence of 7.4 we may assume that i, i' are source idempotents. Let γ, γ' be the local points of P, P' on $\mathcal{O}Gb$ containing i, i', respectively. By [21, 1.2] there is $x \in G$ such that $P'_{\gamma'} = {}^x(P_\gamma)$. We may assume that $i' = {}^x i$. Let $\alpha : A \cong A'$ be the $\mathcal{O}$-algebra isomorphism sending $a \in A$ to ${}^x a \in A'$. Then α induces an isomorphism of A-A-bimodules

$$A \underset{\mathcal{O}Q}{\otimes} A \cong {}_\alpha A' \underset{\mathcal{O}\alpha(Q)}{\otimes} A'_\alpha$$

where we have identified Q to its image in A. Thus the map sending an A-A-bimodule M to the A'-A'-bimodule ${}_{\alpha^{-1}}M_{\alpha^{-1}}$ induces a ring isomorphism $\mathcal{T}(A) \cong \mathcal{T}(A')$. We need to show that this isomorphism does not depend on the choice of x. If y is another element such that $P'_{\gamma'} = {}^y(P_\gamma)$ then $y = xn$ for some $n \in N_G(P_\gamma)$, so it suffices to show that the automorphism of $\mathcal{T}(A)$ induced by n is the identity. By [22, 2.12] there is an element $c \in A^\times$ which normalises the image of P in A such that conjugation by c induces the same group automorphism P as conjugation by n. Denote by β the inner automorphism of A sending $a \in A$ to ${}^ca$. Then β induces for any A-A-bimodule M an isomorphism of A-A-bimodules $M \cong {}_\beta M_\beta$. The result follows. □

Given two $\mathcal{O}$-algebras A, B, the the $\mathcal{O}$-dual $M^* = \mathrm{Hom}_{\mathcal{O}}(M, \mathcal{O})$ of an A-B-bimodule is a B-A-bimodule via $(b.\mu.a)(m) = \mu(amb)$ for $a \in A$, $b \in B$, $m \in M$ and $\mu \in M^*$. This, taking $\mathcal{O}$-duality induces a group homomorphism $\mathcal{R}(A, B) \cong \mathcal{R}(B, A)$. Recall that an $\mathcal{O}$-algebra A is called *symmmetric* if A is free of finite rank as $\mathcal{O}$-module and if $A \cong A^*$ as A-A-bimodule. Group algebras of finite groups, their block algebras and almost source algebras are symmetric.

Proposition 8.6. *Let A, B be symmetric $\mathcal{O}$-algebras. Taking $\mathcal{O}$-duals of bimodules induces a group isomorphism $\mathcal{P}(A, B) \cong \mathcal{P}(B, A)$ and an anti-automorphism on the ring $\mathcal{P}(A)$.*

Proof. Since A, B are symmetric, the $\mathcal{O}$-dual of an A-B-bimodule which is finitely generated projective as left A-module and as right B-module is again finitely generated projective as left B-module and as right A-module. This shows the first statememt. If M, N are A-A-bimodules which are finitely generated projective as left and right A-modules then $(M \underset{A}{\otimes} N)^* \cong N^* \underset{A}{\otimes} M^*$ which implies the second statement. □

Proposition 8.7. *Let A, B be almost source algebras of blocks of finite groups with a common defect group P. Taking $\mathcal{O}$-duals of bimodules induces group isomorphisms $\mathcal{L}(A, B) \cong \mathcal{L}(B, A)$ and $\mathcal{T}(A, B) \cong \mathcal{T}(B, A)$, as well as anti-automorphisms of the rings $\mathcal{L}(A)$ and $\mathcal{T}(A)$.*

Proof. Since A, B, $\mathcal{O}P$ are symmetric, the $\mathcal{O}$-dual of an A-B-bimodule of the form $A \underset{\mathcal{O}Q}{\otimes} {}_\theta B$ for some subgroup Q of P and a group homomorphism $\theta : Q \to \mathcal{O}^\times$ is isomorphic to the B-A-bimodule $B_\theta \underset{\mathcal{O}Q}{\otimes} A \cong B \underset{\mathcal{O}Q}{\otimes} {}_{(\theta^{-1})}A$. The first statement follows. The second statement follows from 8.6. □

Given a finite group G, any p-permutation kG-module lifts uniquely, up to isomorphism, to a p-permutation $\mathcal{O}G$-module. Conversely, a linear source $\mathcal{O}G$-module reduces modulo $J(\mathcal{O})$ to a trivial source kG-module because there are no non trivial group homomorphisms from a finite p-group to $k^\times$. At the source algebra level, this translates to the following statement:

Proposition 8.8. *Let A, B be almost source algebras of blocks with a common defect group P. The map sending an A-B-bimodule M to $k \underset{\mathcal{O}}{\otimes} M$ induces a surjective group homomorphism $\mathcal{L}(A, B) \to \mathcal{T}(k \underset{\mathcal{O}}{\otimes} A, k \underset{\mathcal{O}}{\otimes} B)$ and a group isomorphism $\mathcal{T}(A, B) \cong \mathcal{T}(k \underset{\mathcal{O}}{\otimes} A, k \underset{\mathcal{O}}{\otimes} B)$. For $A = B$ this is a ring isomorphism $\mathcal{T}(A) \cong \mathcal{T}(k \underset{\mathcal{O}}{\otimes} A)$.*

Proof. If A and B are almost source algebras of blocks b and c of finite groups G and H, respectively, with a common defect group P then $A \underset{\mathcal{O}}{\otimes} B$ is an almost source algebra of the block $b \otimes c$ of $G \times H$. Via the Morita equivalence between a block algebra and its almost source algebra any direct summand of $\bar{A} \underset{kQ}{\otimes} \bar{B}$ corresponds to a ΔP-projective p-permutation $k(G \times H)$-module, and hence lifts uniquely, up

to isomorphism, to a direct summand of $A \underset{\mathcal{O}Q}{\otimes} B$, where Q is a subgroup of P and where $\bar{A} = k \underset{\mathcal{O}}{\otimes} A$, $\bar{B} = k \underset{\mathcal{O}}{\otimes} B$. The result follows. □

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§9 Compatibility with the Brauer construction

The first result in this section says that under suitable hypotheses, the Brauer construction sends $\mathcal{T}(A,B)$ to $\mathcal{T}(A(\Delta Q), B(\Delta Q))$. For P abelian this is [4, 3.3].

Theorem 9.1. *Let A, B be almost source algebras of p-blocks b, c of finite groups G, H, respectively, with a common defect group P having the same fusion system $\mathcal{F}$ on P. Let Q be a fully $\mathcal{F}$-centralised subgroup of P. For any $X \in \mathcal{T}(A,B)$ we have $X(\Delta Q) \in \mathcal{T}(A(\Delta Q), B(\Delta Q))$.*

Proof. Let $i \in (\mathcal{O}Gb)^{\Delta P}$ and $j \in (\mathcal{O}Hc)^{\Delta P}$ be almost source idempotents such that $A = i\mathcal{O}Gi$ and $B = j\mathcal{O}Hj$. Denote by c^0 and j^0 the images in $\mathcal{O}H$ of c and j, respectively, of the anti-automorphism of $\mathcal{O}H$ sending $y \in H$ to y^{-1}. Then $B^{op} \cong j^0\mathcal{O}Hj^0$ is an almost source algebra of the block c^0 of $\mathcal{O}H$, by 4.10. Let R be a subgroup of P. We need to show that $(A \underset{\mathcal{O}R}{\otimes} B)(\Delta Q)$ is a direct sum of summands of the $A(\Delta Q)$-$B(\Delta Q)$- bimodules $A(\Delta Q) \underset{kS}{\otimes} B(\Delta Q)$, with S running over the subgroups of $C_P(Q)$. Let Y be an indecomposable direct summand of $(A \underset{\mathcal{O}R}{\otimes} B)(\Delta Q)$. As $A \underset{\mathcal{O}}{\otimes} B^{op}$-modules we have an isomorphism

$$A \underset{\mathcal{O}R}{\otimes} B \cong (A \underset{\mathcal{O}}{\otimes} B^{op}) \underset{\mathcal{O}\Delta R}{\otimes} \mathcal{O}$$

sending $x \otimes y$ to $(x \otimes y) \otimes 1_{\mathcal{O}}$, for $x \in A$ and $y \in B$. By 4.9, the algebra $A \underset{\mathcal{O}}{\otimes} B^{op}$ is an almost source algebra of the block $b \otimes c^0$ of $G \times H$. By the assumptions, A, B and hence B^{op} determine the same fusion system $\mathcal{F}$ on P (cf. 4.10). Since Q is fully $\mathcal{F}$-centralised, ΔQ is fully $\mathcal{F} \times \mathcal{F}$-centralised by 2.2. We apply now 6.2 to $A \underset{\mathcal{O}}{\otimes} B^{op}$, $P \times P$, $\mathcal{F} \times \mathcal{F}$, ΔQ, ΔR instead of A, P, $\mathcal{F}$ Q, R, respectively. The conclusion of 6.2 yields a fully $C_{\mathcal{F}\times\mathcal{F}}(\Delta Q)$-centralised subgroup S of $C_P(Q) \times C_P(Q)$ and a morphism $\psi : \Delta Q \cdot S \to \Delta R$ in $\mathcal{F} \times \mathcal{F}$ such that Y is isomorphic to a direct summand of $(A \underset{\mathcal{O}}{\otimes} B^{op})(\Delta Q) \underset{kS}{\otimes} k$ as $(A \underset{\mathcal{O}}{\otimes} B^{op})(\Delta Q)$-modules. Note that we have obvious isomorphisms

$$(A \underset{\mathcal{O}}{\otimes} B^{op})(\Delta Q) \cong A(\Delta Q) \underset{k}{\otimes} B^{op}(\Delta Q) \cong A(\Delta Q) \underset{k}{\otimes} B(\Delta Q)^{op} .$$

Since $\psi : \Delta Q \cdot S \to \Delta R$ is an injective group homomorphism, the group S must be of the form

$$S = \{(t, \tau(t)) \mid t \in T\}$$

for some subgroup T of $C_P(Q)$ and some injective group homomorphism $\tau : T \to C_P(Q)$. Since ψ is a morphism in $\mathcal{F} \times \mathcal{F}$ there are morphisms $\psi_1 : QT \to P$ and $\psi_2 : Q\tau(T) \to P$ such that

$$\psi(ut, v\tau(s)) = (\psi_1(ut), \psi_2(v\tau(t)))$$

for all $u, v \in Q$ and $t, s \in T$. Since $\psi(\Delta Q) \subseteq \Delta R$ we have $\psi_1(u) = \psi_2(u)$ for all $u \in Q$. Since also $\psi(S) \subseteq \Delta R$ we have $\psi_1(t) = \psi_2(\tau(t))$ for all $t \in T$. Thus $\psi_2^{-1} \circ \psi_1|_Q = \mathrm{Id}_Q$ and $\psi_2^{-1} \circ \psi_1|_T = \tau$. This proves that τ is a morphism in $C_{\mathcal{F}}(Q)$. Thus we have an isomorphism

$$\Delta(QT) \cong \Delta Q \cdot S$$

in $\mathcal{F} \times \mathcal{F}$ mapping (ut, ut) to $(u, u) \cdot (t, \tau(t))$ for all $u \in Q$ and $t \in T$. Now S is fully $C_{\mathcal{F}\times\mathcal{F}}(\Delta Q)$-centralised and $C_{\mathcal{F}\times\mathcal{F}}(\Delta Q) = C_{\mathcal{F}}(Q) \times C_{\mathcal{F}}(Q)$ by 2.3. Thus, by 2.2, in particular T is fully $C_{\mathcal{F}}(Q)$-centralised. But then, by 2.2 again, ΔT is fully $C_{\mathcal{F}}(Q) \times C_{\mathcal{F}}(Q)$-centralised. Thus, by combining 5.1 and 6.2 we may replace S by ΔT. This, however, shows that Y is isomorphic to a direct summand of

$$(A(\Delta Q) \underset{k}{\otimes} B(\Delta Q)^{op}) \underset{k\Delta T}{\otimes} k \cong A(\Delta Q) \underset{kT}{\otimes} B(\Delta Q) \ .$$

Thus the isomorphism class of Y belongs to $\mathcal{T}(A(\Delta Q), B(\Delta Q))$. The result follows. □

The second theorem in this section extends similar statements in [15, 2.4], [17, 2.3.(ii)], [4, 3.6] and includes in particular a proof of Theorem 1.2.

Theorem 9.2. *Let A, B be almost source algebras of p-blocks of finite groups with a common defect group P having the same fusion system $\mathcal{F}$ on P and let C be an interior P-algebra. Let Q be a fully $\mathcal{F}$-centralised subgroup of P and let $\theta : Q \to \mathcal{O}^\times$ be a group homomorphism. For any $X \in \mathcal{T}(A, B)$ and any $Y \in \mathcal{R}(B, C)$ we have*

$$(X \underset{B}{\cdot} Y)(\Delta Q, \Delta\theta) = X(\Delta Q) \underset{B(\Delta Q)}{\cdot} Y(\Delta Q, \Delta\theta)$$

where $\Delta\theta : \Delta Q \to \mathcal{O}^\times$ is the group homomorphism sending (u, u) to $\theta(u)$ for all $u \in Q$. In particular, for any fully $\mathcal{F}$-centralised subgroup Q of P the map sending $X \in \mathcal{T}(A)$ to $X(\Delta Q)$ is a ring homomorphism $\rho_Q : \mathcal{T}(A) \to \mathcal{T}(A(\Delta Q))$.

Proof. Let M be an A-B-bimodule and let N be a B-C-bimodule. Consider the obvious maps

$$M^{\Delta Q} \underset{\mathcal{O}}{\otimes} N^{(\Delta Q, \Delta\theta)} \to (M \underset{\mathcal{O}}{\otimes} N)^{(\Delta Q, \Delta\theta)} \to (M \underset{B}{\otimes} N)^{(\Delta Q, \Delta\theta)} \to (M \underset{B}{\otimes} N)(\Delta Q, \Delta\theta) \ .$$

The composition of these maps induces a homomorphism of $A(\Delta Q)$-$C(\Delta Q)$-bimodules

$$M(\Delta Q) \underset{B(\Delta Q)}{\otimes} N(\Delta Q, \Delta\theta) \longrightarrow (M \underset{B}{\otimes} N)(\Delta Q, \Delta\theta)$$

If we fix N and let M vary, this map is additive functorial in M. Thus, in order to show that this is an isomorphism if M is a finite direct sum of summands of bimodules of the form $A \underset{\mathcal{O}R}{\otimes} B$, we may assume that M is a direct summand of $A \underset{\mathcal{O}R}{\otimes} B$ as $\mathcal{O}Q$-B-bimodule for some subgroup R of P. It suffices to show that this map is a k-linear isomorphism, and so we may ignore the left $A(\Delta Q)$-module structure. Thus we may in fact assume that $M = W \underset{\mathcal{O}R}{\otimes} B$ for some indecomposable direct summand W of A as $\mathcal{O}Q$-$\mathcal{O}R$-bimodule, for some subgroup R of P. By 5.2 we have $W \cong \mathcal{O}Q \underset{\mathcal{O}T}{\otimes} {}_\varphi\mathcal{O}R$ for some subgroup T of Q and some group homomorphism $\varphi : T \to R$ belonging to the common fusion system of A and B on P. If T is a proper subgroup of Q then both sides in the above map are zero. Assume that $Q = T$, hence $M \cong {}_\varphi B$ as $\mathcal{O}Q$-B-bimodule. By decomposing M further we may in fact assume that $M = {}_\varphi nB$ for some primitive idempotent $n \in B^{\Delta\varphi(Q)}$. Then again, if $\mathrm{Br}_{\Delta\varphi(Q)}(n) = 0$, both sides in the above map are easily seen to be zero. Thus we may assume that $\mathrm{Br}_{\Delta\varphi(Q)}(n) \neq 0$. As Q is fully $\mathcal{F}$-centralised, we can apply 5.1, which shows that there is a primitive idempotent $m \in B^{\Delta Q}$ such that ${}_\varphi nB \cong mB$ as $\mathcal{O}Q$-B-bimodules. Thus we may assume $M = mB$. But then M is a direct summand of B itself, so we may assume that $M = B$ as $\mathcal{O}Q$-B-bimodule. But in that case, both sides in the above map are canonically isomorphic to $N(\Delta Q)$. The last statement follows from applying this to $A = B = C$ and $\theta = 1$. □

If one translates 9.2 back to modules over block algebras, one gets in particular the following statement (needed in the proof of 1.4):

Corollary 9.3. *Let G, H be finite groups, let b be a block of $\mathcal{O}G$, and let c be a block of $\mathcal{O}H$. Suppose that b and c have a common defect group P and that there are source idempotents $i \in (\mathcal{O}Gb)^{\Delta P}$ and $j \in (\mathcal{O}Hc)^{\Delta P}$ such that the fusion systems on P determined by i and j coincide. For any subgroup Q of P denote by e_Q the unique block of $kC_G(Q)$ satisfying $\mathrm{Br}_{\Delta Q}(i)e_Q \neq 0$ and by f_Q the unique block of $kC_H(Q)$ satisfying $\mathrm{Br}_{\Delta Q}(j)f_Q \neq 0$. Let M be an $\mathcal{O}Gb$-$\mathcal{O}Hc$-bimodule which is a finite direct sum of summands of the bimodules $\mathcal{O}Gi \underset{\mathcal{O}Q}{\otimes} j\mathcal{O}H$, with Q running over the subgroups of P. For any $\mathcal{O}Hc$-module U, any subgroup Q of P and any group homomorphism $\theta : Q \to \mathcal{O}^\times$ we have an isomorphism of $kC_G(Q)e_Q$-modules*

$$e_Q(M(\Delta Q))f_Q \underset{kC_H(Q)f_Q}{\otimes} f_Q(U(Q,\theta)) \;\cong\; e_Q((M \underset{kHc}{\otimes} U)(Q,\theta)) \; .$$

Proof. By conjugating Q simultaneously we may assume that Q is fully centralised with respect to the common fusion system of $A = i\mathcal{O}Gi$ and $B = j\mathcal{O}Hj$. Then, by making use of the Morita equivalences between $A(\Delta Q)$, $kC_G(e_Q)$ and between $B(\Delta Q)$, $kC_H(Q)f_Q$ from 4.6 it suffices to show that there is an isomorphism of $A(\Delta Q)$-modules

$$\mathrm{Br}_{\Delta Q}(i)M(\Delta Q)\mathrm{Br}_{\Delta Q}(j) \underset{B(\Delta Q)}{\otimes} \mathrm{Br}_{\Delta Q}(j)U(Q,\theta) \;\cong\; \mathrm{Br}_{\Delta Q}(i)((Mj \underset{B}{\otimes} jU)(Q,\theta)) \; .$$

Now the left side is isomorphic to $(iMj)(\Delta Q) \underset{B(\Delta Q)}{\otimes} (jU)(Q,\theta)$ by 3.9, and the right side is isomorphic to $(iMj \underset{B}{\otimes} jU)(Q,\theta)$. Both are isomorphic thanks to 9.2 applied to A, B, $C = \mathcal{O}$, the isomorphism class of iMj instead of X and the isomorphism class of jU instead of Y. □

Remark 9.4. Theorem 9.1 adds to the results in [15, 17] that if X is a splendid tiliting complex of A-B-bimodules then for any $\mathcal{F}$-centric subgroup Q of P the Rickard complex $X(\Delta Q)$ of $A(\Delta Q)$-$B(\Delta Q)$-bimodules is splendid (in the slightly more restrictive sense of the definition given in [15, 1.10] or [17, 1.1]).

Remark 9.5. With the notation of 9.2, if Q is a fully $\mathcal{F}$-centralised subgroup of P and S a fully $C_{\mathcal{F}}(Q)$-centralised subgroup of $C_P(Q)$, then $R = QS$ is fully $\mathcal{F}$-centralised by 2.1. By the usual transitivity properties of the Brauer construction we have $A(\Delta Q)(\Delta S) = A(\Delta R)$ and the following diagram of ring homomorphisms is commutative:

$$\begin{array}{ccc} \mathcal{T}(A) & \xrightarrow{\quad\rho_Q\quad} & \mathcal{T}(A(\Delta Q)) \\ & {\scriptstyle\rho_R}\searrow \quad \swarrow{\scriptstyle\rho_S} & \\ & \mathcal{T}(A(\Delta R)) & \end{array}$$

§10 Isotypies

Let G, H be finite groups, let b be a block of $\mathcal{O}G$ and let c be a block of $\mathcal{O}H$. Suppose that the quotient field K of $\mathcal{O}$ is large enough for b, c and their Brauer pairs. As before, denote by $\mathbb{Z}\mathrm{Irr}_K(G,b)$ the group of generalised characters of G over K associated with the block b, and denote by $\mathbb{Z}\mathrm{IBr}_K(G,b)$ the corresponding group of generalised Brauer characters. Following Broué, a *perfect isometry* between b and c is a group isomorphism

$$\Psi : \mathbb{Z}\mathrm{Irr}_K(H,c) \cong \mathbb{Z}\mathrm{Irr}_K(G,b)$$

sending $\chi \in \mathrm{Irr}_K(H,c)$ to $\epsilon(\chi)\eta_\chi$ for some $\epsilon(\chi) \in \{1,-1\}$ and some $\eta_\chi \in \mathrm{Irr}_K(G,b)$ such that the character $\mu \in \mathbb{Z}\mathrm{Irr}(H \times G)$ defined by

$$\mu(y,x) = \sum_{\chi \in \mathrm{Irr}_K(H,c)} \epsilon(\chi)\chi(y)\eta_\chi(x^{-1})$$

for all $(y,x) \in H \times G$ has the following properties: the character value $\mu(y,x)$ is divisible in $\mathcal{O}$ by the orders of $C_G(x)$, $C_H(y)$, for all $x \in G$, all $y \in H$, and $\mu(x,y) = 0$ if exactly one of x, y is a p'-element. Since the decomposition map $\mathbb{Z}\mathrm{Irr}_K(G,b) \to \mathbb{Z}\mathrm{IBr}_K(G,b)$ given by restricting a generalised character to the set of p'-elements in G is surjective, the arithmetic properties of a perfect isometry Ψ imply that Ψ induces an isomorphism

$$\bar{\Psi} : \mathbb{Z}\mathrm{IBr}_K(H,c) \cong \mathbb{Z}\mathrm{IBr}_K(G,b)\ .$$

Since μ determines the map Ψ by the formula $\Psi(\chi)(x) = \frac{1}{|H|}\sum_{y \in H} \mu(y,x)\chi(y^{-1})$ for all $x \in G$ and all $\chi \in \mathbb{Z}\mathrm{Irr}_K(H,c)$, we will sometimes use abusively the same notation for Ψ and μ.

Suppose now that b and c have a common defect group P, let $i \in (\mathcal{O}Gb)^{\Delta P}$ and $j \in (\mathcal{O}Hc)^{\Delta P}$ be source idempotents. Suppose that the fusion systems of the source algebras $i\mathcal{O}Gi$ and $j\mathcal{O}Hj$ on P are equal. For any subgroup Q of P denote by e_Q the unique block of $kC_G(Q)$ satisfying $\mathrm{Br}_{\Delta Q}(i)e_Q \neq 0$ and by f_Q the unique block of $kC_H(Q)$ satisfying $\mathrm{Br}_{\Delta Q}(j)f_Q \neq 0$. Denote by $\hat{e}_Q$ and $\hat{f}_Q$ the blocks of $\mathcal{O}C_G(Q)$ and $\mathcal{O}C_H(Q)$ which lift e_Q and f_Q, respectively. Again following Broué, an isotypy between b and c is a family of perfect isometries

$$\Psi_Q : \mathbb{Z}\mathrm{Irr}_K(C_H(Q), \hat{f}_Q) \cong \mathbb{Z}\mathrm{Irr}_K(C_G(Q), \hat{e}_Q)$$

for every subgroup Q of P, with the following properties:

(1) for any isomorphism $\varphi : Q \cong R$ in the common fusion system $\mathcal{F}$ we have ${}^{\varphi}\Psi_Q = \Psi_R$, where ${}^{\varphi}\Psi_Q$ is obtained from composing Ψ_Q with the isomorphisms $\mathbb{Z}\mathrm{Irr}_K(C_G(Q), \hat{e}_Q) \cong \mathbb{Z}\mathrm{Irr}_K(C_G(R), \hat{e}_R)$ and $\mathbb{Z}\mathrm{Irr}_K(C_H(Q), \hat{f}_Q) \cong \mathbb{Z}\mathrm{Irr}_K(C_H(R), \hat{f}_R)$ given by conjugation with elements $x \in G$, $y \in H$ satisfying $\varphi(u) = xux^{-1} = yuy^{-1}$ for all $u \in Q$;

(2) for any subgroup Q of P, any element $u \in C_P(Q)$, setting $R = Q\langle u \rangle$, we have an equality of maps

$$d^{(u,e_R)}_{(C_G(Q),e_Q)} \circ \Psi_Q \ = \ \bar{\Psi}_R \circ d^{(u,e_R)}_{(C_H(Q),f_Q)}$$

from $\mathbb{Z}\mathrm{Irr}_K(C_H(Q), \hat{f}_Q)$ to $K \underset{\mathbb{Z}}{\otimes} \mathbb{Z}\mathrm{IBr}_K(C_G(R), \hat{e}_R)$, where $d^{(u,e_R)}_{(C_G(Q),e_Q)}$ is the generalised decomposition map from $\mathbb{Z}\mathrm{Irr}_K(C_G(Q), \hat{e}_Q)$ to $K \underset{\mathbb{Z}}{\otimes} \mathbb{Z}\mathrm{IBr}_K(C_G(R), \hat{e}_R)$ as mentioned in 3.7 and $d^{(u,f_R)}_{(C_H(Q),f_Q)}$ is defined analogously.

Theorem 10.1. *Let G, H be finite groups, let b be a block of $\mathcal{O}G$, and let c be a block of $\mathcal{O}H$. Suppose that b and c have a common defect group P and that there are source idempotents $i \in (\mathcal{O}Gb)^{\Delta P}$ and $j \in (\mathcal{O}Hc)^{\Delta P}$ such that the fusion systems on P of the source algebras $A = i\mathcal{O}Gi$ and $B = j\mathcal{O}Hj$ coincide. Suppose there is a p-permutation equivalence X between A and B. Then for any subgroup Q of P the character Ψ_Q of $C_G(Q) \times C_H(Q)$ determined by $e_Q(\mathcal{O}Gi \underset{A}{\otimes} X \underset{B}{\otimes} j\mathcal{O}H)(\Delta Q)f_Q$ is a perfect isometry between the unique blocks e_Q, f_Q of $C_G(Q)$, $C_H(Q)$ determined by i, j, respectively, and the family $\{\Psi_Q \mid Q \subseteq P\}$ is an isotypy between b and c.*

Proof. Observe the abuse of notation in the statement: we mean by $e_Q((\mathcal{O}Gi \underset{A}{\otimes} X \underset{B}{\otimes} j\mathcal{O}H)(\Delta Q))f_Q$ the image in $T(kC_G(Q)e_Q), kC_H(Q)f_Q)$ of X under the map which is induced by the map sending

an A-B-bimodule M to the $kC_G(Q)e_Q$-$kC_H(Q)f_Q$-bimodule $e_Q((\mathcal{O}Gi \underset{A}{\otimes} M \underset{B}{\otimes} j\mathcal{O}H)(\Delta Q))f_Q$. This could have been denoted more accurately by $e_Q(([\mathcal{O}Gi] \underset{A}{\cdot} X \underset{B}{\cdot} [j\mathcal{O}H])(\Delta Q))f_Q$. Condition **(1)** holds clearly. Thus, in order to show that Ψ_Q induces a perfect isometry, we may assume that Q is fully $\mathcal{F}$-centralised. Set

$$Y_Q = e_Q((\mathcal{O}Gi \underset{A}{\otimes} X \underset{B}{\otimes} j\mathcal{O}H)(\Delta Q))f_Q \ .$$

By 4.11 we have

$$Y_Q = kC_G(Q)\mathrm{Br}_{\Delta Q}(i) \underset{A(\Delta Q)}{\otimes} X(\Delta Q) \underset{B(\Delta Q)}{\otimes} \mathrm{Br}_{\Delta Q}(j)kC_H(Q) \ .$$

By 9.1, $X(\Delta Q)$ belongs to $\mathcal{T}(A(\Delta Q), B(\Delta Q)$ and hence, by 7.5, Y_Q belongs to $T^{\Delta C_P(Q)}(kC_G(Q)e_Q, kC_H(Q)f_Q)$. Since p-permutation modules over k lift uniquely, up to isomorphism, to p-permutation modules over $\mathcal{O}$ preserving vertices, Y_Q lifts to a unique element $\hat{Y}_Q$ in $T^{\Delta C_P(Q)}(\mathcal{O}C_G(Q)\hat{e}_Q, \mathcal{O}C_H(Q)\hat{f}_Q)$. Moreover, Y_Q is a virtual module which is projective on the left and on the right. Therefore, in order to show that this induces a perfect isometry it suffices to show that it induces an isometry. The fact that p-permutation modules lift uniquely as mentioned above implies that it suffices to show that

$$Y_Q \underset{kC_H(Q)f_Q}{\cdot} Y_Q^* = [kC_G(Q)e_Q] \ .$$

Since Q is fully $\mathcal{F}$-centralised, cutting with $\mathrm{Br}_{\Delta Q}(i)$ and $\mathrm{Br}_{\Delta Q}(j)$ yields Morita equivalences. Thus, it suffices to show that

$$\mathrm{Br}_{\Delta Q}(i)Y_Q\mathrm{Br}_{\Delta Q}(i) \underset{B(\Delta Q)}{\cdot} \mathrm{Br}_{\Delta Q}(i)Y_Q^*\mathrm{Br}_{\Delta Q}(i) = [A(\Delta Q)]$$

Using 4.11 we get

$$\mathrm{Br}_{\Delta Q}(i)Y_Q\mathrm{Br}_{\Delta Q}(i) = X(\Delta Q)$$

and hence this is simply equivalent to showing that

$$X(\Delta Q) \underset{B(\Delta Q)}{\cdot} X(\Delta Q)^* = [A(\Delta Q)]$$

Thanks to 9.2 we have

$$X(\Delta Q) \underset{B(\Delta Q)}{\cdot} X(\Delta Q)^* = (X \underset{B}{\cdot} X^*)(\Delta Q) = [A](\Delta Q) = [A(\Delta Q)]$$

as required, and thus Ψ_Q is a perfect isometry. The commutativity **(2)** follows from translating the argument given by Boltje and Xu in [4, 3.8. (e)] with $\mathrm{Br}_P(e)$, $\mathrm{Br}_Q(e)$, $\mathrm{Br}_P(f)$, $\mathrm{Br}_Q(f)$ replaced by e_Q, e_R, f_Q, f_R, respectively, where the notation is now as in **(2)**. Note that f_R is a block of $C_{C_G(Q)}(u) = C_G(R)$. In what follows we write $d^{(u,e_R)}$ instead of $d^{(u,e_R)}_{(C_G(Q),\hat{e}_Q)}$ and smilarly we write $d^{(u,f_R)}$ instead of $d^{(u,f_R)}_{(C_H(Q),\hat{f}_Q)}$ We need to show the equality

$$(d^{(u,e_R)} \circ \Psi_Q)(\chi) = (\bar{\Psi}_R \circ d^{(u,f_R)})(\chi)$$

for any $\chi \in \mathbb{Z}\mathrm{Irr}_K(C_H(Q), \hat{f}_Q)$. By a theorem of Brauer, every generalised character is a $\mathbb{Z}$-linear combination of characters of linear source modules. Thus it suffices to prove the above equality in the case where χ is the character of a linear source $\mathcal{O}C_H(Q)\hat{f}_Q$-module U. Then $\Psi_Q(\chi)$ is the character

of $\hat{Y}_Q \underset{\mathcal{O}C_H(Q)\hat{f}_Q}{\otimes} U$ (which in line with our convention on abuse of notation means $\hat{Y}_Q \underset{\mathcal{O}C_H(Q)\hat{f}_Q}{\cdot} [U]$). This is again a virtual linear source $\mathcal{O}C_G(Q)\hat{e}_Q$-module (cf. 7.8). Thus, by 3.8, we get that

$$d^{(u,e_R)}(\Psi_Q(\chi)) = \sum_{\theta\in\mathrm{Hom}(\langle u\rangle,\mathcal{O}^\times)} \theta(u)\otimes[[f_R((\hat{Y}_Q \underset{\mathcal{O}C_H(Q)\hat{f}_Q}{\otimes} U)(\langle u\rangle,\theta))]]$$

Similarly, again by 3.8, we have $d^{(u,f_R)}(\chi) = \sum_{\theta\in\mathrm{Hom}(\langle u\rangle,\mathcal{O}^\times)} \theta(u)\otimes[[f_R(U(\langle u\rangle,\theta))]]$. Applying $\bar{\Psi}_R$ to this expression yields

$$\bar{\Psi}_R(d^{(u,f_R)}(\chi)) = \sum_{\theta\in\mathrm{Hom}(\langle u\rangle,\mathcal{O}^\times)} \theta(u)\otimes[[Y_R \underset{kC_H(R)f_R}{\otimes} f_R(U(\langle u\rangle,\theta))]]\ .$$

Since $Y_R = e_R(\hat{Y}_Q(\Delta\langle u\rangle))f_R$, the required equality of these two expressions follows now from 9.3. □

§11 Invariance under Morita equivalences with endo-permutation source

If b, c are blocks of finite groups G, H such that the block algebras $\mathcal{O}Gb$, $\mathcal{O}Hc$ are Morita equivalent via an $\mathcal{O}Gb$-$\mathcal{O}Hc$-bimodule M with endo-permutation source V, then by a result of Puig [24], b and c have isomorphic defect groups, and there is an identification of defect groups of b and c such that for some choice of source idempotents the corresponding fusion systems are equal and such that V is stable with respect to this fusion system (see e.g [12] for the terminology regarding fusion stable endo-permutation modules). Thus Theorem 1.6 is a consequence of the following Theorem:

Theorem 11.1. *Let A, B be source algebras of blocks b, c of finite groups G, H with a common defect group P and the same fusion system $\mathcal{F}$ on P. Suppose that there is an indecomposable $\mathcal{F}$-stable endo-permutation $\mathcal{O}P$-module V and an indecomposable direct summand M of the A-B-bimodule $A \underset{\mathcal{O}P}{\otimes} \mathrm{Ind}_{\Delta P}^{P\times P}(V) \underset{\mathcal{O}P}{\otimes} B$ satisfying $M \underset{B}{\otimes} M^* \cong A$ and $M^* \underset{A}{\otimes} M \cong B$. Then the map sending an A-A-bimodule X to the B-B-bimodule $M^* \underset{A}{\otimes} X \underset{A}{\otimes} M$ induces a ring isomorphism $\mathcal{T}(A) \cong \mathcal{T}(B)$.*

Proof. We need to show that for Q a subgroup of P, the isomorphism class of the B-B-bimodule $M^* \underset{A}{\otimes} A \underset{\mathcal{O}Q}{\otimes} A \underset{A}{\otimes} M$ belongs to $\mathcal{T}(B)$. By the assumptions on M, the bimodule $M^* \underset{A}{\otimes} A \underset{\mathcal{O}Q}{\otimes} A \underset{A}{\otimes} M$ is isomorphic to a direct summand of $B \underset{\mathcal{O}P}{\otimes} \mathrm{Ind}_{\Delta P}^{P\times P}(V^*) \underset{\mathcal{O}P}{\otimes} A \underset{\mathcal{O}Q}{\otimes} A \underset{\mathcal{O}P}{\otimes} \mathrm{Ind}_{\Delta P}^{P\times P}(V) \underset{\mathcal{O}P}{\otimes} B$. It follows from 5.2 that every indecomposable direct summand of $A \underset{\mathcal{O}Q}{\otimes} A$ as $\mathcal{O}P$-$\mathcal{O}P$-bimodule is isomorphic to $\mathcal{O}P \underset{\mathcal{O}R}{\otimes} {}_\varphi\mathcal{O}P$ for some subgroup R of P and some $\varphi\in\mathrm{Hom}_{\mathcal{F}}(R,P)$. Thus, if Y is an indecomposable direct summand of the bimodule $M^* \underset{A}{\otimes} A \underset{\mathcal{O}Q}{\otimes} A \underset{A}{\otimes} M$ then Y is isomorphic to a direct summand of $B \underset{\mathcal{O}P}{\otimes} \mathrm{Ind}_{\Delta P}^{P\times P}(V^*) \underset{\mathcal{O}R}{\otimes} {}_\varphi\mathrm{Ind}_{\Delta P}^{P\times P}(V) \underset{\mathcal{O}P}{\otimes} B$. One checks that

$$\mathrm{Ind}_{\Delta P}^{P\times P}(V^*) \underset{\mathcal{O}R}{\otimes} {}_\varphi\mathrm{Ind}_{\Delta P}^{P\times P}(V) \cong \mathrm{Ind}_{\Delta R}^{P\times P}(V^* \underset{\mathcal{O}}{\otimes} {}_\varphi V)$$

and since V is fusion stable, it follows that $V^* \underset{\mathcal{O}}{\otimes} {}_\varphi V$ is a permutation $\mathcal{O}R$-module. Thus the isomorphism class of Y belongs to $\mathcal{T}(B)$. The fact that the map sending an A-A-bimodule X to $M^* \underset{A}{\otimes} X \underset{A}{\otimes} M$ induces a ring homomorphism follows from the isomorphism $M \underset{B}{\otimes} M^* \cong B$. Finally, this is an isomorphism - its inverse is obtained by exchanging the roles of A, B and of M, M^*. □

Corollary 11.2. *Let A be a source algebra of a nilpotent block with defect group P. Then $\mathcal{T}(A) \cong \mathcal{T}(\mathcal{O}P)$.*

Proof. By the main result in [23] we can identify $A = S \underset{\mathcal{O}}{\otimes} \mathcal{O}P$, where $S = \mathrm{End}_{\mathcal{O}}(V)$ for some indecomposable endo-permutation $\mathcal{O}P$-module V with vertex P. Set

$$M = V \underset{\mathcal{O}}{\otimes} \mathcal{O}P \ ,$$

viewed as A-$\mathcal{O}P$-bimodule via the canonical action of S on V and the regular actions of $\mathcal{O}P$ on the left and the right of $\mathcal{O}P$. Using the canonical isomorphism $S \cong V \underset{\mathcal{O}}{\otimes} V^*$ one shows that M induces a Morita equivalence. As A-A-bimodule, A is isomorphic to a direct summand of $A \underset{\mathcal{O}P}{\otimes} A$. Thus, as A-$\mathcal{O}P$-bimodule, M is isomorphic to a direct summand of $A \underset{\mathcal{O}P}{\otimes} M = A \underset{\mathcal{O}P}{\otimes} (V \underset{\mathcal{O}}{\otimes} \mathcal{O}P)$. When considered as $A \underset{\mathcal{O}}{\otimes} \mathcal{O}P$-module, this is isomorphic to $(A \underset{\mathcal{O}}{\otimes} \mathcal{O}P) \underset{\mathcal{O}\Delta P}{\otimes} V$ via the map sending $a \otimes (v \otimes y)$ to $(a \otimes y^{-1}) \otimes v$, for all $a \in A$, $v \in V$ and $y \in P$. This in turn is isomorphic to $A \underset{\mathcal{O}P}{\otimes} \mathrm{Ind}_{\Delta P}^{P \times P}(V) \underset{\mathcal{O}P}{\otimes} \mathcal{O}P$ via the map sending $(a \otimes y^{-1}) \otimes v$ to $a \otimes ((1,1) \otimes v) \otimes y$, so the isomorphism $\mathcal{T}(A) \cong \mathcal{T}(\mathcal{O}P)$ is a particular case of 11.1. □

Remark 11.3. A nilpotent block with defect group P is known to be isotypic to the defect group algebra $\mathcal{O}P$. Does any such isotypy lift to a p-permutation equivalence? The answer is positive whenever a source V of the unique simple module has an *endo-split p-permutation resolution* in the sense of Rickard [25, §7], because then the isotypy is in fact induced by a splendid derived equivalence (cf. [15, 1.3] and [25, 6.3]).

§12 Augmentation homomorphisms

Given a finite p-group P we denote by $B(P)$ the Burnside ring of P.

Proposition 12.1. *Let P be a finite p-group. The map sending for any subgroup Q of P the P-set P/Q to the $\mathcal{O}P$-$\mathcal{O}P$-bimodule $\mathcal{O}P \underset{\mathcal{O}Q}{\otimes} \mathcal{O}P$ induces a ring isomorphism*

$$\delta_P : B(P) \cong \mathcal{T}(\mathcal{O}P) \ .$$

In particular, the ring $\mathcal{T}(\mathcal{O}P)$ is commutative and the $\mathbb{Z}$-rank of $\mathcal{T}(\mathcal{O}P)$ is equal to the number of conjugacy classes of subgroups of P.

Proof. One can verify this directly; alternatively, by [14, 11.4.6], the map sending an $\mathcal{O}P$-module V to the $\mathcal{O}P$-$\mathcal{O}P$-bimodule $\mathrm{Ind}_{\Delta P}^{P \times P}(V)$ induces an injective ring homomorphism from the Green ring of $\mathcal{O}P$-modules to the ring $\mathcal{P}(\mathcal{O}P)$, where here V is considered as module over $\mathcal{O}\Delta P$ via the obvious isomorphism $\Delta P \cong P$. This map sends the isomorphism class of $\mathcal{O}P/Q$ to that of $\mathcal{O}P \underset{\mathcal{O}Q}{\otimes} \mathcal{O}P$, for any subgroup Q of P, and hence induces an isomorphism $B(P) \cong \mathcal{T}(\mathcal{O}P)$ because any bimodule of the form $\mathcal{O}P \underset{\mathcal{O}Q}{\otimes} \mathcal{O}P$ is indecomposable. □

The Burnside ring of a finite p-group P admits for any subgroup Q an augmentation homomorphism $\sigma_Q : B(P) \to \mathbb{Z}$, induced by the map sending a finite P-set X to the number of Q-fixpoints $|X^Q|$. Composed with δ_P this yields an augmentation of $\mathcal{T}(\mathcal{O}P)$ which can be described explicitly as follows:

Proposition 12.2. *Let P be a finite p-group and let Q be a subgroup of P. The map sending an $\mathcal{O}P$-$\mathcal{O}P$-bimodule M to the rational number $\frac{\dim_k(M(\Delta Q)}{|C_P(Q)|}$ induces a ring homomorphism $\rho_Q : \mathcal{T}(\mathcal{O}P) \to \mathbb{Z}$, and we have $\sigma_Q = \rho_Q \circ \delta_P$.*

Proof. Let R be a subgroup of P. The image $P \underset{R}{\times} P$ of $P \times P$ in $\mathcal{O}P \underset{\mathcal{O}R}{\otimes} \mathcal{O}P$ is a P-P-stable $\mathcal{O}$-basis of $\mathcal{O}P \underset{\mathcal{O}R}{\otimes} \mathcal{O}P$. In order to calculate $(\mathcal{O}P \underset{\mathcal{O}R}{\otimes} \mathcal{O}P)(\Delta Q)$ we need to determine the set of Q-fixpoints in this basis. Let $y, z \in P$ and denote by $y \otimes z$ the image of (y, z) in $\mathcal{O}P \underset{\mathcal{O}R}{\otimes} \mathcal{O}P$. This is a ΔQ-fixpoint if and only if for every $u \in Q$ there is $r_u \in R$ such that $uy \otimes z = yr_u \otimes z = y \otimes r_u z = y \otimes zu$, hence if and only if $Q^y \subseteq R$ and $yz \in C_P(Q)$. The condition $Q^y \subseteq R$ is also equivalent to yR being in the Q-fixpoint set $(P/R)^Q$. Note that $C_P(Q)$ acts freely on the left on the set of ΔQ-fixpoints in the basis $P \underset{R}{\times} P$ of $\mathcal{O}P \underset{\mathcal{O}R}{\otimes} \mathcal{O}P$. Thus the map sending $y \in P$ satisfying $Q^y \subseteq R$ to $y \otimes y^{-1}$ induces a bijection between the Q-fixpoints in P/R and the $C_P(Q)$-orbits of ΔQ-fixpoints in $P \underset{R}{\times} P$. The result follows. □

We get in particular an augmentation for the ring $\mathcal{T}(A)$:

Proposition 12.3. *Let A be a source algebra of a block of a finite group with defect group P. The map sending an A-A-bimodule M to to the rational number $\frac{\dim_k(M(\Delta P))}{|Z(P)|}$ induces a unitary ring homomorphism $\alpha_P : \mathcal{T}(A) \to \mathbb{Z}$.*

Proof. By [28, 38.10] we have $A(\Delta P) \cong kZ(P)$. Thus, by 9.2, the Brauer construction with respect to ΔP induces a unitary ring homomorphism $\mathcal{T}(A) \to \mathcal{T}(kZ(P))$. The result follows from 12.2 applied to $kZ(P)$ and 1 instead of $\mathcal{O}P$ and Q, respectively. □

As in the case of Burnside rings of finite groups, there are other homomorphisms from $\mathcal{T}(A)$ to $\mathbb{Z}$. In order to describe them, we need the following terminology.

Definition 12.4. Let $\mathcal{F}$ be a fusion system on a finite p-group P. A subgroup Q of P is called *$\mathcal{F}$-nilpotent* if Q is fully $\mathcal{F}$-centralised and if $C_{\mathcal{F}}(Q)$ is the trivial fusion system $\mathcal{F}_{C_P(Q)}(C_P(Q))$ of the group $C_P(Q)$ on itself.

In particular, if Q is $\mathcal{F}$-centric then Q is $\mathcal{F}$-nilpotent. It is easy to check that if Q, R are fully $\mathcal{F}$-centralised subgroups of P such that Q is $\mathcal{F}$-nilpotent and $\mathrm{Hom}_{\mathcal{F}}(Q, R) \neq \emptyset$ then R is $\mathcal{F}$-nilpotent as well.

Remark 12.5. Let G be a finite group, b a block of $\mathcal{O}G$, P a defect group of b and let $i \in (\mathcal{O}Gb)^{\Delta P}$ be an almost source idempotent of b. Set $A = i\mathcal{O}Gi$, denote by $\mathcal{F}$ the fusion system of A on P and for any subgroup Q of P denote by e_Q the unique block of $kC_G(Q)$ satisfying $\mathrm{Br}_{\Delta P}(i)e_Q \neq 0$. Then a subgroup Q of P is $\mathcal{F}$-nilpotent if and only if e_Q is a nilpotent block of $kC_G(Q)$ having $C_P(Q)$ as defect group (this is well-known and follows, for instance, from [18, §2]). Therefore, if Q is $\mathcal{F}$-nilpotent we have $\mathcal{T}(A(\Delta Q)) \cong \mathcal{T}(kC_P(Q))$ by 11.2. Thus the composition of ring homomorphisms $\mathcal{T}(A) \to \mathcal{T}(A(\Delta Q)) \cong \mathcal{T}(kC_P(Q)) \to \mathbb{Z}$ yields a unitary ring homomorphism $\mathcal{T}(A) \longrightarrow \mathbb{Z}$ for any $\mathcal{F}$-nilpotent subgroup Q of P. We can describe these ring homomorphisms explicitly:

Theorem 12.6. *Let A be an almost source algebra of a block with defect group P and fusion system $\mathcal{F}$. Let Q be an $\mathcal{F}$-nilpotent subgroup of P and let $j \in A^{\Delta Q}$ be a primitive idempotent such that $\mathrm{Br}_{\Delta Q}(j) \neq 0$. The map sending an A-A-bimodule M to the rational number $\frac{\dim_k((jMj)(\Delta Q))}{|C_P(Q)|}$ induces a unitary ring homomorphism $\alpha_Q : \mathcal{T}(A) \to \mathbb{Z}$. Moreover, α_Q is independent of the choice of j.*

Proof. The ring homomorphism $\rho_Q : \mathcal{T}(A) \to \mathcal{T}(A(\Delta Q))$ is induced by the map sending an A-A-bimodule M to $M(\Delta Q)$. The block e_Q of $C_G(Q)$ determined by A is nilpotent because Q is $\mathcal{F}$-nilpotent. Thus e_Q has $C_P(Q)$ as defect group, there is a unique local point μ of $C_P(Q)$ on $A(\Delta Q)$, and for any $m \in \mu$ we have an isomorphism of interior $C_P(Q)$-algebra

$$mA(\Delta Q)m \cong S \underset{k}{\otimes} kC_P(Q)$$

with $S = \mathrm{End}_k(V)$ for some indecomposable endo-permutation $kC_P(Q)$-module V. Every primitive idempotent in S remains primitive in $mA(\Delta Q)m$ and hence lifts to a primitive idempotent j in $A(\Delta Q)^{\Delta Q}$, uniquely up to conjugation, and hence

$$(jAj)(\Delta Q) \cong kC_P(Q)$$

Thus the map sending M to $(jMj)(\Delta Q)$ induces a ring homomorphism $\mathcal{T}(A) \to \mathcal{T}(kC_P(Q))$. Composing this with the augmentation $\mathcal{T}(kC_P(Q)) \to \mathbb{Z}$ obtained from 12.2 (with $C_P(Q)$ and 1 instead of P and Q) yields the ring homomorphism α_Q as stated. □

The ring homomorphisms α_Q depend only on the isomorphism class of Q in the fusion system. More precisely:

Theorem 12.7. *Let A be a source algebra of a block of a finite group G with defect group P and fusion system $\mathcal{F}$. Let Q, R be $\mathcal{F}$-nilpotent subgroups of P. We have $\alpha_Q = \alpha_R$ if and only if $Q \cong R$ in $\mathcal{F}$.*

Proof. Suppose that $\alpha_Q = \alpha_R$. Let $j \in A^{\Delta Q}$ be a primitive idempotent satisfying $\mathrm{Br}_{\Delta Q}(j) \neq 0$. Then $(jA \underset{\mathcal{O}Q}{\otimes} Aj)(\Delta Q)$ is non zero, hence the image of $[A \underset{\mathcal{O}Q}{\otimes} A]$ under α_Q is non zero. But then its image under α_R is non zero, which forces $(A \underset{\mathcal{O}Q}{\otimes} A)(\Delta R) \neq \{0\}$. It follows from 6.4 that $\mathrm{Hom}_{\mathcal{F}}(R, Q)$ is non-empty. Exchanging the roles of Q and R yields an isomorphism $Q \cong R$ in $\mathcal{F}$. Suppose conversely that $Q \cong R$ in $\mathcal{F}$. By [22, 3.1], the existence of an isomorphism $Q \cong R$ in $\mathcal{F}$ is equivalent to the existence of an invertible element $a \in A^\times$ and primitive idempotents $m \in A^{\Delta Q}$, $n \in A^{\Delta R}$ not contained in $\ker(\mathrm{Br}_{\Delta Q})$, $\ker(\mathrm{Br}_{\Delta R})$, respectively, such that ${}^a(Qm) = Rn$. Thus conjugation by a induces an isomorphism $(mAm)(\Delta Q) \cong (nAn)(\Delta R)$, and since the blocks e_Q, e_R of $kC_G(Q)$, $kC_G(R)$, respectively, determined by A are nilpotent, we have $kC_P(Q) \cong (mAm)(\Delta Q)$ and $kC_P(R) \cong (nAn)(\Delta R)$. Thus we get a commutative diagram of ring homomorphisms of the form

$$\begin{array}{ccccc} & & \mathcal{T}(A(\Delta Q)) & \xrightarrow{\cong} & \mathcal{T}((mAm)(\Delta Q)) \\ & \nearrow & & & \\ \mathcal{T}(A) & & & \cong \downarrow & & \searrow & \mathbb{Z} \\ & \searrow & & & & \nearrow & \\ & & \mathcal{T}(A(\Delta R)) & \xrightarrow[\cong]{} & \mathcal{T}((nAn)(\Delta R)) \end{array}$$

where the vertical arrow is induced by conjugation with a. The result follows. □

§13 Further remarks

13.1. One advantage of the concept of a p-permutation equivalence introduced by Boltje and Xu [4] is that it admits the obvious source algebra formulation 1.3 above. The concept of isotypies, however, is formulated at the block algebra level, involving character values of group elements. Some of the technical issues in this paper are caused by the need to switch between block algebras and their source algebras; a version of isotypies in terms of source algebras would simplify some parts. The other advantage of p-permutation equivalences versus isotypies is that the rather technical compatibility conditions of an isotypy get replaced by a condition which is simpler in that it does not refer to the local structure of the blocks. It does not seem to be known whether every isotypy between two blocks can be "lifted" to a p-permutation equivalence.

13.2. Let A, B be symmetric $\mathcal{O}$-algebras. Following [16, 2.9] any A-B-bimodule M which is finitely generated projective as left A-module and as right B-module gives rise to a *transfer map* on Hochschild cohomology

$$t_M : HH^*(B) \longrightarrow HH^*(A) \ .$$

This depends only on the isomorphism class of M. Thus, any element in $X \in \mathcal{P}(A, B)$ determines a transfer map $t_X : HH^*(B) \longrightarrow HH^*(A)$. When applied to $A = B$ this yields a graded $\mathcal{P}(A)$-module structure on $HH^*(A)$ given by

$$X \cdot \zeta = t_X(\zeta)$$

for any $X \in \mathcal{P}(A)$ and any $\zeta \in HH^*(A)$. If A is a source algebra of a block b of some finite group with defect group P then this module structure restricts to a $\mathcal{T}(A)$-module structure on $HH^*(A)$. By [16, 5.6.(iii)], the Hochschild cohomology algebra $HH^*(A)$ contains a canonical copy of the block cohomology $H^*(G, b)$. By [16, 6.7] this is a $\mathcal{T}(A)$-submodule of $HH^*(A)$.

13.3. Unlike the Burnside ring of a finite group, the trivial source bimodule ring of a block need not be commutative. Consider $p = 2$ and $G = A_4 = P \rtimes E$, where $P \cong C_2 \times C_2$ and $|E| = 3$. Then $\mathcal{O}G$ has a unique block. Write $\mathrm{Hom}(E, \mathcal{O}^\times) = \{\mu_0, \mu_1, \mu_2\}$ with $\mu_0 = 1$. Every μ_i extends to an automorphism α_i of $\mathcal{O}G$ defined by $\alpha_i(eu) = \mu_i(e)ue$, for $u \in P$ and $e \in E$. These automorphisms fix P and hence the three $\mathcal{O}G$-$\mathcal{O}G$-bimodules $(\mathcal{O}G)_{\alpha_i}$ have vertex ΔP and trivial source. Thus they determine three elements in $\mathcal{T}(\mathcal{O}G)$. Setting $e_i = \frac{1}{3}\sum_{e \in E} \mu_i(e)e$ we get a primitive decomposition $\{e_1, e_2, e_3\}$ of 1 in $\mathcal{O}G$. One checks that $\alpha_i(e_0) = e_i$ and that $\alpha_2 = (\alpha_1)^{-1}$. Any algebra automorphism α of $\mathcal{O}G$ induces an isomorphism of bimodules ${}_\alpha\mathcal{O}G \cong \mathcal{O}G_{\alpha-1}$ and isomorphisms of left $\mathcal{O}G$-modules ${}_\alpha\mathcal{O}Gj \cong \mathcal{O}G\alpha^{-1}(j)$ for any idempotent j in $\mathcal{O}G$. Therefore

$$[(\mathcal{O}G)_{\alpha_1}] \underset{\mathcal{O}G}{\cdot} [\mathcal{O}Ge_0 \underset{\mathcal{O}}{\otimes} e_0\mathcal{O}G] = [\mathcal{O}Ge_1 \underset{\mathcal{O}}{\otimes} e_0\mathcal{O}G] \ ,$$

$$[\mathcal{O}Ge_0 \underset{\mathcal{O}}{\otimes} e_0\mathcal{O}G] \underset{\mathcal{O}G}{\cdot} [(\mathcal{O}G)_{\alpha_1}] = [\mathcal{O}Ge_0 \underset{\mathcal{O}}{\otimes} e_2\mathcal{O}G] \ .$$

In particular $\mathcal{T}(\mathcal{O}G)$ is not commutative.

Appendix: On stable equivalences and blocks of defect 2

The technology of this paper can be used to show that the proofs of two results due to R. Rouquier [27, 5.6, 6.3] for principal blocks carry over nearly verbatim to arbitrary blocks, giving in particular proofs of some of the results announced by Rouquier in [27, 6.4, Appendix]. For A, B two symmetric $\mathcal{O}$-algebras, a bounded complex X of A-B-bimodules which are projective as left A-modules and

as right B-modules is said to *induce a stable equivalence* if there are isomorphisms of complexes of bimodules $X \underset{B}{\otimes} X^* \cong A \oplus Y$ and $X^* \underset{A}{\otimes} X \cong B \oplus Z$ with Y and Z homotopy equivalent to bounded complexes of projective A-A-bimodules and B-B-bimodules, respectively. If Y and Z are homotopic to zero then X is called a *Rickard complex*. The following two theorems are [27, 5.6] and [27, 6.3] for arbitrary blocks:

Theorem A.1. *Let G, H be finite groups, b, c blocks of $\mathcal{O}G$, $\mathcal{O}H$, respectively, having a common defect group P, let $i \in (\mathcal{O}Gb)^{\Delta P}$ and $j \in (\mathcal{O}Hc)^{\Delta P}$ be source idempotents. Suppose that i and j determine the same fusion system $\mathcal{F}$ on P. For any subgroup Q of P denote by e_Q and f_Q the unique blocks of $kC_G(Q)$ and $kC_H(Q)$ satisfying $\mathrm{Br}_{\Delta Q}(i)e_Q \neq 0$ and $\mathrm{Br}_{\Delta Q}(j)f_Q \neq 0$. Let X be a bounded complex of $\mathcal{O}Gb$-$\mathcal{O}Hc$ bimodules whose terms are finite direct sums of summands of the $\mathcal{O}Gb$-$\mathcal{O}Hc$-bimodules $\mathcal{O}Gi \underset{\mathcal{O}Q}{\otimes} j\mathcal{O}H$, where Q runs over the subgroups of P. The following are equivalent:*

(i) The complex X induces a stable equivalence.

(ii) For every non-trivial subgroup Q of P the complex $e_QX(\Delta Q)f_Q$ is a Rickard complex of $kC_G(Q)e_Q$-$kC_H(Q)f_Q$-bimodules.

Proof. Suppose that (i) holds. Write $X \underset{\mathcal{O}Hc}{\otimes} X^* \cong \mathcal{O}Gb \oplus Y$ for some bounded complex Y of $\mathcal{O}Gb$-$\mathcal{O}Gb$-bimodules which is homotopy equivalent to a complex of projective $\mathcal{O}Gb$-$\mathcal{O}Gb$-bimodules. Since the Brauer construction with respect to a non-trivial p-subgroup sends any projective module to zero and preserves homotopies, we get that $Y(\Delta Q) \simeq \{0\}$ for any non-trivial subgroup Q of P. Thus, by [17, 2.3] (or by 9.2 translated back to block algebras via 4.7) we get that

$$(e_QX(\Delta Q)f_Q) \underset{kC_H(Q)f_Q}{\otimes} (f_QX^*(\Delta Q)e_Q) \cong e_Q(X \underset{\mathcal{O}Hc}{\otimes} X^*)(\Delta Q)e_Q$$

$$\simeq e_Q(\mathcal{O}Gb)(\Delta Q)e_Q = kC_G(Q)e_Q$$

for any non-trivial fully $\mathcal{F}$-centralised subgroup Q of P. Since any subgroup of P is isomorphic, in the fusion system $\mathcal{F}$, to a fully $\mathcal{F}$-centralised subgroup we get this isomorphism for any non-trivial subgroup Q of P. In particular, $e_QX(\Delta Q)f_Q$ is a Rickard complex for any non-trivial subgroup Q of P. Suppose conversely that (ii) holds. Let Y be the mapping cone of the adjunction unit $\mathcal{O}Gb \to X \underset{\mathcal{O}Hc}{\otimes} X^*$. By the assumptions, the induced map

$$kC_G(Q)e_Q \to (e_QX(\Delta Q)f_Q) \underset{kC_H(Q)f_Q}{\otimes} (f_QX^*(\Delta Q)e_Q)$$

is a homotopy equivalence, and so its mapping cone is contractible, for every non-trivial subgroup Q of P. But as before, the right side is canonically isomorphic to $e_Q(X \underset{\mathcal{O}Hc}{\otimes} X^*)(\Delta Q)e_Q$, and so the corresponding mapping cone is $e_QY(\Delta Q)f_Q$. This is contractible, for every non-trivial Q. But then in fact $Y(\Delta Q) \simeq 0$ for every non-trivial subgroup Q of P, thanks to 4.8. A theorem of Bouc [5, 7.9] implies that Y is homotopy equivalent to a bounded complex of projective bimodules, whence (i). □

Theorem A.2. *Let G be a finite group, let b be a block of $\mathcal{O}G$ with a defect group of order p^2, set $H = N_G(P)$ and denote by c the block of $\mathcal{O}H$ satisfying $\mathrm{Br}_{\Delta P}(b) = \mathrm{Br}_{\Delta P}(c)$. Let $i \in (\mathcal{O}Gb)^{\Delta P}$ and $j \in (\mathcal{O}Hc)^{\Delta P}$ be source idempotents such that $\mathrm{Br}_{\Delta P}(i) = \mathrm{Br}_{\Delta P}(j)$. There is a bounded complex of $\mathcal{O}Gb$-$\mathcal{O}Hc$-bimodules whose components are finite direct sums of summands of the bimodules $\mathcal{O}Gi \underset{\mathcal{O}Q}{\otimes} j\mathcal{O}H$, with Q running over the subgroups of P, such that X induces a stable equivalence.*

Proof. If P is cyclic there is a Rickard complex of $\mathcal{O}Gb$-$\mathcal{O}Hc$-bimodules explicitly constructed by Rouquier [26]; it is observed in [17, §7] that the components of this complex are as stated. Thus we

may assume that P is elementary abelian of rank 2. The proof follows the lines of [27, 6.3]. Denote by e_Q and f_Q the unique blocks of $kC_G(Q)$ and $kC_H(Q)$, respectively, satisfying $\mathrm{Br}_{\Delta Q}(i)e_Q \neq 0$ and $\mathrm{Br}_{\Delta Q}(j)f_Q \neq 0$. Since P is abelian, the fusion systems on P determined by i and by j are equal to that of $kN(P, e_P)e_P$. Hence, for any subgroup Q of P, we have

$$N_G(Q, e_Q)/C_G(Q) \cong N_H(Q, f_Q)/C_H(Q)$$

and both sides are p'-groups. The blocks e_Q, f_Q lift to unique blocks $\hat{e}_Q$, $\hat{f}_Q$ of $\mathcal{O}C_G(Q)$, $\mathcal{O}C_H(Q)$, respectively. The images of $\hat{e}_Q$, $\hat{f}_Q$ in $\mathcal{O}C_G(Q)/Q$, $\mathcal{O}C_H(Q)/Q$ are blocks, denoted by $\bar{e}_Q$, $\bar{f}_Q$, respectively. If Q has order p then $\bar{e}_Q$, $\bar{f}_Q$ have the cyclic group P/Q as defect group, and $C_H(Q)/Q$ is the normaliser in $C_G(Q)/Q$ of P/Q. Thus $\bar{f}_Q$ is in fact the Brauer correspondent of $\bar{e}_Q$. By Rouquier's construction of Rickard complexes for blocks with cyclic defect groups in [26, 10.3], there is a Rickard complex of $\mathcal{O}C_G(Q)/Q\bar{e}_Q$-$\mathcal{O}C_H(Q)\bar{f}_Q$-bimodules $\bar{C}_Q$ of the form

$$\bar{C}_Q \;=\; \cdots \to 0 \to \bar{N}_Q \xrightarrow{\bar{\Phi}_Q} \bar{e}_Q\mathcal{O}C_G(Q)/Q\bar{f}_Q \to 0 \to \cdots$$

for some projective bimodule $\bar{N}_Q$. This complex lifts to a Rickard complex of $\mathcal{O}C_G(Q)\hat{e}_Q$-$\mathcal{O}C_H(Q)\hat{f}_Q$-bimodules of the form

$$C_Q \;=\; \cdots \to 0 \to N_Q \xrightarrow{\Phi_Q} e_Q\mathcal{O}C_G(Q)f_Q \to 0 \to \cdots$$

where N_Q is a projective $\mathcal{O}(C_G(Q)\times\mathcal{O}C_H(Q))/\Delta Q$-module lifting $\bar{N}_Q$, inflated to $\mathcal{O}(C_G(Q)\times C_H(Q))$, and where Φ_Q lifts the map $\bar{\Phi}_Q$. By adapting arguments of Marcuş [20, 5.5] this complex extends to the group

$$T = N_{G\times H}(\Delta Q) \cap (N_G(Q, e_Q) \times N_H(Q, f_Q))$$

and T contains $C_G(Q) \times C_H(Q)$ as normal subgroup of p'-index by the above remarks. Thus N_Q remains projective when considered as $\mathcal{O}T/\Delta Q$-module. We set

$$V_Q = \mathrm{Ind}_T^{G\times H}(N_Q)\ .$$

The inclusion $C_G(Q) \subseteq G$ induces an $\mathcal{O}T$-homomorphism $e_Q\mathcal{O}C_G(Q)f_Q \longrightarrow b\mathcal{O}Gc$, which, by adjunction, yields a homomorphism of $\mathcal{O}(G \times H)$-modules

$$\alpha_Q : \mathrm{Ind}_T^{G\times H}(e_Q\mathcal{O}C_G(Q)f_Q) \longrightarrow b\mathcal{O}Gc\ .$$

Set $\psi_Q = \alpha_Q \circ \mathrm{Ind}_T^{G\times H}(\Phi_Q) : V_Q \to b\mathcal{O}Gc$ and define the complex X by

$$X \;=\; \cdots \to 0 \to \bigoplus_Q V_Q \xrightarrow{\sum_Q \psi_Q} b\mathcal{O}Gc \to 0 \to \cdots$$

with $b\mathcal{O}Gc$ in degree zero, where Q runs over a set of representatives of the $N_G(P, e_P)$-conjugacy classes of subgroups of order p of P. If Q, R are two subgroups of order p which are not $N_G(P, e_P)$-conjugate then $V_R(\Delta Q) = \{0\}$. This implies that $e_QX(\Delta Q)f_Q \;\simeq C_Q \underset{\mathcal{O}}{\otimes} k$, and this is a Rickard complex of $kC_G(Q)e_Q$-$C_H(Q)f_Q$-bimodules. Moreover, $b\mathcal{O}Gc$ is a direct summand of $\mathcal{O}Gi \underset{\mathcal{O}P}{\otimes} j\mathcal{O}H$, and V_Q is a direct sum of summands of $\mathcal{O}Gi \underset{\mathcal{O}Q}{\otimes} j\mathcal{O}H$ because it is obtained from lifting, inflating and inducing a projective bimodule. Thus Theorem A.1 applies, showing that X induces a stable equivalence. □

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