A RECIPROCITY FOR SYMMETRIC ALGEBRAS

MARKUS LINCKELMANN, BERNARD STALDER

March 2001

ABSTRACT. The aim of this note is to show, that the reciprocity property of group algebras in [5, (11.5)] can be deduced from formal properties of symmetric algebras, as exposed in [1], for instance.

Let $\mathcal{O}$ be a commutative ring. By an $\mathcal{O}$-algebra we always mean a unitary associative algebra over $\mathcal{O}$. Given an $\mathcal{O}$-algebra A, we denote by A^0 the opposite algebra of A. An A-module is a unitary left module, unless stated otherwise. A right A-module can be considered as a left A^0-module. If A, B are $\mathcal{O}$-algebras, we mean by an A-B-bimodule always a bimodule whose left and right $\mathcal{O}$-module structure coincide; in other words, any A-B-bimodule can be regarded as $A \otimes_{\mathcal{O}} B^0$-module. For an A-A-bimodule M we set $M^A = \{m \in M \mid am = ma \text{ for all } a \in A\}$. In particular, $A^A = Z(A)$, the center of A. If A, B, C are $\mathcal{O}$-algebras, M is an A-B-bimodule and N is an A-C-bimodule, we consider the space $\mathrm{Hom}_A(M, N)$ of left A-module homomorphisms from M to N as B-C-bimodule via $(b.\varphi.c)(m) = \varphi(mb)c$. Similarly, if furthermore N' is a C-B-bimodule, we consider the space $\mathrm{Hom}_{B^0}(M, N')$ of right B-module homomorphisms from M to N' as C-A-bimodule via $(c.\psi.a)(m) = c\psi(am)$. In particular, the $\mathcal{O}$-dual $M^* = \mathrm{Hom}_{\mathcal{O}}(M, \mathcal{O})$ becomes a B-A-bimodule via $(b.\tau.a)(m) = \tau(amb)$. Here $a \in A$, $b \in B$, $c \in C$, $m \in M$, $\varphi \in \mathrm{Hom}_A(M, N)$, $\psi \in \mathrm{Hom}_{B^0}(M, N')$ and $\tau \in M^*$.

An $\mathcal{O}$-algebra A is called *symmetric* if A is finitely generated projective as $\mathcal{O}$-module and if A is isomorphic to its $\mathcal{O}$-dual $A^* = \mathrm{Hom}_{\mathcal{O}}(A, \mathcal{O})$ as A-A-bimodule. The image $s \in A^*$ of 1_A under any A-A-isomorphism $\Phi : A \cong A^*$ fulfills $\Phi(a) = a.s = s.a$ for all $a \in A$; that is, s is symmetric and the map $a \mapsto a.s$ is a bimodule isomorphism $A \cong A^*$. Any such linear form is called a *symmetrising form of* A. The choice of a symmetrising form on A is thus equivalent to the choice of a bimodule isomorphism $A \cong A^*$.

Theorem 1. *Let A, B be symmetric $\mathcal{O}$-algebras and let M, N be A-B-bimodules which are finitely generated projective as left and right modules. We have a bifunctorial $\mathcal{O}$-linear isomorphism*

$$(M^* \underset{A}{\otimes} N)^B \cong (N \underset{B}{\otimes} M^*)^A$$

which is canonically determined by the choice of symmetrising forms of A and B.

Proof. Let $s \in A^*$ and $t \in B^*$ be symmetrising forms on A and B, respectively. It is well-known (see [1] or also the appendix in [3]) that there is an isomorphism of

B-A-bimodules

$$\begin{cases} \mathrm{Hom}_A(M,A) & \cong M^* \\ f & \mapsto s \circ f \end{cases}$$

which is functorial in M. Moreover, since M and N are finitely generated projective as left and right modules, we have an isomorphism of B-B-bimodules

$$\begin{cases} \mathrm{Hom}_A(M,A) \underset{A}{\otimes} N & \cong \mathrm{Hom}_A(M,N) \\ f \otimes n & \mapsto (m \mapsto f(m)n) \end{cases}$$

which is functorial in both M and N. Taking B-fixpoints yields $(M^* \underset{A}{\otimes} N)^B \cong (\mathrm{Hom}_A(M,A) \underset{A}{\otimes} N)^B \cong (\mathrm{Hom}_A(M,N))^B = \mathrm{Hom}_{A\otimes B^0}(M,N)$. Similarly, there is an isomorphism of B-A-bimodules

$$\begin{cases} \mathrm{Hom}_{B^0}(M,B) & \cong M^* \\ g & \mapsto t \circ g \end{cases}$$

and we have an isomorphism of A-A-bimodules

$$\begin{cases} N \underset{B}{\otimes} \mathrm{Hom}_{B^0}(M,B) & \cong \mathrm{Hom}_{B^0}(M,N) \\ n \otimes g & \mapsto (m \mapsto ng(m)) \end{cases}.$$

As before, taking A-fixpoints yields $(N \underset{B}{\otimes} M^*)^A \cong (N \underset{B}{\otimes} \mathrm{Hom}_{B^0}(M,B))^A \cong (\mathrm{Hom}_{B^0}(M,N))^A = \mathrm{Hom}_{A\otimes B^0}(M,N)$. □

Remark. The proof of Theorem 1 shows, that the two expressions in the statement of Theorem 1 are isomorphic to $\mathrm{Hom}_{A\otimes B^0}(M,N)$. In particular, for $M = N$, this induces algebra structures on $(M^* \underset{A}{\otimes} M)^B$ and $(M \underset{B}{\otimes} M^*)^A$.

Taking derived functors of the fixpoint functors in Theorem 1 yields the following consequence on Hochschild cohomology.

Corollary. *With the notation and assumptions of Theorem 1, we have an isomorphism of graded $\mathcal{O}$-modules $HH^*(B, M^* \underset{A}{\otimes} N) \cong HH^*(A, N \underset{B}{\otimes} M^*)$.*

Proof. Let P be a projective resolution of M as A-B-bimodule. Then $P^* = \mathrm{Hom}_{\mathcal{O}}(P,\mathcal{O})$ is an $\mathcal{O}$-injective resolution of M^*. Thus $N \underset{B}{\otimes} P^*$ and $P^* \underset{A}{\otimes} N$ are $\mathcal{O}$-injective resolutions of $N \underset{B}{\otimes} M^*$ and $M^* \underset{A}{\otimes} N$, respectively. Using Theorem 1, we have isomorphisms of cochain complexes $\mathrm{Hom}_{B\underset{\mathcal{O}}{\otimes} B^0}(B, P^* \underset{A}{\otimes} N) \cong (P^* \underset{A}{\otimes} N)^B \cong (N \underset{B}{\otimes} P^*)^A \cong \mathrm{Hom}_{A\underset{\mathcal{O}}{\otimes} A^0}(A, N \underset{B}{\otimes} P^*)$. Taking cohomology yields the statement. □

Let A be an $\mathcal{O}$-algebra. Following the terminology in [2], [3] (which generalises [4]), an *interior* A-algebra is an $\mathcal{O}$-algebra B endowed with a unitary algebra homomorphism $\sigma : A \to B$. If A, B are $\mathcal{O}$-algebras, C is an interior B-algebra and M an A-B-bimodule, we set $\mathrm{Ind}_M(C) = \mathrm{End}_{C^0}(M \underset{B}{\otimes} C)$, considered as interior A-algebra via the homomorphism $A \to \mathrm{Ind}_M(C)$ sending a to the C^0-endomorphism given by left multiplication with a on $M \underset{B}{\otimes} C$.

Theorem 2. *Let A, B be symmetric $\mathcal{O}$-algebras and let M be an A-B-bimodule which is finitely generated projective as left and right module. There is a canonical anti-isomorphism of $\mathcal{O}$-algebras*

$$(\mathrm{Ind}_M(B))^A \cong (\mathrm{Ind}_{M^*}(A))^B \ .$$

Proof. We have $\mathrm{Ind}_M(B) = \mathrm{End}_{B^0}(M)$ and $\mathrm{Ind}_{M^*}(A) = \mathrm{End}_{A^0}(M^*)$. Since taking $\mathcal{O}$-duality is a contravariant functor, this algebra is isomorphic to $\mathrm{End}_A(M)^0$. Taking fixpoints completes the proof. □

The group algebra $\mathcal{O}G$ of a finite group G is a symmetric algebra. More precisely, $\mathcal{O}G$ has a canonical symmetrising form, namely the form $s : \mathcal{O}G \to \mathcal{O}$ mapping a group element $g \in G$ to zero if $g \neq 1$ and to 1 if $g = 1$. Following the terminology of Puig [4], an interior G-algebra is an $\mathcal{O}$-algebra endowed with a group homomorphism $\sigma : G \to A^\times$. Such a group homomorphism extends uniquely to an $\mathcal{O}$-algebra homomorphism $\mathcal{O}G \to A$, and thus A becomes an interior $\mathcal{O}G$-algebra (and vice versa). If H is a subgroup of G and B an interior H-algebra, the *induced algebra* $\mathrm{Ind}_H^G(B)$ defined in [4] is the $\mathcal{O}$-module $\mathcal{O}G \underset{\mathcal{O}H}{\otimes} B \underset{\mathcal{O}H}{\otimes} \mathcal{O}G$ endowed with the multiplication $(x \otimes b \otimes y)(x' \otimes b' \otimes y) = (x \otimes byx'b' \otimes y')$ provided that $yx' \in H$, and 0 otherwise, where $x, y, x', y' \in G$ and $b, b' \in B$. The algebra $\mathrm{Ind}_H^G(B)$ is viewed as interior G-algebra with the structural homomorphism mapping $x \in G$ to $\sum_{y \in [G/H]} xy \otimes 1_B \otimes y^{-1}$. For $B = \mathcal{O}H$, we have the obvious identification $\mathrm{Ind}_H^G(\mathcal{O}H) = \mathcal{O}G \underset{\mathcal{O}H}{\otimes} \mathcal{O}G$, with multiplication given by $(x \otimes y)(x' \otimes y') = x \otimes yx'y'$ if $yx' \in H$ and 0 otherwise, where $x, y, x', y' \in G$. The previous notion of algebra induction is consistent with this concept:

Lemma. *Let G be a finite group, H a subgroup of G and let B be an interior H-algebra. Set $M = \mathcal{O}G_H$. There is an isomorphism of $\mathcal{O}$-algebras*

$$\begin{cases} \mathrm{Ind}_H^G(B) & \cong \mathrm{Ind}_M(B) \\ (x \otimes b \otimes y) & \mapsto (z \otimes c \ \mapsto x \otimes byzc \textit{ if } yz \in H \textit{ and } 0 \textit{ otherwise}) \end{cases},$$

where $x, y, z \in G$ and $b, c \in B$.

Proof. Straightforward verification. □

Theorem 3. *(Stalder [5]) Let G be a finite group, let H, K be subgroups of G. Consider $\mathcal{O}G$ as $\mathcal{O}H$-$\mathcal{O}K$-bimodule via multiplication in $\mathcal{O}G$. Then there is an isomorphism of $\mathcal{O}$-algebras*

$$\begin{cases} (\mathrm{Ind}_H^G(\mathcal{O}H))^K & \xrightarrow{\sim} (\mathrm{Ind}_K^G(\mathcal{O}K))^H \\ \sum_{k \in [K/K_{(x \otimes y)}]} kx \otimes yk^{-1} & \longmapsto \sum_{h \in [H/H_{(x^{-1} \otimes y^{-1})}]} hx^{-1} \otimes y^{-1}h^{-1}, \end{cases}$$

where $K_{(x \otimes y)}$ is the stabilizer in K of $x \otimes y \in \mathrm{Ind}_H^G(\mathcal{O}H)$ under the action of K and $H_{(x^{-1} \otimes y^{-1})}$ is the stabilizer in H of $x^{-1} \otimes y^{-1} \in \mathrm{Ind}_K^G(\mathcal{O}K)$ under the action of H.

There are (at least) three ways to go about the proof of Theorem 3: by explicit verification or by interpreting Theorem 3 as special case of either Theorem 1 or Theorem 2. We sketch the three different proofs.

Proof 1 of Theorem 3. The image of the set $G \times G$ in $\mathrm{Ind}_H^G(\mathcal{O}H) = \mathcal{O}G \underset{\mathcal{O}H}{\otimes} \mathcal{O}G$ is an $\mathcal{O}$-basis which is permuted under the action of K by conjugation. Thus the subalgebra $(\mathrm{Ind}_H^G(\mathcal{O}H))^K$ of K-stable elements has as $\mathcal{O}$-basis the set of relative traces $\mathrm{Tr}_{K_{(x\otimes y)}}^K(x \otimes y)$, where $x,\, y \in G$. If $x, x', y, y' \in G$ and $k \in K$ such that

$$kx \otimes yk^{-1} = x' \otimes y'$$

in $\mathrm{Ind}_H^G(\mathcal{O}H)$, there is a (necessarily unique) $h \in H$ such that $kx = x'h^{-1}$ and $yk^{-1} = hy'$, which in turn is equivalent to the equality

$$hx^{-1} \otimes y^{-1}h^{-1} = (x')^{-1} \otimes (y')^{-1}$$

in $\mathrm{Ind}_K^G(\mathcal{O}K)$. Thus the map $x \otimes y \mapsto x^{-1} \otimes y^{-1}$ induces a bijection between the sets of K-orbits and of H-orbits of the images of $G \times G$ in $\mathrm{Ind}_H^G(\mathcal{O}H)$ and $\mathrm{Ind}_K^G(\mathcal{O}K)$, respectively. In particular, with the notation above, we have $k \in K_{(x\otimes y)}$ if and only if $h \in H_{(x^{-1}\otimes y^{-1})}$, and the correspondence $k \mapsto h$ induces a group isomorphism $K_{(x\otimes y)} \cong H_{(x^{-1}\otimes y^{-1})}$. From this follows that the map given in Theorem 3 is an $\mathcal{O}$-linear isomorphism. It remains to verify that this is an algebra homomorphism. In $\mathrm{Ind}_H^G(\mathcal{O}H)$, multiplication is given by $(x \otimes y)(z \otimes t) = x \otimes yzt$, if $yz \in H$ and 0, otherwise, where $x, y, z, t \in G$. If $yz \in H$, then in $\mathrm{Ind}_K^G(\mathcal{O}K)$, the elements $(yz)z^{-1} \otimes t^{-1}(yz)^{-1}$ and $z^{-1} \otimes t^{-1}$ are in the same H-orbit, and the multiplication in $\mathrm{Ind}_K^G(\mathcal{O}K)$ yields $(x^{-1} \otimes y^{-1})((yz)z^{-1} \otimes t^{-1}(yz)^{-1}) = x^{-1} \otimes t^{-1}z^{-1}y^{-1}$, and this corresponds precisely to the bijection between the sets of K-orbits and H-orbits of the images of the set $G \times G$ in $\mathrm{Ind}_K^G(\mathcal{O}K)$ and $\mathrm{Ind}_H^G(\mathcal{O}H)$, respectively. □

Proof 2 of Theorem 3. We are going to apply Theorem 1 to the particular case where $A = \mathcal{O}H$, $B = \mathcal{O}K$, $M = N = \mathcal{O}G$ viewed as A-B-bimodule (through the inclusions $H \subseteq G$, $K \subseteq G$). This yields an $\mathcal{O}$-linear isomorphism

$$((\mathcal{O}G)^* \underset{\mathcal{O}H}{\otimes} \mathcal{O}G)^K \cong (\mathcal{O}G \underset{\mathcal{O}K}{\otimes} (\mathcal{O}G)^*)^H \ .$$

Composing this with the canonical isomorphism $(\mathcal{O}G)^* \cong \mathcal{O}G$ mapping $f \in (\mathcal{O}G)^*$ to $\sum_{x\in G} f(x^{-1})x$ yields the isomorphism in Theorem 3. □

Proof 3 of Theorem 3. Applying Theorem 2 and the above Lemma to $A = \mathcal{O}K$, $B = \mathcal{O}H$ and $M = \mathcal{O}G$ as A-B-bimodule yields an anti-isomorphism $(\mathrm{Ind}_H^G(\mathcal{O}H))^K \cong (\mathrm{Ind}_K^G(\mathcal{O}K))^H$. The map sending $x \otimes y$ to $y^{-1} \otimes x^{-1}$ is an anti-automorphism of $\mathrm{Ind}_H^G(\mathcal{O}H)$ which induces an anti-automorphism of $(\mathrm{Ind}_H^G(\mathcal{O}H))^K$. Composing both maps yields again the isomorphism in Theorem 3. □

Remark. The proof 3 of Theorem 3 is essentially the proof given in [5, §11].

References

1. M. Broué, *On representations of symmetric algebras: an introduction*, Notes by M. Stricker, Mathematik Department ETH Zürich (1991).
2. B. Külshammer, T. Okuyama, A. Watanabe, *A lifting theorem with applications to blocks and source algebras*, J. Algebra **232** (2000), 299–309.
3. M. Linckelmann, *Induction for interior algebras*, preprint (2001).
4. L. Puig, *Pointed groups and construction of characters*, Math. Z. **176** (1981), 265–292.
5. B. Stalder, *Une extension de l'algèbre de groupe en théorie des représentations modulaires*, Thèse, Université Lausanne (2000).

Markus Linckelmann
CNRS, Université Paris 7
UFR Mathématiques
2, place Jussieu
75251 Paris Cedex 05
FRANCE

Bernard Stalder
Université de Lausanne
Institut de Mathématiques
1015 Lausanne
SUISSE