BLOCKS OF MINIMAL DIMENSION

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Abstract. Any block with defect group $P$ of a finite group $G$ with Sylow-$p$-subgroup $S$ has dimension at least $|S|^2/|P|$; we show that a block which attains this bound is nilpotent, answering a question of G. R. Robinson.

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Theorem. Let $p$ be a prime and let $\mathcal{O}$ be a complete local Noetherian commutative ring with algebraically closed residue field $k$ of characteristic $p$. Let $G$ be a finite group, let $b$ be a block of $\mathcal{O}G$ with a defect group $P$ and let $S$ be a Sylow-$p$-subgroup of $G$. Then $\mathrm{rk}_\mathcal{O}(\mathcal{O}Gb) \geq |S|^2/|P|$, and if $\mathrm{rk}_\mathcal{O}(\mathcal{O}Gb) = |S|^2/|P|$ then $b$ is a nilpotent block, the block algebra $\mathcal{O}Gb$ is isomorphic to the matrix algebra $M_{|S|/|P|}(\mathcal{O}P)$ and the algebra $\mathcal{O}P$ is a source algebra of $b$.

Nilpotent blocks were introduced in [2] as a block theoretic analogue of $p$-nilpotent finite groups. The proof of the Theorem is based on Puig’s results in [6] on the bimodule structure of a source algebra of $\mathcal{O}G$ as $\mathcal{O}P$-$\mathcal{O}P$-bimodule and the structure theory of nilpotent blocks in [7]. Examples of blocks of minimal $\mathcal{O}$-rank include all blocks of $p$-nilpotent finite groups $G$ with abelian $O_{p'}(G)$ and, with $P = 1$, the block of the Steinberg module of a finite group of Lie type in defining characteristic.

We refer to Thévenaz [8] for background material on $p$-blocks of finite groups. In particular, with the notation of the Theorem, by a block of $\mathcal{O}G$ we mean a primitive idempotent $b$ in $\mathbb{Z}(\mathcal{O}G)$, and a defect group of $b$ is a minimal subgroup $P$ of $G$ such that $\mathcal{O}Gb$ is isomorphic to a direct summand of $\mathcal{O}Gb \otimes_{\mathcal{O}P} \mathcal{O}Gb$ as $\mathcal{O}P$-$\mathcal{O}P$-bimodule. This is equivalent to requiring that $P$ is a maximal $p$-subgroup of $G$ such that $\mathrm{Br}_P(b) \neq 0$, where

$$\mathrm{Br}_P : (\mathcal{O}G)^P \rightarrow kC_G(P)$$

is the Brauer homomorphism sending a $P$-stable element $\sum_{x \in G} \lambda_xx$ of the group algebra $\mathcal{O}G$ to the element $\sum_{x \in C_G(P)} \tilde{\lambda}_xx$ in the group algebra $kC_G(P)$, where here $\tilde{\lambda}_x$ is the canonical image of the coefficient $\lambda_x \in \mathcal{O}$ in the residue field $k$. The map $\mathrm{Br}_P$ is well-known to be a surjective algebra homomorphism. In particular, $\mathrm{Br}_P(b)$ is an $\mathcal{O}P$-algebra.
idempotent in \( Z(kC_G(P)) \), hence a sum of blocks of \( kC_G(P) \). The blocks occurring in \( \text{Br}_P(b) \) are all conjugate under \( N_G(P) \). More generally, a \( b \)-Brauer pair is a pair \((Q,e)\) consisting of a \( p \)-subgroup \( Q \) of \( G \) and a block \( e \) of \( kC_G(Q) \) such that \( \text{Br}_Q(b)e \neq 0 \). Following [1], the set of \( b \)-Brauer pairs admits a canonical structure of partially ordered \( G \)-set with respect to the conjugation action of \( G \). This partial order has the property that for any \( b \)-Brauer pair \((Q,e)\) and any subgroup \( R \) of \( Q \) there is a unique block \( f \) of \( kC_G(R) \) such that \( (R,f) \) is a \( b \)-Brauer pair and such that \( (R,f) \subseteq (Q,e) \). The block \( b \) is called nilpotent if \( N_G(Q,e)/C_G(Q) \) is a \( p \)-group for any \( b \)-Brauer pair \((Q,e)\). As a consequence of a theorem of Frobenius, the group \( G \) is \( p \)-nilpotent if and only if the principal block of \( OG \) is nilpotent, which explains the terminology.

**Proof of the Theorem.** The statement on the minimal possible rank of \( OGb \) is well-known, but we include a proof for the convenience of the reader. Choose a Sylow-\( p \)-subgroup \( S \) of \( G \) such that \( P \subseteq S \). Since \( OGb \) is a direct summand of \( OG \) as \( O S \times O S \)-bimodule, there is an \( O \)-basis \( X \) of \( OGb \) which is stable under left and right multiplication with elements in \( S \). For any subgroup \( R \) of \( S \), the set of “diagonal” fixpoints

\[
X^R = \{ x \in X \mid uxu^{-1} = x \text{ for all } u \in R \}
\]

is mapped by \( \text{Br}_R \) to a \( k \)-basis in \( \text{Br}_R((OGb)^R) = kC_G(R)\text{Br}_R(b) \). Since \( P \) is maximal such that \( \text{Br}_P(b) \neq 0 \), the set \( X^P \) is in particular non empty. Also, \( OGb \) has vertex \( \Delta P \) and trivial source as \( O(G \times G) \)-module, hence is a direct summand of \( \text{Ind}_{G^P}^{G \times G}(O) \), where \( \Delta P = \{(u,u) \mid u \in P \} \). Mackey’s formula implies that every indecomposable direct summand of \( OGb \) as \( O S \times O S \)-bimodule is of the form \( \text{Ind}_{Q}^{S \times S}(O) \) for some subgroup \( Q \) of \( S \times S \) of the form \( S \times S \cap (x,y)\Delta P \) with \( x,y \in G \); in particular, \( Q \) has order at most \( |P| \). In other words, the stabiliser of any element \( x \in X \) in \( S \times S \) has at most order \( |P| \).

Let \( x \in X^P \). The stabiliser of \( x \) in \( S \times S \) contains \( \Delta P \) but has at most order \( |P| \), hence is equal to \( \Delta P \). Thus the \( O S \times O S \)-bimodule \( O[SxS] \) generated by \( x \) is a direct summand of \( OGb \) as \( O S \times O S \)-bimodule isomorphic to \( O S \stackrel{\partial}{\otimes} O S \). In particular, \( \text{rk}_O(OGb) \geq \text{rk}_O(O S \otimes \mathcal{O}_P) = |S|^2/|P| \).

In order to show that \( b \) is nilpotent we use a result of Puig [6, 3.1] in the form as described in [4, 7.8]. Let \( i \in (OGb)^P \) be a primitive idempotent in the algebra of fixpoints in \( OGb \) with respect to the conjugation action by \( P \) on \( OGb \) such that \( \text{Br}_P(i) \neq 0 \); that is, \( i \) is a source idempotent for \( b \) and the algebra \( iOGi \) is a source algebra of \( b \). Since \( i \) commutes with the action of \( P \), the source algebra \( iOGi \) is also a direct summand of \( OGb \cong O S \otimes \mathcal{O}_P \) as \( O P \times O P \)-bimodule. As a consequence of results in [1], the choice of the source idempotent \( i \) determines a fusion system \( \mathcal{F} = \mathcal{F}_{(P,e)}(G,b) \) on \( P \), where \( e \) is the unique block of \( kC_G(P) \) such that \( \text{Br}_P(i)e = \text{Br}_P(i) \); this makes sense as \( \text{Br}_P(i) \) is a primitive idempotent in \( kC_G(P) \). More precisely, for any subgroup \( Q \) of \( P \) we have \( \text{Aut}_{\mathcal{F}}(Q) \cong N_G(Q,e_Q)/C_G(Q) \) where \( e_Q \) is the unique block of \( kC_G(Q) \) such that \( (Q,e_Q) \subseteq (P,e) \). See e.g. [3] or [5], for more details on fusion systems of blocks. Now let \( Q \) be a subgroup of \( P \) and let \( \varphi \in \text{Aut}_{\mathcal{F}}(Q) \). Denote by \( \varphi \mathcal{O}Q \) the
\(\mathcal{O}Q\)-\(\mathcal{O}Q\)-bimodule which is, as \(\mathcal{O}\)-module, equal to \(\mathcal{O}Q\) but with \(u \in Q\) acting on the left by multiplication with \(\varphi(u)\) and on the right by multiplication with \(u\). By [4, 7.8], the \(\mathcal{O}Q\)-\(\mathcal{O}Q\)-bimodule \(\varphi\mathcal{O}Q\) is isomorphic to a direct summand of \(i\mathcal{O}G_i\) as \(\mathcal{O}Q\)-\(\mathcal{O}Q\)-bimodule. Thus \(\varphi\mathcal{O}Q\) is isomorphic to a direct summand of \(\mathcal{O}S \otimes \mathcal{O}S\) as \(\mathcal{O}Q\)-\(\mathcal{O}Q\)-bimodule. This forces \(\varphi\) to be induced by conjugation with an element in \(N_S(Q)\). In particular, \(\varphi\) is a \(p\)-automorphism of \(Q\). Thus \(\text{Aut}_F(Q)\) is a \(p\)-group for all subgroups \(Q\) of \(P\), and hence \(b\) is a nilpotent block.

By the general structure theory of nilpotent blocks [7], the block algebra \(\mathcal{O}Gb\) is isomorphic to a matrix algebra \(M_n(\mathcal{O}P)\); in particular, the block \(b\) has a unique isomorphism class of simple modules. If \(V\) is a simple \(\mathcal{O}Gb\)-module then \(V\) has the defect group \(P\) as vertex and an endo-permutation \(kP\)-module \(W\) as source. This source is trivial if and only if the source algebra \(i\mathcal{O}G_i\) is isomorphic to \(\mathcal{O}P\). Dimension counting yields \(\text{rk}_{\mathcal{O}}(\mathcal{O}Gb) = n^2|P| = |S|^2/P|\), hence \(\dim_k(V) = n = [S : P]\). Now \(V\) is a direct summand of \(\text{Ind}^G_S(W)\), hence by Mackey’s formula, \(\text{Res}_S^G(V)\) is a direct sum of direct summands of \(\text{Ind}^S_{S \cap \pi} P(xW)\) with \(x \in G\). Green’s indecomposability theorem [8, (23.6)] forces \(S \cap xP = xP\) and \(\dim_k(W) = 1\), hence \(W\) is the trivial \(kP\)-module. □

**Remark.** If \(\mathcal{O}Gb\) has \(\mathcal{O}\)-rank \(|S|^2/P|\) then the first part in the proof of the Theorem says that \(\mathcal{O}Gb \cong \mathcal{O}S \otimes \mathcal{O}S\) as \(\mathcal{O}S-\mathcal{O}S\)-bimodules for any defect group \(P\) of \(b\) contained in \(S\). Thus, if \(x \in G\) such that \((xP \subset S\) then \(\mathcal{O}S \otimes \mathcal{O}S \cong \mathcal{O}S \otimes \mathcal{O}S\), which forces \((xP = uP\) for some \(u \in S\). It follows that the set \(\text{Hom}_G(P, S)\) of group homomorphisms from \(P\) to \(S\) induced by conjugation with elements in \(G\) is equal to \(\text{Hom}_S(P, S) \circ \text{Aut}_G(P)\) or equivalently, \(N_G(P, S) = SN_G(P)\), where \(N_G(P, S) = \{x \in G \mid xP \subset S\}\). In other words, the fact that \(P\) is a defect group of a block of minimal \(\mathcal{O}\)-rank has implications for the fusion system of the group itself.

**References**


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