# ON THE COXETER COMPLEX AND ALVIS-CURTIS DUALITY FOR PRINCIPAL $\ell$-BLOCKS OF $GL_n(q)$

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Abstract. M. Cabanes and J. Rickard showed in [3] that the Alvis-Curtis character duality of a finite group of Lie type is induced in non defining characteristic $\ell$ by a derived equivalence given by tensoring with a bounded complex $X$, and they further conjecture that this derived equivalence should actually be a homotopy equivalence. Following a suggestion of R. Kessar, we show here for the special case of principal blocks of general linear groups with abelian Sylow-$\ell$-subgroups that this is true, by an explicit verification relating the complex $X$ to the Coxeter complex of the corresponding Weyl group.

Throughout this note, $n$ is a positive integer, $q$ a prime power and $\ell$ a prime divisor of $q-1$ such that $\ell > n$. We denote by $\mathcal{O}$ a complete discrete valuation ring having an algebraically closed residue field $k$ of characteristic $\ell$.

Set $G = GL_n(q)$ and let $b$ be the principal block of $\mathcal{O}G$; that is, $b$ is the unique primitive idempotent in $Z(\mathcal{O}G)$ which acts as the identity on the trivial $\mathcal{O}G$-module. We will say as usual that an irreducible character $\chi$ of $G$ belongs to the principal block $b$ if $\chi(b) = \chi(1)$. See [11] for more block theoretic background material. The $\ell$-blocks of finite linear groups were first described by Fong and Srinivasan in [5]; see also [2]. For any $\ell'$-subgroup $H$ of $G$ let $e_H$ be the idempotent in $\mathcal{O}G$ defined by $e_H = \frac{1}{|H|}\sum_{x\in H} x$. Denote by $T$ the maximal torus of diagonal matrices in $G$, by $U$ the group of upper triangular matrices whose diagonal entries are 1 and set $B = UT$. Let $W$ be the subgroup of permutation matricies of $G$; that is $W \cong S_n$. Denote by $S$ the generating set of $W$ corresponding to the set of permutations $(i-1, i)$, where $2 \le i \le n$. For any subset $I$ of $S$ denote by $W_I$ the subgroup of $W$ generated by $I$, by $P_I$ the standard parabolic subgroup of $G$ generated by $B$ and $W_I$, by $U_I$ the unipotent radical of $P_I$ and by $L_I$ a Levi complement of $U_I$ in $P_I$, with the convention $W_\emptyset = 1$, $U_\emptyset = U$, $P_\emptyset = B$ and $L_\emptyset = T$.

Since $\ell > n$, the torus $T$ contains a Sylow-$\ell$-subgroup $Q$ of $G$, and then $T$ decomposes uniquely as direct product $T = Q \times T'$, where $T' = O_{\ell'}(T)$. The set of subsets of $S$ is viewed as simplicial complex with respect to the order which is reverse to the inclusion of subsets. The complex $X$ defined in [3] inducing the Alvis-Curtis duality is the complex of $\mathcal{O}G$-$\mathcal{O}G$-bimodules associated with the coefficient system sending $I \subseteq S$ to the $\mathcal{O}G$-$\mathcal{O}G$-bimodule $\mathcal{O}Ge_{U_I} \underset{\mathcal{O}P_I}{\otimes} e_{U_I}\mathcal{O}G$; the Coxeter complex (see e.g. [6, Section 1.15]) is the complex $C$ of $\mathcal{O}W$-modules associated with the coefficient system sending $I$ to the $\mathcal{O}W$-module $\mathcal{O}W/W_I$. See [1, Chapter 7] or [3] for more details on coefficient systems. We view $X$ and $C$ as cochain complexes with non zero components in the degrees 0 to $|S|$. The principal block $b$ of $\mathcal{O}G$ has the Sylow-$\ell$-subgroup $Q$ as defect group. By a result of Puig [8, 3.4], the idempotent $i = e_{T'}e_U$ is a source idempotent of $b$ (that is, $i$ is primitive in $(\mathcal{O}Gb)^Q$ and $2_Q(i) \neq 0$) and there is an isomorphism of interior $Q$-algebras

$$\Phi : i\mathcal{O}Gi \cong \mathcal{O}(Q \rtimes W) ;$$

that is, $\Phi$ is an $\mathcal{O}$-algebra isomorphism mapping $iui$ to $u$ for every element $u \in Q$. Denote by $\Delta(Q \rtimes W)$ the diagonal subgroup of $(Q \rtimes W) \times (Q \rtimes W)$ and consider $C$ as cochain complex of $\Delta(Q \rtimes W)$-modules in the obvious way (that is, with $\Delta Q$ acting trivially on the components of $C$).

**Theorem 1.** *With the notation above, there is an isomorphism of complexes of $i\mathcal{O}Gi$-$i\mathcal{O}Gi$-bimodules*

$$iXi \cong \mathrm{Res}_{\Phi}(\mathrm{Ind}_{\Delta(Q\rtimes W)}^{(Q\rtimes W)\times(Q\rtimes W)}(C)) .$$

*In particular, $iXi$ is homotopy equivalent to the bimodule ${}_{\sigma}(i\mathcal{O}Gi)$ viewed as complex concentrated in degree zero, where $\sigma$ is the algebra automorphism of $i\mathcal{O}Gi$ induced via $\Phi$ by the sign representation of $W$.*

Since $\mathcal{O}Gb$ and $i\mathcal{O}Gi$ are Morita equivalent via the bimodules $\mathcal{O}Gi$ and $i\mathcal{O}G$ (cf. [7, 3.5]) this implies immediately the following:

**Corollary 2.** *With the notation above, the functor $X \underset{\mathcal{O}G}{\otimes} -$ induces an equivalence on the homotopy category $K(\mathcal{O}Gb)$ of complexes of $\mathcal{O}Gb$-modules which extends, up to natural isomorphism, the Morita equivalence on* $\mathrm{Mod}(\mathcal{O}Gb)$ *induced by the sign representation of $W$.*

The proof of Theorem 1 is based on the following three Propositions, the first of which is a particular case of a result of Puig [8, 3.4]:

**Proposition 3.** *(Puig) With the notation above, for any subset $I$ of $S$, the idempotent $i$ is a source idempotent of the principal block of $\mathcal{O}P_I$ and there is an isomorphism of interior $Q$-algebras $\Phi_I : i\mathcal{O}P_Ii \cong \mathcal{O}(Q \rtimes W_I)$.*

Since $W = W_S$ we choose notation such that $\Phi = \Phi_S$. The various isomorphisms $\Phi_I$ are compatible in the following sense:

**Proposition 4.** *With the notation above, for any subset $I$ of $S$ there is an element $a_I \in 1 + J((\mathcal{O}(Q \rtimes W))^Q)$ such that $\Phi(i\mathcal{O}P_I i) = (\mathcal{O}(Q \rtimes W_I))^{a_I}$.*

*Proof.* For every $w \in W_I$, the elements $w$ and $w' = \Phi(\Phi_I^{-1}(w))$ act in the same way on $Q$ because $\Phi$, $\Phi_I$ are homomorphisms of interior $Q$-algebras. Thus $w(w')^{-1} \in ((\mathcal{O}(Q \rtimes W))^Q)^\times$. Since the unit element of $\mathcal{O}(Q \rtimes W)$ is primitive in the algebra $(\mathcal{O}(Q \rtimes W))^Q$ it follows that the group $((\mathcal{O}(Q \rtimes W))^Q)^\times$ is isomorphic to $k^\times \times (1 + J((\mathcal{O}(Q \rtimes W))^Q))$. In other words, identifying $k^\times$ to its canonical preimage in $\mathcal{O}^\times$ (cf. [10, Ch. II, §4, Prop. 8]) there is a unique scalar $\zeta(w) \in k^\times$ such that $\zeta(w)^{-1}w(w')^{-1} \in 1 + J((\mathcal{O}(Q \rtimes W))^Q)$. One checks easily that the map sending $w$ to $\zeta(w)$ is in fact a group homomorphism. Setting $a_I = \frac{1}{|W_I|} \sum_{w \in W_I} \zeta(w)^{-1}w(w')^{-1}$ it is clear that $a_I \in 1 + J((\mathcal{O}(Q \rtimes W))^Q)$ and an easy computation shows that $wa_I = \zeta(w)a_I w'$ for any $w \in W_I$, implying the result. □

**Proposition 5.** *Let $I$ be a subset of $S$ and denote by $b_I$ the principal block of $\mathcal{O}P_I$. We have $e_{U_I}b = b_I$.*

*Proof.* Let $c_I$ be the principal block of $\mathcal{O}L_I$. Since $\ell$ divides $q - 1$ it follows from [5] that the principal blocks $b$ and $c_I$ of $\mathcal{O}G$ and $\mathcal{O}L_I$ are the unique unipotent blocks of $\mathcal{O}G$ and $\mathcal{O}L_I$, respectively. Let $\chi$ be an irreducible character of $G$ and let $\psi$ be an irreducible character of $L_I$ such that $\chi$ is a constituent of the Harish-Chandra induced character $R_{L_I}^G(\psi)$. Since Harish-Chandra induction is given by tensoring with the $\mathcal{O}G$-$\mathcal{O}L_I$-bimodule $\mathcal{O}Ge_{U_I}$, this is equivalent to $e_{U_I}e(\chi)e(\psi) \neq 0$ in $KG$, where $K$ is the quotient field of $\mathcal{O}$ and where $e(\chi), e(\psi)$ are the primitive idempotents in $Z(KG), Z(KL_I)$ associated with $\chi$, $\psi$, respectively. Since Harish-Chandra induction preserves Lusztig series, $\chi$ belongs to the principal block $b$ of $\mathcal{O}G$ if and only if $\psi$ belongs to the principal block $c_I$ of $\mathcal{O}L_I$. This implies $e_{U_I}bc = 0$ for any non principal block $c$ of $\mathcal{O}L_I$, and hence the equality $e_{U_I}b = e_{U_I}bc_I$. It implies also that $e_{U_I}b'c_I = 0$ for any non principal block $b'$ of $\mathcal{O}G$, and hence the equality $e_{U_I}bc_I = e_{U_I}c_I$. Clearly $\mathcal{O}P_I e_{U_I} \cong \mathcal{O}L_I$, and since $U_I \subseteq O_{\ell'}(P_I)$ is in the kernel of the principal block $b_I$ of $\mathcal{O}P_I$, we get the equality $b_I = e_{U_I}c_I$. The result follows. □

*Proof of Theorem 1.* Set $Y = \mathrm{Ind}_{\Delta(Q \rtimes W)}^{(Q \rtimes W) \times (Q \rtimes W)}(C)$, viewed as cochain complex of $\mathcal{O}(Q \rtimes W)$-$\mathcal{O}(Q \rtimes W)$-bimodules. For any integer $r$, the degree $r$ term of $Y$ is isomorphic to the direct sum of the bimodules

$$\mathcal{O}(Q \rtimes W) \underset{\mathcal{O}(Q \rtimes W_I)}{\otimes} \mathcal{O}(Q \rtimes W)$$

with $I$ running over the set of subsets of $S$ such that $|I| = r$. The differential of $Y$ is an alternating sum of the canonical maps

$$a_{IJ} : \mathcal{O}(Q \rtimes W) \underset{\mathcal{O}(Q \rtimes W_I)}{\otimes} \mathcal{O}(Q \rtimes W) \longrightarrow \mathcal{O}(Q \rtimes W) \underset{\mathcal{O}(Q \rtimes W_J)}{\otimes} \mathcal{O}(Q \rtimes W)$$

for any $I \subseteq J \subseteq S$ such that $|I| + 1 = |J|$.

Let $I$ be a subset of $S$. By Proposition 5, the $i\mathcal{O}Gi$-$i\mathcal{O}Gi$-bimodule $i\mathcal{O}Ge_{U_I} \underset{\mathcal{O}P_I}{\otimes} e_{U_I}\mathcal{O}Gi$ is isomorphic to $i\mathcal{O}Ge_{U_I} \underset{\mathcal{O}P_I b_I}{\otimes} e_{U_I}\mathcal{O}Gi$. Since $i$ is still a source idempotent of $b_I$ it follows from [7, 3.5] that this bimodule is isomorphic to $i\mathcal{O}Gi \underset{i\mathcal{O}P_I i}{\otimes} i\mathcal{O}Gi$.

Set $Y' = \mathrm{Res}_{\Phi^{-1}}(iXi)$. It follows from combining the Propositions 3 and 4 with the previous paragraph that the degree $r$ term of $Y'$ is isomorphic to the direct sum of bimodules

$$\mathcal{O}(Q \rtimes W) \underset{\mathcal{O}(Q\rtimes W_I)^{a_I}}{\otimes} \mathcal{O}(Q \rtimes W)$$

with $I$ running again over the set of subsets of $S$ such that $|I| = r$. Furthermore, if $I \subseteq J \subseteq S$ then Proposition 4 implies that $\mathcal{O}(Q \rtimes W_I)^{a_I} \subseteq \mathcal{O}(Q \rtimes W_J)^{a_J}$, and so the differential of $Y'$ is again just an alternating sum of the canonical maps

$$a'_{IJ} : \mathcal{O}(Q \rtimes W) \underset{\mathcal{O}(Q\rtimes W_I)^{a_I}}{\otimes} \mathcal{O}(Q \rtimes W) \longrightarrow \mathcal{O}(Q \rtimes W) \underset{\mathcal{O}(Q\rtimes W_J)^{a_J}}{\otimes} \mathcal{O}(Q \rtimes W)$$

for any $I \subseteq J \subseteq S$ such that $|I| + 1 = |J|$.

In order to prove the first isomorphism in Theorem 1 we have to prove that $Y \cong Y'$ as complexes of $\mathcal{O}(Q \rtimes W)$-$\mathcal{O}(Q \rtimes W)$-bimodules. There is an isomorphism

$$\mathcal{O}(Q \rtimes W) \underset{\mathcal{O}(Q\rtimes W_I)}{\otimes} \mathcal{O}(Q \rtimes W) \cong \mathcal{O}(Q \rtimes W) \underset{\mathcal{O}(Q\rtimes W_I)^{a_I}}{\otimes} \mathcal{O}(Q \rtimes W)$$

mapping $x \otimes y$ to $xa_I \otimes a_I^{-1}y$ for any $I \subseteq S$ and any $x, y \in \mathcal{O}(Q \rtimes W)$. Thus the terms of $Y$ and $Y'$ are isomorphic. However, these isomorphisms need not commute to the differentials. In order to show that $Y$ and $Y'$ are actually isomorphic as complexes it suffices to show that they are both split and have cohomology concentrated in degree zero.

Since $\ell$ does not divide the order of $W$ the complex $C$ is split and its cohomology is concentrated in degree zero isomorphic to the sign representation of $W$ (cf. [4, 66.28] or [9, §8]). As induction is exact it follows that $Y$ is split with cohomology concentrated in degree zero isomorphic to ${}_\tau\mathcal{O}(Q \rtimes W)$, where $\tau$ is the automorphism of $\mathcal{O}(Q \rtimes W)$ mapping $uw$ to $sgn(w)uw$ for any $u \in Q$ and any $w \in W$. It remains to show that $Y'$ is split with cohomology concentrated in degree zero. To see that $Y'$ is split we explicitly define a section $s_{IJ}$ for the above map $a'_{IJ}$ by setting

$$s_{IJ}(x \otimes y) = \frac{1}{[W_J : W_I]} \sum_v xva_J^{-1}a_I \otimes a_I^{-1}a_J v^{-1} y$$

for any $I \subseteq J \subseteq S$ such that $|I| + 1 = |J|$, any $x, y \in \mathcal{O}(Q \rtimes W)$, and where $v$ runs over a system of representatives in $W_J$ of the set of cosets $W_J/W_I$. A straightforward verification shows that $s_{IJ}$ is well-defined and that $a'_{IJ} \circ s_{IJ}$ is the identity map.

Knowing that $Y'$ is split, in order to see that the cohomology is concentrated in degree zero it suffices to show that the cohomology of the quotient complex $Y'/J(\mathcal{O}(Q \rtimes W))Y'$ is concentrated in degree zero. The point is here that the elements $a_I$ are in $1 + J((\mathcal{O}(Q \rtimes W))^Q)$ which in turn is contained in $1 + J(\mathcal{O}(Q \rtimes W))$. Thus the quotients $Y/J(\mathcal{O}(Q \rtimes W))Y$ and $Y'/J(\mathcal{O}(Q \rtimes W))Y'$ are actually isomorphic as complexes; in particular, their cohomology is both concentrated in degree zero. Theorem 1 follows. □

**Remark 6.** We expect that it should be possible to extend Theorem 1 to the blocks considered by Puig in [8, 3.4], using similar techniques.

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