A BLOCK THEORETIC ANALOGUE OF A THEOREM OF GLAUBERMAN AND THOMPSON

Radha Kessar, Markus Linckelmann

May 2001

Abstract. If p is an odd prime, G a finite group and P a Sylow-p-subgroup of G, a theorem of Glauberman and Thompson states that G is p-nilpotent if and only if $N_G(Z(J(P)))$ is p-nilpotent, where $J(P)$ is the Thompson subgroup of P generated by all abelian subgroups of P of maximal order. Following a suggestion of G. R. Robinson, we prove a block-theoretic analogue of this theorem.

Theorem. *Let p be an odd prime and let k be an algebraically closed field of characteristic p. Let G be a finite group, b a block of kG, and P a defect group of b. Set $N = N_G(Z(J(P)))$ and let c be the unique block of kN such that $Br_P(c) = \mathrm{Br}_P(b)$; that is, c is the Brauer correspondent of b. Then kGb is nilpotent if and only if kNc is nilpotent.*

We refer to [5] and [7] for accounts on the terminology from group theory and block theory, respectively, involved in the theorem above and its proof. Nilpotent blocks, introduced by Broué and Puig in [3], are the block theoretic analogue of the notion of p-nilpotent groups; the principal block of kG is nilpotent if and only if G is p-nilpotent. Thus, in this case, our theorem is equivalent to the theorem of Glauberman and Thompson. The proof proceeds in two steps. We reduce to the case where G is the normaliser of a b-centric Brauer pair (following the lines of the proof of [8, Ch. 8, Theorem 3.1]), and then we apply results of Külshammer and Puig in [6] to transport the problem back to the analogous group theoretic statement.

Proof. We fix a block e_P of $kC_G(P)$ such that $\mathrm{Br}_P(b)e_P = e_P$; that is, (P, e_P) is a maximal b-Brauer pair. By [1], for any subgroup Q of P there is a unique block e_Q of $kC_G(Q)$ such that $(Q, e_Q) \subseteq (P, e_P)$. Denote by $\mathcal{F}_{G,b}$ the category whose objects are the subgroups of P and whose set of morphisms from a subgroup Q of P to another subgroup R of P is the set of group homomorphisms $\varphi : Q \to R$ for which there exists an element $x \in G$ satisfying $\varphi(u) = xux^{-1}$ for all $u \in Q$ and ${}^x(Q, e_Q) \subseteq (R, e_R)$. Thus the automorphism group of a subgroup Q of P as object of the category $\mathcal{F}_{G,b}$ is canonically isomorphic to $N_G(Q, e_Q)/C_G(Q)$. By Alperin's fusion theorem, the category $\mathcal{F}_{G,b}$ is completely determined by the structure of P and the groups $N_G(Q, e_Q)/C_G(Q)$ where either $Q = P$ or (Q, e_Q) is an essential b-Brauer pair (cf. [7, §48]). Note that $O_p(G) \subseteq Q$ whenever the pair (Q, e_Q) is essential.

2000 Mathematics Subject Classification 20C20

By Brauer's third main theorem (cf. [7, (40.17)]), if b is the principal block of kG, then e_Q is the principal block of $kC_G(Q)$, for any subgroup Q of P. Thus the above condition ${}^x(Q, e_Q) \subseteq (R, e_R)$ is equivalent to ${}^xQ \subseteq R$. Therefore, if b is the principal block of kG, we write $\mathcal{F}_G$ instead of $\mathcal{F}_{G,b}$.

In general, the definition of $\mathcal{F}_{G,b}$ depends on the choice of a maximal b-Brauer pair, but since all maximal b-Brauer pairs are G-conjugate, it is easy to see that $\mathcal{F}_{G,b}$ is unique up to isomorphism of categories. Note that we allways have $\mathcal{F}_P \subseteq \mathcal{F}_{G,b}$. Following [3], the block b is called *nilpotent* if $\mathcal{F}_P = \mathcal{F}_{G,b}$.

If H is any subgroup of G containing $PC_G(P)$, the block e_P determines a unique block d of kH by $\mathrm{Br}_P(d)e_P = e_P$. Then (P, e_P) is also a maximal d-Brauer pair, and this gives rise to the Brauer category $\mathcal{F}_{H,d}$ of kHd, defined as above for H and d instead of G and b.

We are going to use frequently the following fact:

1. *If Q is a normal subgroup of P and H a subgroup of G such that $PC_G(Q) \subseteq H \subseteq N_G(Q)$, then*

$$\mathcal{F}_{H,d} \subseteq \mathcal{F}_{G,b} \ ,$$

where d is the unique block of kH such that $\mathrm{Br}_P(d)e_P = e_P$. In particular, if kGb is nilpotent, then kHd is nilpotent.

Proof. If (R, f_R) is an essential d-Brauer pair contained in (P, e_P), then R contains Q as Q is normal in H. But then $C_G(R) = C_H(R)$, and hence $f_R = e_R$. Thus $N_H(R, f_R)/C_H(R)$ is a subgroup of $N_G(R, e_R)/C_G(R)$. □

Statement 1 applies to N, c and $Z(J(P))$ instead of H, d, Q, respectively. Thus if kGb is nilpotent, so is kNc. In order to show the converse, we consider now a minimal counter example; that is, we assume that kGb is not nilpotent while kNc is nilpotent and that $|G|$ is minimal with this property. Under this assumption, 1 implies the following statement:

2. *If Q is a normal subgroup of P and H a subgroup of G such that $PC_G(Q) \subseteq H \subseteq N_G(Q)$, then either $H = G$ or kHd is nilpotent, where d is the unique block of kH such that $\mathrm{Br}_P(d)e_P = e_P$.*

Proof. Let e be the unique block of $N \cap H$ such that $\mathrm{Br}_P(e)e_P = e_P$. We have $PC_N(Q) \subseteq N \cap H \subseteq N_N(Q)$, and thus statement 1 implies that $\mathcal{F}_{N\cap H,e} \subseteq \mathcal{F}_{N,c}$. But then $k(N \cap H)e$ is nilpotent, as kNc is so. Therefore, if H is a proper subgroup of G, then the induction hypothesis implies that the block kHd is nilpotent. □

3. *We have $O_p(G) \neq \{1\}$.*

Proof. Since the block b of kG is not nilpotent, there exists a b-Brauer pair (Q, e_Q) with $Q \neq 1$ such that $kN_G(Q, e_Q)e_Q$ is not nilpotent. This is because for some non-trivial Brauer pair (Q, e_Q), $N_G(Q, e_Q)/QC_G(Q)$ is not a p-group. Amongst all such b-Brauer pairs, choose (Q, e_Q) such that a defect group R of $kN_G(Q, e_Q)e_Q$ has maximal order. After replacing, if necessary, (Q, e_Q) by a suitable G-conjugate, we may assume that $R = N_P(Q)$. We are going to show that $R = P$, or equivalently that $P \subseteq N_G(Q, e_Q)$. We assume that R is a proper subgroup of P, and derive a contradiction. Set $H = N_G(Q, e_Q)$. Clearly $R \subseteq H$. Since $Q \subset R$, we have $C_G(R) \subset C_G(Q) \subset H$. Now $(Q, e_Q) \subseteq (R, e_R)$, and Q is normal in R, hence e_Q is the unique block of $kC_G(Q)$ which is R-stable and for which $\mathrm{Br}_R(e_Q)e_R = e_R$ (cf. [1]).

Set $M = N_G(Z(J(R)))$. Since $C_G(R)$ centralises Q and centralises $Z(J(R))$, we have $C_G(R) \subset M \cap H$. Let d be the unique block of $k(M \cap H)$ (having R as defect group) such that $\mathrm{Br}_R(d)e_R = e_R$. Let f be the unique block of kM (having R as defect group) such that $\mathrm{Br}_R(f)e_R = e_R$. Since $Z(J(R))$ is a normal p-subgroup of M, f is a central idempotent of $kC_G(Z(J(R))$ (cf. [1]). Thus there exists a block f_0 of $C_G(Z(J(R)))$ such that $ff_0 = f_0$ and $(Z(J(R)), f_0) \subseteq (R, e_R)$ in M, and hence in G. Since $(R, e_R) \subseteq (P, e_P)$, by the uniqueness of inclusion of Brauer pairs, we must have $f_0 = e_{Z(J(R))}$. Let M' be the stabiliser of $e_{Z(J(R))}$ in M. Then $N_P(Z(J(R)))$, and hence $N_P(R)$ is contained in a defect group of $kM'e_{Z(J(R))}$. In particular, the defect groups of $kM'e_{Z(J(R))}$ have order strictly greater than $|R|$. By the maximality of $|R|$, we have that $kM'e_{Z(J(R))}$ is nilpotent. Since kMf is the induced algebra $\mathrm{Ind}_{M'}^{M}(kM'e_{Z(J(R))})$, it follows that kMf is nilpotent. Now $RC_G(Q) \subseteq M \cap H \subseteq N_M(Q)$, and by statement 1 again, it follows that $k(M \cap H)d$ is nilpotent. By the minimality of $|G|$, and the fact that kHe_Q is not nilpotent, it follows that $H = G$ and hence $R = P$, contradicting the assumption $R \neq P$. If $R = P$, then H satisfies the hypothesis of 2 with $d = e_Q$, and kHe_Q is not nilpotent, thus $G = H$. In particular, $Q \subseteq O_p(G) \neq 1$. □

From now on set $Q = O_p(G)$.

4. *We have $G = N_G(Q, e_Q)$ and $b = e_Q$.*

Proof. Since $G = N_G(Q)$, the block b is contained in $kC_G(Q)$ (cf. [1]) and hence $b = \mathrm{Tr}_{N_G(Q,e_Q)}^{G}(e_Q)$. Thus $kGb \cong \mathrm{Ind}_{N_G(Q,e_Q)}^{G}(kN_G(Q, e_Q)e_Q)$, so that in particular, $kN_G(Q, e_Q)e_Q$ is not nilpotent. Since P is contained in $N_G(Q, e_Q)$, it follows from 2 that $G = N_G(Q, e_Q)$ and hence $b = e_Q$. □

Note that b is a block of any subgroup of G containing $C_G(Q)$. We want to show that actually the pair (Q, b) is *b-centric* (or *self-centralising* in the terminology of Puig, cf. [7, §41]); that is, the block $kC_G(Q)b$ is nilpotent with $Z(Q)$ as defect group. This notion goes back to Brauer [2]. We need the following technical statement.

5. *Let H be a subgroup of G containing P and let d be a block of kH having P as defect group. Put $\bar{H} = H/Q$ and for any element a of kH let $\bar{a}$ denote the image of a under the canonical surjection of kH onto $k\bar{H}$. Then $\overline{\mathrm{Br}_P(d)} = \mathrm{Br}_{\bar{P}}(\bar{d})$.*

Proof. Since Q is normal in H, the block idempotent d is a k-linear combination over the set $C_H(Q)_{p'}$ of p'-elements in $C_H(Q)$. Write $d = \sum_{g \in C_H(Q)_{p'}} \alpha_g g$ with coefficients $\alpha_g \in k$. So $\bar{d} = \sum_{g \in C_H(Q)_{p'}} \alpha_g \bar{g}$ and $\mathrm{Br}_{\bar{P}}(\bar{d}) = \sum_{g \in C_H(Q)_{p'} \cap C_H(\bar{P})} \alpha_g \bar{g}$, where $C_H(\bar{P})$ denotes the inverse image in H of $C_{\bar{H}}(\bar{P})$.

We claim that $C_H(Q)_{p'} \cap C_H(\bar{P}) = C_H(P)_{p'}$. To see this, consider the action of an element $g \in C_H(Q)_{p'} \cap C_H(\bar{P})$ on an element u of P. Since g normalises P and centralises P/Q, ${}^g(u) = uv$ for some v in Q. Let n be the order of g. Since g centralises Q, it follows that $u = {}^{g^n}u = uv^n$. But p and n are relatively prime, hence $v = 1$, thereby proving the claim.

The statement is immediate from the above expression for $\bar{d}$ □

6. *The blocks $kPC_G(Q)b$ and $kC_G(Q)b$ are nilpotent.*

Proof. By a result of Cabanes [4], normal p-extensions of nilpotent blocks are nilpotent; thus $kPC_G(Q)b$ is nilpotent if and only if $kC_G(Q)b$ is nilpotent. If $PC_G(Q)$ is a proper subgroup of G, then, by 2, b is nilpotent as block of $PC_G(Q)$, and hence of $C_G(Q)$. Thus we may assume that $G = PC_G(Q)$. We have to show that kGb is a nilpotent block. Set $\bar{G} = G/Q$ and let $\bar{b}$ denote the image of b under the canonical surjection of kG onto $k\bar{G}$. Identify $C_G(Q)/Z(Q)$ with its canonical image in $\bar{G}$; this is a normal subgroup of $\bar{G}$ of index a p-power. Since b is a k-linear combination of p'-elements in $C_G(Q)$ and $Z(Q) = Q \cap C_G(Q)$ is a central subgroup of $C_G(Q)$, it is clear that $\bar{b}$ is a block of $kC_G(Q)/Z(Q)$ and hence of $k\bar{G}$. Furthermore, $\bar{P}$ is a defect group of $k\bar{G}\bar{b}$. Let Z be the inverse image in G of $Z(J(\bar{P}))$ and set $H = N_G(Z)$. Then H is the inverse image in G of the group $\bar{H} = kN_{\bar{G}}(Z(J(\bar{P})))$. Let f be the block of $k\bar{H}$ which corresponds to the block $\bar{b}$ of $k\bar{G}$; that is, $\mathrm{Br}_{\bar{P}}(\bar{b}) = \mathrm{Br}_{\bar{P}}(f)$. Clearly, P and $C_G(Z)$ are both subgroups of H. Since Z properly contains Q and $Q = O_p(G)$, H is a proper subgroup of G. Thus by 2, the block kHd is nilpotent where d is the block of kH satisfying $\mathrm{Br}_P(d)e_P = e_P$. Since $N_G(P)$ is contained in H, we have in fact that $\mathrm{Br}_P(d) = \mathrm{Br}_P(b)$.

Now, it follows from 5 that $\mathrm{Br}_{\bar{P}}(\bar{d}) = \overline{\mathrm{Br}_P(d)} = \overline{\mathrm{Br}_P(b)} = \mathrm{Br}_{\bar{P}}(\bar{b}) = \mathrm{Br}_{\bar{P}}(f)$. In particular $\bar{d}f \neq 0$. Since kHd is nilpotent, this means that $f = \bar{d}$ and hence that $k\bar{H}f$ is nilpotent. As G is a minimal counter example to the Theorem, it follows that $k\bar{G}\bar{b}$ is nilpotent, which implies that kGb is nilpotent. □

7. *The group Q is a defect group of $kQC_G(Q)b$.*

Proof. Let R be a defect group of $kQC_G(Q)b$. We may assume that $R = QC_P(Q)$. The pair (R, e_R) is a maximal Brauer pair for the block $kQC_G(Q)b$, and hence, by the Frattini argument, $G = N_G(R, e_R)QC_G(Q) = N_G(R, e_R)C_G(Q)$. Suppose, if possible, that Q is a proper subgroup of R. Then, $N_G(R, e_R)$ is a proper subgroup of G because $Q = O_p(G)$. On the other hand $N_G(R, e_R)$ satisfies the hypothesis of 2 with R instead of Q, since P normalises R, and consequently (R, e_R). So $kN_G(R, e_R)e_R$ is nilpotent. In particular, $N_G(R, e_R)/C_G(R)$ is a p-group, and hence so is $G/C_G(Q)$. In other words, $G = PC_G(Q)$, and hence kGb is nilpotent by 6, a contradiction. □

We are now in the situation where kGb is an extension of the nilpotent block $kQC_G(Q)b$, and this is where the results of Külshammer and Puig in [6] come in.

8. *There exists a short exact sequence of groups*

$$1 \longrightarrow Q \longrightarrow L \longrightarrow G/QC_G(Q) \longrightarrow 1$$

such that P is a Sylow p-subgroup of L and such that we have $\mathcal{F}_{G,b} = \mathcal{F}_L$.

Proof. Note first that P is also a defect group of $\{b\}$ viewed as point of G on $\mathcal{O}QC_G(Q)$ because P is maximal with the property $\mathrm{Br}_P(b) \neq 0$. The existence of a canonical short exact sequence of finite groups as stated such that P is a Sylow-p-subgroup of L is a particular case of [6, 1.8]. The equality $\mathcal{F}_{G,b} = \mathcal{F}_L$ is a translation of the statement [6, 1.8.2], which requires a brief explanation. Since Q is normal in L and in G, it suffices to show that the images in $\mathrm{Aut}(R)$ of $N_G(R, e_R)/C_G(R)$ and $N_L(R)/C_L(R)$ are equal, where R is a subgroup of P containing Q. As (Q, e_Q) is b-centric and Q is

p-centric in L, it follows from a result of Puig (cf. [7, (41.1), (41.4)]) that (R, e_R) is b-centric and R is p-centric in L (that is, $Z(R)$ is a Sylow-p-subgroup of $C_L(R)$). In particular, $kC_G(R)e_R$ has a unique conjugacy class of primitive idempotents. Setting $H = QC_G(Q)$, we have $C_G(R) = C_H(R)$, hence there is a unique point γ_R of R on kH such that $\mathrm{Br}_R(i)e_R = i$ for some (and hence any) element i of γ_R. In this way, we get an inclusion preseving bijection, $R_{\gamma_R} \to (R, e_R)$ between local pointed groups R_{γ_R} on kHb for which $Q_{\gamma_Q} \subseteq R_{\gamma_R} \subseteq P_{\gamma_P}$ and kGb-Brauer pairs, (R, e_R) with $(Q, e_Q) \subseteq (R, e_R) \subseteq (P, e_P)$. Further, it is clear that $N_G(R, e_R) = N_G(R_{\gamma_R})$. Thus, setting $\bar{G} = G/QC_G(Q)$, with the notation in [6, 1.8] (which is defined in [6, 2.8]), we have $E_{G,\bar{G}}(R, e_R) = E_{L,\bar{G}}(R)$ for any subgroup R such that $Q \leq R \leq P$. By [6, (2.8.1)], the canonical maps $E_{G,\bar{G}}(R, e_R) \to E_G(R, e_R)$ and $E_{L,\bar{G}}(R) \to E_L(R)$ are surjective. Thus $E_G(R, e_R) = E_L(R)$, which implies the equality $\mathcal{F}_{G,b} = \mathcal{F}_L$. □

9. *We have* $\mathcal{F}_{N,c} = \mathcal{F}_{N_L(Z(J(P)))}$.

Proof. Since $Z(J(P))$ is normal in both N and $N_L(Z(J(P))$, it suffices to show that the images of $N_G(S, f) \cap N$ and $N_L(S) \cap N_L(Z(J(P)))$ in $\mathrm{Aut}(S)$ are equal, where (S, f) is a c-Brauer pair contained in (P, e_P) such that $Z(J(P)) \subseteq S$. Note that then $C_G(S) \subseteq N$ and hence $f = e_S$. Also, by 8 we have $\mathcal{F}_{G,b} = \mathcal{F}_L$. Thus for any $x \in N_G(S, e_S)$ there is $y \in N_L(S)$ such that ${}^x u = {}^y u$ for all $u \in S$. Since $Z(J(P)) \subseteq S$ we have $x \in N_G(S, e_S) \cap N$ if and only if $y \in N_L(S) \cap N_L(Z(J(P)))$, from which the equality 9 follows. □

We conclude the proof of the Theorem as follows. Since kGb is not nilpotent, L is not a p-nilpotent group by 8. However, kNc is nilpotent and hence $N_L(Z(J(P)))$ is p-nilpotent by 9. This contradicts the normal p-complement theorem [5, Ch. 8, Theorem 3.1] of Glauberman and Thompson. □

Acknowledgements. This work was done while the second author was a visitor at the Mathematical Institute of the University of Oxford and he would like to thank the institute for its hospitality.

References

1. J. L. Alperin, M. Broué, *Local methods in block theory*, Ann. Math. **110** (1979), 143–157.
2. R. Brauer, *On the structure of blocks of characters of finite groups*, Lecture Notes in Mathematics **372** (1974), 103–130.
3. M. Broué, L. Puig, *A Frobenius theorem for blocks*, Invent. Math. **56** (1980), 117–128.
4. M. Cabanes, *Extensions of p-groups and construction of characters*, Comm. Alg. **15** (1987), 1297–1311.
5. G. Gorenstein, *Finite Groups*, Chelsea Publishing Company, New York, 1980.
6. B. Külshammer, L. Puig, *Extensions of nilpotent blocks*, Invent. Math. **102** (1990), 17–71.
7. J. Thévenaz, *G-Algebras and Modular Representation Theory*, Oxford Science Publications,

Clarendon Press, Oxford, 1995.

Radha Kessar
University College
High Street
Oxford OX14BH
U.K.

Markus Linckelmann
CNRS, Université Paris 7
UFR Mathématiques
2, place Jussieu
75251 Paris Cedex 05
FRANCE