



## City Research Online

### City, University of London Institutional Repository

---

**Citation:** Kessar, R. (2009). On duality inducing automorphisms and sources of simple modules in classical groups. *Journal of Group Theory*, 12(3), pp. 331-349. doi: 10.1515/jgt.2008.081

This is the unspecified version of the paper.

This version of the publication may differ from the final published version.

---

**Permanent repository link:** <https://openaccess.city.ac.uk/id/eprint/1892/>

**Link to published version:** <https://doi.org/10.1515/jgt.2008.081>

**Copyright:** City Research Online aims to make research outputs of City, University of London available to a wider audience. Copyright and Moral Rights remain with the author(s) and/or copyright holders. URLs from City Research Online may be freely distributed and linked to.

**Reuse:** Copies of full items can be used for personal research or study, educational, or not-for-profit purposes without prior permission or charge. Provided that the authors, title and full bibliographic details are credited, a hyperlink and/or URL is given for the original metadata page and the content is not changed in any way.

---

---



# ON DUALITY INDUCING AUTOMORPHISMS AND SOURCES OF SIMPLE MODULES IN CLASSICAL GROUPS

RADHA KESSAR

## 1. INTRODUCTION

Let  $p$  be a prime number,  $k$  an algebraically closed field of characteristic  $p$ ,  $P$  a finite  $p$ -group and  $W$  an indecomposable  $kP$ -module. Given a finite group  $G$  and an indecomposable  $kG$ -module  $V$ , we say that  $(P, W)$  is a vertex source pair for  $(G, V)$  if there is an inclusion  $P \hookrightarrow G$  of groups, under which  $P$  is a vertex of  $V$  and  $W$  is a source of  $V$ .

Endo-permutation modules occur frequently as sources of simple modules of finite groups. For instance, every simple module of a  $p$ -soluble group has endo-permutation source; if  $V$  is a simple  $kG$ -module, where  $G$  is a finite group, lying in a nilpotent block of  $kG$ , then  $V$  has endo-permutation source, and any 2-block of a finite group whose defect groups are isomorphic to the Klein 4-group possesses a simple module with endo-permutation source.

By results of T. Berger-W. Feit and independently of L. Puig, the proof of which invokes the classification of finite simple groups, if  $W$  is an endo-permutation  $kP$  module which occurs as a source of a simple  $kG$ -module, for a  $p$ -soluble group  $G$ , then the class of  $W$  is a torsion element in the Dade group of  $P$ . Further, Mazza [8] has shown that any endo-permutation  $kP$ -module whose isomorphism class is a torsion element of the Dade group of  $P$  and which satisfies certain structural constraints identified by Puig does occur as source of some simple module of some  $p$ -nilpotent group. By contrast, the question of which endo-permutation modules occur as sources of simple modules in simple, quasi-simple or almost simple groups has not been extensively studied.

The smallest interesting case is the case where  $P$  is elementary abelian of rank 2, since if  $P$  is a finite cyclic group, there are no non-torsion endo-permutation  $kP$ -modules. In this paper, we study two situations wherein simple modules of groups related to the finite classical groups having vertex source pairs  $(P, W)$ , where  $P$  is elementary abelian of order  $p^2$  and where  $W$  is endo-permutation occur and we prove that in both cases,  $W$  must be a self-dual module and hence corresponds to an element of order at most 2 in the corresponding Dade group.

NOTATION- Let  $H$  be a finite group. For a finite dimensional  $kH$ -module  $M$ , we denote by  $M^\vee$  the contragradient dual of  $M$ , that is  $M^\vee$  is the  $kH$ -module given by  $M^\vee := \text{Hom}_k(M, k)$  as  $k$ -vector space, and  $h.\alpha(m) = \alpha(h^{-1}m)$ , for  $\alpha \in M^\vee$ ,  $h \in H$  and  $m \in M$ .

If  $\phi : G \rightarrow H$  is an isomorphism of groups, and  $M$  is a  $kG$ -module, we will denote by  ${}^\phi M$  the  $kH$ -module  $\text{Res}_{\phi^{-1}} M$ .

For a finite  $p$ -group, we denote by  $D_k(P)$  the Dade group of  $P$  over  $k$ . For the definition and basic properties of the Dade group, we refer to [12]. Here, we merely recall that if

$W$  is an indecomposable  $kP$ -module, with vertex  $P$ , then the (isomorphism class) of  $W$  defines a unique element  $[W]$  of  $D_k(P)$ , the identity of  $D_k(P)$  is  $[k]$ , and that the inverse of the element  $[W]$  in  $D_k(P)$  is the element  $[W^\vee]$ . In particular, if  $W$  is self-dual, then  $[W]$  has order at most 2 in  $D_k(P)$ .

**Theorem 1.1.** *Let  $L$  be a finite symmetric group  $S_n$  or one of the finite classical groups  $GL_n(q)$ ,  $GU_n(q)$ ,  $O_{2n+1}(q)$ ,  $O_{2n}^+(q)$ ,  $O_{2n}^-(q)$ , or  $Sp_{2n}(q)$ , where  $q$  is a prime power. Let  $G$  be a subgroup of  $L$  containing the derived subgroup of  $L$  and let  $Z$  be a subgroup of  $Z(G)$ . Let  $P$  be an elementary abelian group of order  $p^2$ , and let  $W$  be an indecomposable endo-permutation  $kP$ -module. Suppose that there exists a simple  $kG/Z$  module with  $(P, W)$  as vertex source pair. Then  $W \cong W^\vee$ , and if  $p$  is 2, then  $W \cong k$ .*

Before stating our next result, we recall the following standard facts from modular representation theory. Let  $G$  be a finite group and  $N$  a normal subgroup of  $G$  such that  $G/N$  is a  $p$ -group. Suppose that  $U$  is a simple projective  $kN$ -module whose isomorphism class is stable under the conjugation action of  $G$  on  $N$ . Then the  $kN$ -module structure on  $U$  extends uniquely to a  $kG$ -module structure. The  $kG$  module  $U$  is simple, and if  $(P, W)$  is a vertex source pair of  $(G, U)$ , then  $P$  is isomorphic to  $G/N$  and  $W$  is an endo-permutation  $kP$ -module.

The question of whether any  $W$  that appears in the above context is torsion in the Dade group of  $P$  has been investigated by Salminen in [10] and [11] for odd primes  $p$ . He has reduced the problem to the case where  $P$  is elementary abelian of order  $p^2$  and where  $N$  is a central  $p'$ -extension of a projective special linear or unitary group, or  $p = 3$  and  $N$  is a central extension of the simple group  $D_4(q)$ .

In this paper, we prove that groups of type  $A$  do not pose a problem.

**Theorem 1.2.** *Let  $p$  be an odd prime. Let  $G$  be a finite group with a normal subgroup  $N$  such that  $[G : N]$  is elementary abelian of order  $p^2$ . Suppose that  $N$  is a quasi-simple group, with  $Z(N)$  a  $p'$ -group and such that  $N/Z(N)$  is isomorphic to  $PSL_n(q)$  or to  $PSU_n(q)$  where  $q$  is a prime power not divisible by  $p$ . Suppose that  $U$  is a simple projective  $kN$ -module which is  $G$ -stable and let  $(P, W)$  be a vertex source pair for the  $kG$ -module  $U$ . Then  $W \cong W^\vee$ .*

Combining the above with Salminen's work thus proves that if  $p \geq 5$ , then for a finite group  $G$ , and a simple  $kG$ -module  $U$  with vertex-source pair  $(P, W)$  such that  $\text{Res}_N^G(U)$  is a simple projective  $kN$ -module for some normal subgroup  $N$  of  $p$ -power order,  $[W]$  is a torsion element in  $D_k(P)$ . We remark that this result is a special case of a long standing conjecture on the finiteness of the number of source algebra equivalences of nilpotent blocks of finite groups, a proof of which has been recently announced by Puig.

The proof of both Theorem 1.1 and Theorem 1.2 is based on the following elementary proposition.

**Proposition 1.3.** *Suppose that  $G$  is a finite group,  $M$  is an indecomposable  $kG$  module with vertex source pair  $(P, W)$  such that  $P$  is elementary abelian of order  $p^2$  and  $W$  is an endo-permutation  $kP$ -module. If there exists an automorphism  $\phi : G \rightarrow G$  such that  $\phi M \cong M^\vee$  as  $kG$ -modules, then  $W$  is self dual and if  $p = 2$ ,  $W = k$ .*

Theorem 1.1 is an easy consequence of the above proposition, once one observes that the symmetric and classical groups have automorphisms which invert conjugacy classes.

However, as will be explained in more detail later on in the paper, the proof of Theorem 1.2 is somewhat more subtle and relies on the following rather curious fact, which we state below.

NOTATION- For our next result,  $G$  will denote either the general linear group  $GL_n(q)$  or the general unitary group  $GU_n(q)$  for some prime power  $q$ .

If  $G = GL_n(q)$  we set  $\epsilon = 1$  and if  $G = GU_n(q)$ , we set  $\epsilon = -1$ . If  $G = GL_n(q)$ , set  $N := SL_n(q)$  and if  $G = GU_n(q)$ , set  $N = SU_n(q)$ . If  $q$  is a prime power not divisible by  $p$  such that  $q = q'^p$  for some  $q'$ , then if  $G = GL_n(q)$ , we let  $\phi : G \rightarrow G$  be the automorphism  $(a_{ij}) \rightarrow (a_{ij}^{q'})$  and if  $G = GU_n(q)$ , we let  $\phi : G \rightarrow G$  be the automorphism  $(a_{ij}) \rightarrow (a_{ij}^{q'})^{\epsilon-1}$ . Note that since  $N$  is  $\phi$ -stable and  $G/N$  is cyclic, any subgroup  $I$  of  $G$  containing  $N$  is  $\phi$ -stable, and we have a natural inclusion of groups  $I \rtimes \langle \phi \rangle \leq G \rtimes \langle \phi \rangle$ . This is not, in general, a normal inclusion.

**Theorem 1.4.** *With the notation above suppose that  $p$  is an odd prime such that  $p|q - \epsilon$  and  $p|n$ . Suppose further that  $q = q'^p$  for some prime power  $q'$ . Let  $c$  be a block of  $kN$  with central defect group and let  $I(c) = \text{Stab}_G(c)$ . If  $c$  is stable under  $\phi$ , then  $I(c) \rtimes \langle \phi \rangle$  is a normal subgroup of  $G \rtimes \langle \phi \rangle$ .*

The paper is organized into six sections. In section 2, we prove Proposition 1.3 as well as some other general results which are needed for the proofs of the main theorems. In section 3, we will prove Theorem 1.1. Section 4 contains some block theoretic results which will be needed for the proof of Theorem 1.2. In section 5, we recall some facts from the representation theory of the finite general linear and unitary groups and their commutator subgroups and prove Theorem 1.4. Theorem 1.2 is proved in the final section.

**Acknowledgments.** We thank Cedric Bonnafé and Markus Linckelmann for many helpful discussions and suggestions and the referee for a careful reading of the manuscript.

## 2. A CRITERION FOR TORSION

NOTATION- We keep the notation of the introduction. In addition, let  $(K, \mathcal{O}, k)$  be a  $p$ -modular system.

**Definition 2.1.** *Let  $H$  be a finite group and let  $V$  be a  $kH$ -module. Then  $V$  is automorphically dual if there exists an automorphism  $\phi : H \rightarrow H$  such that  ${}^\phi V \cong V^\vee := \text{Hom}_k(V, k)$ .*

Clearly, any self-dual module is automorphically dual. Our results are based on the observation that the converse holds for endo-permutation modules for elementary abelian  $p$  groups of order  $p^2$ .

**Lemma 2.2.** *Let  $P$  be an elementary abelian  $p$ -group of order  $p^2$  and let  $W$  be an indecomposable endo-permutation  $kP$ -module with vertex  $P$ . If  $W$  is automorphically dual, then  $W$  is self-dual.*

*Proof.* Let  $\phi : P \rightarrow P$  be an automorphism of  $P$  such that  ${}^\phi W \cong W^\vee$ . The result is an easy consequence of Dade's classification of endo-permutation modules for abelian  $p$ -groups [2, 10.1 and 12.5]. By this classification, if  $p = 2$ , then  $W$  is isomorphic to  $\Omega^n(P)$  for some uniquely determined  $n \in \mathbb{Z}$  whence  ${}^\phi W$  is isomorphic to  $\Omega^n(k)$  and  $W^\vee$

is isomorphic to  $\Omega^{-n}(P)$ . But we are given that  ${}^\phi W \cong W^\vee$ . Hence  $n = 0$ . Now suppose that  $p$  is odd. Then

$$[W] = [\Omega^n(k) \otimes_k V]$$

for some  $n \in \mathbb{Z}$  and some self-dual  $kP$ -module  $V$ , with  $n$  and  $[V]$  uniquely determined by  $W$  whence

$$[{}^\phi W] = [\Omega^n(k) \otimes_k {}^\phi V]$$

and

$$[W^\vee] = [\Omega^{-n}(k) \otimes_k V^\vee] = [\Omega^{-n}(k) \otimes_k V].$$

Thus,  ${}^\phi W \cong W^\vee$  implies that  $n = 0$  thanks to the uniqueness of  $n$  and  $[V]$ .  $\square$

**Lemma 2.3.** *Let  $M$  be an indecomposable  $kG$  module and let  $(S, V)$  be a vertex-source pair for  $M$ . Then  $(S, V^\vee)$  is a vertex-source pair for  $M^\vee$ .*

*Proof.* Let  $H$  be a subgroup of  $G$  and let  $U$  be a  $kH$ -module such that  $M$  is a summand of  $\text{Ind}_H^G(U)$ . Then  $M^\vee$  is a summand of  $(\text{Ind}_H^G(U))^\vee \cong \text{Ind}_H^G(U^\vee)$ . The result follows  $\square$

**Lemma 2.4.** *Suppose that  $G$  is a finite group,  $M$  is an indecomposable  $kG$  module with vertex source pair  $(P, W)$ . If  $M$  is automorphically dual, then so is  $W$ .*

*Proof.* Let  $\phi : G \rightarrow G$  be such that  ${}^\phi M \cong M^\vee$ . By Lemma 2.3,  $(P, W^\vee)$  is a vertex source pair for the  $kG$ -module  $M^\vee$ . On the other hand, by transport of structure,  $({}^\phi P, {}^\phi W)$  is a vertex source pair for the  $kG$ -module  ${}^\phi M$ . Thus, by hypothesis, both  $(P, W^\vee)$  and  $({}^\phi P, {}^\phi W)$  are vertex source pairs for  $M^\vee$ . So, there exists  $g \in G$  such that  $P = {}^g \phi P$  and  $W^\vee \cong {}^g \phi W$ . The result follows by considering the automorphism  $(y \rightarrow g\phi(y)g^{-1})$  of  $P$ .  $\square$

Now we can prove Proposition 1.3:

*Proof.* By hypothesis,  $M$  is automorphically dual. Hence by Lemma 2.4,  $W$  is automorphically dual. The result follows by Lemma 2.2  $\square$

### 3. PROOF OF THEOREM 1.1

We keep all the notation of the previous sections.

**Lemma 3.1.** *Let  $G$  be a finite group and  $\tau : G \rightarrow G$  an automorphism such that  $\tau(g)$  is conjugate to  $g^{-1}$  for all  $g \in G_{p'}$ . If  $N$  is a normal subgroup of  $G$  which is  $\tau$ -stable, then every simple  $kN$ -module is automorphically dual.*

*Proof.* First, let  $M$  be a simple  $kG$ -module and let  $\phi_M$  be the Brauer character of  $M$ . Then by hypothesis, the Brauer character  $\phi_{\tau M}$  of  ${}^\tau M$  satisfies  $\phi_{\tau M}(g) = \phi_M(g^{-1}) = \phi_{M^\vee}(g)$  for all  $g \in G_{p'}$ . Hence  $M$  is automorphically dual. Now, let  $U$  be a simple  $kN$ -module and let  $M$  be a simple  $kG$ -module such that  $U$  is a summand of  $\text{Res}_N M$ . Then  $U^\vee$  is a summand of  $\text{Res}_N M^\vee$  and  ${}^\tau U$  is a summand of  $\text{Res}_N {}^\tau M$ . But as shown above,  ${}^\tau M \cong M^\vee$ , hence  ${}^\tau U$  and  $U^\vee$  are covered by the same simple  $kG$ -module. Thus there exists  $g \in G$  such that  ${}^{g\tau} U \cong U^\vee$ . The result follows by considering the automorphism  $y \rightarrow g\tau(y)g^{-1}$  of  $N$ .  $\square$

**Lemma 3.2.** *Let  $L$  be a finite symmetric group  $S_n$  or one of the finite classical groups  $GL_n(q)$ ,  $GU_n(q)$ ,  $O_{2n+1}(q)$ ,  $O_{2n}^+(q)$ ,  $O_{2n}^-(q)$ , or  $Sp_{2n}(q)$ , where  $q$  is a prime power. Let  $G$  be a subgroup of  $L$  such that  $[L, L] \leq G$  and let  $Z$  be a central subgroup of  $G$ . Every simple module  $kG/Z$ -module is automorphically dual.*

*Proof.* First, note that since  $Z(G)$  is cyclic, it suffices to show that every simple  $kG$ -module is automorphically dual. In each case above, there is an automorphism  $\tau : L \rightarrow L$  such that  $\tau(x)$  is conjugate to  $x^{-1}$  for all  $x \in L$ : If  $L$  is a symmetric group, then take  $\tau$  to be the identity map. If  $L$  is an orthogonal group, or  $L$  is a symplectic group  $Sp_{2n}(q)$ , where  $q$  is a power of 2, then by results of Gow (Lemma 2.1 and Lemma 2.2 of [6]) and Wonenburger [13], every element of  $L$  is conjugate to its inverse, hence again  $\tau$  may be taken to be the identity map. If  $L = Sp_{2n}(q)$  where  $q$  is odd, then by Lemma 2.1(b) of [6], every element of  $L$  is the product of two skew symplectic involutions. Hence, we may take  $\tau$  to be conjugation by any fixed skew-symplectic involution. If  $L = GL_n(q)$  we take  $\tau$  to be the transpose inverse map. Finally, if  $L = GU_n(q)$  we let  $\tau$  be the map which raises every entry of every matrix to its  $q$ -th power (considering  $GU_n(q)$  as the fixed point subgroup of  $GL_n(\overline{\mathbb{F}}_q)$  under the map which sends a matrix  $(a_{ij})$  to the matrix  $((a_{ij}^q))^{-t}$ ). It is easy to check that in each case  $G$  is  $\tau$ -stable. Now the result follows from Lemma 3.1.  $\square$

Now we can prove Theorem 1.1:

*Proof.* Immediate from Lemma 3.2 and Proposition 1.3.  $\square$

#### 4. SOME RESULTS FROM BLOCK THEORY

NOTATION We keep the notation of the previous sections. Let  $G$  be a finite group. By a block of  $kG$  (or of  $\mathcal{O}G$ ), we will mean a primitive idempotent of the center  $Z(kG)$  of the group algebra  $kG$  (or of the center  $Z(\mathcal{O}G)$  of the group algebra  $\mathcal{O}G$ ).

**Lemma 4.1.** *Let  $H$  be a finite group, and let  $N, J$  be normal subgroups of  $H$  such that  $N \leq J$ . Suppose that  $H/J$  is a  $p$ -group and that  $J/N$  is a cyclic  $p'$ -group. Suppose further that  $b$  is an  $H$ -stable block of  $kN$  of defect 0. Then there is a block  $f$  of  $kJ$  which is  $H$  stable and such that  $bf = f$ . Further, if  $J/N$  is central in  $H/N$ , then any block  $f$  of  $kJ$  such that  $bf = f$  is  $H$ -stable.*

*Proof.* Let  $W$  be the unique simple  $kNb$ -module. Then  $W$  is  $H$ -stable and hence  $W$  is  $J$ -stable. Now  $J/N$  being a cyclic  $p'$ -group means that  $W$  extends in  $[J : N]$  ways to a  $kJ$ -module and these are all the simple  $kJ$ -modules covering  $W$ . Let  $S$  be the set of isomorphism classes of these extensions. Since  $W$  is  $H$ -stable,  $H$  acts on  $S$  by conjugation. Since the normal subgroup  $J$  of  $H$  is clearly in the kernel of this action,  $H/J$  acts on  $S$ . But  $H/J$  is a  $p$ -group and  $|S| = [J : N]$  is prime to  $p$ , so this action must have a fixed point. In other words, there exists a  $kJ$ -module  $V$  such that  $\text{Res}_N^J(V) \cong W$  and such that  $^xV \cong V$  for all  $x \in H$ . Let  $f$  be the block of  $kJ$  containing  $V$ . Then  $f$  has the required properties.

Now, suppose that  $J/N$  is central in  $H/N$ , let  $V$  be as above be an  $H$ -stable extension of  $W$  to  $J$ , and let  $U$  be any extension of  $W$  to  $J$ . Then, there is a 1-dimensional  $kJ/N$ -module  $Z$  such that  $U \cong V \otimes_k \text{Inf}_{J/N}^J(Z)$ , where  $\text{Inf}_{J/N}^J Z$  is the inflation to  $J$ , via the

canonical map  $J \rightarrow J/N$ , and where  $V \otimes_k \text{Inf}_{J/N}^J(Z)$  has the  $kJ$ -module structure given by  $x.(v \otimes z) = xv \otimes xz$ , for  $x \in J$ ,  $v \in V$  and  $z \in \text{Inf}_{J/N}^J(Z)$ . Since  $J/N$  is central in  $H/N$ ,  ${}^g J \cong {}^g J$ , and hence  ${}^g U \cong U$  for any  $g \in H$ . The result now follows as above.  $\square$

In the sequel, we will use the following fact about block idempotents without comment: If  $N$  is a normal subgroup of a finite group  $G$ , such that  $G/N$  is a  $p$ -group, then any block of  $kG$  is a central idempotent of  $kN$ , and consequently, if  $b$  is a block of  $kN$  which is stable under the conjugation action of  $G$  on  $N$ , then  $b$  is a block of  $kG$ .

**Lemma 4.2.** *Let  $L$  be a finite group, and let  $G$  be a subgroup of  $L$  and  $N, J$  be normal subgroups of  $L$ . Suppose that  $L = JG$ ,  $J \cap G = N$ , that  $G/N$  is a  $p$ -group and that  $J/N$  is a cyclic  $p'$ -group. Suppose further that  $b$  is an  $L$ -stable block of  $kN$  of defect 0 and that  $f$  is an  $L$ -stable block of  $kJ$  such that  $bf = f$ . Then,*

(i) *The blocks  $kGb$  and  $kLf$  are nilpotent.*

*Furthermore, letting  $U$  be the unique (upto isomorphism)  $kGb$ -module and  $V$  the unique (upto isomorphism) simple  $kLf$ -module, and letting  $P$  be a defect group of  $kGb$ , we have*

(ii)  *$\text{Res}_G^L(V) \cong U$ .*

(iii)  *$P$  is a defect group of  $kLf$ .*

(iv) *The map*

$$kGb \rightarrow kLf, \quad a \rightarrow af$$

*is an interior  $P$ -algebra isomorphism, and induces an isomorphism of interior  $P$ -algebras between a source algebra of  $kGb$  and that of  $kLf$ .*

(v) *There is a pair  $(P, W)$  which is a vertex-source pair of the  $kG$ -module  $U$  and which is also a vertex-source pair for the  $kL$ -module  $V$ .*

*Proof.* Since  $G/N$  is a  $p$ -group, and  $b$  is a  $G$ -stable block of  $kN$ , it is clear that  $kGb$  is a nilpotent block, and that  $P \cong G/N$ . Since  $J/N$  is a  $p'$  group, and  $f$  covers  $b$ ,  $kJf$  is a block of defect 0. Also,  $L/J = GJ/J \cong G/N$ , and  $G/N$  is a  $p$ -group, so it follows that  $kLf$  is nilpotent with defect groups isomorphic to  $G/N \cong P$ . Now,  $b = \text{Tr}_P^G(x)$ , for some  $x \in (kG)^P$ . Since  $f \in Z(kL)$  this yields

$$f = bf = \text{Tr}_P^G(xf).$$

The fact that  $[L : G]$  is prime to  $p$ , then gives

$$f = \text{Tr}_G^L\left(\frac{1}{[L : G]}f\right) = \text{Tr}_P^L\left(\frac{1}{[L : G]}xf\right),$$

from which it follows that  $P$  is contained in a defect group of  $kLf$ . But  $P$  is isomorphic to the defect groups of  $kLf$ , proving (iii).

Now,  $\text{Res}_N^G(U)$  is the unique (upto isomorphism) simple  $kNb$ -module, so the fact that  $b$  is  $L$ -stable, means that  $\text{Res}_N^G(U)$  is  $J$ -stable. Since  $J/N$  is a cyclic  $p'$ -group,  $\text{Res}_N^G(U)$  extends in precisely  $[J : N]$  ways to a  $kJ$ -module, each of which has the same dimension as  $U$ , and furthermore, these  $[J : N]$  extensions are the only simple  $kJ$ -modules whose restriction to  $N$  contain direct summands isomorphic to  $\text{Res}_N^G(U)$ . On the other hand,  $\text{Res}_J^L(V)$  is a simple  $kNf$ -module, so the fact that  $fb = f$  means that  $\text{Res}_J^L(V)$  is one of these  $[J : N]$  extensions. Thus,

$$\text{Res}_N^G(\text{Res}_G^L(V)) = \text{Res}_N^J(\text{Res}_J^L(V)) = \text{Res}_N^G(U).$$



But  $U$  is the unique extension of  $\text{Res}_N^G(U)$  to a  $kG$ -module, hence  $\text{Res}_G^L(V) \cong U$ , proving (ii).

We now prove (iv). Note that the second assertion of (iv) is an immediate consequence of the first assertion of (iv) and (iii). Thus, it suffices to prove that the map  $a \rightarrow af$  is an isomorphism of interior  $P$ -algebras between  $kGb$  and  $kLf$ . Since  $f$  is a central idempotent of  $kL$ , the map is clearly a  $P$ -algebra homomorphism. Furthermore, since by (i), (ii) and (iii) both  $kLf$  and  $kGb$  are nilpotent, with defect groups of the same order and with simple modules of the same  $k$ -dimension, it follows that  $kLf$  and  $kGb$  have the same  $k$ -dimension, thus it suffices to prove that the map  $a \rightarrow af$  is injective. So, let  $a \in kGb$  be such that  $af = 0$ . Let  $P$  be the projective cover of the simple  $kLf$ -module  $V$ . Then  $af = 0$  implies that  $aP = afP = 0$ . But it is easy to see that  $\text{Res}_G^L(P)$  is a projective cover of  $U$ . Thus,  $a$  annihilates the unique projective  $kGb$ -module, which means  $akGb = 0$ , that is  $a = 0$ . Thus, the map  $a \rightarrow af$  is injective, finishing the proof of (iv). Statement (v) is a consequence of (iv).  $\square$

**Lemma 4.3.** *Let  $G$  be a finite group with  $O_p(G) = 1$ . Let  $N$  be a normal subgroup of  $G$ . If  $G/N$  is a  $p$ -group, then  $C_G(N) = Z(N)$  is a  $p'$ -group. Consequently, the canonical homomorphism of  $G/N$  into  $\text{Out}(N)$  is injective.*

*Proof.* Let  $P$  be a Sylow  $p$ -subgroup of  $G$ . Then  $G = NP$  from which it follows that  $C_P(N)$  is a normal subgroup of  $G$  (as  $C_P(N)$  is normalised both by  $N$  and by  $P$ ). By hypothesis,  $C_P(N) = 1$ . Thus,  $C_G(N)$  is a  $p'$ -group from which it also follows that  $C_G(N) \leq Z(N)$ .  $\square$

**Lemma 4.4.** *Let  $N, G, G'$  be finite groups such that both  $G$  and  $G'$  contain  $N$  as a normal subgroup. Let  $\gamma_G : G \rightarrow \text{Aut}(N)$  and  $\gamma_{G'} : G' \rightarrow \text{Aut}(N)$  be the canonical homomorphisms and let  $\pi : \text{Aut}(N) \rightarrow \text{Out}(N)$  be the canonical surjection. Suppose that  $G/N$  and  $G'/N$  are both  $p$ -groups, that  $O_p(G) = 1 = O_p(G')$ , and that  $kN$  has a  $G$ -stable block of defect 0. If  $\pi \circ \gamma_G(G)$  and  $\pi \circ \gamma_{G'}(G')$  are conjugate subgroups of  $\text{Out}(N)$ , then  $G$  and  $G'$  are isomorphic through an isomorphism which stabilises  $N$ . Furthermore, if  $\pi \circ \gamma_G(G) = \pi \circ \gamma_{G'}(G')$ , then the above isomorphism may be chosen to be the identity on  $N$ .*

*Proof.* Since  $G$  is a  $p$  extension of  $N$  and  $b$  is  $G$ -stable,  $b$  is a block of  $kG$ . Let  $P$  be a defect group of  $kGb$ . Then  $G = NP$ , a semi-direct product. Set  $P_0 := \gamma_G(P)$  and let

$$\phi : G \rightarrow N \rtimes P_0$$

be defined by

$$\phi(nx) = n\gamma_G(x), \quad n \in N, x \in P.$$

By Lemma 4.3, it follows that  $\gamma_G$  induces an isomorphism between  $P$  and  $P_0$ , that  $\phi$  is an isomorphism and that  $\gamma_G(G) = \text{Inn}(N)P_0$ , a semidirect product.

Now  $G$  and  $G'$  both contain  $N$ . Hence, by hypothesis, there exists  $\alpha \in \text{Aut}(N)$  such that

$$\gamma_{G'}(G') = {}^\alpha \gamma_G(G) = \text{Inn}(N) {}^\alpha P_0.$$

If  $\pi \circ \gamma_G(G) = \pi \circ \gamma_{G'}(G')$ , then choose  $\alpha$  to be the identity.

Let  $U$  be the full inverse image of  ${}^\alpha P_0$  in  $G'$  through  $\gamma_{G'}$  and let  $S$  be a Sylow  $p$ -subgroup of  $U$ . By Lemma 4.3, applied to  $G'$ ,  $\gamma_{G'}$  induces an isomorphism between  $S$  and

${}^\alpha P_0$ . Now,

$$\gamma_{G'}(NS) = \text{Inn}(N) {}^\alpha P_0 = \gamma_{G'}(G')$$

and by Lemma 4.3  $\text{Ker}(\gamma_{G'}) = Z(N) \leq N$ , hence  $G' = NS$ . On the other hand,

$$|N||S| = |N| {}^\alpha P_0 = |Z(N)||\text{Inn}(N)| {}^\alpha P_0 = |\text{Ker}(\gamma_{G'})||\gamma_{G'}(G')| = |G'|.$$

Thus  $G = NS$ , a semidirect product and the map

$$\varphi : G' \rightarrow N \rtimes {}^\alpha P_0$$

defined by

$$\varphi(nx) = n\gamma_{G'}(x), \quad n \in N, x \in S,$$

is an isomorphism.

Let

$$\psi : N \rtimes P_0 \rightarrow N \rtimes {}^\alpha P_0$$

be defined by

$$\psi(nx) = \alpha(n) {}^\alpha x, \quad n \in N, x \in P_0.$$

Then,  $\varphi^{-1} \circ \psi \circ \phi : G \rightarrow G'$  is an isomorphism with the required properties.  $\square$

**Remarks.** (i) The condition that  $kN$  has a  $G$ -stable block of defect 0 in the above may be replaced by the weaker condition that  $N$  has a complement in  $G$ .

(ii) The above proposition may be also understood more structurally as a consequence of the fact that, as  $Z(N)$  is an abelian  $p'$ -group, and as  $G/N$  (respectively  $G'/N$ ) is a  $p$ -group, restriction from  $G/N$  (respectively  $G'/N$ ) induces an injective map from  $H^2(G/Z(N), Z(N))$  (respectively  $G'/N$ ) to  $H^2(N/Z(N), Z(N))$  and of the fact that  $G/Z(N)$  and  $G'/Z(N)$  are isomorphic groups.

## 5. ON CHARACTERS AND BLOCKS OF FINITE GENERAL LINEAR AND UNITARY GROUPS AND THEIR COMMUTATOR SUBGROUPS

The aim of this section is to prove Theorem 1.4. For this, we recall some facts from the Lusztig parametrization of characters of finite groups of Lie type in our special situation. We will follow [1] and [3].

NOTATION. We keep the notation of the previous sections and that introduced for the statement of Theorem 1.4. In particular,  $G$  will denote either a finite general linear group or a finite unitary group. For  $s$  a semi-simple element of  $G$ , we let  $[s]$  denote the  $G$ -conjugacy class of  $s$ .

**5.A. On the ordinary characters of  $G$ .** For  $s$  a semi-simple element of  $G$ , we let  $\mathcal{E}(G, [s])$  denote the rational Lusztig series corresponding to  $[s]$ . The subsets  $\mathcal{E}(G, [s])$  as  $[s]$  runs over the semi-simple classes of  $G$  partition the set of ordinary irreducible characters of  $G$ ; if an irreducible character  $\chi$  belongs to the subset  $\mathcal{E}(G, [s])$ , we will say that  $\chi$  has semisimple label  $[s]$ . Now, let  $\phi$  be the automorphism of  $G$  which appears in the statement of Theorem 1.4 (so  $\phi$  is only defined when  $q$  is a  $p$ -th power).

The above labelling of characters is compatible with the action of  $\phi$  in the following sense.

**Lemma 5.1.** *Let  $\chi$  be an irreducible character of  $G$ . Let  $[s]$  be the semi-simple label of  $\chi$  and  $[t]$  be the semi-simple label of  $\phi\chi$ . Then  $[s] = [\phi(t)]$ .*

*Proof.* This is immediate from Corollary 2.4 of [9].  $\square$

**5.B. On the blocks of  $kG$ .** For the rest of this section, we will assume that  $p|q - \epsilon$ . Then by the Fong-Srinivasan classification of the blocks of the finite general linear and unitary groups [4], two ordinary irreducible characters  $\chi$  and  $\chi'$  of  $G$  lie in the same  $p$ -block of  $G$  if and only if the  $p'$ -parts of  $s$  and of  $s'$  are conjugate in  $G$  where  $[s]$  and  $[s']$  are the semi-simple labels of  $\chi$  and  $\chi'$  respectively. Thus the  $p$ -blocks of  $G$  are partitioned by  $G$  conjugacy classes of  $p'$ -semi-simple elements of  $G$ . If  $b$  is a block of  $G$ , the  $p'$ -part of the semi-simple label of whose irreducible characters is the conjugacy class  $[t]$ , we say that  $b$  has semi-simple label  $[t]$ . If  $b$  has label  $[t]$ , then any Sylow  $p$ -subgroup of  $C_G(t)$  is a defect group of  $b$ .

**5.C. On characters of  $N$ .** Let  $\psi$  be an ordinary irreducible character of  $N$ , let  $\chi$  be an irreducible character of  $G$  covering  $\psi$  and let  $[s]$  be the semi-simple label of  $\chi$ . Let  $I(\psi)$  be the stabiliser of  $\psi$  in  $G$ . If  $\chi'$  is another character of  $G$  covering  $\psi$ , the semisimple label of  $\chi'$  is  $[sz]$  for some  $z \in Z(G)$  and conversely, for any  $z \in Z(G)$ , there exists some irreducible character  $\chi'$  of  $G$  with label  $[sz]$  covering  $\psi$  (see for instance [1, Proposition 11.7]). If we set  $d_{[s]}$  to be the number of distinct conjugacy classes of  $G$  of the form  $[sz]$ , where  $z \in Z(G)$ , we get that  $[I(\psi) : N]$  is divisible by  $d_{[s]}$  (see [1, Corollary 11.13]).

Now let  $s$  be as above. Multiplication yields a transitive action of  $Z(G)$  on the set of conjugacy classes of  $G$  of the form  $[sz]$ ,  $z \in Z(G)$ . Denote by  $Z(s)$  the stabiliser of  $[s]$  under this action, so that  $d_{[s]} = \frac{q-\epsilon}{|Z(s)|}$  and the number of  $G$ -conjugates of  $\psi$  divides  $|Z(s)|$ .

Suppose  $z \in Z(s)$ . If  $G = GL_n(q)$ , then  $s$  being semi-simple there is a diagonal matrix  $\text{diag}(\alpha_1, \dots, \alpha_n)$ ,  $\alpha_i \in \mathbb{F}_q^\times$ , such that  $s$  is conjugate to  $\text{diag}(\alpha_1, \dots, \alpha_n)$  in  $GL_n(\mathbb{F}_q)$ . Thus  $sz$  is conjugate to  $\text{diag}(\zeta\alpha_1, \zeta\alpha_2, \dots, \zeta\alpha_n)$ , where  $z = \text{diag}(\zeta, \dots, \zeta)$ ,  $\zeta \in \mathbb{F}_q^\times$ . Hence  $\text{diag}(\alpha_1, \alpha_2, \dots, \alpha_n)$  is conjugate to  $\text{diag}(\zeta\alpha_1, \dots, \zeta\alpha_n)$ . Taking determinants, one sees that  $\zeta^n = 1$ . Furthermore, letting  $o(s)$  denote the order of  $s$ , we have that  $1 = (\text{diag}(\alpha_1, \alpha_2, \dots, \alpha_n))^{o(s)}$  is conjugate to  $z^{o(s)}$ , hence  $z^{o(s)} = 1$ . Of course  $z^{q-\epsilon} = 1$ . Thus  $Z(s)$  is a cyclic group of order dividing  $\gcd(n, q-1, o(s))$ . Arguing similarly when  $G = GU_n(q)$  and noting that  $Z(GU_n(q))$  consists of scalar matrices of order dividing  $q+1$ , we obtain that  $Z(s)$  is a cyclic group of order dividing  $\gcd(n, q+1, o(s))$ .

Summarising the above discussion we get :

**Proposition 5.2.** *Let  $\chi$  be an ordinary irreducible character  $G$  and let  $[s]$  be the semi-simple label of  $\chi$ . Let  $\psi$  be an irreducible character of  $N$  covered by  $\chi$ .*

(i) *The number of  $G$ -conjugates of  $\psi$  is a divisor of  $|Z(s)|$ , and  $I(\psi)/N$  has order divisible by  $\frac{q-\epsilon}{|Z(s)|}$ .*

(ii)  *$|Z(s)|$  is a divisor of  $\gcd(n, q - \epsilon, o(s))$ .*

We also get an analogous result for blocks:

**Proposition 5.3.** *Let  $b$  be a block of  $G$  and let  $[t]$  be the semi-simple label of  $b$ . Let  $c$  be a block of  $N$  covered by  $b$ . If  $b'$  is a block of  $G$  covering  $c$ , then the semi-simple label of  $b'$  is of the form  $[tz]$  for some  $z \in Z(G)$ ; the number of elements in the  $G$ -orbit of  $c$  is a divisor of  $|Z(t)|$  and in particular, this number is prime to  $p$ .*

*Proof.* Since  $b$  and  $b'$  cover the same blocks of  $N$ , there is a sequence  $b =: b_1, b_2, \dots, b_r := b'$  of blocks of  $G$  and ordinary characters  $\chi_i$  in  $b_i$ ,  $1 \leq i \leq r$  such that for each  $i$ ,  $1 \leq i \leq r-1$ ,

$\chi_i$  and  $\chi_{i+1}$  cover a common character of  $N$ . Let  $[s_i]$  be the semi-simple label of  $\chi_i$ ,  $1 \leq i \leq r$ . Then from the above discussion, it follows that for each  $i$ ,  $1 \leq i \leq r-1$  there is a  $z_i \in Z(G)$  such that  $[s_{i+1}] = [s_1 z_i]$ . Setting  $t_i$  to be the  $p'$ -part of  $s_i$  and  $v_i$  to be the  $p'$ -part of  $z_i$ , it follows that  $[t_{i+1}] = [t_i v_i]$  for all  $i$ ,  $1 \leq i \leq r-1$ . This proves the first assertion.

The block  $b$  contains an irreducible character, say  $\chi$  with semi-simple label  $[t]$  and  $c$  contains a character, say  $\psi$  covered by  $\chi$ . Since  $I(b)$  contains  $I(\psi)$ , the second statement follows from Proposition 5.2(i). Since  $t$  is a  $p'$ -element, the last assertion is immediate from Proposition 5.2(ii).  $\square$

In what follows we will identify  $GL_n(q)$  with the group of invertible linear transformations of an  $n$ -dimensional vector space over a field of  $q$ -elements, and identify  $GU_n(q)$  with a subgroup of  $GL_n(q^2)$  in the natural way. As before, we assume that  $p$  divides  $q - \epsilon$ . In addition, we assume from now on that  $q = q^p$ , and  $\phi$  is the automorphism of  $G$  appearing in the statement of Theorem 1.4.

**Lemma 5.4.** *Let  $t$  be a semi-simple element of  $GL_n(q)$  and suppose that the minimal polynomial of  $t$  over  $\mathbb{F}_q$  has an irreducible factor whose degree is distinct from the degrees of all other irreducible factors of the minimal polynomial of  $t$ . Suppose that  $[\phi(t)] = [tz]$  for some  $z = \text{diag}(\eta, \eta, \dots, \eta) \in Z(GL_n(q))$ . Then,  $|Z(t)||q' - 1$ .*

*Proof.* Let  $p(x)$  be an irreducible factor of the minimal polynomial of  $t$  with degree distinct from that of every other irreducible factor. Let  $\lambda$  be a root of  $p(x)$  in  $\overline{\mathbb{F}}_q$ . and let  $r$  be the order of  $\lambda$ . We claim that there exists a positive integer  $v$  such that  $v$  is relatively prime to  $p$  and such that the order  $r$  is a factor of  $(q'^v - 1)(q - 1)$ . Indeed, by hypothesis,  $\lambda^{q'}$  is an eigen value of  $tz$ , i.e.  $\lambda^{q'} = \lambda' \eta$  for some eigen value  $\lambda'$  of  $t$ . The minimal polynomials of  $\lambda$  and  $\lambda^{q'}$  over  $\mathbb{F}_q$  have the same degree and the minimal polynomials of  $\lambda' \eta$  and  $\lambda'$  over  $\mathbb{F}_q$  also have the same degree. Hence, it follows that  $\lambda'$  is a root of  $p(x)$ , that is  $\lambda^{q'} = \lambda^{q^u} \eta$  for some  $u$ . This gives  $\lambda^{q'(q'^{pu-1}-1)} = \eta \in \mathbb{F}_q$ , whence  $\lambda^{(q'^{pu-1}-1)} \in \mathbb{F}_q$ . The claim follows by setting  $v = pu - 1$ .

Now, let  $y = \text{diag}(\zeta, \zeta, \dots, \zeta) \in Z(t)$ , and let  $\lambda$  be a root of  $p(x)$ . We claim that  $\zeta = \lambda^{q^m - 1}$ , for some integer  $m$ . Indeed, the eigen values of  $yt$  are of the form  $\zeta \alpha$ , where  $\alpha$  is an eigen value of  $t$ . Hence  $[t] = [yt]$  implies  $\lambda \zeta$  is also an eigen value of  $t$ . Again, since  $\zeta \in \mathbb{F}_q$ , the minimal polynomial of  $\lambda \zeta$  over  $\mathbb{F}_q$  has the same degree as the minimal polynomial of  $\lambda$  over  $\mathbb{F}_q$  which means that  $\lambda \zeta$  is a root of  $p(x)$ , so  $\lambda \zeta = \lambda^{q^m}$  for some  $m$ , proving the claim.

Since  $\zeta^{q'-1} = 1$ , it follows from the claim that  $r$  is a divisor of  $(q^m - 1)(q - 1) = (q'^{pm} - 1)(q - 1)$ . Combining this with what we showed previously, it follows that  $r$  is a factor of  $\gcd((q'^{pm} - 1)(q - 1), (q'^v - 1)(q - 1)) = (q'^{\gcd(m,v)} - 1)(q - 1)$ , the last equality holding because  $v$  is relatively prime to  $p$ . Since  $q^m - 1$  is divisible by  $q'^{\gcd(m,v)} - 1$ , and by  $q - 1$  and since  $\gcd(q'^{\gcd(m,v)} - 1, q - 1) = q' - 1$ , this gives that  $r$  is a factor of  $(q^m - 1)(q' - 1)$ . Thus,

$$\zeta^{(q'-1)} = \lambda^{(q^m-1)(q'-1)} = 1.$$

Thus  $Z(t)$  has order dividing  $q' - 1$ .  $\square$

We need an analogous result for the unitary groups.

**Lemma 5.5.** *Let  $t$  be a semi-simple element of  $GU_n(q)$ , and suppose that  $t$  has an eigen value  $\lambda$  satisfying the following:*

*For any eigen value  $\lambda'$  and any element  $\text{diag}(\zeta, \zeta, \dots, \zeta) \in Z(GU_n(q))$  such that either  $\lambda^{-q'}$  or  $\lambda$  equals  $\zeta\lambda'$ , we have  $\lambda' = \lambda^{(-q)^m}$  for some non-negative integer  $m$ .*

*If  $[\phi(t)] = [tz]$  for some  $z = (\eta, \eta, \dots, \eta) \in Z(GU_n(q))$ , then  $|Z(t)| \mid q' + 1$ .*

*Proof.* Let  $r$  be the order of  $\lambda$ . We claim that there exists a positive integer  $v$  such that  $v$  is relatively prime to  $p$  and such that the order  $r$  is a factor of  $((-q')^v - 1)(q + 1)$ . Indeed, by hypothesis,  $\lambda^{-q'} = \lambda'\eta$  for some eigen value  $\lambda'$  of  $t$ , and some  $\eta \in \overline{\mathbb{F}}$  such that  $\eta^{q+1} = 1$ . Thus,  $\lambda^{-q'} = \lambda^{(-q)^u}\eta$  for some  $u$ . This gives  $\lambda^{q'((-q)^{pu-1}-1)} = \eta$ , and  $\eta^{q+1} = 1$ . The claim follows by setting  $v = pu - 1$ .

Now, let  $y = \text{diag}(\zeta, \zeta, \dots, \zeta) \in Z(t)$ . So  $[t] = [yt]$  implies  $\lambda\zeta$  is also an eigen value of  $t$ . Again, the nature of  $\lambda$  gives that  $\lambda\zeta = \lambda^{(-q)^m}$  for some  $m$ . Since  $\zeta^{q+1} = 1$ , it follows from the above that  $r$  is a divisor of  $((-q)^m - 1)(q + 1) = ((-q')^{pm} - 1)(q + 1)$ . Combining this with what we showed previously, it follows that  $r$  is a factor of  $\gcd((-q')^{pm} - 1)(q + 1), ((-q')^v - 1)(q + 1)$ .

We claim that  $d := \gcd(q + 1, (-q')^v - 1) = q' + 1$ . Indeed suppose first that  $v$  is even, say  $v = 2w$ . Then  $(-q')^v = q'^v$  and  $d$  is a factor of  $\gcd(q'^{2p} - 1, q'^{2w} - 1) = q'^2 - 1$ , the last equality holding because  $v$  is relatively prime to  $p$ . Thus  $d \mid q'^2 - 1$  and  $d \mid q'^p + 1$ . Now  $\gcd(q' - 1, q'^p + 1) = 1$  if  $q'$  is even and is 2 otherwise. On the other hand, since  $p$  is odd,  $\frac{q'^p+1}{q'+1}$  is odd if  $q$  is odd. The claim follows. Suppose next that  $v$  is odd. In this case,  $(-q')^v = -q'^v$ , and  $d = \gcd(q + 1, q'^v + 1)$ . Again, the fact that  $v$  is relatively prime to  $p$ , and that  $p$  is odd, will imply the claim.

Since  $q + 1$  is a factor of  $(-q')^{pm} - 1$ , it follows from the claim that  $\gcd(((q')^{pm} - 1)(q + 1), ((-q')^v - 1)(q + 1))$  is a factor of  $(-q')^{pm} - 1$ , from which we get that

$$\zeta^{(q'+1)} = \lambda^{((-q')^{pm}-1)(q'+1)} = 1.$$

The above shows that  $Z(t)$  has order dividing  $q' + 1$ . □

NOTATION - For any positive integer  $m$ , let  $m_+$  denote the  $p$ -part of  $m$  and let  $m_-$  denote the  $p'$  part of  $m$ .

**Lemma 5.6.** *Suppose that  $p \mid (q - 1)$  and  $p \mid n$ . Suppose also that  $SL_n(q)$  has a block  $c$  with central defect group. Let  $b$  be a block of  $GL_n(q)$  covering  $SL_n(q)$ , let  $[t]$  be the semi-simple label of  $b$ , and let  $f(x)$  be the characteristic polynomial of  $t$ . One of the following holds:*

- (i)  $f(x)$  is irreducible and  $n_+ \leq (q - 1)_+$ .
- (ii)  $n_+ \geq (q - 1)_+$ , and  $f(x)$  is a product  $f(x) = p_1(x)p_2(x)$ , where  $p_1(x)$  and  $p_2(x)$  are irreducible polynomials such that  $\deg(f_1(x)) \neq \deg(f_2(x))$  and neither  $\deg(f_1(x))$  nor  $\deg(f_2(x))$  is divisible by  $p$ .

*In particular,  $f(x)$  has an irreducible factor whose degree is distinct from any other irreducible factor of  $f(x)$ .*

*Proof.* Let  $f(x) = \prod_{1 \leq i \leq u} p_i(x)^{m_i}$  be a prime factorisation of  $f(x)$  in  $\mathbb{F}_q[x]$ . Let  $R$  be a defect group of the block  $b$  of  $kGL_n(q)$ . Since  $p$  divides  $q - 1$ ,  $R$  is conjugate to a Sylow  $p$ -subgroup of  $C_G(t)$ . Then  $R \cap SL_n(q)$  is a defect group of  $kSL_n(q)c$ . But by hypothesis, the Sylow  $p$ -subgroup of  $Z(SL_n(q))$  is the (unique) defect group of  $kSL_n(q)c$ . Since the Sylow  $p$ -subgroup of  $Z(SL_n(q))$  is a cyclic group of order  $d_+$ , where  $d = \gcd(n, q - 1)$

and since  $R/R \cap SL_n(q)$  is a cyclic group of order dividing  $(q-1)$ , it follows that  $R$  is meta-cyclic and has  $p$ -rank at most 2. Now by the prime decomposition of  $f(x)$  above, if  $n_i$  is the degree of  $p_i(x)$ ,  $1 \leq i \leq u$  we have

$$C_{GL_n(q)}(t) \cong \prod GL_{m_i}(q^{n_i}).$$

Since  $p$  divides  $q-1$ , the fact that  $R$  is a Sylow  $p$ -subgroup of  $C_{GL_n(q)}(t)$  and that  $R$  is metacyclic, forces  $u \leq 2$  and either  $m_i = 1$  for all  $i, 1 \leq i \leq u$  or  $u = 1$ , and  $m_1 = 2$ .

Suppose  $u = 1$  and that  $f(x)$  is irreducible and  $|R| = (q^n - 1)_+ = (q-1)_+ n_+$ . Suppose, if possible that  $n_+ > (q-1)_+$ . Then  $|R| > (q-1)d_+$ , a contradiction. Hence in this case, case (i) of the proposition holds.

Next, suppose  $u = 2$ , so that  $f(x) = p_1(x)p_2(x)$ , and  $R \cong R_1 \times R_2$  where  $R_i$  is a Sylow  $p$  subgroup of  $GL_{m_i}(q^{n_i} - 1)$ , for  $i = 1, 2$ . So, we get

$$(q-1)_+ d_+ \geq |R| = (q^{n_1} - 1)_+ (q^{n_2} - 1)_+ = (q-1)_+^2 n_{1+} n_{2+},$$

$$d_+ = (q-1)_+ n_{1+} n_{2+},$$

which implies that  $n_+ \geq (q-1)_+$  and that  $n_1$  and  $n_2$  are not divisible by  $p$ . Now, if  $n_1 = n_2$ , then  $n$  would be a  $p'$  number, a contradiction. Finally, consider the case that  $u = 1$  and  $m_1 = 2$ . Then  $f(x) = p_1^2(x)$ . The same argument as above will lead to a contradiction.

The last assertion is immediate from the description of  $f(x)$ .  $\square$

**NOTATION** - For  $p(x) \in \mathbb{F}_{q^2}[x]$  an irreducible polynomial, different from  $x$ , we let  $p_-(x)$  be the irreducible polynomial over  $\mathbb{F}_{q^2}[x]$  whose roots are of the form  $\lambda^{-q}$ , where  $\lambda$  is a root of  $p(x)$ . We say that  $p(x)$  is of unitary type if  $p(x) = p_-(x)$  and we say that  $f(x)$  is of non-unitary type otherwise.

Note that if  $t$  is a semi-simple element of  $GU_n(q)$ , and if  $f(x)$  is the characteristic polynomial of  $t$  over  $\mathbb{F}_{q^2}$ , then for any irreducible  $p(x) \in \mathbb{F}_{q^2}[x]$ , the multiplicity of  $p(x)$  as a divisor of  $f(x)$  is the same as that of  $p_-(x)$ .

**Lemma 5.7.** *Suppose that  $p|(q+1)$  and  $p|n$ . Suppose also that  $c$  is a block of  $SU_n(q)$  with central defect group. Let  $b$  be a block of  $GU_n(q)$  covering  $SU_n(q)$ , let  $[t]$  be the semi-simple label of  $b$ , and let  $f(x)$  be the characteristic polynomial of  $t$  over  $\mathbb{F}_{q^2}$ . One of the following holds:*

- (i)  $f(x)$  is irreducible, of unitary type and  $n_+ \leq (q+1)_+$ .
  - (ii)  $n_+ \geq (q+1)_+$ , and  $f(x)$  is a product  $f(x) = p_1(x)p_2(x)$ , where  $p_1(x)$  and  $p_2(x)$  are irreducible polynomials of unitary type such that  $\deg(p_1(x)) \neq \deg(p_2(x))$  and neither  $\deg(p_1(x))$  nor  $\deg(p_2(x))$  is divisible by  $p$ .
  - (iii)  $f(x) = p(x)p_-(x)$  where  $p(x)$  is irreducible of non-unitary type.
  - (iv)  $f(x) = p(x)p_-(x)g(x)g_-(x)$ , where  $p(x)$  and  $g(x)$  are irreducible polynomials of non-unitary type such that  $\deg(p(x)) \neq \deg(g(x))$  and neither  $\deg(p(x))$  nor  $\deg(g(x))$  is divisible by  $p$ .
  - (v)  $f(x) = p(x)g(x)g_-(x)$ , where  $p(x)$  is of unitary type and  $g(x)$  is of non-unitary type.
- In particular,  $t$  has an eigen value  $\lambda$  satisfying the conditions of Proposition 5.5.

*Proof.* The proof is entirely similar to that for the linear case. Let

$$f(x) = \prod_{1 \leq i \leq u} p_i(x)^{m_i} \prod_{1 \leq i \leq v} [(g_i(x)g_{i-}(x))^{n_i}]$$

be a prime factorisation of  $f(x)$  in  $\mathbb{F}_q[x]$ , where the  $p_i(x)$  are of unitary type and the  $g_i(x)$  are of non-unitary type.

Then, if  $d_i$  is the degree of  $p_i(x)$ ,  $1 \leq i \leq u$  and  $e_i$  is the degree of  $g_i(x)$ ,  $1 \leq i \leq v$ , we have (see for instance, Proposition 1A of [4])

$$C_{GU_n(q)}(t) \cong \prod_i GU_{m_i}(q^{d_i}) \prod_i GL_{n_i}(q^{2e_i}).$$

Furthermore, if  $R$  is a defect group of the block  $b$  of  $GU_n(q)$  then since  $p$  divides  $q+1$ ,  $R$  is conjugate to a Sylow  $p$ -subgroup of  $C_G(t)$ . On the other hand,  $R$  is metacyclic of order at most  $(q+1)^2$ . Now proceeding as for the general linear groups, we get that  $f(x)$  satisfies one of (i)-(v). Now, if  $f(x)$  is of types (i), (ii), (iii) or (iv) take for  $\lambda$  a root of  $f(x)$ . Let  $\lambda'$  be any eigen value of  $f(x)$  and let  $\zeta \in \mathbb{F}_{q^2}$  satisfy  $\zeta^{q+1} = 1$ . Since  $\mathbb{F}_{q^2}[\lambda^{-q'}] = \mathbb{F}_{q^2}[\lambda]$  and  $\mathbb{F}_{q^2}[\lambda'] = \mathbb{F}_{q^2}[\zeta\lambda']$ , if either  $\lambda$  or  $\lambda^{-q'}$  equal  $\zeta\lambda'$ , then  $\mathbb{F}_{q^2}[\lambda'] = \mathbb{F}_{q^2}[\lambda]$ , and the degree constraints on (i)-(iv) will yield that  $\lambda' = \lambda^{(-q)^m}$  for some  $m$ . If  $f(x)$  is of type (v), take for  $\lambda$  a root of  $p(x)$ . Let  $\lambda'$  be any eigen value of  $f(x)$  and let  $\zeta \in \mathbb{F}_{q^2}$  be such that  $\zeta^{q+1} = 1$  and such that either  $\lambda$  or  $\lambda^{-q'}$  equals  $\zeta\lambda'$ . Since  $\lambda$  is  $\mathbb{F}_{q^2}$ -conjugate to  $\lambda^{-q}$ ,  $\lambda^{-q'}$  is  $\mathbb{F}_{q^2}$ -conjugate to  $(\lambda^{-q'})^{-q}$ , hence the same is true of  $\lambda'\zeta$  and of  $\lambda'$ . But  $p(x)$  is the only unitary factor of  $f(x)$ , hence  $\lambda' = \lambda^{(-q)^m}$  for some  $m$ .  $\square$

We now prove Theorem 1.4.

*Proof.* Assume the conditions of Theorem 1.4 hold. Let  $b$  be a block of  $G$  covering  $c$  and let  $[t]$  be the semi-simple label of  $b$ . Since  $c$  is  $\phi$ -stable,  $\phi^{-1}(b)$  also covers  $c$ . By Lemma 5.1,  $\phi^{-1}(b)$  has semi-simple label  $[\phi(t)]$ . Thus, by Proposition 5.3 we have that  $[\phi(t)] = [tz]$  for some  $z = \text{diag}(\eta, \eta, \dots, \eta) \in Z(G)$ . It follows from Lemmas 5.4, 5.5, 5.6 and 5.7 that  $|Z(t)|$  is a factor of  $q' - \epsilon$ . On the other hand, by Proposition 5.3  $[G : I(c)]$  is a factor of  $|Z(t)|$ . This means that if  $h \in G$  is such that  $\det(h) = \alpha^{q'-\epsilon}$  for some  $\alpha \in \mathbb{F}_q$ , then  $h \in I(c)$ . Now let  $g \in G$  and set  $h = \phi(g)g^{-1}$ . Then  $\det(h) = \det(g)^{q'-\epsilon}$ , hence  $h \in I$ . But this means exactly that  $I(c) \rtimes \langle \phi \rangle$  is normal in  $G \rtimes \langle \phi \rangle$ .  $\square$

## 6. PROOF OF THEOREM 1.2

NOTATION- We keep the notation of Theorem 1.2. In addition, if the simple quotient of  $N$  is  $PSL_n(q)$ , let  $K_0 := GL_n(q)$  and let  $N_0 := SL_n(q)$ . If the simple quotient of  $N$  is  $PSU_n(q)$ , let  $K_0 := GU_n(q)$  and let  $N_0 := SU_n(q)$ . Set  $Z := Z(K_0)$ .

If  $q = q'^p$  and  $K_0 = GL_n(q)$ , we define  $\phi : K_0 \rightarrow K_0$  to be the automorphism  $(a_{ij}) \rightarrow (a_{ij}^{q'})$ . If  $q = q'^p$  and  $K_0 = GU_n(q)$ , we define  $\phi : K_0 \rightarrow K_0$  to be the automorphism  $(a_{ij}) \rightarrow (a_{ij}^{q'})^{t^{-1}}$ . Note that  $\phi$  is an automorphism of  $K_0$  of order  $p$ ,  $N_0$  is  $\phi$ -stable, and since  $K_0/N_0$  is cyclic any subgroup of  $K_0$  containing  $N_0$  is also  $\phi$ -stable.

If  $K_0 = GL_n(q)$  we set  $\epsilon = 1$  and if  $K_0 = GU_n(q)$ , we set  $\epsilon = -1$ .

For an abelian group  $H$ , we will let  $H_+$  denote the Sylow  $p$ -subgroup of  $H$  and let  $H_-$  denote the Hall  $p'$ -subgroup of  $H$ .

We first prove the following result detailing the structure of  $G$ .

**Proposition 6.1.** *With the notation and assumptions of Theorem 1.2, suppose that  $W$  is not self dual. Then there exists a subgroup  $Z_0$  of  $Z(N_0)$  containing the Sylow  $p$ -subgroup of  $Z(N_0)$  such that  $N$  is isomorphic to  $N_0 Z_+ / Z_+ Z_0$ . Further,  $q = q'^p$  for some prime power*

$q'$ , the index of  $N_0Z_+/Z_+Z_0$  in  $K_0/Z_+Z_0$  is divisible by  $p$  and letting  $M$  be the unique subgroup of  $K_0/Z_+Z_0$  containing  $N_0Z_+/Z_+Z_0$  as a subgroup of index  $p$ , there exists an isomorphism,  $G \cong M \rtimes \langle \phi \rangle$  sending  $N$  to  $N_0Z_+/Z_+Z_0$ .

*Proof.* We make a series of reductions.

**6.2.**  $O_p(G) = 1$ .

*Proof.* If not, then  $W$  is isomorphic to  $\text{Inf}_{P/O_p(G)}^P W'$  for some endo-permutation module for the cyclic group  $P/O_p(G)$ , and clearly  $W$  is self-dual.

**6.3.**  $p$  is a divisor of  $\gcd(q - \epsilon, n)$  and  $q = q'^p$  for some prime power  $q'$ .

*Proof.* By Lemma 4.3,  $G/N$  is isomorphic to a subgroup of  $\text{Out}(N)$ . Since  $N$  is quasi-simple,  $\text{Out}(N)$  is in turn isomorphic to a subgroup of  $\text{Out}(N/Z(N))$ . In particular,  $\text{Out}(N/Z(N))$  has an elementary abelian subgroup of order  $p^2$ . Since  $p$  is odd, the result is immediate from the nature of the outer automorphism groups of  $PSL_n(q)$  and of  $PSU_n(q)$  [5, Theorem 2.5.1].

**6.4.**  $N$  is isomorphic to  $N_0/Z_0$ , where  $Z_0$  is a central subgroup of  $N_0$  containing the Sylow  $p$ -subgroup of  $Z(N_0)$ .

*Proof.* By 6.3, and since  $p$  is odd,  $N/Z(N)$  is not one of the groups  $PSL_2(4)$ ,  $PSL_3(2)$ ,  $PSL_3(4)$ ,  $PSL_4(2)$ , or  $PSL_2(9)$ ,  $PSU_4(2)$ ,  $PSU_6(2)$  or  $PSU_4(3)$ . Hence, the exceptional part of the Schur multiplier of  $N/Z(N)$  is trivial (see [5, Table 6.1.3]). By [5, Table 6.1.2],  $N_0$  is a universal covering group of  $N/Z(N)$ . By [5, Corollary 5.1.5],  $N$  is a quotient of  $N_0$  by a central subgroup, say  $Z_0$ . Finally, since  $Z(N)$  is assumed to be a  $p'$ -group, it follows that  $Z_0$  contains the Sylow  $p$ -subgroup of  $Z(N_0)$ .

It follows from 6.3 that  $|K_0/Z_+Z_0 : N_0Z_+/Z_+Z_0|$  is divisible by  $p$  and that  $\phi$  is defined. Let  $M$  be the unique subgroup of  $K_0/Z_+Z_0$  containing  $N_0Z_+/Z_+Z_0$  as a subgroup of index  $p$ .

**6.5.** There exists an isomorphism,  $G \cong M \rtimes \langle \phi \rangle$  sending  $N$  to  $N_0Z_+/Z_+Z_0$ .

*Proof.* Since  $Z_0$  contains  $Z_+ \cap N_0$ , the inclusion of  $N_0$  in  $K_0$  induces an isomorphism between  $N_0/Z_0$  and  $N_0Z_+/Z_+Z_0$ . Henceforth, we will identify  $N$  with the subgroup  $N_0Z_+/Z_+Z_0$  of  $K_0/Z_+Z_0$ .

Let  $G' = M \rtimes \langle \phi \rangle$ . Then, clearly  $O_p(G') = 1$  and  $G'/N$  is elementary abelian of order  $p^2$ . Now, since  $p$  is odd it follows from the structure of the outer automorphism group of  $PSL_n(q)$  and of  $PSU_n(q)$  that  $\text{Out}(N)$  has metacyclic Sylow  $p$ -subgroups. On the other hand, for odd  $p$ , metacyclic  $p$ -groups have at most one elementary abelian subgroup of order  $p^2$  (see for instance [7, Lemma 2.1]). Thus, using the notation of Lemma 4.4,  $\pi \circ \gamma_G(G)$  and  $\pi \circ \gamma_{G'}(G')$  are conjugate subgroups of  $\text{Out}(N)$ . The claim follows from Lemma 4.4.  $\square$

NOTATION- If  $N/Z(N) \cong PSL_n(q)$ , let  $\tau : GL_n(q) \rightarrow GL_n(q)$  be the transpose automorphism. If  $N/Z(N) \cong PSU_n(q)$ , let  $\tau : GU_n(q) \rightarrow GU_n(q)$  be the automorphism which raises every entry to the  $q$ -th power.

We now prove Theorem 1.2.



*Proof.* Suppose if possible that  $W$  is not self-dual. By Proposition 6.1, we may assume that  $G = M \rtimes \langle \phi \rangle$ , where  $Z_0$  and  $M$  are as in the statement of Proposition 6.1. We identify  $N$  with  $N_0 Z_+ / Z_+ Z_0$  as before.

Let  $c$  be the block (necessarily of defect 0) of  $kN$  containing the simple  $kN$ -module  $U$  and let  $c_0$  be the unique block of  $kN_0 Z_+$  whose image under the canonical surjection of  $kN_0 Z_+$  onto  $kN$  is  $c$ . Let  $I$  be the inertial subgroup of  $c$  in  $K_0 / Z_0 Z_+$  and let  $I_0$  be the inertial subgroup of  $c_0$  in  $K_0$ .

**6.6.** *For any  $g \in K_0$ ,  $g^\phi g^{-1} \in I_0$ . Further,  $K_{0+} \leq I_0$ .*

*Proof.* Since  $Z_+$  is a central  $p$ -group,  $c_0$  is also a block of  $kN_0$ . Furthermore, since  $c$  is a block of defect 0 of  $kN_0 Z_+ / Z_+ Z_0$ , as block of  $kN_0$ ,  $c_0$  has central defect group. Thus,  $I_0$  is the inertial subgroup in  $K_0$  of a central defect block of  $kN_0$ . By hypothesis,  $U$  is  $G$ -stable, from which it follows that  $c_0$  is  $\phi$ -stable. The first assertion follows from Theorem 1.4 applied to  $K_0$ ,  $N_0$  and  $c_0$  and the second follows from Proposition 5.3.

Set  $J = N_0 I_{0-} Z_+ / Z_+ Z_0$  and set  $L = M_0 I_{0-} / Z_+ Z_0 \rtimes \langle \phi \rangle$ , where  $M_0$  is the inverse image of  $M$  in  $K_0$ . Then we have the following diagram of group inclusions:

$$\begin{array}{ccc}
 & (I_0 / Z_+ Z_0) \rtimes \langle \phi \rangle & \\
 & \downarrow & \\
 & L = M_0 I_{0-} / Z_+ Z_0 \rtimes \langle \phi \rangle & \\
 \swarrow & & \searrow \\
 G = M \rtimes \langle \phi \rangle & & J = N_0 I_{0-} Z_+ / Z_+ Z_0 \\
 \searrow & & \swarrow \\
 & N = N_0 Z_+ / Z_+ Z_0 &
 \end{array}$$

Note that  $J$  and  $N$  are normal in  $(I_0 / Z_+ Z_0) \rtimes \langle \phi \rangle$ , that  $(I_0 / Z_+ Z_0) \rtimes \langle \phi \rangle / J$  is a  $p$ -group and that  $J/N$  is isomorphic to a quotient of  $I_{0-} / I_{0-} \cap N_0 Z_+$ , hence is a cyclic  $p'$ -group, thus by Lemma 4.1 (applied with  $H = (I_0 / Z_+ Z_0) \rtimes \langle \phi \rangle$ ) there is a block  $f$  of  $kJ$  such that  $f = bf$  and such that  $f$  is  $(I_0 / Z_+ Z_0) \rtimes \langle \phi \rangle$ -stable.

Further, we see that the conditions of Lemma 4.2 hold, hence for some  $W'$ ,  $(P, W')$  is a vertex-source pair for the unique simple module, say  $V$ , of  $kLf$  and  $(P, W')$  is also a vertex-source pair of the  $kG$ -module  $U$ . We may assume without loss that  $W' = W$ .

Note that  $K_0$  acts by conjugation on  $J$ .

**6.7.** *The group  $K_{0-}$  acts transitively on the  $K_0$ -orbit of  $f$ .*

*Proof.* By choice of  $f$ ,  $I_0 / Z_+ Z_0$ , and hence  $I_0$  stabilizes  $f$ . On the other hand, by 6.6  $I_0$  contains  $K_{0+}$ . The claim follows.

**6.8.**  *$K_{0-}$  normalizes  $L$ .*

*Proof.* Let  $g \in K_{0-}$ . By 6.6,  $g^\phi g^{-1} \in I_0$ , but since  $g \in K_{0-}$ , in fact  $g^\phi g^{-1} \in I_{0-}$ , proving the claim.

**6.9.** *The simple  $kL$ -module  $V$  is automorphically dual.*

*Proof.* Let  $e$  be a block of  $K_0/Z_+Z_0$  such that  $ef^\vee \neq 0$ . Then  ${}^\tau e {}^\tau f^\vee \neq 0$ . But by Lemma 3.1 and its proof,  ${}^\tau e = e^\vee$  and  $e^\vee f \neq 0$ , from which it follows that  $f$  and  ${}^\tau f^\vee$  are covered by the same block of  $K_0$ , namely  $e^\vee$ . By 6.7 we get that  ${}^\tau f^\vee = {}^g f$  for some  $g \in K_{0-}$ . Now, by definition,  $\tau$  and  $\phi$  commute as automorphisms of  $K_0$ , hence the action of  $\tau$  on  $K_0$  extends to an automorphism of  $K_0 \rtimes \langle \phi \rangle$ , and the group  $L$  is clearly invariant under this automorphism.

Let  $\omega : L \rightarrow L$  be the map  $x \rightarrow g^{-1}\tau(x)g$ ,  $x \in L$ . By 6.8,  $\omega$  is well defined and is an automorphism of  $L$ . The claim follows by setting  $\psi = \omega^{-1}$ .

Now, as observed above  $(P, W)$  is a vertex source pair of the  $kL$ -module  $V$ . So, by 6.9 and by Proposition 1.3 (applied to the simple  $kL$  module  $V$ ), it follows that  $W$  is self-dual.  $\square$

## REFERENCES

- [1] C. BONNAFÉ, Sur les caractères des groupes réductifs finis à centre non connexe : applications aux groupes spéciaux linéaires et unitaires, *Astérisque* **306**(2006).
- [2] E.C. Dade, Endo-permutation modules over  $p$ -groups, II *Ann. of Math* **108** (1978), 317–346.
- [3] F. Digne and J. Michel *Representations of finite groups of Lie type*, London Mathematical Society Student Texts **21** (1991).
- [4] P. Fong and B. Srinivasan, The blocks of finite general linear and unitary groups, *Invent. Math.* **69** (1982) 109–153.
- [5] D. Gorenstein, R. Lyons and R. Solomon, *The classification of finite simple groups 3*, Mathematical Surveys and Monographs **40**, American Mathematical Society (1998).
- [6] R. Gow, On the Schur indices of characters of finite classical groups, *J. London Math. Soc. (2)* **24** (1981), no. 1, 135–147.
- [7] N. Mazza, The Dade group of a metacyclic  $p$ -group, *J. Algebra* **266**(2003), 102–111.
- [8] N. Mazza, Endo-permutation modules as sources of simple modules, *J. Group Theory* **6** (2003), 477–497.
- [9] G. Navarro, P.H. Tiep and A. Turull, Brauer characters with cyclotomic field of values, *Journal of Pure and Applied Algebra* **212** (2008), 628–635.
- [10] A. Salminen, On the sources of simple modules in nilpotent blocks, (preprint).
- [11] A. Salminen, On the sources of simple modules in nilpotent blocks II, (preprint).
- [12] J. Thevenaz, Endo-permutation modules: A guided tour, *Group representation theory* (M. Geck, D. Testerman, J. Thevenaz), EPFL press, (2007), 115–147.
- [13] M. Wonenburger, Transformations which are products of two involutions, *J. Math. Mech.* **16** (1966) 327–338.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF ABERDEEN, ABERDEEN, AB243UE  
*E-mail address:* kessar@maths.abdn.ac.uk