# BLOCKS OF MINIMAL DIMENSION

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Abstract. Any block with defect group $P$ of a finite group $G$ with Sylow-$p$-subgroup $S$ has dimension at least $|S|^2/|P|$; we show that a block which attains this bound is nilpotent, answering a question of G. R. Robinson.
**Mathematics Subject Classification 2000:** 20C20

**Theorem.** *Let $p$ be a prime and let $\mathcal{O}$ be a complete local Noetherian commutative ring with algebraically closed residue field $k$ of characteristic $p$. Let $G$ be a finite group, let $b$ be a block of $\mathcal{O}G$ with a defect group $P$ and let $S$ be a Sylow-$p$-subgroup of $G$. Then $\mathrm{rk}_{\mathcal{O}}(\mathcal{O}Gb) \geq |S|^2/|P|$, and if $\mathrm{rk}_{\mathcal{O}}(\mathcal{O}Gb) = |S|^2/|P|$ then $b$ is a nilpotent block, the block algebra $\mathcal{O}Gb$ is isomorphic to the matrix algebra $M_{|S|/|P|}(\mathcal{O}P)$ and the algebra $\mathcal{O}P$ is a source algebra of $b$.*

Nilpotent blocks were introduced in [2] as a block theoretic analogue of $p$-nilpotent finite groups. The proof of the Theorem is based on Puig's results in [6] on the bimodule structure of a source algebra of $\mathcal{O}Gb$ as $\mathcal{O}P$-$\mathcal{O}P$-bimodule and the structure theory of nilpotent blocks in [7]. Examples of blocks of minimal $\mathcal{O}$-rank include all blocks of $p$-nilpotent finite groups $G$ with abelian $O_{p'}(G)$ and, with $P = 1$, the block of the Steinberg module of a finite group of Lie type in defining characteristic.

We refer to Thévenaz [8] for background material on $p$-blocks of finite groups. In particular, with the notation of the Theorem, by a block of $\mathcal{O}G$ we mean a primitive idempotent $b$ in $Z(\mathcal{O}G)$, and a defect group of $b$ is a minimal subgroup $P$ of $G$ such that $\mathcal{O}Gb$ is isomorphic to a direct summand of $\mathcal{O}Gb \underset{\mathcal{O}P}{\otimes} \mathcal{O}Gb$ as $\mathcal{O}Gb$-$\mathcal{O}Gb$-binodule. This is equivalent to requiring that $P$ is a maximal $p$-subgroup of $G$ such that $\mathrm{Br}_P(b) \neq 0$, where

$$\mathrm{Br}_P : (\mathcal{O}G)^P \longrightarrow kC_G(P)$$

is the *Brauer homomorphism* sending a $P$-stable element $\sum_{x\in G} \lambda_x x$ of the group algebra $\mathcal{O}G$ to the element $\sum_{x\in C_G(P)} \bar{\lambda}_x x$ in the group algebra $kC_G(P)$, where here $\bar{\lambda}_x$ is the canonical image of the coefficient $\lambda_x \in \mathcal{O}$ in the residue field $k$. The map $\mathrm{Br}_P$ is well-known to be a surjective algebra homomorphism. In particular, $\mathrm{Br}_P(b)$ is an

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idempotent in $Z(kC_G(P))$, hence a sum of blocks of $kC_G(P)$. The blocks occurring in $\mathrm{Br}_P(b)$ are all conjugate under $N_G(P)$. More generally, a *b-Brauer pair* is a pair $(Q,e)$ consisting of a $p$-subgroup $Q$ of $G$ and a block $e$ of $kC_G(Q)$ such that $\mathrm{Br}_Q(b)e \neq 0$. Following [1], the set of $b$-Brauer pairs admits a canonical structure of partially ordered $G$-set with respect to the conjugation action of $G$. This partial order has the property that for any $b$-Brauer pair $(Q,e)$ and any subgroup $R$ of $Q$ there is a unique block $f$ of $kC_G(R)$ such that $(R,f)$ is a $b$-Brauer pair and such that $(R,f) \subseteq (Q,e)$. The block $b$ is called *nilpotent* if $N_G(Q,e)/C_G(Q)$ is a $p$-group for any $b$-Brauer pair $(Q,e)$. As a consequence of a theorem of Frobenius, the group $G$ is $p$-nilpotent if and only if the principal block of $\mathcal{O}G$ is nilpotent, which explains the terminology.

*Proof of the Theorem.* The statement on the minimal possible rank of $\mathcal{O}Gb$ is well-known, but we include a proof for the convenience of the reader. Choose a Sylow-$p$-subgroup $S$ of $G$ such that $P \subseteq S$. Since $\mathcal{O}Gb$ is a direct summand of $\mathcal{O}G$ as $\mathcal{O}S$-$\mathcal{O}S$-bimodule, there is an $\mathcal{O}$-basis $X$ of $\mathcal{O}Gb$ which is stable under left and right multiplication with elements in $S$. For any subgroup $R$ of $S$, the set of "diagonal" fixpoints

$$X^R = \{x \in X \mid uxu^{-1} = x \text{ for all } u \in R\}$$

is mapped by $\mathrm{Br}_R$ to a $k$-basis in $\mathrm{Br}_R((\mathcal{O}Gb)^R) = kC_G(R)\mathrm{Br}_R(b)$. Since $P$ is maximal such that $\mathrm{Br}_P(b) \neq 0$, the set $X^P$ is in particular non empty. Also, $\mathcal{O}Gb$ has vertex $\Delta P$ and trivial source as $\mathcal{O}(G\times G)$-module, hence is a direct summand of $\mathrm{Ind}_{\Delta P}^{G\times G}(\mathcal{O})$, where $\Delta P = \{(u,u) \mid u \in P\}$. Mackey's formula implies that every indecomposable direct summand of $\mathcal{O}Gb$ as $\mathcal{O}S$-$\mathcal{O}S$-bimodule is of the form $\mathrm{Ind}_Q^{S\times S}(\mathcal{O})$ for some subgroup $Q$ of $S\times S$ of the form $S \times S \cap {}^{(x,y)}\Delta P$ with $x, y \in G$; in particular, $Q$ has order at most $|P|$. In other words, the stabiliser of any element $x \in X$ in $S \times S$ has at most order $|P|$.

Let $x \in X^P$. The stabiliser of $x$ in $S \times S$ contains $\Delta P$ but has at most order $|P|$, hence is equal to $\Delta P$. Thus the $\mathcal{O}S$-$\mathcal{O}S$-bimodule $\mathcal{O}[SxS]$ generated by $x$ is a direct summand of $\mathcal{O}Gb$ as $\mathcal{O}S$-$\mathcal{O}S$-bimodule isomorphic to $\mathcal{O}S \underset{\mathcal{O}P}{\otimes} \mathcal{O}S$. In particular, $\mathrm{rk}_{\mathcal{O}}(\mathcal{O}Gb) \geq \mathrm{rk}_{\mathcal{O}}(\mathcal{O}S \underset{\mathcal{O}P}{\otimes} \mathcal{O}S) = |S|^2/|P|$.

In order to show that $b$ is nilpotent we use a result of Puig [6, 3.1] in the form as described in [4, 7.8]. Let $i \in (\mathcal{O}Gb)^P$ be a primitive idempotent in the algebra of fixpoints in $\mathcal{O}Gb$ with respect to the conjugation action by $P$ on $\mathcal{O}Gb$ such that $\mathrm{Br}_P(i) \neq 0$; that is, $i$ is a source idempotent for $b$ and the algebra $i\mathcal{O}Gi$ is a source algebra of $b$. Since $i$ commutes with the action of $P$, the source algebra $i\mathcal{O}Gi$ is also a direct summand of $\mathcal{O}Gb \cong \mathcal{O}S \underset{\mathcal{O}P}{\otimes} \mathcal{O}S$ as $\mathcal{O}P$-$\mathcal{O}P$-bimodule. As a consequence of results in [1], the choice of the source idempotent $i$ determines a fusion system $\mathcal{F} = \mathcal{F}_{(P,e)}(G,b)$ on $P$, where $e$ is the unique block of $kC_G(P)$ such that $\mathrm{Br}_P(i)e = \mathrm{Br}_P(i)$; this makes sense as $\mathrm{Br}_P(i)$ is a primitive idempotent in $kC_G(P)$. More precisely, for any subgroup $Q$ of $P$ we have $\mathrm{Aut}_{\mathcal{F}}(Q) \cong N_G(Q,e_Q)/C_G(Q)$ where $e_Q$ is the unique block of $kC_G(Q)$ such that $(Q,e_Q) \subseteq (P,e)$. See e.g. [3] or [5], for more details on fusion systems of blocks. Now let $Q$ be a subgroup of $P$ and let $\varphi \in \mathrm{Aut}_{\mathcal{F}}(Q)$. Denote by ${}_\varphi\mathcal{O}Q$ the

$\mathcal{O}Q$-$\mathcal{O}Q$-bimodule which is, as $\mathcal{O}$-module, equal to $\mathcal{O}Q$ but with $u \in Q$ acting on the left by multiplication with $\varphi(u)$ and on the right by multiplication with $u$. By [4, 7.8], the $\mathcal{O}Q$-$\mathcal{O}Q$-bimodule ${}_\varphi\mathcal{O}Q$ is isomorphic to a direct summand of $i\mathcal{O}Gi$ as $\mathcal{O}Q$-$\mathcal{O}Q$-bimodule. Thus ${}_\varphi\mathcal{O}Q$ is isomorphic to a direct summand of $\mathcal{O}S \underset{\mathcal{O}P}{\otimes} \mathcal{O}S$ as $\mathcal{O}Q$-$\mathcal{O}Q$-bimodule. This forces $\varphi$ to be induced by conjugation with an element in $N_S(Q)$. In particular, $\varphi$ is a $p$-automorphism of $Q$. Thus $\mathrm{Aut}_{\mathcal{F}}(Q)$ is a $p$-group for all subgroups $Q$ of $P$, and hence $b$ is a nilpotent block.

By the general structure theory of nilpotent blocks [7], the block algebra $\mathcal{O}Gb$ is isomorphic to a matrix algebra $M_n(\mathcal{O}P)$; in particular, the block $b$ has a unique isomorphism class of simple modules. If $V$ is a simple $\mathcal{O}Gb$-module then $V$ has the defect group $P$ as vertex and an endo-permutation $kP$-module $W$ as source. This source is trivial if and only if the source algebra $i\mathcal{O}Gi$ is isomorphic to $\mathcal{O}P$. Dimension counting yields $\mathrm{rk}_{\mathcal{O}}(\mathcal{O}Gb) = n^2|P| = |S|^2/|P|$, hence $\dim_k(V) = n = [S : P]$. Now $V$ is a direct summand of $\mathrm{Ind}_P^G(W)$, hence by Mackey's formula, $\mathrm{Res}_S^G(V)$ is a direct sum of direct summands of $\mathrm{Ind}_{S\cap {}^xP}^S({}^xW)$ with $x \in G$. Green's indecomposability theorem [8, (23.6)] forces $S \cap {}^xP = {}^xP$ and $\dim_k(W) = 1$, hence $W$ is the trivial $kP$-module. $\square$

**Remark.** If $\mathcal{O}Gb$ has $\mathcal{O}$-rank $|S|^2/|P|$ then the first part in the proof of the Theorem says that $\mathcal{O}Gb \cong \mathcal{O}S \underset{\mathcal{O}P}{\otimes} \mathcal{O}S$ as $\mathcal{O}S$-$\mathcal{O}S$-bimodules for any defect group $P$ of $b$ contained in $S$. Thus, if $x \in G$ such that ${}^xP \subseteq S$ then $\mathcal{O}S \underset{\mathcal{O}P}{\otimes} \mathcal{O}S \cong \mathcal{O}S \underset{\mathcal{O}^xP}{\otimes} \mathcal{O}S$, which forces ${}^xP = {}^uP$ for some $u \in S$. It follows that the set $\mathrm{Hom}_G(P, S)$ of group homomorphisms from $P$ to $S$ induced by conjugation with elements in $G$ is equal to $\mathrm{Hom}_S(P, S) \circ \mathrm{Aut}_G(P)$ or equivalently, $N_G(P, S) = SN_G(P)$, where $N_G(P, S) = \{x \in G \mid {}^xP \subseteq S\}$. In other words, the fact that $P$ is a defect group of a block of minimal $\mathcal{O}$-rank has implications for the fusion system of the group itself.

## References

1. J. L. Alperin, M. Broué, *Local methods in block theory*, Ann. Math. **110** (1979), 143–157.
2. M. Broué, L. Puig, *A Frobenius theorem for blocks*, Invent. Math. **56** (1980), 117–128.
3. R. Kessar, M. Linckelmann, G. R. Robinson, *Local control in fusion systems of p-blocks of finite groups*, J. Algebra **257** (2002), 393–413.
4. M. Linckelmann, *On splendid derived and stable equivalences between blocks of finite groups*, J. Algebra **242** (2001), 819–843.
5. M. Linckelmann, *Simple fusion systems and the Solomon 2-local groups*, J. Algebra **296** (2006), 385–401.
6. L. Puig, *Local fusion in block source algebras*, J. Algebra 104 (1986), 358–369.
7. L. Puig, *Nilpotent blocks and their source algebras*, Invent. Math. **93** (1986), 77–116.
8. J. Thévenaz, *G−Algebras and Modular Representation Theory*, Oxford Science Publications, Clarendon Press, Oxford, 1995.

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