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# ZJ-THEOREMS FOR FUSION SYSTEMS

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ABSTRACT. For  $p$  an odd prime, we generalise the Glauberman-Thompson  $p$ -nilpotency theorem [5, Ch. 8, Theorem 3.1] to arbitrary fusion systems. We define a notion of  $Qd(p)$ -free fusion systems and show that if  $\mathcal{F}$  is a  $Qd(p)$ -free fusion system on some finite  $p$ -group  $P$  then  $\mathcal{F}$  is controlled by  $W(P)$  for any Glauberman functor  $W$ , generalising Glauberman's  $ZJ$ -theorem [3] to arbitrary fusion systems.

## 1 INTRODUCTION

Let  $p$  be an odd prime, let  $G$  be a finite group, let  $P$  be a Sylow- $p$ -subgroup of  $G$  and let  $J(P)$  be the Thompson subgroup of  $P$  (generated by the set of abelian subgroups of  $P$  of maximal order). The  $p$ -nilpotency theorem of Glauberman and Thompson [5, Ch. 8, Theorem 3.1] states that  $G$  is  $p$ -nilpotent if and only if  $N_G(Z(J(P)))$  is  $p$ -nilpotent. By a theorem of Frobenius [4, 8.6],  $G$  is  $p$ -nilpotent if and only if  $P$  controls  $G$ -fusion in  $P$ , or equivalently, if and only if  $\mathcal{F}_P(G) = \mathcal{F}_P(P)$ , where the notation is as described in §2 below. The  $p$ -nilpotency theorem has been generalised to  $p$ -blocks of finite groups in [7], and the following theorem proves an analogue for arbitrary fusion systems.

**Theorem A.** *Let  $p$  be an odd prime and let  $\mathcal{F}$  be a fusion system on a finite  $p$ -group  $P$ . We have  $\mathcal{F} = \mathcal{F}_P(P)$  if and only if  $N_{\mathcal{F}}(Z(J(P))) = \mathcal{F}_P(P)$ .*

The proof of Theorem A is given in §4. A finite group  $A$  is said to be *involved* in another finite group  $G$  if there are subgroups  $H, K$  of  $G$  such that  $K \trianglelefteq H$  and  $A \cong H/K$ . Glauberman's  $ZJ$ -Theorem in [3] asserts that if  $p$  is odd and  $Qd(p)$  is not involved in  $G$  then  $N_G(Z(J(P)))$  controls strong  $p$ -fusion in  $G$ , where  $P$  is Sylow- $p$ -subgroup of  $G$  and where  $Qd(p)$  is the semi-direct product  $(C_p \times C_p) \rtimes SL_2(p)$  with  $SL_2(p)$  acting naturally on the elementary abelian group of rank 2. By [4, 14.8] the conclusion holds in fact with  $ZJ$  replaced by any Glauberman functor (cf. 1.2 below). In order to extend this to arbitrary fusion systems, we introduce the following notation and terminology. By [1, 4.3], if  $\mathcal{F}$  is a fusion system on a finite  $p$ -group  $P$  and  $Q$  an  $\mathcal{F}$ -centric fully normalised

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subgroup of  $P$ , there is, up to isomorphism, a unique finite group  $L = L_Q^{\mathcal{F}}$  having  $N_P(Q)$  as Sylow- $p$ -subgroup such that  $C_L(Q) = Z(Q)$  and  $N_{\mathcal{F}}(Q) = \mathcal{F}_{N_P(Q)}(L)$ .

**Definition 1.1.** A fusion system  $\mathcal{F}$  on a finite  $p$ -group  $P$  is called  $Qd(p)$ -free, if  $Qd(p)$  is not involved in any of the groups  $L_Q^{\mathcal{F}}$ , with  $Q$  running over the set of  $\mathcal{F}$ -centric radical fully normalised subgroups of  $P$ .

**Definition 1.2.** (cf. [8, 1.3]) A *positive characteristic  $p$ -functor* is a map sending any finite  $p$ -group  $P$  to a characteristic subgroup  $W(P)$  of  $P$  such that  $W(P) \neq 1$  if  $P \neq 1$  and such that any isomorphism of finite  $p$ -groups  $P \cong Q$  maps  $W(P)$  onto  $W(Q)$ . A *Glauberman functor* is a positive characteristic  $p$ -functor with the following additional property: whenever  $P$  is a Sylow- $p$ -subgroup of a finite group  $L$  which satisfies  $C_L(O_p(L)) = Z(O_p(L))$  and which does not involve  $Qd(p)$ , then  $W(P)$  is normal in  $L$ .

Any of the maps sending a finite  $p$ -group  $P$  to  $Z(J(P))$  or  $K_{\infty}(P)$  or  $K^{\infty}(P)$  are Glauberman functors, where  $J(P)$  is the Thompson subgroup of  $P$ , and where  $K_{\infty}, K^{\infty}$  are as defined in [4, Section 12].

**Theorem B.** *Let  $p$  be an odd prime, let  $W$  be a Glauberman functor and let  $\mathcal{F}$  be a fusion system on a finite  $p$ -group  $P$ . If  $\mathcal{F}$  is  $Qd(p)$ -free then  $\mathcal{F} = N_{\mathcal{F}}(W(P))$ .*

For fusion systems of finite groups this is Glauberman's  $ZJ$ -theorem; for fusion systems of  $p$ -blocks of finite groups this has also been noted by G. R. Robinson, generalising [8, 1.4] where it was shown that the conclusion of Theorem B holds under the slightly stronger assumption that  $SL_2(p)$  is not involved in any of the automorphism groups  $\text{Aut}_{\mathcal{F}}(Q)$ , with  $Q$  running over the set of  $\mathcal{F}$ -centric radical subgroups of  $P$ . The proof of Theorem B, given in §7, follows the pattern of the proof of [8, 1.4].

Since there exist Glauberman functors mapping  $P$  to a subgroup  $W(P)$  satisfying  $C_P(W(P)) = Z(W(P))$  (for example,  $K_{\infty}, K^{\infty}$  have this property), the above Theorem in conjunction with [1, 4.3] implies that a  $Qd(p)$ -free fusion system on a finite  $p$ -group  $P$  is in fact equal to the fusion system of a finite group  $L$  having  $P$  as Sylow- $p$ -subgroup and satisfying  $C_L(O_p(L)) \subseteq O_p(L)$ . In particular, a  $Qd(p)$ -free fusion system is the underlying fusion system of a unique  $p$ -local finite group in the sense of [2].

## 2 BACKGROUND MATERIAL ON FUSION SYSTEMS

Let  $p$  be a prime and let  $P$  be a finite  $p$ -group. Following the terminology of [10], a *category on  $P$*  is a category  $\mathcal{F}$  with the subgroups of  $P$  as objects and with morphism sets  $\text{Hom}_{\mathcal{F}}(Q, R)$  consisting of injective group homomorphisms, for any two subgroups  $Q, R$  of  $P$ , such that the following hold. Composition of morphisms in  $\mathcal{F}$  is the usual composition of group homomorphisms, and for any morphism  $\varphi : Q \rightarrow R$  in  $\mathcal{F}$ , the induced isomorphism  $Q \cong \varphi(Q)$ , its inverse and the inclusion  $\varphi(Q) \subseteq R$  are all morphisms in  $\mathcal{F}$  as well. Given a category  $\mathcal{F}$  on  $P$  and a subgroup  $Q$  of  $P$ , we say that

- $Q$  is *fully  $\mathcal{F}$ -normalised* if  $|N_P(Q)| \geq |N_P(\varphi(Q))|$  for every morphism  $\varphi : Q \rightarrow P$  in  $\mathcal{F}$ ;
- $Q$  is *fully  $\mathcal{F}$ -centralised* if  $|C_P(Q)| \geq |C_P(\varphi(Q))|$  for every morphism  $\varphi : Q \rightarrow P$  in  $\mathcal{F}$ ;
- $Q$  is  *$\mathcal{F}$ -centric* if  $C_P(\varphi(Q)) = Z(\varphi(Q))$  for every morphism  $\varphi : Q \rightarrow P$  in  $\mathcal{F}$ ; and
- $Q$  is  *$\mathcal{F}$ -radical* if  $O_p(\text{Aut}_{\mathcal{F}}(Q)) = \text{Aut}_Q(Q)$ .

Following [2, 1.2], if  $\varphi : Q \rightarrow P$  is a morphism in a category  $\mathcal{F}$  on  $P$ , we denote by  $N_\varphi$  the subgroup of  $N_P(Q)$  consisting of all  $y \in N_P(Q)$  for which there is  $z \in N_P(\varphi(Q))$  such that  $\varphi(yuy^{-1}) = z\varphi(u)z^{-1}$  for all  $u \in Q$ . If  $Q, R$  are subgroups of  $P$  we denote by  $\text{Hom}_P(Q, R)$  the set of all group homomorphisms from  $Q$  to  $R$  induced by conjugation with elements in  $P$ . If  $Q = R$  we write  $\text{Aut}_P(Q) = \text{Hom}_P(Q, Q)$ ; note that  $\text{Aut}_P(Q) \cong N_P(Q)/C_P(Q)$ .

A *fusion system on  $P$*  is a category  $\mathcal{F}$  on  $P$  whose morphism sets contain all morphisms induced by conjugation with elements in  $P$ , and which has furthermore the following properties.

(I-S)  $\text{Aut}_P(P)$  is a Sylow- $p$ -subgroup of  $\text{Aut}_{\mathcal{F}}(P)$ ;

(II-S) for every morphism  $\varphi : Q \rightarrow P$  such that  $\varphi(Q)$  is fully  $\mathcal{F}$ -normalised, there is a morphism  $\psi : N_\varphi \rightarrow P$  such that  $\varphi = \psi|_Q$ .

This concept is due to Puig [11]; the above definition appears in [12] and is equivalent to the definition of what is called a *saturated fusion system* in [2, 1.2]; in particular, it is shown in [12] that the axioms (I-S) and (II-S) imply the axioms used in [2, 1.2],

(I-BLO) if  $Q$  is a fully  $\mathcal{F}$ -normalised subgroup of  $P$  then  $Q$  is fully  $\mathcal{F}$ -centralised and  $\text{Aut}_P(Q)$  is a Sylow- $p$ -subgroup of  $\text{Aut}_{\mathcal{F}}(Q)$ ;

(II-BLO) for every morphism  $\varphi : Q \rightarrow P$  such that  $\varphi(Q)$  is fully  $\mathcal{F}$ -centralised, there is a morphism  $\psi : N_\varphi \rightarrow P$  such that  $\varphi = \psi|_Q$ .

If  $G$  is a finite group having  $P$  as Sylow- $p$ -subgroup we denote by  $\mathcal{F}_P(G)$  the category on  $P$  whose morphism sets are the group homomorphisms induced by conjugation with elements in  $G$ ; that is,  $\text{Hom}_{\mathcal{F}_P(G)}(Q, R) = \text{Hom}_G(Q, R)$  for any two subgroups  $Q, R$  of  $G$ . It is well-known and easy to verify that  $\mathcal{F}_P(G)$  is a fusion system; we call  $\mathcal{F}_P(G)$  the fusion system of the finite group  $G$  on  $P$ . In particular,  $\mathcal{F}_P(P)$  is the fusion system on  $P$  whose morphisms are exactly those induced by inner automorphisms in  $P$ . Note that  $\mathcal{F}_P(P) \subseteq \mathcal{F}$  for any fusion system on  $P$ .

Let  $\mathcal{F}$  be a fusion system on  $P$  and let  $Q$  be a subgroup of  $P$ . We denote by  $C_{\mathcal{F}}(Q)$  the category on  $C_P(Q)$  such that, for any two subgroups  $R, R'$  of  $C_P(Q)$ , the morphism set in  $C_{\mathcal{F}}(Q)$  from  $R$  to  $R'$  consists of all group homomorphisms  $\varphi : R \rightarrow R'$  such that there exists a morphism  $\psi : RQ \rightarrow RQ'$  in  $\mathcal{F}$  satisfying  $\psi|_R = \varphi$  and  $\psi|_Q = \text{Id}_Q$ . Similarly, we denote by  $N_{\mathcal{F}}(Q)$  the category on  $N_P(Q)$  such that, for any two subgroups  $R, R'$  of  $N_P(Q)$ , the morphism set in  $N_{\mathcal{F}}(Q)$  from  $R$  to  $R'$  consists of all group homomorphisms  $\varphi : R \rightarrow R'$  such that there exists a morphism  $\psi : RQ \rightarrow RQ'$  in  $\mathcal{F}$  satisfying  $\psi|_R = \varphi$  and  $\psi(Q) = Q$ . By a result of Puig, if  $Q$  is fully  $\mathcal{F}$ -centralised then  $C_{\mathcal{F}}(Q)$  is a fusion

system on  $C_P(Q)$ , and if  $Q$  is fully  $\mathcal{F}$ -normalised then  $N_{\mathcal{F}}(Q)$  is a fusion system on  $N_P(Q)$ . Both statements are in fact particular cases of a more general result; see e.g. [2, Appendix, Prop. A6] for a proof. If  $\mathcal{F} = N_{\mathcal{F}}(Q) = N_{\mathcal{F}}(R)$  for normal subgroups  $Q, R$  of  $P$  then one easily checks that  $\mathcal{F} = N_{\mathcal{F}}(QR)$ . Thus the following definition makes sense:

**Definition 2.1.** Let  $\mathcal{F}$  be a fusion system on a finite  $p$ -group  $P$ . We denote by  $O_p(\mathcal{F})$  the largest normal subgroup of  $P$  such that  $\mathcal{F} = N_{\mathcal{F}}(O_p(\mathcal{F}))$ .

**Lemma 2.2.** *Let  $\mathcal{F}$  be a fusion system on a finite  $p$ -group  $P$ . For any subgroup  $Q$  of  $P$  there is a morphism  $\varphi : N_P(Q) \rightarrow P$  in  $\mathcal{F}$  such that  $\varphi(Q)$  is fully  $\mathcal{F}$ -normalised.*

*Proof.* Let  $\psi : Q \rightarrow P$  be a morphism in  $\mathcal{F}$  such that  $\psi(Q)$  is fully  $\mathcal{F}$ -normalised. Thus  $\text{Aut}_P(\psi(Q))$  is a Sylow- $p$ -subgroup of  $\text{Aut}_{\mathcal{F}}(Q)$ . Therefore  $\psi \circ \text{Aut}_P(Q) \circ \psi^{-1}$  is conjugate to a subgroup of  $\text{Aut}_P(\psi(Q))$ ; say  $\tau \circ \psi \circ \text{Aut}_P(Q) \circ \psi^{-1} \circ \tau^{-1} \subset \text{Aut}_P(\psi(Q))$  for some  $\tau \in \text{Aut}_{\mathcal{F}}(\psi(Q))$ . Thus  $\varphi = \tau \circ \psi$  has the property that  $N_{\varphi} = N_P(Q)$ , hence  $\varphi$  extends to a morphism  $N_P(Q) \rightarrow P$ , and  $\varphi(Q) = \psi(Q)$  is fully  $\mathcal{F}$ -normalised.  $\square$

**Lemma 2.3.** *Let  $\mathcal{F}$  be a fusion system on a finite  $p$ -group  $P$ . For any fully  $\mathcal{F}$ -normalised subgroup  $Q$  of  $P$  and any morphism  $\varphi : N_P(Q) \rightarrow P$  the subgroup  $\varphi(Q)$  is fully  $\mathcal{F}$ -normalised.*

*Proof.* If  $\varphi : N_P(Q) \rightarrow P$  is a morphism in  $\mathcal{F}$  then  $\varphi(N_P(Q)) \subseteq N_P(\varphi(Q))$ , hence this inclusion is an equality whenever  $Q$  is fully  $\mathcal{F}$ -normalised.  $\square$

Given two subgroups  $Q, R$  of  $P$  such that  $Q \leq R$  we denote by  $\text{Aut}_{\mathcal{F}}(Q \leq R)$  the subgroup of  $\text{Aut}_{\mathcal{F}}(R)$  consisting of all  $\varphi \in \text{Aut}_{\mathcal{F}}(R)$  satisfying  $\varphi(Q) = Q$ . Restriction to  $Q$  induces a group homomorphism  $\text{Aut}_{\mathcal{F}}(Q \leq R) \rightarrow \text{Aut}_{\mathcal{F}}(Q)$ . The following Lemma is a reformulation of a well-known fact following from the extension axiom for fusion systems.

**Lemma 2.4.** *Let  $\mathcal{F}$  be a fusion system on a finite  $p$ -group  $P$  and let  $Q, R$  be subgroups of  $P$  such that  $QC_P(Q) \leq R$ . Suppose that  $Q$  is fully  $\mathcal{F}$ -centralised and that  $\text{Aut}_R(Q)$  is normal in  $\text{Aut}_{\mathcal{F}}(Q)$ . The group homomorphism  $\text{Aut}_{\mathcal{F}}(Q \leq R) \rightarrow \text{Aut}_{\mathcal{F}}(Q)$  induced by restriction from  $R$  to  $Q$  is surjective.*

*Proof.* For any  $y \in N_P(Q)$  denote by  $c_y$  the automorphism of  $Q$  given by conjugation with  $y$ . Let  $\varphi \in \text{Aut}_{\mathcal{F}}(Q)$ . Since  $\text{Aut}_R(Q)$  is normal in  $\text{Aut}_{\mathcal{F}}(Q)$ , for any  $y \in R$  there is  $z \in R$  such that  $\varphi \circ c_y \circ \varphi^{-1} = c_z$ . In particular, we have  $R \subseteq N_{\varphi}$ . Thus  $\varphi$  extends to a morphism  $\psi : R \rightarrow P$ . Then, for all  $u \in Q$ , we have  $(\varphi \circ c_y \circ \varphi^{-1})(u) = (\psi \circ c_y \circ \psi^{-1})(u) = \psi(y(\psi^{-1}(u))) = \psi^{(y)}u$ , or equivalently,  $\varphi \circ c_y \circ \varphi^{-1} = c_{\psi(y)}$ . Since  $C_P(Q) \subseteq R$  we get  $\psi(R) = R$  and hence  $\psi \in \text{Aut}_{\mathcal{F}}(Q \leq R)$ .  $\square$

**Lemma 2.5.** *Let  $\mathcal{F}$  be a fusion system on a finite  $p$ -group  $P$ . Suppose that  $\mathcal{F} = N_{\mathcal{F}}(Q)$  for some normal subgroup  $Q$  of  $P$ . Then  $Q$  is contained in any  $\mathcal{F}$ -centric radical subgroup  $R$  of  $P$ .*

*Proof.* Let  $R$  be a fully  $\mathcal{F}$ -normalised centric radical subgroup of  $P$ . The hypothesis  $\mathcal{F} = N_{\mathcal{F}}(Q)$  implies that  $\text{Aut}_{QR}(R)$  is normal in  $\text{Aut}_{\mathcal{F}}(R)$ . Since  $R$  is radical, this forces  $\text{Aut}_{QR}(R) = \text{Aut}_R(R)$ , hence  $N_{QR}(R) \subseteq RC_P(R)$ . As  $R$  is also centric, we get  $N_{QR}(R) = R$ , and hence  $QR = R$ , or equivalently,  $Q \subseteq R$ .  $\square$

Besides  $C_{\mathcal{F}}(Q)$  and  $N_{\mathcal{F}}(Q)$  we need another particular case of Puig's result in [2, Appendix, Prop. A6].

**Proposition 2.6.** *Let  $\mathcal{F}$  be a fusion system on a finite  $p$ -group  $P$  and let  $Q$  be a fully  $\mathcal{F}$ -normalised subgroup of  $P$ . Then there is a fusion system  $N_P(Q)C_{\mathcal{F}}(Q)$  on  $N_P(Q)$  contained in  $N_{\mathcal{F}}(Q)$  such that, for any two subgroups  $R, R'$  of  $N_P(Q)$ , the morphism set  $\text{Hom}_{N_P(Q)C_{\mathcal{F}}(Q)}(R, R')$  consists of all group homomorphisms  $\varphi : R \rightarrow R'$  such that there exists a homomorphism  $\psi : RQ \rightarrow R'Q$  in  $N_{\mathcal{F}}(Q)$  and an element  $y \in P$  satisfying  $\psi|_R = \varphi$  and  $\psi(u) = {}^y u$  for all  $u \in Q$ .*

*Proof.* In the notation of [2, Appendix, Prop. A 6], this is the case where  $K = \text{Aut}_P(Q)$  applied to the fusion system  $N_{\mathcal{F}}(Q)$  on  $N_P(Q)$ .  $\square$

We will frequently use Alperin's fusion theorem in the following form (see e.g. [2, Appendix, Theorem A 10] for a proof):

**Theorem 2.7.** *(Alperin's fusion theorem for fusion systems) Let  $\mathcal{F}$  be a fusion system on a finite  $p$ -group  $P$ . Any isomorphism in  $\mathcal{F}$  can be written as a composition of isomorphisms  $\varphi : Q \cong Q'$  for which there exists a fully  $\mathcal{F}$ -normalised centric radical subgroup  $R$  of  $P$  containing  $Q, Q'$  and an automorphism  $\alpha \in \text{Aut}_{\mathcal{F}}(R)$  such that  $\alpha|_Q = \varphi$ .*

Let  $\mathcal{F}$  be a fusion system on a finite  $p$ -group  $P$  such that  $\mathcal{F} = N_{\mathcal{F}}(Q)$  for some normal subgroup  $Q$  of  $P$ . We define a category  $\mathcal{F}/Q$  on  $P/Q$  as follows: for any two subgroups  $R, S$  of  $P$  containing  $Q$ , a group homomorphism  $\psi : R/Q \rightarrow S/Q$  is a morphism in  $\mathcal{F}/Q$  if there is a morphism  $\varphi : R \rightarrow S$  in  $\mathcal{F}$  satisfying  $\psi(uQ) = \varphi(u)Q$  for all  $u \in R$ . The following result is due to Puig [11].

**Proposition 2.8.** *Let  $\mathcal{F}$  be a fusion system on a finite  $p$ -group  $P$  such that  $\mathcal{F} = N_{\mathcal{F}}(Q)$  for some normal subgroup  $Q$  of  $P$ . Then the category  $\mathcal{F}/Q$  is a fusion system on  $P/Q$ .*

## 3 ON CENTRAL EXTENSIONS OF FUSION SYSTEMS

The following result is well-known.

**Proposition 3.1.** *Let  $\mathcal{F}$  be a fusion system on a finite  $p$ -group  $P$  and let  $Z$  be a subgroup of  $Z(P)$  such that  $\mathcal{F} = C_{\mathcal{F}}(Z)$ . Set  $\bar{P} = P/Z$  and  $\bar{\mathcal{F}} = \mathcal{F}/Z$ . For any subgroup  $Q$  of  $P$  containing  $Z$  the canonical map  $Q \rightarrow \bar{Q}$  induces a surjective group homomorphism*

$$\mathrm{Aut}_{\mathcal{F}}(Q) \rightarrow \mathrm{Aut}_{\bar{\mathcal{F}}}(\bar{Q})$$

whose kernel is an abelian  $p$ -group, where  $\bar{Q}$  is the image of  $Q$  in  $\bar{P}$ . In particular, if  $Q$  is  $\mathcal{F}$ -radical then  $\bar{Q}$  is  $\bar{\mathcal{F}}$ -radical, if  $\bar{Q}$  is  $\bar{\mathcal{F}}$ -centric then  $Q$  is  $\mathcal{F}$ -centric, and if  $Q$  is  $\mathcal{F}$ -centric radical then  $\bar{Q}$  is  $\bar{\mathcal{F}}$ -centric radical.

*Proof.* If  $\varphi$  is an automorphism of  $Q$  inducing the identity on  $Z$  and on  $\bar{Q}$  then, for all  $u \in Q$ , we have  $\varphi(u) = u\zeta(u)$  for some group homomorphism  $\zeta : Q \rightarrow Z$ , and hence the group of all such automorphisms is isomorphic to the abelian  $p$ -group  $\mathrm{Hom}(Q, Z)$  with group structure induced by that of  $Z$ . Thus if  $Q$  is  $\mathcal{F}$ -radical, the kernel of the map  $\mathrm{Aut}_{\mathcal{F}}(Q) \rightarrow \mathrm{Aut}_{\bar{\mathcal{F}}}(\bar{Q})$  is contained in  $\mathrm{Aut}_Q(Q)$ . Therefore, in that case,  $\mathrm{Aut}_{\mathcal{F}}(Q)/\mathrm{Aut}_Q(Q) \cong \mathrm{Aut}_{\bar{\mathcal{F}}}(\bar{Q})/\mathrm{Aut}_{\bar{Q}}(\bar{Q})$ . Finally, if  $\bar{Q}$  is  $\bar{\mathcal{F}}$ -centric, for every subgroup  $R$  of  $P$  isomorphic to  $Q$  in  $\mathcal{F}$  we have  $C_{\bar{P}}(\bar{R}) = Z(\bar{R}) \subseteq \bar{R}$ , and therefore  $C_P(R) \subseteq R$ , which implies that  $C_P(R) = Z(R)$  and hence that  $Q$  is  $\mathcal{F}$ -centric. For the last statement we may assume that  $\bar{Q}$  is fully  $\bar{\mathcal{F}}$ -centralised. The kernel  $K$  of the canonical map  $\mathrm{Aut}_{\mathcal{F}}(Q) \rightarrow \mathrm{Aut}_{\bar{\mathcal{F}}}(\bar{Q})$  is a  $p$ -group. Thus, if  $Q$  is  $\mathcal{F}$ -radical then  $K \subseteq \mathrm{Aut}_Q(Q)$ . Let  $C$  be the inverse image in  $P$  of  $C_{\bar{P}}(\bar{Q})$ . That is, the image in  $\bar{P}$  of any element in  $C$  centralises  $\bar{Q}$ , and hence  $\mathrm{Aut}_C(Q) \subseteq K$ . This implies  $C \subseteq QC_P(Q)$ . Thus, if in addition  $Q$  is  $\mathcal{F}$ -centric we get that  $C \subseteq Q$ , and hence  $C_{\bar{P}}(\bar{Q}) \subseteq \bar{Q}$ , which shows that  $\bar{Q}$  is  $\bar{\mathcal{F}}$ -centric.  $\square$

**Proposition 3.2.** *Let  $\mathcal{F}, \mathcal{F}'$  be fusion systems on a finite  $p$ -group  $P$  and let  $Z$  be a subgroup of  $Z(P)$  such that  $\mathcal{F} = C_{\mathcal{F}}(Z)$  and  $\mathcal{F}' = C_{\mathcal{F}'}(Z)$ . Suppose that  $\mathcal{F} \subseteq \mathcal{F}'$ . Then  $\mathcal{F} = \mathcal{F}'$  if and only if  $\mathcal{F}/Z = \mathcal{F}'/Z$ .*

*Proof.* Suppose that  $\mathcal{F}/Z = \mathcal{F}'/Z$ . Let  $Q$  be an  $\mathcal{F}'$ -centric radical subgroup of  $P$ . Then, by 3.1, the kernel  $K$  of the canonical map  $\mathrm{Aut}_{\mathcal{F}'}(Q) \rightarrow \mathrm{Aut}_{\mathcal{F}'/Z}(Q/Z)$  is contained in  $\mathrm{Aut}_Q(Q)$ . Thus  $K$  is also the kernel of the canonical map  $\mathrm{Aut}_{\mathcal{F}}(Q) \rightarrow \mathrm{Aut}_{\mathcal{F}/Z}(Q/Z)$ . Since  $\mathrm{Aut}_{\mathcal{F}/Z}(Q/Z) = \mathrm{Aut}_{\mathcal{F}'/Z}(Q/Z)$  it follows that  $\mathrm{Aut}_{\mathcal{F}}(Q)$  and  $\mathrm{Aut}_{\mathcal{F}'}(Q)$  have the same order. The assumption  $\mathcal{F} \subseteq \mathcal{F}'$  implies  $\mathrm{Aut}_{\mathcal{F}}(Q) = \mathrm{Aut}_{\mathcal{F}'}(Q)$ . The equality  $\mathcal{F} = \mathcal{F}'$  follows then from Alperin's fusion theorem 2.7. The converse is trivial.  $\square$

**Corollary 3.3.** *Let  $\mathcal{F}$  be a fusion systems on a finite  $p$ -group  $P$ , let  $Z$  be a subgroup of  $Z(P)$  such that  $\mathcal{F} = C_{\mathcal{F}}(Z)$  and let  $\mathcal{G}$  be a fusion system on  $P$  such that  $\mathcal{G} \subseteq \mathcal{F}$ . Set  $\bar{\mathcal{F}} = \mathcal{F}/Z$  and  $\bar{\mathcal{G}} = \mathcal{G}/Z$ . Let  $Q$  be a normal subgroup of  $P$  containing  $Z$  and set  $\bar{Q} = Q/Z$ . We have  $\mathcal{G} = N_{\mathcal{F}}(Q)$  if and only if  $\bar{\mathcal{G}} = N_{\bar{\mathcal{F}}}(\bar{Q})$ .*

*Proof.* Suppose that  $\bar{\mathcal{G}} = N_{\bar{\mathcal{F}}}(\bar{Q})$ . Then  $\bar{Q}$  is contained in any  $\bar{\mathcal{G}}$ -centric radical subgroup of  $\bar{P}$ . It follows from the last statement in 3.1 that  $Q$  is contained in any  $\mathcal{G}$ -centric radical subgroup of  $P$ , and hence  $\mathcal{G} = N_{\mathcal{G}}(Q)$ . Since  $\mathcal{G} \subseteq \mathcal{F}$  this implies  $\mathcal{G} \subseteq N_{\mathcal{F}}(Q)$ . Clearly  $N_{\mathcal{F}}(Q)/Z = N_{\bar{\mathcal{F}}}(\bar{Q})$ , and thus  $\mathcal{G} = N_{\mathcal{F}}(Q)$  by 3.2. The converse is trivial.  $\square$

**Proposition 3.4.** *Let  $P$  be a finite  $p$ -group, let  $Q$  be a normal subgroup of  $P$  and let  $\mathcal{F}, \mathcal{G}$  be fusion systems on  $P$  such that  $\mathcal{F} = PC_{\mathcal{F}}(Q)$  and such that  $\mathcal{G} \subseteq \mathcal{F}$ . Let  $R$  be a normal subgroup of  $P$  containing  $Q$ . We have  $\mathcal{G} = N_{\mathcal{F}}(R)$  if and only if  $\mathcal{G}/Q = N_{\mathcal{F}/Q}(R/Q)$ .*

*Proof.* Suppose  $\mathcal{G}/Q = N_{\mathcal{F}/Q}(R/Q)$ . In order to show the equality  $\mathcal{G} = N_{\mathcal{F}}(R)$  we proceed by induction over the order of  $Q$ . If  $Q = 1$  there is nothing to prove. Suppose  $Q \neq 1$ . Since  $Q$  is normal in  $P$ , the group  $Z = Q \cap Z(P)$  is non trivial. The assumption  $\mathcal{F} = PC_{\mathcal{F}}(Q)$  implies that  $\mathcal{F} = C_{\mathcal{F}}(Z)$ . Set  $\bar{\mathcal{F}} = \mathcal{F}/Z$  and  $\bar{\mathcal{G}} = \mathcal{G}/Z$ . Similarly, set  $\bar{P} = P/Z$ ,  $\bar{Q} = Q/Z$  and  $\bar{R} = R/Z$ . Then  $\bar{\mathcal{F}}, \bar{\mathcal{G}}$  are fusion systems on  $\bar{P}$  satisfying  $\bar{\mathcal{F}} = \bar{P}C_{\bar{\mathcal{F}}}(\bar{Q})$  and  $\bar{\mathcal{G}} \subseteq \bar{\mathcal{F}}$ . We have isomorphisms of fusion systems  $\bar{\mathcal{G}}/\bar{Q} \cong \mathcal{G}/Q$  and  $N_{\bar{\mathcal{F}}/\bar{Q}}(\bar{R}/\bar{Q}) \cong N_{\mathcal{F}/Q}(R/Q)$  induced by the canonical isomorphism  $\bar{P}/\bar{Q} \cong P/Q$ . Thus  $\bar{\mathcal{G}}/\bar{Q} = N_{\bar{\mathcal{F}}/\bar{Q}}(\bar{R}/\bar{Q})$ . By induction we get that  $\bar{\mathcal{G}} = N_{\bar{\mathcal{F}}}(\bar{R})$ , where  $\bar{R} = R/Z$ . But then 3.3 implies  $\mathcal{G} = N_{\mathcal{F}}(R)$  as required. The converse is trivial.  $\square$

#### 4 PROOF OF THEOREM A

Let  $p$  be an odd prime, let  $P$  be a finite  $p$ -group and let  $\mathcal{F}$  be a fusion system on  $P$ . We have  $\mathcal{F}_P(P) \subseteq N_{\mathcal{F}}(Z(J(P))) \subseteq \mathcal{F}$ . Thus if  $\mathcal{F} = \mathcal{F}_P(P)$  then trivially  $N_{\mathcal{F}}(Z(J(P))) = \mathcal{F}_P(P)$ . In order to prove the converse, let  $\mathcal{F}$  be a minimal counter example to Theorem A; that is, the number of morphisms  $|\mathcal{F}|$  of  $\mathcal{F}$  is minimal such that  $N_{\mathcal{F}}(Z(J(P))) = \mathcal{F}_P(P)$  but  $\mathcal{F} \neq \mathcal{F}_P(P)$ . We proceed in a series of steps as in [7].

**4.1.** *Any fusion system  $\mathcal{G}$  on  $P$  which is properly contained in  $\mathcal{F}$  is equal to  $\mathcal{F}_P(P)$ .*

*Proof.* Since  $N_{\mathcal{G}}(Z(J(P))) \subseteq N_{\mathcal{F}}(Z(J(P))) = \mathcal{F}_P(P)$  we have  $N_{\mathcal{G}}(Z(J(P))) = \mathcal{F}_P(P)$  and hence  $\mathcal{G} = \mathcal{F}_P(P)$  by the minimality assumption on  $\mathcal{F}$ .  $\square$

**4.2.** *We have  $O_p(\mathcal{F}) \neq 1$ .*

*Proof.* Since  $\mathcal{F} \neq \mathcal{F}_P(P)$ , Alperin's fusion theorem implies that there is a fully  $\mathcal{F}$ -normalised subgroup  $Q$  of  $P$  such that  $N_{\mathcal{F}}(Q) \neq \mathcal{F}_R(R)$ , where  $R = N_P(Q)$ . Amongst all such subgroups choose  $Q$  such that  $R = N_P(Q)$  has maximal possible order. We are going to show that  $R = P$ . Assume that  $R \neq P$ . We may choose  $Q$  such that  $Z(J(R))$  is also fully  $\mathcal{F}$ -normalised; indeed, by 2.2 there is a morphism  $\varphi : N_P(Z(J(R))) \rightarrow P$  such that  $\varphi(Z(J(R)))$  is fully  $\mathcal{F}$ -normalised, and since  $N_P(Q) = R \subseteq N_P(R) \subseteq N_P(Z(J(R)))$  it follows from 2.3 that  $\varphi(Q)$  is still fully  $\mathcal{F}$ -normalised. Having replaced  $Q$  by  $\varphi(Q)$ , consider the fusion system  $N_{\mathcal{F}}(Z(J(R)))$  on  $N_P(Z(J(R)))$ . Note that since  $R$  is a proper



subgroup of  $P$  it is also a proper subgroup of  $N_P(R)$ , hence of  $N_P(Z(J(R)))$ . The choice of  $Q$  implies that  $N_{\mathcal{F}}(Z(J(R))) = \mathcal{F}_{N_P(Z(J(R)))}(N_P(Z(J(R))))$ . Then in particular  $N_{N_{\mathcal{F}}(Q)}(Z(J(R))) = \mathcal{F}_R(R)$ . The minimality assumption on  $\mathcal{F}$  implies the contradiction  $N_{\mathcal{F}}(Q) = \mathcal{F}_R(R)$ . Thus  $R = P$ , or equivalently,  $Q \trianglelefteq P$ . Since  $N_{\mathcal{F}}(Q) \neq \mathcal{F}_P(P)$  we get that  $N_{\mathcal{F}}(Q) = \mathcal{F}$  by 4.1, thus  $Q \subseteq O_p(\mathcal{F})$ .  $\square$

Set now  $Q = O_p(\mathcal{F})$ . Note that  $Q$  is a proper subgroup of  $P$  as the equality  $Q = P$  would imply the contradiction  $\mathcal{F} = N_{\mathcal{F}}(P) = N_{\mathcal{F}}(Z(J(P))) = \mathcal{F}_P(P)$ , where the second equality uses the fact that  $Z(J(P))$  is characteristic in  $P$ . In particular,  $\text{Aut}_{\mathcal{F}}(P) = \text{Aut}_{N_{\mathcal{F}}(Z(J(P)))}(P) = \text{Aut}_P(P)$ .

**4.3.** *We have  $PC_{\mathcal{F}}(Q) = \mathcal{F}_P(P)$ .*

*Proof.* Assume that  $PC_{\mathcal{F}}(Q) \neq \mathcal{F}_P(P)$ . Then  $\mathcal{F} = PC_{\mathcal{F}}(Q)$  by 4.1. It follows from 3.4 that  $\mathcal{F}/Q \neq \mathcal{F}_P(P)/Q = \mathcal{F}_{P/Q}(P/Q)$ . Since  $Q \neq 1$ , the minimality assumption on  $\mathcal{F}$  implies that  $N_{\mathcal{F}/Q}(Z(J(P/Q))) \neq \mathcal{F}_{P/Q}(P/Q)$ . Let  $R$  be the inverse image of  $Z(J(P/Q))$  in  $P$ . Then  $R \trianglelefteq P$  and  $N_{\mathcal{F}}(R) \neq \mathcal{F}_P(P)$ , by 3.4. But then  $N_{\mathcal{F}}(R) = \mathcal{F}$  by 4.1, contradicting the fact that  $R$  contains  $Q = O_p(\mathcal{F})$  properly.  $\square$

**4.4.** *The subgroup  $Q$  of  $P$  is  $\mathcal{F}$ -centric.*

*Proof.* Set  $R = QC_P(Q)$ . In order to show that  $Q$  is  $\mathcal{F}$ -centric it suffices to show that  $R = Q$ , since  $Q = O_p(\mathcal{F})$ . Assume that  $R \neq Q$ . Then  $R \trianglelefteq P$  but  $N_{\mathcal{F}}(R) \neq \mathcal{F}$ . Thus  $N_{\mathcal{F}}(R) = \mathcal{F}_P(P)$  by 4.1. Since  $Q$  is normal in  $P$  it is in particular fully  $\mathcal{F}$ -normalised, and hence the restriction map  $\text{Aut}_{\mathcal{F}}(R) \rightarrow \text{Aut}_{\mathcal{F}}(Q)$  is surjective, by the extension axiom (II-S). Since  $N_{\mathcal{F}}(R) = \mathcal{F}_P(P)$  we have  $\text{Aut}_{\mathcal{F}}(R) = \text{Aut}_P(R)$ , hence  $\text{Aut}_{\mathcal{F}}(Q) = \text{Aut}_P(Q)$ . Let  $S$  be a fully  $\mathcal{F}$ -normalised centric radical subgroup of  $P$ . Then  $Q \subseteq S$  by 2.5. Let  $\sigma : \text{Aut}_{\mathcal{F}}(S) \rightarrow \text{Aut}_{\mathcal{F}}(Q) = \text{Aut}_P(Q)$  be the restriction homomorphism. Then  $\ker(\sigma)$  is a subgroup of  $\text{Aut}_{PC_{\mathcal{F}}(Q)}(S) = \text{Aut}_P(S)$ . Thus  $\text{Aut}_{\mathcal{F}}(S) = \text{Aut}_P(S)$  as claimed. Alperin's fusion theorem yields the contradiction  $\mathcal{F} = \mathcal{F}_P(P)$ . Thus  $R = Q$ , or equivalently,  $Q$  is  $\mathcal{F}$ -centric.  $\square$

We conclude the proof of Theorem A as in [7]. Since  $\mathcal{F} = N_{\mathcal{F}}(Q)$  for some  $\mathcal{F}$ -centric normal subgroup  $Q$  of  $P$  it follows from [1, 4.3] that there is a finite group  $L$  having  $P$  as Sylow- $p$ -subgroup such that  $Q \trianglelefteq L$ ,  $C_L(Q) = Z(Q)$  and  $\mathcal{F} = \mathcal{F}_P(L)$ . Then  $N_{\mathcal{F}}(Z(J(P))) = \mathcal{F}_P(N_L(Z(J(P))))$ . Since  $N_{\mathcal{F}}(Z(J(P))) = \mathcal{F}_P(P)$  it follows from Frobenius' theorem [4, 8.6] that  $N_L(Z(J(P)))$  is  $p$ -nilpotent. But the nilpotency theorem [5, Ch. 8, Theorem 3.1] of Glauberman and Thompson implies then that  $L$  itself is  $p$ -nilpotent, or equivalently,  $\mathcal{F}_P(L) = \mathcal{F}_P(P)$ . This however yields the contradiction  $\mathcal{F} = \mathcal{F}_P(P)$ , and the proof of Theorem A is complete.

5 LOCAL CONTROL OF CHARACTERISTIC  $p$ -FUNCTORS

**Definition 5.1.** Let  $\mathcal{F}$  be a fusion system on a finite  $p$ -group  $P$  and let  $W$  be a positive characteristic  $p$ -functor. Let  $Q$  be a subgroup of  $P$ . Set  $W_1(Q) = Q$  and  $P_1(Q) = N_P(Q)$ . For any positive integer  $i$  define inductively  $W_{i+1}(Q) = W(P_i(Q))$  and  $P_{i+1}(Q) = N_P(W_{i+1}(Q))$ . We will say that  $Q$  is  $(\mathcal{F}, W)$ -well-placed if  $W_i(Q)$  is fully  $\mathcal{F}$ -normalised for all positive integers  $i$ .

Note that for all positive integers  $i$  we have  $W_i(Q) \subseteq P_i(Q)$ , and if  $P_i(Q)$  is a proper subgroup of  $P$  then in fact  $P_i(Q)$  is a proper subgroup of  $P_{i+1}(Q)$ . In particular,  $P_i(Q) = P$  for  $i$  large enough. The following result generalises [4, 5.2], [8, 3.1].

**Proposition 5.2.** *Let  $\mathcal{F}$  be a fusion system on a finite  $p$ -group  $P$ , let  $W$  be a positive characteristic  $p$ -functor and let  $Q$  be a subgroup of  $P$ . There is a morphism  $\varphi : N_P(Q) \rightarrow P$  such that  $\varphi(Q)$  is  $(\mathcal{F}, W)$ -well-placed.*

*Proof.* Set  $W_i = W_i(Q)$  and  $P_i = P_i(Q)$  for any positive integer  $i$ . Note that  $P_i = N_P(W_i)$  for any positive integer  $i$ . Let  $\varphi_1 : P_1 = N_P(Q) \rightarrow P$  be a morphism in  $\mathcal{F}$  such that  $W_1 = Q$  is fully  $\mathcal{F}$ -normalised. Thus, after replacing  $Q$  by  $\varphi_1(Q)$  we may assume that  $W_1 = Q$  is fully normalised. Assume now that for some positive integer  $n$  the subgroups  $W_i$  are fully  $\mathcal{F}$ -normalised for  $1 \leq i \leq n$ . Let  $\varphi_{n+1} : P_{n+1} = N_P(W_{n+1}) \rightarrow P$  be a morphism in  $\mathcal{F}$  such that  $W_{i+1}$  is fully  $\mathcal{F}$ -normalised. Since  $P_i = N_P(W_i) \subseteq P_{n+1}$  the subgroups  $\varphi_{n+1}(W_i)$  are still all fully  $\mathcal{F}$ -normalised. Note that in particular  $P_1 = N_P(Q) \subseteq P_{n+1}$ . Thus we may in fact assume that  $W_i$  is fully  $\mathcal{F}$ -normalised for  $1 \leq i \leq n+1$ . The result follows by induction.  $\square$

The next result generalises [4, 5.5], [8, 3.2] to arbitrary fusion systems, saying that if a positive characteristic  $p$ -functor controls fusion locally, it does so globally.

**Proposition 5.3.** *Let  $\mathcal{F}$  be a fusion system on a finite  $p$ -group  $P$  and let  $W$  be a positive characteristic  $p$ -functor. Assume that for any non-trivial fully  $\mathcal{F}$ -normalised subgroup  $Q$  of  $P$  we have  $N_{\mathcal{F}}(Q) = N_{N_{\mathcal{F}}(Q)}(W(N_P(Q)))$ . Then  $\mathcal{F} = N_{\mathcal{F}}(W(P))$ .*

*Proof.* Suppose the conclusion does not hold. Then there is a fully  $\mathcal{F}$ -normalised non-trivial subgroup  $Q$  of  $P$  such that  $\text{Aut}_{N_{\mathcal{F}}(W(P))}(Q)$  is a proper subgroup of  $\text{Aut}_{\mathcal{F}}(Q)$ . By the previous Proposition we may assume that  $Q$  is  $(\mathcal{F}, W)$ -well-placed. For any positive integer set  $W_i = W_i(Q)$  and  $P_i = P_i(Q)$ . Set  $\mathcal{F}_i = N_{\mathcal{F}}(W_i)$  and  $\mathcal{G}_i = N_{\mathcal{F}_i}(W_{i+1})$ . Since  $W_i$  is fully  $\mathcal{F}$ -normalised, the category  $\mathcal{F}_i$  is a fusion system on  $P_i = N_P(W_i)$ . Clearly  $W_{i+1} \subseteq P_i$ , and since  $W_{i+1}$  is fully  $\mathcal{F}$ -normalised,  $W_{i+1}$  is also fully  $\mathcal{F}_i$ -normalised, and so  $\mathcal{G}_i$  is a fusion system as well. Note that  $\mathcal{G}_i \subseteq \mathcal{F}_{i+1}$ . Clearly  $\mathcal{F}_1 \subseteq \mathcal{F}_i$ . In fact, by the assumptions, we have  $\mathcal{G}_i = \mathcal{F}_i$ . Thus  $\mathcal{F}_1 \subseteq \mathcal{F}_i \subseteq \mathcal{F}_{i+1}$ . If  $i$  is large enough then  $P_i = P$  and hence  $\mathcal{F}_i = N_{\mathcal{F}}(W(P))$ . But then also  $\mathcal{F}_1 \subseteq N_{\mathcal{F}}(W(P))$ . In particular,  $\text{Aut}_{\mathcal{F}_1}(Q) = \text{Aut}_{\mathcal{F}}(Q)$  is contained in  $\text{Aut}_{N_{\mathcal{F}}(W(P))}(Q)$ , contradicting our choice of  $Q$ .  $\square$

6 ON  $Qd(p)$ -FREE FUSION SYSTEMS

In this Section we prove that if a fusion system  $\mathcal{F}$  on a finite  $p$ -group  $P$  is  $Qd(p)$ -free, then so are  $N_{\mathcal{F}}(Q)$ ,  $N_P(Q)C_{\mathcal{F}}(Q)$  and  $N_{\mathcal{F}}(Q)/Q$  for any fully  $\mathcal{F}$ -normalised subgroup  $Q$  of  $P$ . In fact, the statements in this section remain true with  $Qd(p)$  replaced by any finite group, but we state them as needed in the proof of Theorem B. Given a fusion system  $\mathcal{F}$  on a finite  $p$ -group  $P$  and a fully  $\mathcal{F}$ -normalised centric subgroup  $Q$  of  $P$  we denote as before by  $L_Q^{\mathcal{F}}$  the  $p'$ -reduced  $p$ -constrained group from [1, 4.3] for which there is a short exact sequence

$$1 \longrightarrow Z(Q) \longrightarrow L_Q^{\mathcal{F}} \longrightarrow \text{Aut}_{\mathcal{F}}(Q) \longrightarrow 1$$

such that  $N_P(Q)$  is a Sylow- $p$ -subgroup of  $L_Q^{\mathcal{F}}$  and such that  $N_{\mathcal{F}}(Q) = \mathcal{F}_{N_P(Q)}(L_Q^{\mathcal{F}})$ . This short exact sequence is represented by an element in  $H^2(\text{Aut}_{\mathcal{F}}(Q); Z(Q))$ . Since  $\text{Aut}_P(Q)$  is a Sylow- $p$ -subgroup of  $\text{Aut}_{\mathcal{F}}(Q)$  and since  $Z(Q)$  is an abelian  $p$ -group, the restriction map  $H^2(\text{Aut}_{\mathcal{F}}(Q); Z(Q)) \rightarrow H^2(\text{Aut}_P(Q); Z(Q))$  is injective. In other words, the group  $L_Q^{\mathcal{F}}$  is, up to isomorphism determined by the group  $\text{Aut}_{\mathcal{F}}(Q)$  and the  $p$ -group extension

$$1 \longrightarrow Z(Q) \longrightarrow N_P(Q) \longrightarrow \text{Aut}_P(Q) \longrightarrow 1$$

**Proposition 6.1.** *Let  $\mathcal{F}$  be a fusion system on a finite  $p$ -group  $P$ . If  $Qd(p)$  is involved in the group  $L_Q^{\mathcal{F}}$  for some fully  $\mathcal{F}$ -normalised centric subgroup  $Q$  of  $P$  then  $Qd(p)$  is involved in the group  $L_R^{\mathcal{F}}$  for some fully  $\mathcal{F}$ -normalised centric radical subgroup  $R$  of  $P$ .*

*Proof.* Let  $Q$  be a fully  $\mathcal{F}$ -normalised centric subgroup of  $P$ . We proceed by induction over the order of  $Q$ . If  $Q$  is  $\mathcal{F}$ -radical there is nothing to prove. Otherwise, let  $R$  be the unique subgroup of  $N_P(Q)$  containing  $Q$  such that  $\text{Aut}_R(Q) = O_p(\text{Aut}_{\mathcal{F}}(Q))$ . Then  $R$  is an  $\mathcal{F}$ -centric subgroup of  $P$  which properly contains  $Q$ . In particular,  $\text{Aut}_R(Q)$  is normal in  $\text{Aut}_{\mathcal{F}}(Q)$ , hence  $N_P(Q) \subseteq N_P(R)$ . Let  $\psi : N_P(R) \rightarrow P$  be a morphism in  $\mathcal{F}$  such that  $\psi(R)$  is fully  $\mathcal{F}$ -normalised. Then  $\psi(Q)$  is still fully  $\mathcal{F}$ -normalised. Thus we may assume that both  $Q$  and  $R$  are fully  $\mathcal{F}$ -normalised. Let  $L_{Q \leq R}$  be the inverse image of  $\text{Aut}_{\mathcal{F}}(Q \leq R)$  in  $L_R^{\mathcal{F}}$ . That is, we have a short exact sequence of groups

$$1 \longrightarrow Z(R) \longrightarrow L_{Q \leq R} \longrightarrow \text{Aut}_{\mathcal{F}}(Q \leq R) \longrightarrow 1$$

By 2.4 restriction from  $R$  to  $Q$  induces a surjective group homomorphism  $\text{Aut}_{\mathcal{F}}(Q \leq R) \rightarrow \text{Aut}_{\mathcal{F}}(Q)$ . Since  $Q$  is  $\mathcal{F}$ -centric, the kernel of this group homomorphism is  $\text{Aut}_{Z(Q)}(R) \cong Z(Q)/Z(R)$  (cf. [10, 1.12]). Thus we have an exact commutative di-



**Proposition 6.3.** *Let  $\mathcal{F}$  be a fusion system on a finite  $p$ -group  $P$ , let  $Q$  be a fully  $\mathcal{F}$ -normalised subgroup of  $P$  and let  $\mathcal{G}$  be a fusion system on  $N_P(Q)$  such that  $N_P(Q)C_{\mathcal{F}}(Q) \subseteq \mathcal{G} \subseteq N_{\mathcal{F}}(Q)$ . If  $\mathcal{F}$  is  $Qd(p)$ -free, so is  $\mathcal{G}$ . In particular, if  $\mathcal{F}$  is  $Qd(p)$ -free, so are  $N_{\mathcal{F}}(Q)$  and  $N_P(Q)C_{\mathcal{F}}(Q)$ .*

*Proof.* Suppose that  $\mathcal{F}$  is  $Qd(p)$ -free. Let  $R$  be a fully  $\mathcal{G}$ -normalised radical centric subgroup of  $N_P(Q)$ . By 2.5 we have  $Q \subseteq R$ . By 6.1 it suffices to show that  $L_R^{\mathcal{G}}$  is  $Qd(p)$ -free. By 6.2,  $R$  is  $\mathcal{F}$ -centric. By 2.3 there is a morphism  $\varphi : N_P(R) \rightarrow P$  such that  $\varphi(R)$  is fully  $\mathcal{F}$ -normalised. The plan is to show that  $L_R^{\mathcal{G}}$  is isomorphic to a subgroup of  $L_{\varphi(R)}^{\mathcal{F}}$ . Conjugation by  $\varphi$  induces a group isomorphism  $\text{Aut}_{\mathcal{F}}(R) \cong \text{Aut}_{\mathcal{F}}(\varphi(R))$  sending  $\rho \in \text{Aut}_{\mathcal{F}}(R)$  to  $\varphi \circ \rho \circ \varphi^{-1}|_{\varphi(R)}$ . Restricting this to the subgroup  $\text{Aut}_{\mathcal{G}}(R)$  of  $\text{Aut}_{\mathcal{F}}(R)$  yields an injective group homomorphism

$$\Phi : \text{Aut}_{\mathcal{G}}(R) \rightarrow \text{Aut}_{\mathcal{F}}(\varphi(R)) .$$

Consider the canonical group extension

$$1 \longrightarrow Z(\varphi(R)) \longrightarrow L_{\varphi(R)}^{\mathcal{F}} \xrightarrow{\lambda} \text{Aut}_{\mathcal{F}}(\varphi(R)) \longrightarrow 1$$

Let  $L$  be the pullback of  $\lambda$  and  $\Phi$ ; that is,

$$L = \{(y, \alpha) \in L_{\varphi(R)}^{\mathcal{F}} \times \text{Aut}_{\mathcal{G}}(R) \mid \lambda(y) = \Phi(\alpha)\}$$

The canonical projections yield a commutative diagram

$$\begin{array}{ccccccc} 1 & \longrightarrow & Z(\varphi(R)) & \longrightarrow & L & \xrightarrow{\mu} & \text{Aut}_{\mathcal{G}}(R) & \longrightarrow & 1 \\ & & \parallel & & \downarrow & & \downarrow \Phi & & \\ 1 & \longrightarrow & Z(\varphi(R)) & \longrightarrow & L_{\varphi(R)}^{\mathcal{F}} & \xrightarrow[\lambda]{} & \text{Aut}_{\mathcal{F}}(\varphi(R)) & \longrightarrow & 1 \end{array}$$

Since  $L_{\varphi(R)}^{\mathcal{F}}$  is  $Qd(p)$ -free, so is  $L$ . It suffices thus to show that  $L \cong L_R^{\mathcal{G}}$ . Note that  $\text{Aut}_P(\varphi(R))$  is a Sylow- $p$ -subgroup of  $\text{Aut}_{\mathcal{F}}(\varphi(R))$  because  $\varphi(R)$  is fully  $\mathcal{F}$ -normalised, and similarly,  $\text{Aut}_P(Q \leq R) = \text{Aut}_{N_P(Q)}(R)$  is a Sylow- $p$ -subgroup of  $\text{Aut}_{\mathcal{G}}(R)$  because  $R$  is fully  $\mathcal{G}$ -normalised. Let  $S$  be the inverse image of  $\text{Aut}_P(Q \leq R)$  in  $L$ . Then  $S$  is a Sylow- $p$ -subgroup of  $L$ . The group  $L$  is determined, up to isomorphism, by  $\text{Aut}_{\mathcal{G}}(R)$  and the short exact sequence of  $p$ -groups

$$1 \longrightarrow Z(\varphi(R)) \longrightarrow S \longrightarrow \text{Aut}_P(Q \leq R) \longrightarrow 1$$

obtained from restricting the first row in the above diagram to Sylow- $p$ -subgroups. In order to show that  $L \cong L_R^{\mathcal{G}}$  it suffices to show that this short exact sequence is equivalent to

$$1 \longrightarrow Z(R) \longrightarrow N_P(Q \leq R) \longrightarrow \text{Aut}_P(Q \leq R) \longrightarrow 1$$

As  $S$  is the inverse image in  $L$  of  $\text{Aut}_P(Q \leq R)$ , we have

$$S = \{(u, c_v) \in N_P(\varphi(R)) \times \text{Aut}_P(Q \leq R) \mid c_u = \Phi(c_v)\},$$

where  $c_u, c_v$  are the automorphisms of  $\varphi(R)$ ,  $R$ , induced by conjugation with  $u, v$ , respectively. The equality  $c_u = \Phi(c_v)$  is equivalent to  $c_u = \varphi \circ c_v \circ \varphi^{-1}|_{\varphi(R)} = c_{\varphi(v)}$ . Thus  $S = \{(\varphi(v), c_v) \mid v \in N_P(Q \leq R)\}$ . Hence the map  $\psi : N_P(Q \leq R) \rightarrow S$  sending  $v \in N_P(Q \leq R)$  to  $(\varphi(v), c_v)$  is an isomorphism making the diagram

$$\begin{array}{ccccccc} 1 & \longrightarrow & Z(R) & \longrightarrow & N_P(Q \leq R) & \longrightarrow & \text{Aut}_P(Q \leq R) \longrightarrow 1 \\ & & \varphi|_{Z(R)} \downarrow & & \downarrow \psi & & \parallel \\ 1 & \longrightarrow & Z(\varphi(R)) & \longrightarrow & S & \longrightarrow & \text{Aut}_P(Q \leq R) \longrightarrow 1 \end{array}$$

is commutative. The isomorphism  $L \cong L_R^{\mathcal{G}}$  follows.  $\square$

**Proposition 6.4.** *Let  $\mathcal{F}$  be a fusion system on a finite  $p$ -group  $P$  and let  $Q$  be a normal subgroup of  $P$  such that  $\mathcal{F} = N_{\mathcal{F}}(Q)$ . If  $\mathcal{F}$  is  $Qd(p)$ -free then  $\mathcal{F}/Q$  is  $Qd(p)$ -free.*

*Proof.* Let  $R$  be a subgroup of  $P$  containing  $Q$  such that  $R/Q$  is  $\mathcal{F}/Q$ -centric fully normalised. We show first that then  $R$  is  $\mathcal{F}$ -centric fully normalised. Let  $\varphi : R \rightarrow P$  be a morphism in  $\mathcal{F}$  and let  $y \in C_P(\varphi(R))$ . Then  $yQ \in C_{P/Q}(\varphi(R)/Q) = Z(\varphi(R)/Q)$ , hence  $y \in \varphi(R)$ , which implies that  $R$  is  $\mathcal{F}$ -centric. Since  $N_{P/Q}(R/Q) = N_P(R)/Q$  it follows also that  $R$  is fully  $\mathcal{F}$ -normalised. Consider the group extension

$$1 \longrightarrow Z(R) \longrightarrow L_R^{\mathcal{F}} \longrightarrow \text{Aut}_{\mathcal{F}}(R) \longrightarrow 1$$

The fusion system of  $L = L_R^{\mathcal{F}}$  on  $N_P(R)$  is equal to  $N_{\mathcal{F}}(R)$ , and hence the fusion system of  $\bar{L} = L_R^{\mathcal{F}}/Q$  is equal to  $N_{\mathcal{F}}(R)/Q = N_{\mathcal{F}/Q}(R/Q)$ . Now  $R/Q$  is centric in  $\mathcal{F}/Q$ , hence  $R/Q$  is a  $p$ -centric subgroup of the finite group  $\bar{L}$ . Thus

$$C_{\bar{L}}(R/Q) = Z(R/Q) \times C$$

where  $C = O_{p'}(C_{\bar{L}}(R/Q))$ . Let  $S = C_R(R/Q)$  be the inverse image of  $Z(R/Q)$  in  $R$ , let  $K$  be the inverse image of  $C$  in  $L$  and let  $C_L(R/Q)$  be the inverse image of  $C_{\bar{L}}(R/Q)$  in  $L$ . Then  $C_{\bar{L}}(R/Q)$  is the kernel of the composition of canonical maps

$$L \longrightarrow \text{Aut}_{\mathcal{F}}(R) \longrightarrow \text{Aut}_{\mathcal{F}/Q}(R/Q)$$

and we have  $C_{\bar{L}}(R/Q) = SK$  and  $S \cap K = Q$ . It follows that the canonical maps induce an exact commutative diagram of the form

$$\begin{array}{ccccccc} 1 & \longrightarrow & Z(R) & \longrightarrow & L & \longrightarrow & \text{Aut}_{\mathcal{F}}(R) \longrightarrow 1 \\ & & \downarrow & & \downarrow \pi & & \downarrow \\ 1 & \longrightarrow & Z(R/Q) & \longrightarrow & L/K & \longrightarrow & \text{Aut}_{\mathcal{F}/Q}(R/Q) \longrightarrow 1 \end{array}$$

The second row, when restricted to Sylow- $p$ -subgroups, yields the exact sequence

$$1 \longrightarrow Z(R/Q) \longrightarrow N_{P/Q}(R/Q) \longrightarrow \text{Aut}_{P/Q}(R/Q) \longrightarrow 1$$

and hence  $L/K \cong L_{R/Q}^{\mathcal{F}/Q}$ . This group is obviously  $Qd(p)$ -free as it is a quotient of the  $Qd(p)$ -free group  $L = L_R^{\mathcal{F}}$ . Thus the fusion system  $\mathcal{F}/Q$  on  $P/Q$  is  $Qd(p)$ -free.  $\square$

**Remark 6.5.** For future reference we point out that the proofs of 6.1, 6.3 and 6.4 yield the following slightly more precise statements about the groups  $L_Q^{\mathcal{F}}$ , where  $\mathcal{F}$  is a fusion system on a finite  $p$ -group  $P$ .

(1) If  $Q$  is a fully  $\mathcal{F}$ -normalised centric subgroup of  $P$  and  $Q \subseteq R \subseteq N_P(R)$  such that  $\text{Aut}_R(Q) = O_p(\text{Aut}_{\mathcal{F}}(Q))$ , then there is a morphism  $\varphi : R \rightarrow P$  such that both  $\varphi(Q)$ ,  $\varphi(R)$  are fully  $\mathcal{F}$ -normalised, and for any such  $\varphi$  the group  $L_{\varphi(Q)}^{\mathcal{F}}$  is isomorphic to a subgroup of  $L_{\varphi(R)}^{\mathcal{F}}$ .

(2) If  $Q$  is a fully  $\mathcal{F}$ -normalised subgroup of  $P$  and  $\mathcal{G}$  a fusion system on  $N_P(Q)$  such that  $N_P(Q)C_{\mathcal{F}}(Q) \subseteq \mathcal{G} \subseteq N_{\mathcal{F}}(Q)$ , then for any fully  $\mathcal{G}$ -normalised centric radical subgroup  $R$  of  $N_P(Q)$  there is a morphism  $\varphi : R \rightarrow P$  in  $\mathcal{F}$  such that  $\varphi(R)$  is fully  $\mathcal{F}$ -normalised centric, and for any such  $\varphi$  the group  $L_R^{\mathcal{G}}$  is isomorphic to a subgroup of  $L_{\varphi(R)}^{\mathcal{F}}$ .

(3) If  $Q$  is a normal subgroup of  $P$  such that  $\mathcal{F} = N_{\mathcal{F}}(Q)$ , then for any fully  $\mathcal{F}/Q$ -normalised centric subgroup  $R/Q$  of  $P/Q$ , its inverse image  $R$  in  $P$  is fully  $\mathcal{F}$ -normalised centric, and the group  $L_R^{\mathcal{F}}$  has a quotient isomorphic to  $L_{R/Q}^{\mathcal{F}/Q}$ .

## 7 PROOF OF THEOREM B

Given a fusion system  $\mathcal{F}$  on a finite  $p$ -group  $P$  we denote as before by  $|\mathcal{F}|$  the number of morphisms in  $\mathcal{F}$ . We argue by induction over  $|\mathcal{F}|$ . Let  $\mathcal{F}$  be a counterexample to Theorem A with  $|\mathcal{F}|$  minimal. That is,  $\mathcal{F}$  is  $Qd(p)$ -free,  $N_{\mathcal{F}}(W(P)) \neq \mathcal{F}$ , where  $W$  is a Glauberman functor, but  $N_{\mathcal{F}}(W(P')) = \mathcal{F}'$  for any  $Qd(p)$ -free fusion system  $\mathcal{F}'$  on some finite  $p$ -group  $P'$  such that  $|\mathcal{F}'| < |\mathcal{F}|$ . We show first that

**7.1.**  $O_p(\mathcal{F}) \neq 1$ .

*Proof.* If  $O_p(\mathcal{F}) = 1$  then for any non trivial fully  $\mathcal{F}$ -normalised subgroup  $Q$  of  $P$  we have  $N_{\mathcal{F}}(Q) < \mathcal{F}$ . Using 6.3 we get that  $N_{\mathcal{F}}(Q)$  is  $Qd(p)$ -free. Hence  $N_{\mathcal{F}}(Q) = N_{N_{\mathcal{F}}(Q)}(W(N_P(Q)))$  by the induction hypothesis. Then 5.3 implies the contradiction  $\mathcal{F} = N_{\mathcal{F}}(W(P))$ . This proves 7.1.  $\square$

We set now  $Q = O_p(\mathcal{F})$  and  $R = QC_P(Q)$ . We observe next that

### 7.2. $Q < R$ .

*Proof.* If  $Q = R = QC_P(Q)$  then  $Q$  is  $\mathcal{F}$ -centric, and we have a short exact sequence of finite groups

$$1 \longrightarrow Z(Q) \longrightarrow L_Q \longrightarrow \text{Aut}_{\mathcal{F}}(Q) \longrightarrow 1$$

where  $L_Q$  is a finite group having  $P$  as Sylow- $p$ -subgroup such that  $C_{L_Q}(O_p(L_Q)) \subseteq O_p(L_Q)$  and such that  $\mathcal{F}_P(L_Q) = \mathcal{F}$ . But  $L_Q$  is  $Qd(p)$ -free by the assumptions. Glauberman's Theorem implies that  $\mathcal{F}_P(L_Q) = \mathcal{F}_P(N_{L_Q}(W(P)))$  and hence  $\mathcal{F} = N_{\mathcal{F}}(W(P))$ , contradicting our choice of  $\mathcal{F}$ .  $\square$

The next step is to prove that

### 7.3. $\mathcal{F} = PC_{\mathcal{F}}(Q)$ .

*Proof.* Assume that  $PC_{\mathcal{F}}(Q) < \mathcal{F}$ . Note that then  $PC_{\mathcal{F}}(Q) = N_{PC_{\mathcal{F}}(Q)}(W(P))$  by induction. We will show that this implies that  $\mathcal{F} = N_{\mathcal{F}}(W(P))$ , contradicting our choice of  $\mathcal{F}$ . To show this, we will prove by induction over  $[P : S]$  that for any  $\mathcal{F}$ -centric radical subgroup  $S$  of  $P$  we have  $\text{Aut}_{\mathcal{F}}(S) = \text{Aut}_{N_{\mathcal{F}}(W(P))}(S)$ . The equality  $\text{Aut}_{\mathcal{F}}(P) = \text{Aut}_{N_{\mathcal{F}}(W(P))}(P)$  is clear because  $W(P)$  is a characteristic subgroup of  $P$ . Let  $S$  be an  $\mathcal{F}$ -centric radical subgroup of  $P$ . Note that then  $S$  contains  $Q$ . By Alperin's fusion theorem, any automorphism of  $S$  can be written as product of automorphisms of fully  $\mathcal{F}$ -normalised centric radical subgroups of  $P$  of order at least  $|S|$ , and hence we may assume that  $S$  is fully  $\mathcal{F}$ -normalised. Restriction from  $S$  to  $Q$  induces a group homomorphism

$$\rho : \text{Aut}_{\mathcal{F}}(S) \longrightarrow \text{Aut}_{\mathcal{F}}(Q) .$$

Set  $A = \rho^{-1}(\text{Aut}_S(Q))$ . Since  $\text{Aut}_S(Q)$  is normal in  $\text{Im}(\rho)$  it follows that  $A$  is a normal subgroup of  $\text{Aut}_{\mathcal{F}}(S)$ , and clearly  $A \subseteq \text{Aut}_{PC_{\mathcal{F}}(Q)}(S)$ . Also, since  $\text{Aut}_P(S)$  is a Sylow- $p$ -subgroup of  $\text{Aut}_{\mathcal{F}}(S)$ , the intersection  $\text{Aut}_P(S) \cap A$  is a Sylow- $p$ -subgroup of  $A$ . Setting  $T = N_P(S) \cap SC_P(Q) = N_{SR}(S)$  yields

$$\text{Aut}_P(S) \cap A = \text{Aut}_T(S) .$$

The Frattini argument implies that

$$\text{Aut}_{\mathcal{F}}(S) = A \cdot N_{\text{Aut}_{\mathcal{F}}(S)}(\text{Aut}_T(S)) .$$

By our initial induction on fusion systems, we get  $A \subseteq \text{Aut}_{PC_{\mathcal{F}}(Q)}(S) \subseteq \text{Aut}_{N_{\mathcal{F}}(W(P))}(S)$ . Thus, in order to prove 7.3 we have to prove that  $N_{\text{Aut}_{\mathcal{F}}(S)}(\text{Aut}_T(S)) \subseteq \text{Aut}_{N_{\mathcal{F}}(W(P))}(S)$ . We have to consider two cases, depending on whether  $S$  contains  $R$  or not.

Suppose first that  $S$  does not contain  $R$ . Then  $T$  has bigger order than  $S$ . By induction, we get  $\text{Aut}_{\mathcal{F}}(T) = \text{Aut}_{N_{\mathcal{F}}(W(P))}(T)$ . Now every automorphism of  $S$  which normalises  $\text{Aut}_T(S)$  extends to  $T$ , by the extension axiom, and so  $N_{\text{Aut}_{\mathcal{F}}(S)}(\text{Aut}_T(S))$



is contained in the image of the restriction map  $\text{Aut}_{\mathcal{F}}(S < T) \rightarrow \text{Aut}_{\mathcal{F}}(S)$ . Thus indeed  $\text{Aut}_{\mathcal{F}}(S) = \text{Aut}_{N_{\mathcal{F}}(W(P))}(S)$  in this case.

Consider second the case where  $R \subseteq S$ . Set  $B = \rho^{-1}(\text{Aut}_Q(Q))$ . Then  $B$  is normal in  $\text{Aut}_{\mathcal{F}}(S)$  and  $B \subseteq \text{Aut}_{PC_{\mathcal{F}}(Q)}(S) \subseteq \text{Aut}_{N_{\mathcal{F}}(W(P))}(S)$ . As before, since  $\text{Aut}_P(S)$  is a Sylow- $p$ -subgroup of  $\text{Aut}_{\mathcal{F}}(S)$ , the intersection  $\text{Aut}_P(S) \cap B$  is a Sylow- $p$ -subgroup of  $B$ . We claim that  $\text{Aut}_P(S) \cap B = \text{Aut}_R(S)$ . Indeed, if we denote by  $c_y$  the automorphism of  $S$  given by conjugation with an element  $y \in N_P(S)$ , we have  $\text{Aut}_P(S) \cap B = \{c_y \mid y \in N_P(S), \text{ there is } x \in Q \text{ such that } c_y|_Q = c_x|_Q\} = \{c_y \mid y \in R\} = \text{Aut}_R(S)$ . The Frattini argument implies that

$$\text{Aut}_{\mathcal{F}}(S) = B \cdot N_{\text{Aut}_{\mathcal{F}}(S)}(\text{Aut}_R(S)) .$$

Since we know already that  $B$  is contained in  $\text{Aut}_{N_{\mathcal{F}}(W(P))}(S)$  we need to show that  $N_{\text{Aut}_{\mathcal{F}}(S)}(\text{Aut}_R(S)) \subseteq \text{Aut}_{N_{\mathcal{F}}(W(P))}(S)$ . To see this we prove first that  $N_{\text{Aut}_{\mathcal{F}}(S)}(\text{Aut}_R(S)) \subseteq \text{Aut}_{N_{\mathcal{F}}(R)}(S)$ . Indeed, let  $\varphi \in N_{\text{Aut}_{\mathcal{F}}(S)}(\text{Aut}_R(S))$  and let  $x \in R$ . As before, denote by  $c_x$  the automorphism of  $S$  given by conjugation with  $x$ . Since  $\varphi$  normalises  $\text{Aut}_R(S)$  we have  $\varphi \circ c_x \circ \varphi = c_z$  for some  $z \in R$ . But since  $R \subseteq S$  we also have  $\varphi \circ c_x \circ \varphi^{-1} = c_{\varphi(x)}$ . Thus  $z^{-1}\varphi(x) \in C_P(S) = Z(S) \subseteq R$ , and hence  $\varphi(R) = R$ . This shows the inclusion  $N_{\text{Aut}_{\mathcal{F}}(S)}(\text{Aut}_R(S)) \subseteq \text{Aut}_{N_{\mathcal{F}}(R)}(S)$ . From here we observe that, by induction applied to the fusion system  $N_{\mathcal{F}}(R)$ , we get  $\text{Aut}_{N_{\mathcal{F}}(R)}(S) \subseteq \text{Aut}_{N_{\mathcal{F}}(W(P))}(S)$  as required.

Thus the assumption  $PC_{\mathcal{F}}(Q) < \mathcal{F}$  yields the contradiction  $\mathcal{F} = N_{\mathcal{F}}(W(P))$ , which concludes the proof of 7.3.  $\square$

We use 3.4 to finish the proof of the Theorem. Since  $\mathcal{F} = PC_{\mathcal{F}}(Q)$  is  $Qd(p)$ -free, so is the fusion system  $\mathcal{F}/Q$  on  $P/Q$  by 6.4. Thus, by induction, we have  $\mathcal{F}/Q = N_{\mathcal{F}/Q}(W(P/Q))$ . Denoting by  $V$  the inverse image of  $W(P/Q)$  in  $P$  we get from 3.4 that  $\mathcal{F} = N_{\mathcal{F}}(V)$ . Hence  $V \subseteq Q = O_p(\mathcal{F})$ . But  $W(P/Q) \neq 1$ , hence  $V$  contains  $Q$  properly. This contradiction concludes the proof of Theorem B.  $\square$

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