BLOCKS WITH A QUATERNION DEFECT GROUP OVER A 2-ADIC RING: THE CASE $\tilde{A}_4$

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Abstract. Except for blocks with a cyclic or Klein four defect group, it is not known in general whether the Morita equivalence class of a block algebra over a field of prime characteristic determines that of the corresponding block algebra over a p-adic ring. We prove this to be the case when the defect group is quaternion of order 8 and the block algebra over an algebraically closed field k of characteristic 2 is Morita equivalent to $k\tilde{A}_4$. The main ingredients are Erdmann's classification of tame blocks [6] and work of Cabanes and Picaronny [4, 5] on perfect isometries between tame blocks.

Introduction

Throughout these notes, $\mathcal{O}$ is a complete discrete valuation ring with algebraically closed residue field k of characteristic 2 and with quotient field K of characteristic 0. According to Erdmann's classification in [6], if G is a finite group and if b is a block of $\mathcal{O}G$ having the quaternion group Q_8 of order 8 as defect group, then the block algebra $kG\bar{b}$ is Morita equivalent to either kQ_8 or $k\tilde{A}_4$ or the principal block algebra of $k\tilde{A}_5$, where here $\bar{b}$ is the canonical image of b in kG. In the first case the block is it nilpotent (cf. [3]), and it follows from Puig's structure theorem of nilpotent blocks in [8] that $\mathcal{O}Gb$ is Morita equivalent to $\mathcal{O}Q_8$. In the remaining two cases one should expect that $\mathcal{O}Gb$ is Morita equivalent to $\mathcal{O}\tilde{A}_4$ or the principal block algebra of $\mathcal{O}\tilde{A}_5$, respectively. We show this to be true in one of these two cases under the assumption that K is large enough:

Theorem A. *Let G be a finite group, and let b be a block of $\mathcal{O}G$ having a quaternion defect group of order 8. Denote by $\bar{b}$ the image of b in kG. Assume that KGb is split. If $kG\bar{b}$ is Morita equivalent to $k\tilde{A}_4$ then $\mathcal{O}Gb$ is Morita equivalent to $\mathcal{O}\tilde{A}_4$.*

By Cabanes-Picaronny [4, 5], in the situation of Theorem A there is a perfect isometry between the character groups of $\mathcal{O}Gb$ and of $\mathcal{O}\tilde{A}_4$. Thus Theorem A is a consequence of the following slightly more general Theorem which characterises $\mathcal{O}Gb$ in terms of its center, its character group and $k\tilde{A}_4$; see the end of this section for more details regarding the notation.

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Theorem B. *Let A be an $\mathcal{O}$-free $\mathcal{O}$-algebra such that $K \underset{\mathcal{O}}{\otimes} A$ is split semi-simple and such that $k \underset{\mathcal{O}}{\otimes} A$ is Morita equivalent to $k\tilde{A}_4$. Assume that there is an isometry $\Phi : \mathbb{Z}\mathrm{Irr}_K(A) \cong \mathbb{Z}\mathrm{Irr}_K(\mathcal{O}\tilde{A}_4)$ which maps $\mathrm{Proj}(A)$ to $\mathrm{Proj}(\mathcal{O}\tilde{A}_4)$ such that the map sending $e(\chi)$ to $e(\Phi(\chi))$ for every $\chi \in \mathrm{Irr}_K(A)$ induces an $\mathcal{O}$-algebra isomorphism of the centers $Z(A) \cong Z(\mathcal{O}\tilde{A}_4)$. Then A is Morita equivalent to $\mathcal{O}\tilde{A}_4$.*

Theorem B is in turn a consequence of the more precise Theorem C, describing A in terms of generators and relations:

Theorem C. *Let A be a basic $\mathcal{O}$-free $\mathcal{O}$-algebra such that $K \underset{\mathcal{O}}{\otimes} A$ is split semi-simple and such that $k \underset{\mathcal{O}}{\otimes} A$ is isomorphic to $k\tilde{A}_4$. Assume that there is an isometry $\Phi : \mathbb{Z}\mathrm{Irr}_K(A) \cong \mathbb{Z}\mathrm{Irr}_K(\mathcal{O}\tilde{A}_4)$ which maps $\mathrm{Proj}(A)$ to $\mathrm{Proj}(\mathcal{O}\tilde{A}_4)$ such that the map sending $e(\chi)$ to $e(\Phi(\chi))$ for every $\chi \in \mathrm{Irr}_K(A)$ induces an $\mathcal{O}$-algebra isomorphism of the centers $Z(A) \cong Z(\mathcal{O}\tilde{A}_4)$. Then A is isomorphic to the unitary $\mathcal{O}$-algebra with set of generators $\{e_0, e_1, e_2, \beta, \gamma, \delta, \eta, \lambda, \kappa\}$ of A, such that e_0, e_1, e_2 are pairwise orthogonal idempotents whose sum is 1 and satisfying the following relations:*

$$\beta = e_0\beta = \beta e_1,\ \gamma = e_1\gamma = \gamma e_0\ ;$$

$$\delta = e_1\delta = \delta e_2,\ \eta = e_2\eta = \eta e_1\ ;$$

$$\lambda = e_2\lambda = \lambda e_0,\ \kappa = e_0\kappa = \kappa e_2\ ;$$

$$\beta\delta = -2\kappa + \kappa\lambda\kappa\ ;\quad \eta\gamma = -2\lambda + \lambda\kappa\lambda\ ;\quad \delta\lambda = -2\gamma + \gamma\beta\gamma\ ;$$

$$\kappa\eta = -2\beta + \beta\gamma\beta\ ;\quad \lambda\beta = -2\eta + \eta\delta\eta\ ;\quad \gamma\kappa = -2\delta + \delta\eta\delta\ ;$$

$$\gamma\beta\delta = -4\delta + 2\delta\eta\delta\ ;\quad \delta\eta\gamma = -4\gamma + 2\gamma\beta\gamma\ ;\quad \lambda\kappa\eta = -4\eta + 2\eta\delta\eta\ ;$$

$$\beta\gamma\kappa = -4\kappa + 2\kappa\lambda\kappa\ ;\quad \eta\delta\lambda = -4\lambda + 2\lambda\kappa\lambda\ ;\quad \kappa\lambda\beta = -4\beta + 2\beta\gamma\beta\ ;$$

$$\eta\gamma\beta = -4\eta + 2\eta\delta\eta\ ;\quad \beta\delta\eta = -4\beta + 2\beta\gamma\beta\ ;\quad \delta\lambda\kappa = -4\delta + 2\delta\eta\delta\ ;$$

$$\lambda\beta\gamma = -4\lambda + 2\lambda\kappa\lambda\ ;\quad \kappa\eta\delta = -4\kappa + 2\kappa\lambda\kappa\ ;\quad \gamma\kappa\lambda = -4\gamma + 2\gamma\beta\gamma\ ;$$

$$\beta\delta\lambda\beta = -8\beta + 4\beta\gamma\beta\ ;\quad \delta\lambda\beta\delta = -8\delta + 4\delta\eta\delta\ ;\quad \lambda\beta\delta\lambda = -8\lambda + 4\lambda\kappa\lambda\ ;$$

When reduced modulo 2, these relations seem to be more than those occuring in Erdmann's work [6] over k (we recall these more precisely in §2); but they are not, since the extra relations over k can be deduced from those given by Erdmann. We need to add in extra relations over $\mathcal{O}$ in order to ensure that the algebra we construct is $\mathcal{O}$-free of the right rank.

Since $\mathcal{O}\tilde{A}_4$ fulfills the hypotheses of Theorem C it follows that $A \cong \mathcal{O}\tilde{A}_4$, hence Theorem C indeed implies Theorem B. The proof of Theorem C is given at the end of Section 2.

Notation. If A is an $\mathcal{O}$-algebra such that $K \underset{\mathcal{O}}{\otimes} A$ is split semi-simple, we denote by $\mathrm{Irr}_K(A)$ the set of characters of the simple $K \underset{\mathcal{O}}{\otimes} A$-modules, viewed as central functions from A to $\mathcal{O}$ and we denote by $\mathrm{Irr}_k(k \underset{\mathcal{O}}{\otimes} A)$ the set of isomorphism classes of simple $k \underset{\mathcal{O}}{\otimes} A$-modules. We denote by $\mathbb{Z}\mathrm{Irr}_K(A)$ the group of characters of A, and we denote by $\mathrm{Proj}(A)$ the subgroup of $\mathbb{Z}\mathrm{Irr}_K(A)$ generated by the characters of the projective indecomposable A-modules. We denote by $L^0(A)$ the subgroup of $\mathbb{Z}\mathrm{Irr}_K(A)$ of all elements which are orthogonal to $\mathrm{Proj}(A)$ with respect to the usual scalar product in $\mathbb{Z}\mathrm{Irr}_K(A)$. For any $\chi \in \mathrm{Irr}_K(A)$, we denote by $e(\chi)$ the corresponding primitive idempotent in $Z(K \underset{\mathcal{O}}{\otimes} A)$. If $A = \mathcal{O}G$ for some finite group G we have the well-known formula

$$e(\chi) = \frac{\chi(1)}{|G|} \sum_{x \in G} \chi(x^{-1})x \ .$$

We refer to [1, 2] for the concept and basic properties of perfect isometries, and to [9] for general block theoretic background material.

1 Characters and perfect isometries of $\mathcal{O}\tilde{A}_4$

We identify $\tilde{A}_4 = Q_8 \rtimes C_3$. Let t be a generator of C_3 and let y be an element of order 4 in Q_8. Set $z = y^2$; that is, z is the unique central involution of $\tilde{A}_4$. Then the seven elements 1, z, y, t, t^2, tz, t^2z are a complete set of representatives of the conjugacy classes in $\tilde{A}_4$.

Let ω be a primitive third root of unity in $\mathcal{O}$. The character table of $\tilde{A}_4$ is as follows:

	1	z	y	t	t^2	tz	t^2z
η_0	1	1	1	1	1	1	1
η_1	1	1	1	ω	ω^2	ω	ω^2
η_2	1	1	1	ω^2	ω	ω^2	ω
η_3	3	3	-1	0	0	0	0
η_4	2	-2	0	$-\omega^2$	$-\omega$	ω^2	ω
η_5	2	-2	0	$-\omega$	$-\omega^2$	ω	ω^2
η_6	2	-2	0	-1	-1	1	1

The algebra $\mathcal{O}\tilde{A}_4$ has three simple modules T_0, T_1, T_2, up to isomorphism. Choosing for T_0 the trivial module and after possibly exchanging the notation for T_1, T_2, the ordinary decomposition matrix of $\mathcal{O}\tilde{A}_4$ is as follows:

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}$$

The Cartan matrix of $\mathcal{O}\tilde{A}_4$ is the product of the decomposition matrix with its transpose, hence equal to

$$\begin{pmatrix} 4 & 2 & 2 \\ 2 & 4 & 2 \\ 2 & 2 & 4 \end{pmatrix}$$

Let e_0, e_1, e_2 be primitive idempotents in $\mathcal{O}\tilde{A}_4$ such that $\mathcal{O}\tilde{A}_4 e_i$ is a projective cover of T_i, $0 \leq i \leq 2$. By the above decomposition matrix, the characters of the projective indecomposable $\mathcal{O}\tilde{A}_4$-modules $\mathcal{O}\tilde{A}_4 e_i$ are

$$\eta_0 + \eta_3 + \eta_4 + \eta_5 \ ,$$
$$\eta_1 + \eta_3 + \eta_4 + \eta_6 \ ,$$
$$\eta_2 + \eta_3 + \eta_5 + \eta_6 \ ,$$

respectively. Their norm is 4, and the differences of any two different characters of projective indecomposable $\mathcal{O}\tilde{A}_4$-modules yields the following further elements in $\mathrm{Proj}(\mathcal{O}\tilde{A}_4)$ having also norm 4:

$$\eta_0 - \eta_1 + \eta_5 - \eta_6 \ ,$$
$$\eta_0 - \eta_2 + \eta_4 - \eta_6 \ ,$$
$$\eta_1 - \eta_2 + \eta_4 - \eta_5 \ .$$

It is easy to check, that up to signs, these are all elements in $\mathrm{Proj}(\mathcal{O}\tilde{A}_4)$ having norm 4.

A self-isometry Φ of $\mathbb{Z}\mathrm{Irr}_K(\mathcal{O}\tilde{A}_4)$ maps every η_i to $\epsilon_i \eta_{\pi(i)}$ for some signs $\epsilon_i \in \{1, -1\}$ and a permutation π of $\{0, 1, \ldots, 6\}$. In other words, Φ is determined by the permutation τ of the set $\{1, -1\} \times \{0, 1, \ldots, 6\}$ satisfying $\tau(1, i) = (\epsilon_i, \pi(i))$ and $\tau(-1, i) = (-\epsilon_i, \pi(i))$ for all i, $0 \leq i \leq 6$. If we write i, $-i$ instead of $(1, i)$, $(-1, i)$, respectively, this becomes $\tau(i) = \epsilon_i \pi(i)$ and $\tau(-i) = -\epsilon_i \pi(i)$, with the usual cancellation rules for signs. In this way, every self-isometry Φ of $\mathbb{Z}\mathrm{Irr}_K(\mathcal{O}\tilde{A}_4)$ gets identified to a permutation of the set of symbols $\{i, -i | 0 \leq i \leq 6\}$.

A perfect self-isometry of $\mathbb{Z}\mathrm{Irr}_K(\mathcal{O}\tilde{A}_4)$ preserves necessarily $\mathrm{Proj}(\mathcal{O}\tilde{A}_4)$. The next Proposition implies that the converse is true, too:

Proposition 1.1. *The group of all perfect self-isometries of* $\mathbb{Z}\mathrm{Irr}_K(\mathcal{O}\tilde{A}_4)$ *is equal to the group of all self-isometries of* $\mathbb{Z}\mathrm{Irr}_K(\mathcal{O}\tilde{A}_4)$ *which preserve* $\mathrm{Proj}(\mathcal{O}\tilde{A}_4)$*. This group is generated by* $-\mathrm{Id}$ *together with the set of permutations*

$$(0,1,2)(4,6,5)\ ,$$

$$(1,2)(4,5)\ ,$$

$$(2,-3)(5,-6)\ .$$

Every algebra automorphism of $\mathcal{O}\tilde{A}_4$ induces a permutation on $\mathrm{Irr}_K(\mathcal{O}\tilde{A}_4)$ which is in fact a perfect isometry on $\mathbb{Z}\mathrm{Irr}_K(\mathcal{O}\tilde{A}_4)$. Since η_1 has degree 1, it is an algebra homomorphism from $\mathcal{O}\tilde{A}_4$ to $\mathcal{O}$, and hence the map sending $x \in \mathcal{O}\tilde{A}_4$ to $\eta_1(x)x$ is an algebra automorphism of $\mathcal{O}\tilde{A}_4$ whose inverse sends $x \in \mathcal{O}\tilde{A}_4$ to $\eta_2(x)x$. The following statement is an immediate consequence from the character table of $\mathcal{O}\tilde{A}_4$:

Lemma 1.2. *Let* γ *be the algebra automorphism of* $\mathcal{O}\tilde{A}_4$ *defined by* $\gamma(x) = \eta_1(x)x$ *for all* $x \in \mathcal{O}\tilde{A}_4$*. The permutation* π *of* $\{0, 1, \ldots, 6\}$ *defined by* $\eta_i \circ \gamma = \eta_{\pi(i)}$ *is equal to* $\pi = (0,1,2)(4,6,5)$*.*

The anti-automorphism of $\mathcal{O}\tilde{A}_4$ sending $x \in \tilde{A}_4$ to x^{-1} induces also a permutation of the set $\mathrm{Irr}_K(\mathcal{O}\tilde{A}_4)$, and this is also a perfect isometry (this holds for any finite group). This permutation can also be read off the character table:

Lemma 1.3. *Let* ι *be the algebra anti-automorphism of* $\mathcal{O}\tilde{A}_4$ *mapping* $x \in \tilde{A}_4$ *to* x^{-1}*. The permutation* π *of* $\{0, 1, \ldots, 6\}$ *defined by* $\eta_i \circ \iota = \eta_{\pi(i)}$ *is equal to* $\pi = (1,2)(4,5)$*.*

Proof of 1.1. The first two permutations are perfect isometries by 2.2 and 2.3, respectively. An easy but painfully long verification shows that the bicharacter sending $(g,h) \in \tilde{A}_4 \times \tilde{A}_4$ to

$$\eta_0(g)\eta_0(h)+\eta_1(g)\eta_1(h)-\eta_2(g)\eta_3(h)-\eta_3(g)\eta_2(h)+\eta_4(g)\eta_4(h)-\eta_5(g)\eta_6(h)-\eta_6(g)\eta_5(h)$$

is perfect; that is, its value at any (g,h) is divisible in $\mathcal{O}$ by the orders of $C_{\tilde{A}_4}(g)$ and $C_{\tilde{A}_4}(h)$ and it vanishes if exactly one of g, h has odd order. Thus the isometry given by the permutation $(2,-3)(5,-6)$ is perfect. It remains to show that these permutations, together with $-\mathrm{Id}$, generate the group of all self-isometries which preserve $\mathrm{Proj}(\mathcal{O}\tilde{A}_4)$.

We described above a complete list of all elements in $\mathrm{Proj}(\mathcal{O}\tilde{A}_4)$ having norm 4. Since the characters of the projective indecomposable modules are in that list, a self-isometry Φ of $\mathbb{Z}\mathrm{Irr}_K(\mathcal{O}\tilde{A}_4)$ preserves $\mathrm{Proj}(\mathcal{O}\tilde{A}_4)$ if and only if it permutes this set of norm 4 elements.

Let Φ be a self-isometry of $\mathbb{Z}\mathrm{Irr}_K(\mathcal{O}\tilde{A}_4)$ which preserves $\mathrm{Proj}(\mathcal{O}\tilde{A}_4)$. Then Φ preserves also the group $L^0(\mathcal{O}\tilde{A}_4)$ of generalised characters which are orthogonal to all characters in $\mathrm{Proj}(\mathcal{O}\tilde{A}_4)$. Up to signs, the complete list of elements in $L^0(\mathcal{O}\tilde{A}_4)$ having norm 3 is

$$\eta_0+\eta_1-\eta_4\ ,\ \eta_0+\eta_2-\eta_5\ ,\ \eta_0-\eta_3+\eta_6\ ,$$

$$\eta_1+\eta_2-\eta_6\ ,\ \eta_1-\eta_3+\eta_5\ ,\ \eta_2-\eta_3+\eta_4\ .$$

Up to signs again, the complete list of elements in $L^0(\mathcal{O}\tilde{A}_4)$ having norm 4 is

$$\eta_0+\eta_1+\eta_2-\eta_3\ ,$$

$$\eta_0-\eta_1-\eta_5+\eta_6\ ,\ \eta_0-\eta_2-\eta_4+\eta_6\ ,\ \eta_0+\eta_3-\eta_4-\eta_5\ ,$$

$$\eta_1-\eta_2-\eta_4+\eta_5\ ,\ \eta_1+\eta_3-\eta_4-\eta_6\ ,\ \eta_2+\eta_3-\eta_5-\eta_6\ .$$

The first norm 4 element in this list, $\eta_0+\eta_1+\eta_2-\eta_3$, is the only norm 4 element which is orthogonal to all other norm 4 elements in $L^0(\mathcal{O}\tilde{A}_4)$. Thus Φ has to permute the characters η_0, η_1, η_2, η_3 amongst each other.

Suppose first that Φ fixes η_3. Then, by composing Φ with a suitable product of powers of the first two permutations in the statement, we may assume that Φ fixes η_0, η_1, η_2 up to signs. By considering the first of the above norm 4 elements in $L^0(\mathcal{O}\tilde{A}_4)$ we get that Φ fixes η_0, η_1, η_2 all with positive signs. By considering the norm 3 elements in $L^0(\mathcal{O}\tilde{A}_4)$, it follows that Φ fixes also η_4, η_5 and η_6 with positive signs. Thus a self-isometry of $\mathbb{Z}\mathrm{Irr}_K(\mathcal{O}\tilde{A}_4)$ which preserves $\mathrm{Proj}(\mathcal{O}\tilde{A}_4)$ and which fixes η_3 is in the group generated by the set of two permutations $(0,1,2)(4,6,5)$ and $(1,2)(4,5)$.

Suppose next that Φ does not fix η_3. By precomposing Φ with a suitable of power of $(0,1,2)(4,6,5)$ we may assume that Φ sends η_2 to $-\eta_3$. By composing Φ with a suitable power of $(0,1,2)(4,5,6)$ we may assume that Φ fixes η_0, up to a sign. Since Φ preserves the norm 4 element $\eta_0+\eta_1+\eta_2-\eta_3$, we necessarily have $\Phi(\eta_0)=\eta_0$. Then Φ maps η_1 either to η_1 or η_2 (with positive signs, again because of that same norm 4 element). In the first case, Φ fixes both η_0, η_1, and by checking the norm 3 elements in $L^0(\mathcal{O}\tilde{A}_4)$ one gets $\Phi=(2,-3)(5,-6)$. In the second case, again checking on norm 3 elements, one gets $\Phi=(1,2,-3)(4,5,-6)$, but this is already the product of $(1,2)(4,5)$ and $(2,-3)(5,-6)$. □

2 The algebra A

Let A be a basic $\mathcal{O}$-algebra fulfilling the hypotheses of Theorem B; that is, $K\underset{\mathcal{O}}{\otimes}A$ is split semi-simple, $k\underset{\mathcal{O}}{\otimes}A$ is isomorphic to $k\tilde{A}_4$, and there is an isometry $\mathbb{Z}\mathrm{Irr}_K(A)\cong\mathbb{Z}\mathrm{Irr}_K(\mathcal{O}\tilde{A}_4)$ mapping $\mathrm{Proj}(A)$ to $\mathrm{Proj}(\mathcal{O}\tilde{A}_4)$ and inducing an isomorphism $Z(A)\cong Z(\mathcal{O}\tilde{A}_4)$. There is a "compatible choice" for these isomorphisms:

Proposition 2.1. *There is an algebra isomorphism $\alpha : k \underset{\mathcal{O}}{\otimes} A \cong k\tilde{A}_4$ and an isometry $\Phi : \mathbb{Z}\mathrm{Irr}_K(A) \cong \mathbb{Z}\mathrm{Irr}_K(\mathcal{O}\tilde{A}_4)$ mapping* $\mathrm{Proj}(A)$ *to* $\mathrm{Proj}(\mathcal{O}\tilde{A}_4)$ *with the following properties:*

(i) Φ maps $\mathrm{Irr}_K(A)$ onto $\mathrm{Irr}_K(\mathcal{O}\tilde{A}_4)$; that is, all signs are $+1$.

(ii) The map sending $e(\chi)$ to $e(\Phi(\chi))$ for every $\chi \in \mathrm{Irr}_K(A)$ induces an isomorphism $Z(A) \cong Z(\mathcal{O}\tilde{A}_4)$.

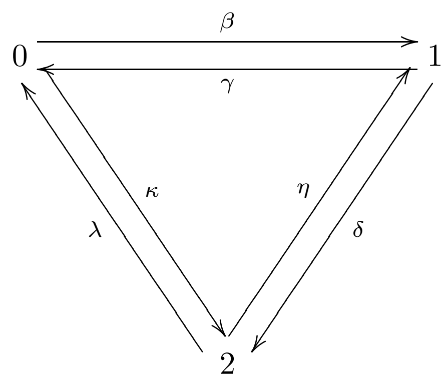
(iii) For any primitive idempotents $e \in A$ and $f \in \mathcal{O}\tilde{A}_4$ and every $\chi \in \mathrm{Irr}_K(A)$ such that $\alpha(\bar{e}) = \bar{f}$ we have $\chi(e) = \Phi(\chi)(f)$; that is, A and $\mathcal{O}\tilde{A}_4$ have the same decomposition matrices through α and Φ.

Proof. The $\mathcal{O}$-rank of A is 24 and also the sum of the squares of the seven irreducible K-linear characters of A; thus every irreducible character of A has degree smaller than 5. Also, there is no character of degree 4 because $24 - 4^2 = 8$ cannot be written as a sum of six squares of the six remaining characters. But there must be a character of degree 3; if not, 24 would be the sum of seven squares all either 1 or 4, which is not possible. Thus the squares of the six remaining characters add up to $24 - 3^2 = 15$, and the only way to do this is with three characters of degree 1 and three characters of degree 2.

This proves that the character degrees of the irreducible characters of A and of $\mathcal{O}\tilde{A}_4$ coincide for some bijection $\mathrm{Irr}_K(A) \cong \mathrm{Irr}_K(\mathcal{O}\tilde{A}_4)$. Since the decomposition matrix of A multiplied with its transpose yields the Cartan matrix of A - which is equal to that of $k\tilde{A}_4$ - the algebra A has in fact the same decomposition matrix as $\mathcal{O}\tilde{A}_4$ for a suitable bijection $\Phi : \mathrm{Irr}_K(A) \cong \mathrm{Irr}_K(\mathcal{O}\tilde{A}_4)$ and the bijection $\mathrm{Irr}_k(k \underset{\mathcal{O}}{\otimes} A) \cong \mathrm{Irr}_k(k\tilde{A}_4)$ induced by α. Extend Φ to a $\mathbb{Z}$-linear isomorphism $\mathbb{Z}\mathrm{Irr}_K(A) \cong \mathbb{Z}\mathrm{Irr}_K(\mathcal{O}\tilde{A}_4)$, still denoted by Φ. By construction, Φ sends the characters of the projective indecomposable A-modules to the characters of the projective indecomposable $\mathcal{O}\tilde{A}_4$-modules; in particular, Φ maps $\mathrm{Proj}(A)$ to $\mathrm{Proj}(\mathcal{O}\tilde{A}_4)$. It remains to see that the map sending $e(\chi)$ to $e(\Phi(\chi))$ for every $\chi \in \mathrm{Irr}_K(A)$ induces an isomorphism $Z(A) \cong Z(\mathcal{O}\tilde{A}_4)$. For any i, $0 \le i \le 6$, denote by χ_i the irreducible character of A such that $\Phi(\chi_i) = \eta_i$. As in the proof of 1.1, we have a distinguished norm 4 element in $L^0(A)$ which is orthogonal to all other norm 4 elements in $L^0(A)$, namely $\chi_0 + \chi_1 + \chi_2 - \chi_3$. Thus, if $\Psi : \mathbb{Z}\mathrm{Irr}_K(A) \cong \mathbb{Z}\mathrm{Irr}_K(\mathcal{O}\tilde{A}_4)$ is some isometry mapping $\mathrm{Proj}(A)$ to $\mathrm{Proj}(\mathcal{O}\tilde{A}_4)$ and inducing an isomorphism $Z(A) \cong Z(\mathcal{O}\tilde{A}_4)$, then $\Psi(\chi_0 + \chi_1 + \chi_2 - \chi_3) = \pm(\eta_0 + \eta_1 + \eta_2 - \eta_3)$. By Proposition 1.1, there is a perfect self-isometry μ of $\mathbb{Z}\mathrm{Irr}_K(\mathcal{O}\tilde{A}_4)$ such that $\Phi = \mu \circ \Psi$. □

Remark 2.2. If we assume that A is Morita equivalent to some block algebra with Q_8 as defect group, then Proposition 2.1 follows also from the work of Cabanes and Picaronny in [4, 5].

Since $k \underset{\mathcal{O}}{\otimes} A \cong k\tilde{A}_4$, the quiver of A is the same as that of $k\tilde{A}_4$, thus of the following form:



Write $\bar{a}$ for the image of $a \in A$ in $\bar{A} = k \underset{\mathcal{O}}{\otimes} A \cong k\tilde{A}_4$. The generators β, γ, δ, κ, λ, η can be chosen such that their images in $\bar{A}$ fulfill the following relations:

$$\bar{\beta}\bar{\delta} = \bar{\kappa}\bar{\lambda}\bar{\kappa}\ ,$$

$$\bar{\eta}\bar{\gamma} = \bar{\lambda}\bar{\kappa}\bar{\lambda}\ ,$$

$$\bar{\delta}\bar{\lambda} = \bar{\gamma}\bar{\beta}\bar{\gamma}\ ,$$

$$\bar{\kappa}\bar{\eta} = \bar{\beta}\bar{\gamma}\bar{\beta}\ ,$$

$$\bar{\lambda}\bar{\beta} = \bar{\eta}\bar{\delta}\bar{\eta}\ ,$$

$$\bar{\gamma}\bar{\kappa} = \bar{\delta}\bar{\eta}\bar{\delta}$$

and

$$\bar{\gamma}\bar{\beta}\bar{\delta} = \bar{\delta}\bar{\eta}\bar{\gamma} = \bar{\lambda}\bar{\kappa}\bar{\eta} = 0\ .$$

In order to determine the algebra structure of A, we have to "lift" these relations over $\mathcal{O}$.

We fix an algebra isomorphism $\alpha : k \underset{\mathcal{O}}{\otimes} A \cong k\tilde{A}_4$ and an isometry $\Phi : \mathbb{Z}\mathrm{Irr}_K(A) \cong \mathbb{Z}\mathrm{Irr}_K(\mathcal{O}\tilde{A}_4)$ satisfying the conclusions of Proposition 2.1. We denote by χ_i the unique irreducible K-linear character of A such that $\Phi(\chi_i) = \eta_i$ for all i, $0 \leq i \leq 6$.

The characters η_0, η_1, η_2, η_3 of $\mathcal{O}\tilde{A}_4$ have height zero, the characters η_4, η_5, η_6 have height one. Thus, via the isomorphism of the centers induced by Φ, it follows that for $0 \leq i \leq 3$ we have $8e(\chi_i) \in A$, and for $4 \leq j \leq 6$ we have $4e(\chi_j) \in A$. We can in fact describe an $\mathcal{O}$-basis of $Z(A)$ in terms of the centrally primitive idempotents $e(\chi_i)$. The strategy is now to play off the descriptions of $Z(k \underset{\mathcal{O}}{\otimes} A)$ in terms of the generators in the quiver and of $Z(A)$ in terms of the centrally primitive idempotents $e(\chi_i)$.

Lemma 2.3. *The following elements of $Z(K \underset{\mathcal{O}}{\otimes} A)$ are all contained in the radical $J(Z(A))$:*

$$s = 2e(\chi_4) + 2e(\chi_5) + 2e(\chi_6) \ ,$$

$$z_0 = 4e(\chi_2) + 4e(\chi_3) + 2e(\chi_4) \ ,$$

$$z_1 = 4e(\chi_1) + 4e(\chi_3) + 2e(\chi_5) \ ,$$

$$z_2 = 4e(\chi_0) + 4e(\chi_3) + 2e(\chi_6) \ ,$$

$$y_0 = 4e(\chi_1) + 4e(\chi_2) + 2e(\chi_4) + 2e(\chi_5) \ ,$$

$$y_1 = 4e(\chi_0) + 4e(\chi_2) + 2e(\chi_4) + 2e(\chi_6) \ ,$$

$$y_2 = 4e(\chi_0) + 4e(\chi_1) + 2e(\chi_5) + 2e(\chi_6) \ .$$

Moreover, for any two different i, j in $\{0,1,2\}$ the set

$$\{1, z_i, z_j, s, 8e(\chi_3), 4e(\chi_{i+4}), 4e(\chi_{j+4})\}$$

is an $\mathcal{O}$-basis of $Z(A)$.

Proof. In view of Proposition 2.1 we may assume that $A = \mathcal{O}\tilde{A}_4$. This is just an explicit verification, using the character table of $\tilde{A}_4$. One verifies first that $z_0 \in A$. By symmetry, this implies that z_1, z_2 are also in A. Then $y_0 = z_0 + z_1 - 8e(\chi_3)$ is in A, similarly for the y_1, y_2. An equally easy computation shows that $s \in A$. Thus all the given elements belong to $Z(A)$. None of these elements is invertible, so they all belong to $J(Z(A))$ because $Z(A)$ is local.

In order to see the last statement on the basis of $Z(A)$, we may assume that $i = 0$ and $j = 1$. For any $x \in \tilde{A}_4$ denote by $\underline{x}$ the conjugacy class sum of x in $\mathcal{O}\tilde{A}_4$. The orthogonality relations imply the well-known formula

$$\underline{x} = \sum_{0 \le m \le 6} \frac{\chi_m(\underline{x}^{-1})}{\chi_m(1)} e(\chi_m) \ .$$

Thus, for the seven conjugacy classes in $\tilde{A}_4$, we have

$$\underline{1} = e(\chi_0) + e(\chi_1) + e(\chi_2) + e(\chi_3) + e(\chi_4) + e(\chi_5) + e(\chi_6) \ ;$$

$$\underline{z} = e(\chi_0) + e(\chi_1) + e(\chi_2) + e(\chi_3) - e(\chi_4) - e(\chi_5) - e(\chi_6) \ ;$$

$$\underline{y} = 6e(\chi_0) + 6e(\chi_1) + 6e(\chi_2) - 2e(\chi_3) \ ;$$

$$\underline{t} = 4e(\chi_0) + 4\omega^2 e(\chi_1) + 4\omega e(\chi_2) - 2\omega e(\chi_4) - 2\omega^2 e(\chi_5) - 2e(\chi_6) \ ;$$

$$\underline{t}^2 = 4e(\chi_0) + 4\omega e(\chi_1) + 4\omega^2 e(\chi_2) - 2\omega^2 e(\chi_4) - 2\omega e(\chi_5) - 2e(\chi_6) \ ;$$

$$\underline{tz} = 4e(\chi_0) + 4\omega^2 e(\chi_1) + 4\omega e(\chi_2) + 2\omega e(\chi_4) + 2\omega^2 e(\chi_5) + 2e(\chi_6) \ ;$$

$$\underline{t^2z} = 4e(\chi_0) + 4\omega e(\chi_1) + 4\omega^2 e(\chi_2) + 2\omega^2 e(\chi_4) + 2\omega e(\chi_5) + 2e(\chi_6)\ .$$

We show that they are all in the $\mathcal{O}$-linear span of the elements in the set

$$\{1, z_0, z_1, s, 8e(\chi_3), 4e(\chi_4), 4e(\chi_5)\}\ .$$

Note first that

$$z_2 = 4\cdot 1 - z_0 - z_1 - s + 8e(\chi_3),$$
$$4e(\chi_6) = 2s - 4e(\chi_4) - 4e(\chi_5)$$

are in the $\mathcal{O}$-linear span of this set. One easily verifies now that

$$\underline{z} = 1 - s\ ,$$

$$\underline{y} = 6\cdot 1 - 3s - 8e(\chi_3)\ ,$$

$$\underline{t} = \omega z_0 + \omega^2 z_1 + z_2 - 4\omega e(\chi_4) - 4\omega^2 e(\chi_5) - 4e(\chi_6)\ ,$$

$$\underline{t}^2 = \omega^2 z_0 + \omega z_1 + z_2 - 4\omega^2 e(\chi_4) - 4\omega e(\chi_5) - 4e(\chi_6)\ ,$$

$$\underline{tz} = \omega z_0 + \omega^2 z_1 + z_2\ ,$$

$$\underline{t^2z} = \omega^2 z_0 + \omega z_1 + z_2\ .$$

This concludes the proof of 2.3 □

The center of $\bar{A} = k \underset{\mathcal{O}}{\otimes} A$ can easily be described in terms of the generators in the quiver of A:

Lemma 2.4. *The following set is a k-basis of $Z(\bar{A})$.*

$$\{1,\ \bar\beta\bar\gamma + \bar\gamma\bar\beta,\ \bar\kappa\bar\lambda + \bar\lambda\bar\kappa,\ \bar\eta\bar\delta + \bar\delta\bar\eta,\ \bar\beta\bar\delta\bar\lambda, \bar\delta\bar\lambda\bar\beta,\ \bar\lambda\bar\beta\bar\delta\}\ .$$

Proof. Straightforward verification, using $(\bar\beta\bar\gamma)^2 = \bar\beta\bar\delta\bar\lambda$ and the similar relations for the other elements in the given set. □

Proposition 2.5. *For any primitive idempotent e in A we have $Z(A)e = eAe$. Moreover,*
(i) the set $\{e_0, z_0e_0, z_1e_0, 4e(\chi_4)e_0\}$ is an $\mathcal{O}$-basis of e_0Ae_0.
(ii) the set $\{e_1, z_0e_1, z_2e_1, 4e(\chi_4)e_1\}$ is an $\mathcal{O}$-basis of e_1Ae_1;
(iii) the set $\{e_2, z_1e_2, z_2e_2, 4e(\chi_5)e_2\}$ is an $\mathcal{O}$-basis of e_2Ae_2.

Proof. Since $Z(A) \cong Z(\mathcal{O}\tilde{A}_4)$ and $Z(\bar{A}) \cong Z(k\tilde{A}_4)$, the canonical map $A \to \bar{A}$ maps $Z(A)$ onto $Z(\bar{A})$ and hence $Z(A)e$ onto $Z(\bar{A})\bar{e}$. By Nakayama's Lemma, it suffices to show that $Z(\bar{A})\bar{e} = \bar{e}\bar{A}\bar{e}$. Now $\dim_k(\bar{e}\bar{A}\bar{e}) = 4$ by the Cartan matrix, and so we have only to show that $\dim_k(Z(\bar{A})\bar{e}) = 4$. By the symmetry of the quiver of A, we may assume that e corresponds to the vertex labelled 0. Then the set $\{\bar{e}, \bar\beta\bar\gamma, \bar\kappa\bar\lambda, \bar\beta\bar\delta\bar\lambda\}$ is a

k-basis of $Z(\bar{A})\bar{e}$ by 2.4; in particular, $\dim_k(Z(\bar{A})\bar{e}) = 4$ as required. This shows that $eAe = Z(A)e$.

In order to prove (i), note that the set

$$\{e_0, z_0e_0z_1e_0, se_0, 8e(\chi_3)e_0, 4e(\chi_4)e_0, 4e(\chi_5)e_0\}$$

generates e_0Ae_0 as $\mathcal{O}$-module, by the first statement and by the $\mathcal{O}$-basis of $Z(A)$ described in 2.3. Now we have

$$8e(\chi_3)e_0 = 2z_0e_0 - 4e(\chi_4)e_0 \ ,$$

$$4e(\chi_5)e_0 = 2z_0e_0 - 2z_1e_0 + 4e(\chi_4)e_0 \ ,$$

$$se_0 = (z_1 - z_0 + 4e(\chi_4))e_0 \ .$$

Thus the set given in (i) generates e_0Ae_0 as $\mathcal{O}$-module, and hence is a basis since the $\mathcal{O}$-rank of e_0Ae_0 is 4. The same arguments show (ii), (iii). □

Proposition 2.6. *We can choose the generators β, γ, δ, η, λ, κ in such a way that*

(i) $A\gamma$ is the unique $\mathcal{O}$-pure submodule of Ae_0 with character $\chi_3 + \chi_4$;
(ii) $A\lambda$ is the unique $\mathcal{O}$-pure submodule of Ae_0 with character $\chi_3 + \chi_5$;
(iii) $A\eta$ is the unique $\mathcal{O}$-pure submodule of Ae_1 with character $\chi_3 + \chi_6$;
(iv) $A\beta$ is the unique $\mathcal{O}$-pure submodule of Ae_1 with character $\chi_3 + \chi_4$;
(v) $A\kappa$ is the unique $\mathcal{O}$-pure submodule of Ae_2 with character $\chi_3 + \chi_5$;
(vi) $A\delta$ is the unique $\mathcal{O}$-pure submodule of Ae_2 with character $\chi_3 + \chi_6$.

Proof. We are going to prove (i); by the symmetry of the quiver of A one gets all other statements. Observe first that $\bar{A}\bar{\gamma}$ is the unique 5-dimensional submodule of Ae_0 with composition factors $2[S_0], 2[S_1], [S_2]$. Indeed, the set $\{\bar{\gamma}, \bar{\beta}\bar{\gamma}, \bar{\eta}\bar{\gamma}, \bar{\gamma}\bar{\beta}\bar{\gamma}, \bar{\beta}\bar{\gamma}\bar{\beta}\bar{\gamma}\}$ is a k-basis of $\bar{A}\bar{\gamma}$, and we have $\bar{\gamma}, \bar{\gamma}\bar{\beta}\bar{\gamma} \in \bar{e}_0\bar{A}\bar{e}_0$, yielding the two composition factors isomorphic to S_0, we have $\bar{\beta}\bar{\gamma}, \bar{\beta}\bar{\gamma}\bar{\beta}\bar{\gamma} \in \bar{e}_1\bar{A}\bar{e}_0$, yielding the two composition factors isomorphic to S_1, and finally $\bar{\eta}\bar{\gamma} \in \bar{e}_2\bar{A}\bar{e}_0$, yielding the remaining composition factor isomorphic to S_2. One checks that there is no other submodule with exactly these composition factors. Now there is exactly one $\mathcal{O}$-pure submodule U of Ae_0 whose reduction modulo $J(\mathcal{O})$ has composition series $2[S_0] + 2[S_1] + [S_2]$, namely the unique $\mathcal{O}$-pure submodule of Ae_0 with character $\chi_3 + \chi_4$; this is a direct consequence of the decomposition matrix. One constructs U as follows: write $K \underset{\mathcal{O}}{\otimes} Ae_0 = X_0 \oplus X_3 \oplus X_4 \oplus X_5$, where X_j is the unique submodule of $K \underset{\mathcal{O}}{\otimes} Ae_0$ with character χ_j for $j \in \{0, 3, 4, 5\}$, and then $U = Ae_0 \cap (X_3 \oplus X_4)$. Take now for γ any inverse image in U of $\bar{\gamma}$. Then $A\gamma \subseteq U$ and $U \subseteq A\gamma + J(\mathcal{O})U$. Thus $A\gamma = U$ by Nakayama's Lemma. □

Corollary 2.7. *If the generators β, γ, δ η, λ, κ are chosen such that they fulfill the conclusions of 2.6 then, with the notation of 2.3, the following hold.*

(i) $y_0\delta = y_0\eta = 0$.

(ii) $y_1\lambda = y_1\kappa = 0$.

(iii) $y_2\gamma = y_2\beta = 0$.

Proposition 2.8. *We can choose the generators β, γ, δ, η, λ, κ such that the following holds:*

$$\beta\gamma = z_0e_0 = 4e(\chi_3)e_0 + 2e(\chi_4)e_1;$$

$$\gamma\beta = z_0e_1 = 4e(\chi_3)e_1 + 2e(\chi_4)e_1;$$

$$\delta\eta = z_2e_1 = 4e(\chi_3)e_1 + 2e(\chi_6)e_1;$$

$$\eta\delta = z_2e_2 = 4e(\chi_3)e_2 + 2e(\chi_6)e_2;$$

$$\lambda\kappa = z_1e_2 = 4e(\chi_3)e_2 + 2e(\chi_5)e_2;$$

$$\kappa\lambda = z_1e_0 = 4e(\chi_3)e_0 + 2e(\chi_5)e_0;$$

$$\beta\delta\lambda = \kappa\eta\gamma = 8e(\chi_3)e_0;$$

$$\delta\lambda\beta = \gamma\kappa\eta = 8e(\chi_3)e_1;$$

$$\lambda\beta\delta = \eta\gamma\kappa = 8e(\chi_3)e_2.$$

Proof. In view of the decomposition matrix of A we have $e_0 = e(\chi_0)e_0 + e(\chi_3)e_0 + e(\chi_4)e_0 + e(\chi_5)e_0$. Moreover, the elements $e(\chi_0)e_0$, $e(\chi_3)e_0$, $e(\chi_4)e_0$, $e(\chi_5)e_0$ are K-linearly independent because they are pairwise orthogonal idempotents in $K \underset{\mathcal{O}}{\otimes} A$. Similar statements hold for e_1, e_2.

We assume a choice of generators fulfilling 2.6. We have $A\beta\gamma \subseteq A\gamma$, and the submodule $A\gamma$ of Ae_0 has character $\chi_3+\chi_4$ by 2.6. Thus $\beta\gamma$ is a K-linear combination of $e(\chi_3)e_1$ and $e(\chi_4)e_1$. But also $\beta\gamma$ is an $\mathcal{O}$-linear combination of the basis elements e_1, z_0e_1 z_1e_1, $4e(\chi_4)e_1$ given in 2.5 in which none of χ_1, χ_5 shows up. Therefore $\beta\gamma$ is in fact an $\mathcal{O}$-linear combination of the elements z_0e_0, $4e(\chi_4)e_0$; say

$$\beta\gamma = (\mu_0z_0e_0 + 4\nu_0e(\chi_4))e_0 = (4\mu_0e(\chi_3) + 2(\mu_0 + 2\nu_0)e(\chi_4))e_0$$

for some coefficients μ_0, $\nu_0 \in \mathcal{O}$. Hence

$$(\beta\gamma)^2 = (16\mu_0^2e(\chi_3) + 4(\mu_0 + 2\nu_0)^2e(\chi_4))e_0 \ .$$

Now $(\bar{\beta}\bar{\gamma})^2 \neq 0$, and therefore $\mu_0 \in \mathcal{O}^\times$. Set now

$$a_0 = 1 + \nu_0\mu_0^{-1}y_0 \ .$$

Since $y_0 \in J(Z(A))$ by 2.3 we have $a_0 \in Z(A)^\times$. A trivial verification, comparing coefficients, shows that we have

$$\beta\gamma = \mu_0 z_0 a_0 e_0 \ .$$

Since $\gamma = e_1\gamma = \gamma e_0$, multiplying this with γ on the left yields

$$\gamma\beta\gamma = \mu_0 z_0 a_0 e_1 \gamma \ .$$

Now both $\gamma\beta$ and $\mu_0 z_0 a_0 e_1$ are contained in the pure submodule $A\beta$ of Ae_1 with character $\chi_3+\chi_4$, by 2.6 and the nature of the element z_0. Right multiplication by γ on this submodule is therefore injective (the annihilator of γ in Ae_1 is the pure submodule with character $\chi_1+\chi_6$). Hence the previous equality implies also the equality

$$\gamma\beta = \mu_0 z_0 a_0 e_1 \ .$$

In an entirely analogous way one finds scalars μ_1, $\mu_2 \in \mathcal{O}^\times$ such that, setting $a_1 = 1+\nu_1\mu_1^{-1}y_1$ and $a_2 = 1+\nu_2\mu_2^{-1}y_2$, one gets the equalities

$$\delta\eta = \mu_2 z_2 a_2 e_1 \ , \ \eta\delta = \mu_2 z_2 a_2 e_2 \ ,$$

$$\lambda\kappa = \mu_1 z_1 a_1 e_2 \ , \ \kappa\lambda = \mu_1 z_1 a_1 e_0 \ .$$

Moreover, the equalities in 2.7 imply the following equalities:

$$a_0\delta = \delta \ , \ a_0\eta = \eta \ ,$$

$$a_1\lambda = \lambda \ , \ a_1\kappa = \kappa \ ,$$

$$a_2\gamma = \gamma \ , \ a_2\beta = \beta \ .$$

If we replace now β by $a_0\beta$, this is not going to change the properties stated in 2.6 and also this is not changing the relations over k of the quiver. Similarly, we can replace δ by $a_2\delta$ and λ by $a_1\lambda$. Then the generators β, γ, δ, η, λ, κ still fulfill 2.6, and in addition, we have now the following equalities:

$$\beta\gamma = \mu_0 z_0 e_0 \ , \ \gamma\beta = \mu_0 z_0 e_1 \ ,$$

$$\delta\eta = \mu_2 z_2 e_1 \ , \ \eta\delta = \mu_2 z_2 e_2 \ ,$$

$$\lambda\kappa = \mu_1 z_1 e_2 \ , \ \kappa\lambda = \mu_1 z_1 e_0 \ .$$

We have to get rid of the scalars μ_0, μ_1, μ_2. Since χ_3 is the only character appearing in the characters of all projective indecomposable A-modules we have

$$\beta\delta\lambda = 8\mu e(\chi_3)e_0$$

for some $\mu \in \mathcal{O}$. Then actually $\mu \in \mathcal{O}^\times$ because $\bar\beta\bar\delta\bar\lambda \neq 0$. Moreover, $\beta\delta\lambda\beta = 8\mu e(\chi_3)\beta$, and hence also

$$\delta\lambda\beta = 8\mu e(\chi_3)e_1 \ .$$

The same argument applied again yields

$$\lambda\beta\delta = 8\mu e(\chi_3)e_2 \ .$$

Applying this argument to the arrows in the quiver in the opposite direction implies that there is $\mu' \in \mathcal{O}^\times$ such that

$$\kappa\eta\gamma = 8\mu' e(\chi_3)e_0 \ ,$$

$$\eta\gamma\kappa = 8\mu' e(\chi_3)e_2 \ ,$$

$$\gamma\kappa\eta = 8\mu' e(\chi_3)e_1 \ .$$

Now $\bar\beta\bar\delta\bar\lambda = \bar\kappa\bar\lambda\bar\kappa\bar\lambda = \bar\kappa\bar\eta\bar\gamma$, and hence $\mu' = \mu(1+\nu)$ for some $\nu \in J(\mathcal{O})$. Note that we can always multiply any of the generators by any scalar in $1+J(\mathcal{O})$ without modifying the relations over k. Thus, if we replace κ by $(1+\nu)\kappa$, we may assume that $\mu' = \mu$.

Since the set $\{\kappa, \kappa\lambda\kappa\}$ is an $\mathcal{O}$-basis of e_0Ae_2, we can write

$$\beta\delta = a\kappa + b\kappa\lambda\kappa$$

for some unique scalars $a, b \in \mathcal{O}$. Multiplying this by λ yields

$$8\mu e(\chi_3)e_0 = \beta\delta\lambda = a\kappa\lambda + b(\kappa\lambda)^2 = (a\mu_1 z_1 + b\mu_1^2 z_1^2)e_0 \ .$$

By comparing the coefficients at $e(\chi_3)e_0$ and $e(\chi_5)e_0$ of the left and right expression in this equality, we get the equations

$$8\mu = 4a\mu_1 + 16b\mu_1^2 \ ,$$

$$0 = 2a\mu_1 + 4b\mu_1^2 \ .$$

An easy computation shows that $b = \frac{\mu}{\mu_1^2}$. Moreover, since $\bar\beta\bar\delta\bar\lambda = (\bar\kappa\bar\lambda)^2$ we have $\bar a = 0$ and $\bar b = 1_k$, hence $b = \frac{\mu}{\mu_1^2} \in 1 + J(\mathcal{O})$. By repeating the same argument we find also that the coefficients $\frac{\mu}{\mu_0^2}$, $\frac{\mu}{\mu_2^2}$ are in $1 + J(\mathcal{O})$.

Next, we compute $\beta\delta\lambda\kappa\eta\gamma$ in two different ways: on one hand we have

$$(\beta\delta\lambda)(\kappa\eta\gamma) = 64\mu^2 e(\chi_3)e_0 \ ,$$

and on the other hand we have

$$\beta(\delta(\lambda\kappa)\eta)\gamma = \mu_0\mu_1\mu_2 z_0 z_1 z_2 e(\chi_3)e_0 = 64\mu_0\mu_1\mu_2 e(\chi_3)e_0 \ .$$

Together we get
$$\mu^2 = \mu_0\mu_1\mu_2 \ .$$
Thus $\frac{\mu}{\mu_0^2}\frac{\mu}{\mu_1^2} = \frac{\mu_2}{\mu_0\mu_1} \in 1 + J(\mathcal{O})$. Similarly, $\frac{\mu_1}{\mu_0\mu_2}, \frac{\mu_0}{\mu_1\mu_2} \in 1 + J(\mathcal{O})$. But then also $\frac{\mu_1\mu_2}{\mu_0}\frac{\mu_1}{\mu_0\mu_2} = \frac{\mu_1^2}{\mu_0^2} \in 1 + J(\mathcal{O})$. Since $2 \in J(\mathcal{O})$ this implies that $\frac{\mu_1}{\mu_0} \in 1 + J(\mathcal{O})$. But then actually $\mu_2 = \frac{\mu_1\mu_2}{\mu_0}\frac{\mu_0}{\mu_1} \in 1 + J(\mathcal{O})$. Similarly, μ_0, $\mu_1 \in 1 + J(\mathcal{O})$. So we can replace β by $\mu_0^{-1}\beta$, or equivalently, we can assume that $\mu_0 = 1$. Similarly, we can assume that $\mu_1 = \mu_2 = 1$. Then $\mu^2 = 1$. If $\mu = -1$ we multiply all generators by -1; since $2 \in J(\mathcal{O})$, this does not change the relations over k, but it does change the sign of any of the above expressions $\beta\delta\lambda$ etc. involving three generators. Therefore, we can also assume that $\mu = 1$.
□

We can now prove Theorem C from the introduction.

Proof of Theorem C. We assume a choice of generators of A fulfilling Proposition 2.8. We show that A satisfies the relations given in Theorem C. Those in the first three lines are obvious. Since the set $\{\kappa, \kappa\lambda\kappa\}$ is an $\mathcal{O}$-basis of e_0Ae_2, we can write

$$\beta\delta = a\kappa + b\kappa\lambda\kappa$$

for some unique scalars $a, b \in \mathcal{O}$. Multiplying this by λ yields

$$8e(\chi_3)e_0 = \beta\delta\lambda = a\kappa\lambda + b(\kappa\lambda)^2 = (4a + 16b)e(\chi_3)e_0 + (2a + 4b)e(\chi_5)e_0 \ .$$

By comparing the coefficients at $e(\chi_3)e_0$ and $e(\chi_5)e_0$ of the left and right expression in this equality, we get the equations

$$8 = 4a + 16b \ ,$$

$$0 = 2a + 4b \ .$$

Thus the coefficients a, b have values

$$a = -2 \ , \ b = 1 \ ,$$

and from this we get the following relation in the statement of Theorem C:

$$\beta\delta = -2\kappa + \kappa\lambda\kappa \ .$$

In exactly the same way we get the following five relations in the Theorem:

$$\eta\gamma = -2\lambda + \lambda\kappa\lambda \ ,$$

$$\delta\lambda = -2\gamma + \gamma\beta\gamma \ ,$$

$$\kappa\eta = -2\beta + \beta\gamma\beta \ ,$$

$$\lambda\beta = -2\eta + \eta\delta\eta \ ,$$

$$\gamma\kappa = -2\delta + \delta\eta\delta \ .$$

A similar technique is going to yield the remaining relations: write $\gamma\beta\delta = c\delta + d\delta\eta\delta$ for some unique $c, d \in \mathcal{O}$; as before, this is possible since $\{\delta, \delta\eta\delta\}$ is an $\mathcal{O}$-basis of e_1Ae_2. Multiplying by η yields

$$\gamma\beta\delta\eta = c\delta\eta + d(\delta\eta)^2 = cz_2e_1 + dz_2^2e_1 \ .$$

The left side is equal to $(\gamma\beta)(\delta\eta) = z_0z_2e_1$, so comparing coefficients yields now

$$16 = 4c + 16d \ ,$$

$$0 = 2c + 4d \ ,$$

and this implies $c = -4$ and $d = 2$. Thus we get indeed

$$\gamma\beta\delta = -4\delta + 2\delta\eta\delta$$

as claimed. The remaining relations of this type follow in exactly the same way.

Now consider the last three relations. Write $\beta\delta\lambda\beta = r\beta + s\beta\gamma\beta$, for $r, s \in \mathcal{O}$. Then $\beta\delta\lambda\beta\gamma = r\beta\gamma + s\beta\gamma\beta\gamma$. So

$$32e(\chi_3)e_0 = (4r + 16s)e(\chi_3)e_0 + (2r + 4s)e(\chi_4)e_0$$

which yields $s = 4$ and $r = -8$. The remaining two relations follow in exactly the same way. Thus A satisfies all relations given in Theorem C.

Let $\tilde{A}$ be the $\mathcal{O}$-algebra described by the generators and relations given in Theorem C. There is a surjective algebra morphism from $\tilde{A}$ to A. In order to show that $\tilde{A}$ and A are isomorphic it suffices therefore to show that the cardinality of a minimal generating set for A as an $\mathcal{O}$-module is at most 24. Thus it suffices to check that the set

$$\begin{aligned}\mathcal{S} :=\{&e_0, e_1, e_2, \beta, \gamma, \delta, \eta, \lambda, \kappa,\\ &\beta\gamma, \gamma\beta, \delta\eta, \eta\delta, \lambda\kappa, \kappa\lambda,\\ &\beta\gamma\beta, \gamma\beta\gamma, \delta\eta\delta, \eta\delta\eta, \lambda\kappa\lambda, \kappa\lambda\kappa,\\ &\beta\delta\lambda, \delta\lambda\beta, \lambda\beta\delta\}\end{aligned}$$

spans $\tilde{A}$ as $\mathcal{O}$-module. This is an easy consequence of the given relations; we give some details for the convenience of the reader: Let

$$\mathcal{G} = \{e_0, e_1, e_2, \beta, \gamma, \delta, \eta, \lambda, \kappa\}$$

From the given relations it is immediate that for any two elements x, y of $\mathcal{G}$, xy is in the $\mathcal{O}$-span of $\mathcal{S}$. Thus it suffices to show that for any two elements x, y of $\mathcal{G} - \{e_0, e_1, e_2\}$ and any element u of $\mathcal{S} - \{e_0, e_1, e_2, \beta, \gamma, \delta, \eta, \lambda, \kappa\}$, xu and uy are in the $\mathcal{O}$-span of $\mathcal{S}$. From the given relations we may also assume that u is one of $\beta\gamma\beta, \gamma\beta\gamma, \delta\eta\delta, \eta\delta\eta, \lambda\kappa\lambda, \kappa\lambda\kappa$ or one of $\beta\delta\lambda, \delta\lambda\beta, \lambda\beta\delta$.

First, note that the relations $\kappa\eta = -2\beta + \beta\gamma\beta$ and $\delta\lambda = -2\gamma + \gamma\beta\gamma$ give that $\kappa\eta\gamma = \beta\delta\lambda$. Similarly, we get $\eta\gamma\kappa = \lambda\beta\delta$ and $\gamma\kappa\eta = \delta\lambda\beta$.

Now suppose $u = \beta\gamma\beta$. Then we may assume that x is one of γ or λ and that y is one of γ or δ. The relation $\kappa\eta = -2\beta + \beta\gamma\beta$ gives $\gamma\kappa\eta = -2\gamma\beta + \gamma\beta\gamma\beta$, hence $\gamma\beta\gamma\beta$ is in the $\mathcal{O}$-span of $\mathcal{S}$. The relation $\kappa\eta = -2\beta + \beta\gamma\beta$ also gives $\lambda\kappa\eta = -2\lambda\beta + \lambda\beta\gamma\beta$. It follows from the relation $\lambda\kappa\eta = -4\eta + 2\eta\delta\eta$ that $\lambda\beta\gamma\beta$ is in the $\mathcal{O}$-span of $\mathcal{S}$. We show similarly that $\beta\gamma\beta\gamma$ and $\beta\gamma\beta\delta$ are in the $\mathcal{O}$-span of $\mathcal{S}$.

The cases $u = \gamma\beta\gamma, \delta\eta\delta, \eta\delta\eta, \lambda\kappa\lambda, \kappa\lambda\kappa$ are handled analogously.

Now suppose $u = \beta\delta\lambda$. Then we may assume that x is one of λ or γ and y is one of β or κ. The relation $\lambda\beta\delta\lambda = -8\lambda + 4\lambda\kappa\lambda$ shows that $\lambda\beta\delta\lambda$ is in the $\mathcal{O}$-span of $\mathcal{S}$. From the relation $\gamma\beta\delta = -4\delta + 2\delta\eta\delta$ we get $\gamma\beta\delta\lambda = -4\delta\lambda + 2\delta\eta\delta\lambda$. From $\gamma\kappa = -2\delta + \delta\eta\delta$, we get $\delta\eta\delta\lambda = \gamma\kappa\lambda + 2\delta\lambda$. Hence $\delta\eta\delta\lambda$ is in the $\mathcal{O}$-span of $\mathcal{S}$, and so is $\gamma\beta\delta\lambda$. We argue similarly to show that $\beta\delta\lambda\beta$ and $\beta\delta\lambda\kappa$ are in the $\mathcal{O}$-span of $\mathcal{S}$.

The cases $u = \delta\lambda\beta$ and $u = \lambda\beta\delta$ are handled in the same fashion. □

Remark 2.9. An interesting consequence of 2.5 is the structure of eAe for any primitive idempotent e in A. We have an $\mathcal{O}$-algebra isomorphism

$$eAe \cong \mathcal{O}[X,Y]/ < X^2 - Y^2 - 2(X-Y)\ ,\ XY - 2X^2 + 4X > ;$$

indeed, we may assume that $e = e_0$, and then the assignment $X \mapsto z_0 e_0$, $Y \mapsto z_1 e_0$ induces the required isomorphism. In particular, we have an isomorphism of k-algebras

$$\bar{e}\bar{A}\bar{e} \cong k[X,Y]/ < X^2 - Y^2\ ,\ XY > \ .$$

This is, by Erdmann [6, III.1, III.3], up to isomorphism the unique 4-dimensional symmetric k-algebra which is not isomorphic to the group algebra of the Klein four group. One might be tempted to ask whether any symmetric $\mathcal{O}$-algebra is the endomorphism algebra of some projective module of some block algebra.

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