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# LOCAL CONTROL IN FUSION SYSTEMS OF $p$ -BLOCKS OF FINITE GROUPS

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ABSTRACT. If  $p$  is an odd prime,  $b$  a  $p$ -block of a finite group  $G$  such that  $SL(2, p)$  is not involved in  $N_G(Q, e)/C_G(Q)$  for any  $b$ -subpair  $(Q, e)$ , then  $N_G(Z(J(P)))$  controls  $b$ -fusion, where  $P$  is a defect group of  $b$ . This is a block theoretic analogue of Glauberman's  $ZJ$ -Theorem [6].

## 1 INTRODUCTION

Glauberman's  $ZJ$ -Theorem [6, Theorem B] states that if  $p$  is an odd prime and  $G$  is a finite group such that  $Qd(p)$  is not involved in  $G$ , then  $N_G(Z(J(P)))$  controls  $p$ -fusion in  $G$ , for  $P$  a Sylow  $p$ -subgroup of  $G$ . Here,  $J(P)$  denotes the Thompson subgroup of  $P$  (that is, the subgroup generated by all abelian subgroups of  $P$  of maximal order) and  $Qd(p)$  denotes the semi-direct product of  $C_p \times C_p$  with  $SL(2, p)$  (with the natural action). This has proved to be an extremely powerful tool in local group-theoretic analysis, as it gives a general condition which ensures that  $p$ -fusion is controlled by a single  $p$ -local subgroup.

In this paper, we establish block-theoretic analogues of this and other similar results. Along the way, we will obtain results which seem to be new even in the group-theoretic case. A key ingredient, allowing us to exploit the existing group-theoretic methods, is a result of Külshammer and Puig [11] on extensions of nilpotent blocks. We also show (both in a group-theoretic and in a block-theoretic context) that if a normal subgroup of a given group  $G$  has a single local subgroup which controls fusion, then  $G$  itself has a single local subgroup with the same property. We discuss some consequences of such control of fusion to other problems in block theory.

Throughout the paper,  $k$  will denote an algebraically closed field of prime characteristic  $p$ . A *block of a finite group  $G$*  is a primitive idempotent  $b$  in  $Z(kG)$ ; following Alperin-Broué [1], a  $(G, b)$ -*subpair* is a pair  $(Q, e)$  consisting of a  $p$ -subgroup  $Q$  of  $G$  and a block  $e$  of  $C_G(Q)$  such that  $\text{Br}_Q(b)e = e$ , where  $\text{Br}_Q : (kG)^Q \rightarrow kC_G(Q)$  is the *Brauer*

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*homomorphism* [5]. The set of  $(G, b)$ -subpairs is a partially ordered set on which  $G$  acts by conjugation, and the maximal  $(G, b)$ -subpairs with respect to this partial order are all  $G$ -conjugate. If  $(P, e)$  is a maximal  $(G, b)$ -subpair, then  $P$  is called a *defect group of the block  $b$*  (this notion is due to Brauer [2]); moreover, for any subgroup  $Q$  of  $P$  there is a unique block  $e_Q$  of  $C_G(Q)$  such that  $(Q, e_Q) \subseteq (P, e)$  (cf. [1]). A detailed account of subpairs and their properties may be found in [14] (where subpairs are referred to as Brauer pairs). The local structure of  $b$  is the  $G$ -set of  $(G, b)$ -subpairs viewed as category; the following definition makes this precise.

**Definition 1.1.** Let  $G$  be a finite group, let  $b$  be a block of  $G$  and let  $(P, e)$  be a maximal  $(G, b)$ -subpair. For any subgroup  $Q$  of  $P$  denote by  $e_Q$  the unique block of  $C_G(Q)$  such that  $(Q, e_Q) \subseteq (P, e)$ . We denote by  $\mathcal{F}_{(P,e)}(G, b)$  the category whose objects are the subgroups of  $P$  and whose sets of morphisms  $\text{Hom}_{\mathcal{F}_{(P,e)}(G, b)}(Q, R)$  are the sets of group homomorphisms  $\varphi : Q \rightarrow R$  for which there exists an element  $x \in G$  satisfying  ${}^x(Q, e_Q) \subseteq (R, e_R)$  and  $\varphi(u) = xux^{-1}$  for all  $u \in Q$ , where  $Q, R$  run over the set of subgroups of  $P$ .

Since all maximal  $(G, b)$ -subpairs are  $G$ -conjugate, the category  $\mathcal{F}_{(P,e)}(G, b)$  does not depend on the choice of  $(P, e)$  up to isomorphism of categories. If  $b$  is the principal block of  $G$  then  $P$  is a Sylow- $p$ -subgroup of  $G$  and  $e_Q$  is the principal block of  $C_G(Q)$  for any subgroup  $Q$  of  $P$ ; in this case we write  $\mathcal{F}_P(G) = \mathcal{F}_{(P,e)}(G, b)$ . Glauberman's  $ZJ$ -Theorem reads then  $\mathcal{F}_P(G) = \mathcal{F}_P(N_G(Z(J(P))))$ , provided that  $p$  is odd and  $Qd(p)$  is not involved in  $G$ .

We need a block-theoretic replacement for the hypothesis on  $Qd(p)$ . Recall that if  $G$  is a finite group and  $b$  is a block of  $G$ , then a  $(G, b)$ -subpair  $(Q, f)$  is called *centric* if  $Z(Q)$  is a defect group of  $f$  and  $(Q, f)$  is called *radical* if  $O_p(N_G(Q, f)/QC_G(Q)) = 1$ . The notion of centric subpairs - frequently called self-centralising pairs in the literature - goes back to Brauer [3].

**Definition 1.2.** Let  $G$  be a finite group. A block  $b$  of  $G$  is called  *$SL(2, p)$ -free* if  $SL(2, p)$  is not isomorphic to a subquotient of any of the groups  $N_G(Q, f)/C_G(Q)$ , where  $(Q, f)$  is a centric and radical  $(G, b)$ -subpair.

The definition of an  $SL(2, p)$ -free block is really a local condition on the block, in that it can be formulated purely in terms of the category  $\mathcal{F}_{(P,e)}(G, b)$ , where  $(P, e)$  is a maximal subpair of a block  $b$  of  $G$ . Indeed,  $b$  is  $SL(2, p)$ -free if and only if  $SL(2, p)$  is not involved in the automorphism group in  $\mathcal{F}_{(P,e)}(G, b)$  of any subgroup  $Q$  of  $P$  such that  $(Q, e_Q)$  is centric and radical for the unique  $e_Q$  such that  $(Q, e_Q) \subseteq (P, e)$ . It may well happen that a non principal block  $b$  of  $G$  is  $SL(2, p)$ -free even though  $SL(2, p)$  is involved in  $G$ . If, however, the principal block of  $G$  is  $SL(2, p)$ -free, then  $Qd(p)$  is not involved in  $G$  (cf. Proposition 5.1 and [7, Lemma 10.6]). In this case, our hypothesis “ $SL(2, p)$ -free” is in fact slightly more restrictive, since (in the principal block case) it effectively excludes faithful action of  $SL(2, p)$  on any  $p$ -subgroup of  $G$ , not just the natural action of  $SL(2, p)$  on  $C_p \times C_p$ .

Examples of  $SL(2, p)$ -free blocks include all blocks with abelian defect groups and, for  $p \geq 5$ , all blocks of finite  $p$ -solvable groups, or more generally, all blocks for which the groups  $N_G(Q, f)/C_G(Q)$  occurring in 1.2 are  $p$ -solvable.

Since Glauberman's control of fusion theorems also apply to some characteristic subgroups of  $p$ -groups other than the center of the Thompson subgroup, we make the following definitions, the first of which is given in [9, §5].

**Definition 1.3** A *positive characteristic  $p$ -functor* is a map  $W$  sending any finite  $p$ -group  $P$  to a subgroup  $W(P)$  of  $P$ , with the property that  $W(P) \neq 1$  if  $P \neq 1$  and that any isomorphism of finite  $p$ -groups  $P \cong Q$  maps  $W(P)$  onto  $W(Q)$ . A *Glauberman functor* is a positive characteristic  $p$ -functor  $W$  with the following additional property: whenever  $P$  is a Sylow- $p$ -subgroup of a finite group  $L$  which satisfies  $C_L(O_p(L)) = Z(O_p(L))$  and which does not have a subquotient isomorphic to  $Qd(p)$ , then  $W(P)$  is normal in  $L$ .

Of course, by Glauberman's  $ZJ$ -Theorem the map sending a finite  $p$ -group  $P$  to  $Z(J(P))$  is a Glauberman functor; in fact showing that this map is a Glauberman functor is the essential ingredient of the  $ZJ$ -Theorem. By [7, Theorem 14.8] any of the maps sending a finite  $p$ -group  $P$  to  $K_\infty(P)$  or  $K^\infty(P)$  are Glauberman functors, where  $K_\infty(P)$ ,  $K^\infty(P)$  are defined in [7, Section 12].

If  $W$  is a positive characteristic  $p$ -functor, then  $W(P)$  is characteristic in  $P$ , for any finite  $p$ -group  $P$ ; in particular, if  $P$  is a  $p$ -subgroup of a finite group  $G$ , then  $N_G(W(P))$  contains  $N_G(P)$ . If  $H$  is any subgroup of  $G$  containing  $N_G(P)$ , there is a unique block  $c$  of  $H$  such that  $\text{Br}_P(b) = \text{Br}_P(c)$ , the *Brauer correspondent* of  $b$  (cf. [1] or [14]). Then  $P$  is again a defect group of  $c$ , and since  $C_G(P) \subseteq H$ , every maximal  $(G, b)$ -subpair  $(P, e)$  is also a maximal  $(H, c)$ -subpair.

We are now ready to state our results. In what follows, refer to 2.1 and 2.3 for the exact definition of control of fusion that we are using.

**Theorem 1.4.** *Let  $G$  be a finite group, let  $b$  be a block of  $G$  and let  $(P, e)$  be a maximal  $(G, b)$ -subpair. Let  $W$  be a Glauberman functor, set  $N = N_G(W(P))$  and denote by  $c$  the unique block of  $N$  such that  $\text{Br}_P(b) = \text{Br}_P(c)$ . If  $p$  is odd and  $b$  is  $SL(2, p)$ -free, then  $\mathcal{F}_{(P, e)}(G, b) = \mathcal{F}_{(P, e)}(N, c)$ . In other words, the group  $N$  controls fusion in  $\mathcal{F}_{(P, e)}(G, b)$ .*

The proof of 1.4 is given in section 6. If we specialise Theorem 1.4 to the case of principal blocks and  $W(P) = Z(J(P))$ , we obtain the conclusion of Glauberman's  $ZJ$ -Theorem (but, as mentioned above, our hypothesis " $SL(2, p)$ -free" is slightly more restrictive).

Our next result shows that the property of being locally controlled by the normaliser of a single non-trivial subgroup of a defect group carries through normal extensions of blocks.

**Theorem 1.5.** *Let  $G$  be a finite group,  $H$  a normal subgroup of  $G$ ,  $c$  a  $G$ -stable block of  $H$  and  $b$  a block of  $G$  such that  $bc = b$ . Let  $(P, e)$  be a maximal  $(G, b)$ -subpair. There is a  $P$ -stable maximal  $(H, c)$ -subpair  $(Q, f)$  such that  $Q = P \cap H$  and  $fe_Q \neq 0$ , where  $(Q, e_Q)$  is the unique  $(G, b)$ -subpair contained in  $(P, e)$ .*

*Furthermore, if there is a normal subgroup  $V$  of  $Q$  such that  $N_H(V)$  controls fusion in  $\mathcal{F}_{(Q, f)}(H, c)$ , then  $N_G(W)$  controls fusion in  $\mathcal{F}_{(P, e)}(G, b)$  where  $W$  is the subgroup of  $P$  generated by the set of  $N_G(Q, f)$ -conjugates of  $V$ .*

An interesting consequence of Theorem 1.5 is that it allows us to prove that any block  $b$  of a finite group  $G$  lying over an  $SL(2, p)$ -free block of a normal subgroup  $N$  of  $G$  with non-trivial defect groups has again a local structure which is controlled by the normaliser of a single non-trivial  $p$ -subgroup of  $G$ , even though  $b$  itself need not be  $SL(2, p)$ -free:

**Corollary 1.6.** *Let  $G$  be a finite group, let  $b$  be a block of  $G$  and let  $(P, e)$  be a maximal  $(G, b)$ -subpair. If there is a normal subgroup  $H$  of  $G$  such that  $H \cap P \neq 1$  and such that  $b$  covers an  $SL(2, p)$ -free block  $c$  of  $H$ , then there is a non-trivial normal subgroup  $W$  in  $P$  such that  $N_G(W)$  controls fusion in  $\mathcal{F}_{(P, e)}(G, b)$ .*

In [7, Section 12] Glauberman showed that for  $W = K_\infty$  or  $W = K^\infty$ , the subgroup  $W(P)$  of  $P$  is self-centralising; that is,  $C_P(W(P)) = Z(W(P))$ . Thus, in the situation of Theorem 1.4, the  $(G, b)$ -subpair  $(W(P), e_{W(P)})$  is centric; in other words, the normaliser in  $G$  of some centric  $(G, b)$ -subpair controls  $b$ -fusion. The next Theorem shows that there is a canonical choice for such a centric subpair. By results of Külshammer and Puig in [11, Theorem 1.8], associated with any centric  $(G, b)$ -subpair  $(Q, f)$  and any choice of a maximal  $(N_G(Q, f), f)$ -subpair  $(R, g)$ , there is a canonical group extension

$$1 \longrightarrow Q \longrightarrow L \longrightarrow N_G(Q, f)/QC_G(Q) \longrightarrow 1$$

having the property that  $R$  is a Sylow- $p$ -subgroup of  $L$  and  $\mathcal{F}_{(R, g)}(N_G(Q, f), f) = \mathcal{F}_R(L)$  (we explain this in some more detail in 2.4 below); moreover,  $O_{p'}(L) = 1$  and  $C_L(Q) = Z(Q)$ . Thus, if  $b$  is  $SL(2, p)$ -free, then  $Qd(p)$  is not involved in  $L$ , and hence  $W(R)$  is normal in  $L$  for any Glauberman functor  $W$ .

**Theorem 1.7.** *Let  $G$  be a finite group, let  $b$  be a block of  $G$  and let  $(P, e)$  be a maximal  $(G, b)$ -subpair. Assume that  $p$  is odd and that  $b$  is  $SL(2, p)$ -free. There is a unique minimal subgroup  $Q$  of  $P$  such that  $(Q, f)$  is centric and radical, where  $f$  is the unique block of  $C_G(Q)$  such that  $(Q, f) \subseteq (P, e)$ . Moreover,  $Q$  is normal in  $P$  and we have  $\mathcal{F}_{(P, e)}(G, b) = \mathcal{F}_P(L)$ , where  $L$  is the middle term of the Külshammer-Puig extension associated with  $(Q, f)$ .*

**Remark 1.8.** Theorems 1.5 and 1.7 seem to add some new information even in the principal block case. Theorem 1.5 implies that if  $N$  is a normal subgroup of a

finite group  $G$  such that  $N_N(V)$  controls strong  $p$ -fusion in  $P \cap N$  with respect to  $N$  for some normal subgroup  $V$  of  $P \cap N$  then the subgroup,  $W$ , of  $P$  generated by all  $N_G(P \cap N)$ -conjugates of  $V$  has the property that  $N_G(W)$  controls strong  $p$ -fusion in  $P$  with respect to  $G$ . Theorem 1.7 translates to the following statement: given a finite group  $G$  with a Sylow- $p$ -subgroup  $P$  such that  $SL(2, p)$  is not involved in  $N_G(Q)/C_G(Q)$  for any  $p$ -subgroup  $Q$  of  $G$ , there is a unique minimal subgroup  $Q$  of  $P$  such that  $Z(Q)$  is a Sylow- $p$ -subgroup of  $C_G(Q)$  and such that  $O_p(N_G(Q)/QC_G(Q)) = 1$ ; moreover,  $N_G(Q)$  controls strong  $p$ -fusion in  $P$  with respect to  $G$ .

A *classifying space* of  $b$  is a  $p$ -complete space  $B(G, b)$  having the homotopy type of the  $p$ -completion of an  $\mathcal{L}$ -system associated with  $\mathcal{F}_{(P,e)}(G, b)$  in the sense of Broto, Levi and Oliver [4]. Note that in the situation of Theorem 1.7, the local structure of  $b$  is the same as the local structure of the principal block of  $L$ . Thus, if we take for  $B(G, b)$  the  $p$ -completion  $BL_p^\wedge$  of the classifying space  $BL$  of  $L$  we obtain the following immediate consequence.

**Corollary 1.9.** *If  $p$  is odd, any  $SL(2, p)$ -free block has a classifying space, which is unique up to homotopy.*

Theorems 1.4 and 1.5 provide many examples of blocks whose fusion pattern is determined by the normaliser of a single non-trivial  $p$ -subgroup. The existence of such controlling subgroups has ramifications for the Dade Projective Conjectures (DPC).

**Theorem 1.10.** *Let  $G$  be a finite group, let  $b$  be a block of  $G$  and let  $(P, e)$  be a maximal  $(G, b)$ -subpair. Assume that there is a normal subgroup  $R$  in  $P$  such that  $N_G(P, e) \subseteq N_G(R)$  and such that  $N_G(R)$  controls fusion in  $\mathcal{F}_{(P,e)}(G, b)$ . Let  $c$  be the block of  $N_G(R)$  which satisfies  $\text{Br}_P(c)e = e$ ; that is,  $c$  is the Brauer correspondent in  $N_G(Q)$  of  $b$ .*

(i) *If every section of  $G$  satisfies DPC, then there is a defect preserving bijection between the sets of irreducible characters of  $b$  and irreducible characters of  $c$ .*

(ii) *If every proper section of  $G$  satisfies DPC, then DPC holds for  $b$  if and only if there is a defect preserving bijection between the sets of irreducible characters of  $b$  and irreducible characters of  $c$ .*

## 2 ON LOCAL CATEGORIES OF BLOCKS

We collect in this Section some standard terminology and properties of local categories of blocks. We fix a finite group  $G$ , a block  $b$  of  $G$  and a maximal  $(G, b)$ -subpair  $(P, e)$ . For any subgroup  $Q$  of  $P$ , denote by  $e_Q$  the unique block of  $C_G(Q)$  such that  $(Q, e_Q) \subseteq (P, e)$  (in particular,  $e = e_P$ ).

By the uniqueness of the inclusion of subpairs (cf. [1]) we have  $\mathcal{F}_P(P) \subseteq \mathcal{F}_{(P,e)}(G, b)$ . If we choose a Sylow- $p$ -subgroup  $S$  of  $G$  containing  $P$ , we have also  $\mathcal{F}_{(P,e)}(G, b) \subseteq \mathcal{F}_S(G)$ .

Two subgroups  $Q, R$  of  $P$  are isomorphic as objects in  $\mathcal{F}_{(P,e)}(G, b)$  if there is  $x \in G$  such that  ${}^x(Q, e_Q) = (R, e_R)$ . Any subgroup  $Q$  of  $P$  is isomorphic in  $\mathcal{F}_{(P,e)}(G, b)$  to a subgroup  $R$  of  $P$  such that  $N_P(R)$  is a defect group of  $e_R$  viewed as block of  $N_G(R, e_R)$  (cf. [1] or [14]). We say that  $(Q, e_Q)$  is an *Alperin-Goldschmidt-pair* (for  $\mathcal{F}_{(P,e)}(G, b)$ ), if  $(Q, e_Q)$  is centric, radical and  $N_P(Q)$  is a defect group of  $kN_G(Q, e_Q)e_Q$ . If  $Q$  is normal in  $P$ , then  $P$  is a defect group of  $e_Q$  as block of  $N_G(Q, e_Q)$ , and hence  $(P, e_P)$  is also a maximal  $(N_G(Q, e_Q), e_Q)$ -subpair. It has been shown by Puig, that  $(Q, e_Q)$  is centric if and only if  $C_P(R) = Z(R)$  for any subgroup  $R$  of  $P$  which is isomorphic to  $Q$  in  $\mathcal{F}_{(P,e)}(G, b)$ . Thus the property of being centric can be read off the category  $\mathcal{F}_{(P,e)}(G, b)$ . Furthermore, the automorphism group of  $Q$  in  $\mathcal{F}_{(P,e)}(G, b)$  is canonically isomorphic to  $N_G(Q, e_Q)/C_G(Q)$ .

A *conjugation family* for  $\mathcal{F}_{(P,e)}(G, b)$  is a set  $\mathcal{C}$  of subgroups of  $P$  with the following property: every isomorphism in  $\mathcal{F}_{(P,e)}(G, b)$  is the composition of isomorphisms of the form  $\varphi : Q \rightarrow R$ , where  $Q, R$  are subgroups of  $P$ , such that there exists a subgroups  $S$  in  $\mathcal{C}$  containing both  $Q, R$  and an element  $x \in N_G(S, e_S)$  satisfying  $\varphi(u) = xux^{-1}$  for all  $u \in Q$ .

It is well-known and easy to check that if  $\mathcal{C}$  is a conjugation family for  $\mathcal{F}_{(P,e)}(G, b)$ , then any subset  $\mathcal{C}'$  of  $\mathcal{C}$  such that any object in  $\mathcal{C}$  is isomorphic to an object of  $\mathcal{C}'$  in  $\mathcal{F}_{(P,e)}(G, b)$  is again a conjugation family.

By Alperin's fusion theorem (in its refined version by Goldschmidt and adapted to blocks, cf. [1, §4]), the set of subgroups  $Q$  of  $P$  for which  $(Q, e_Q)$  is an Alperin-Goldschmidt pair is a conjugation family for  $\mathcal{F}_{(P,e)}(G, b)$ , called the *Alperin-Goldschmidt conjugation family* for  $\mathcal{F}_{(P,e)}(G, b)$ .

**Definition 2.1** A subgroup  $H$  of  $G$  *controls fusion* in  $\mathcal{F}_{(P,e)}(G, b)$  if  $H$  contains  $P$  and if  $\mathcal{F}_{(P,e)}(G, b) \subseteq \mathcal{F}_S(H)$  for some Sylow- $p$ -subgroup  $S$  of  $H$  which contains  $P$ .

By Alperin's fusion theorem, a subgroup  $H$  of  $G$  containing  $P$  controls fusion in  $\mathcal{F}_{(P,e)}(G, b)$  if and only if  $N_G(Q, e_Q) = N_H(Q, e_Q)C_G(Q)$  for any subgroup  $Q$  of  $P$ .

**Lemma 2.2.** *Let  $W$  be a normal subgroup in  $P$ , and let  $H$  be a subgroup of  $G$  such that  $P \subseteq H \subseteq N_G(W)$ . Assume that  $H$  controls fusion in  $\mathcal{F}_{(P,e)}(G, b)$ . Then  $W$  is contained in any subgroup  $Q$  of  $P$  such that  $(Q, e_Q)$  is centric and radical.*

*Proof.* Let  $Q$  be a subgroup of  $P$  such that  $(Q, e_Q)$  is centric and radical. Since  $N_G(Q, e_Q) = N_H(Q, e_Q)C_G(Q)$  and  $W$  is normal in  $H$ , the image of  $N_W(Q)$  is normal in  $N_G(Q, e_Q)/QC_G(Q)$ , hence  $N_W(Q) \subseteq QC_G(Q)$  as  $(Q, e_Q)$  is radical. Thus  $N_W(Q) \subseteq Q$  because  $(Q, e_Q)$  is centric, and therefore  $W \subseteq Q$ .  $\square$

The first statement of the following Proposition is a variation of [10, Statement 1]. The second statement makes precise what it means, in certain circumstances, for a subgroup to control fusion.



**Proposition 2.3.** *Let  $Q$  be a subgroup of  $P$ , let  $H$  be a subgroup of  $N_G(Q)$  containing  $QC_G(Q)$ , and let  $c$  be the unique block of  $H$  such that  $\text{Br}_Q(c)e_Q = e_Q$ . Assume that  $c$  has a defect group  $R$  contained in  $P$ . Then  $(R, e_R)$  is a maximal  $(H, c)$ -subpair, and we have  $\mathcal{F}_{(R, e_R)}(H, c) \subseteq \mathcal{F}_{(P, e)}(G, b)$ ; moreover, this inclusion is an equality if and only if  $H$  controls fusion in  $\mathcal{F}_{(P, e)}(G, b)$ .*

*Proof.* Since  $Q$  is normal in  $H$ ,  $Q$  is contained in any defect group of  $H$ . If  $R$  is a defect group of  $c$  contained in  $P$ , then  $C_G(R) \subseteq C_G(Q) \subseteq H$ , and thus  $(R, e_R)$  is a - necessarily maximal -  $(H, c)$ -subpair. Let  $(S, f)$  be a centric radical  $(H, c)$ -subpair contained in  $(R, e_R)$ . Again, since  $Q$  is normal in  $H$ , we have  $Q \subseteq S$  by 2.2. Then  $C_G(S) = C_H(S)$ , and so  $f = e_S$ . Thus  $N_H(S, f) = N_H(S, e_S) \subseteq N_G(S, e_S)$ . The inclusion  $\mathcal{F}_{(R, e_R)}(H, c) \subseteq \mathcal{F}_{(P, e)}(G, b)$  follows, using Alperin's fusion theorem.

Assume that  $H$  controls fusion in  $\mathcal{F}_{(P, e)}(G, b)$ . Then in particular  $R = P$  is a defect group of  $c$ , as  $Q$  is normal in  $H$  and  $P$  is contained in  $H$ . Thus  $(P, e)$  is also a maximal  $(H, c)$ -subpair. Let now  $S$  be a subgroup of  $P$  such that  $(S, e_S)$  is a radical centric  $(G, b)$ -subpair. Thus  $Q \subseteq S$  by 2.2. But then  $C_G(S) \subseteq H$ , and so  $(S, e_S)$  is also a centric  $(H, c)$ -subpair. Thus the inclusion  $N_G(S, e_S) \subseteq N_H(S, e_S)C_G(S)$  translates to  $\mathcal{F}_{(P, e)}(G, b) \subseteq \mathcal{F}_{(P, e)}(H, c)$ , hence equality by the first statement. The rest is clear.  $\square$

Proposition 2.3 applies in the following two situations. If  $H$  contains  $N_G(P)$  and if  $c$  is the unique block of  $H$  such that  $\text{Br}_P(c) = \text{Br}_P(b)$ , then  $(P, e)$  is also a maximal  $(H, c)$ -subpair. Thus if  $P$  has a subgroup  $Q$  such that  $C_G(Q) \subseteq H \subseteq N_G(Q)$ , we have  $\mathcal{F}_{(P, e)}(H, c) \subseteq \mathcal{F}_{(P, e)}(G, b)$ . The second situation, in which we are going to apply 2.3 arises if  $H = N_G(Q, e_Q)$  for some subgroup  $Q$  of  $P$  and if  $c = e_Q$  such that  $N_P(Q)$  is a defect group of  $c$  (viewed as block of  $H$ ).

The next Proposition is a particular case of Külshammer-Puig [11, Theorem 1.8], translated to our terminology (see also [10, Statement 8]).

**Proposition 2.4.** *Assume that  $G = N_G(Q, e_Q)$  for some subgroup  $Q$  of  $P$  such that  $(Q, e_Q)$  is centric. Then  $b = e_Q$ , and there is a short exact sequence of finite groups*

$$1 \longrightarrow Q \longrightarrow L \longrightarrow G/QC_G(Q) \longrightarrow 1$$

*such that  $P$  is a Sylow- $p$ -subgroup of  $L$  and such that  $\mathcal{F}_{(P, e)}(G, b) = \mathcal{F}_P(L)$ . Moreover, we have  $O_{p'}(L) = 1$  and  $C_L(Q) = Z(Q)$ ; in particular,  $L$  is  $p$ -constrained.*

*Proof.* As  $Q$  is normal in  $G$ , the block idempotent  $b$  is contained in  $kC_G(Q)$ , and as  $G$  stabilises  $e_Q$ , we have  $b = e_Q$  (this is a standard argument; see [1]). To establish the link with the terminology in [11, 1.8], note first that  $P$  is also a defect group of  $\{b\}$  viewed as point of  $G$  on  $kC_G(Q)$ , because  $P$  is maximal with respect to the property  $\text{Br}_P(b) \neq 0$ . The existence of a canonical exact sequence as stated such that  $P$  is a Sylow- $p$ -subgroup of  $L$  is a particular case of [11, 1.8]. This extension has the property, that for any  $y \in L$ , the outer automorphisms of  $Q$  induced by conjugation with  $y$  and by conjugation with some element  $x \in G$  such that  $xQC_G(Q)$  is the image of  $y$  in  $G/QC_G(Q)$  coincide.

In particular, if  $y \in C_L(Q)$  then  $x \in QC_G(Q)$ , and hence  $y \in Q$ . This shows that  $C_L(Q) = Z(Q)$ , and since  $Q$  is normal in  $L$ , we have  $O_{p'}(L) = O_{p'}(C_L(Q)) = 1$ . The equality  $\mathcal{F}_{(P,e)}(G, b) = \mathcal{F}_P(L)$  is essentially a reformulation of [11, 1.8.2]; we reproduce the argument from [10, Statement 8]. Since  $Q$  is normal in  $L$  and in  $G$ , it suffices to show that the images in  $\text{Aut}(R)$  of  $N_G(R, e_R)$  and  $N_L(R)$  are equal, where  $R$  is a subgroup of  $P$  containing  $Q$ . As  $(Q, e_Q)$  is centric, so is  $(R, e_R)$ . Similarly, as  $C_L(Q) = Z(Q)$ , we have  $C_L(R) = Z(R)$ . Setting  $\bar{G} = G/QC_G(Q)$ , with the notation of [11, 1.8] (which is defined in [11, 2.8]) we have  $E_{G, \bar{G}}(R, e_R) = E_{L, \bar{G}}(R)$ . By [11, (2.8.1)], the canonical maps  $E_{G, \bar{G}}(R, e_R) \rightarrow E_G(R, e_R)$  and  $E_{L, \bar{G}}(R) \rightarrow E_L(R)$  are surjective. Thus  $E_G(R, e_R) = E_L(R)$ . This implies the equality  $\mathcal{F}_{(P,e)}(G, b) = \mathcal{F}_P(L)$ .  $\square$

We need the following generalisation of [10, Statement 9].

**Proposition 2.5.** *Let  $Q$  be a normal subgroup of  $P$ , set  $H = N_G(Q)$  and denote by  $c$  the unique block of  $H$  such that  $e_Qc = e_Q$ . Suppose there is a finite group  $L$  having  $P$  as Sylow- $p$ -subgroup such that  $\mathcal{F}_{(P,e)}(G, b) = \mathcal{F}_P(L)$ . Then  $(P, e)$  is a maximal  $(H, c)$ -subpair,  $P$  is a Sylow- $p$ -subgroup of  $N_L(Q)$ , and we have  $\mathcal{F}_{(P,e)}(H, c) = \mathcal{F}_P(N_L(Q))$ .*

*Proof.* Since  $Q$  is normal in  $P$ , the pair  $(P, e_P)$  is also a maximal  $(H, c)$ -subpair, and clearly  $P$  is a Sylow- $p$ -subgroup of  $N_L(Q)$ . By 2.3, we have  $\mathcal{F}_{(P,e)}(H, c) \subseteq \mathcal{F}_{(P,e)}(G, b)$ . In order to show the equality  $\mathcal{F}_{(P,e)}(H, c) = \mathcal{F}_P(N_L(Q))$ , it suffices to show that  $N_H(S, f)$  and  $N_L(S) \cap N_L(Q)$  have the same images in  $\text{Aut}(S)$ , where  $(S, f)$  is an  $(H, c)$ -Brauer pair contained in  $(P, e)$ . Since  $Q$  is normal in  $H$  and  $N_L(Q)$ , we may assume that  $Q \subseteq S$ , by 2.2. Then  $C_G(S) \subseteq H$  and  $f = e_S$ . The assumption  $\mathcal{F}_{(P,e)}(G, b) = \mathcal{F}_P(L)$  implies that given any  $x \in N_G(S, e_S)$ , there is  $y \in N_L(S)$  such that  ${}^x u = {}^y u$  for all  $u \in S$ . Since  $Q \subseteq S$ , clearly  $x \in N_H(S, e_S)$  if and only if  $y \in N_L(S) \cap N_L(Q)$ . The equality  $\mathcal{F}_{(P,e)}(H, c) = \mathcal{F}_P(N_L(Q))$  follows.  $\square$

The following Lemma appears in a slightly more general version in Puig [12].

**Lemma 2.6.** *Let  $G$  be a finite group, let  $b$  be a block of  $G$  and let  $(Q, e)$ ,  $(R, f)$  be centric  $(G, b)$ -subpairs such that  $(Q, e) \subseteq (R, f)$ . We have*

$$N_G(R, f) \cap C_G(Q) = Z(Q)C_G(R) .$$

*Proof.* Clearly the right side is contained in the left side. For the converse, assume first that  $Q$  is normal in  $R$ . Let  $x \in N_G(R, f) \cap C_G(Q)$ . It is easy to check that  $[R, x] \subseteq C_R(Q) = Z(Q)$ . Thus  $[R, x, x] = 1$ . If  $x$  is a  $p'$ -element, this forces  $x \in C_G(R)$  by standard properties of coprime group actions (cf. [8]). Note that the image of a defect group of  $f$  as block of  $N_G(R, f)$  is a Sylow- $p$ -subgroup of  $N_G(R, f)/C_G(R)$ . Thus if  $x$  is a  $p$ -element, we may assume that  $x$  belongs to a defect group of  $f$  as block of  $N_G(R, f)$ , which implies  $x \in Z(Q)$ , as  $(Q, e)$  is centric. The general case follows by induction.  $\square$

**Proposition 2.7.** *Assume that there is a unique minimal subgroup  $R$  of  $P$  such that  $(R, e_R)$  is centric and radical. Then  $R$  is normal in  $P$ , the pair  $(P, e)$  is a maximal  $(N_G(R, e_R), e_R)$ -subpair, and we have  $\mathcal{F}_{(P, e)}(G, b) = \mathcal{F}_{(P, e)}(N_G(R, e_R), e_R)$ .*

*Proof.* The uniqueness of  $R$  implies that  $R$  is normal in  $P$ , and hence  $(P, e)$  is also a maximal  $(N_G(R, e_R), e_R)$ -subpair. Let  $S$  be a subgroup of  $P$  such that  $(S, e_S)$  is centric and radical. Then  $R \subseteq S$  by the uniqueness of  $(R, e_R)$ . If  $x \in N_G(S, e_S)$ , then  ${}^x(R, e_R) \subseteq (S, e_S)$ , and again, by the uniqueness of  $(R, e_R)$ , we deduce that  ${}^x(R, e_R) = (R, e_R)$ . In other words,  $N_G(S, e_S) \subseteq N_G(R, e_R)$ , which implies  $\mathcal{F}_{(P, e)}(G, b) \subseteq \mathcal{F}_{(P, e)}(N_G(R, e_R), e_R)$ , hence the equality by 2.3.  $\square$

We provide a criterion for when the Alperin-Goldschmidt conjugation family has a unique minimal element.

**Proposition 2.8.** *Assume that  $\mathcal{F}_{(P, e)}(G, b) = \mathcal{F}_{(P, e)}(N_G(Q, e_Q), e_Q)$  for some normal subgroup  $Q$  of  $P$  such that  $(Q, e_Q)$  is centric. Then there is a unique subgroup  $R$  of  $P$  containing  $Q$  such that  $O_p(N_G(Q, e_Q)/QC_G(Q)) = RC_G(Q)/QC_G(Q)$ . The group  $R$  is then the unique minimal subgroup of  $P$  such that  $(R, e_R)$  is centric and radical. In particular,  $R$  is normal in  $P$  and  $\mathcal{F}_{(P, e)}(G, b) = \mathcal{F}_{(P, e)}(N_G(R, e_R), e_R)$ .*

*Proof.* We may assume that  $G = N_G(Q, e_Q)$  and hence that  $b = e_Q$ . The image of  $P$  in  $G/QC_G(Q)$  is a Sylow- $p$ -subgroup; since  $(Q, e_Q)$  is centric, this image is isomorphic to  $P/Q$ . Therefore, there is a unique subgroup  $R$  of  $P$  containing  $Q$  such that the image of  $R$  in  $G/QC_G(Q)$  is  $O_p(G/QC_G(Q))$ . The uniqueness of  $R$  implies that  $R$  is normal in  $P$ . Note that  $b$  is still a block of  $RC_G(Q)$ , and then  $(R, e_R)$  is a maximal  $(RC_G(Q), b)$ -subpair. By our choice of  $R$ , the group  $RC_G(Q)$  is normal in  $G$ , and since  $RC_G(Q)$  acts transitively on the set of maximal  $(RC_G(Q), b)$ -subpairs, the Frattini argument shows that  $G = N_G(R, e_R)C_G(Q)$ .

Let  $S$  be a subgroup of  $P$  such that  $(S, e_S)$  is centric and radical. By Lemma 2.6, we have  $N_G(S, e_S) \cap QC_G(Q) = QC_G(S)$ . Thus the inclusion  $N_G(S, e_S) \subset G$  induces an injective group homomorphism  $N_G(S, e_S)/QC_G(S) \rightarrow G/QC_G(Q)$ . The image of  $R$  in  $G/QC_G(Q)$  is  $O_p(G/QC_G(Q))$ ; thus the image of  $N_R(S)$  in  $N_G(S, e_S)/QC_G(S)$  is contained in  $O_p(N_G(S, e_S)/QC_G(S))$ , and hence the image of  $N_R(S)$  in  $N_G(S, e_S)/SC_G(S)$  is contained in  $O_p(N_G(S, e_S)/SC_G(S)) = 1$ . This forces  $N_R(S) \subseteq SC_G(S)$ . As  $(S, e_S)$  is centric, we get  $N_R(S) \subseteq S$ , hence  $R \subseteq S$ .

By Lemma 2.6 again, we have  $N_G(R, e_R) \cap RC_G(Q) = RC_G(R)$ . As  $G = N_G(R, e_R)C_G(Q)$ , it follows that  $N_G(R, e_R)/RC_G(R) \cong G/RC_G(Q)$ , and hence  $O_p(N_G(R, e_R)/RC_G(R)) = 1$  by our choice of  $R$ . This shows that  $R$  is indeed the unique minimal subgroup of  $P$  such that  $(R, e_R)$  is centric and radical. The rest is clear by 2.7.  $\square$

3 LOCAL CONTROL OF CHARACTERISTIC  $p$ -FUNCTORS

Let  $G$  be a finite group, let  $b$  be a block of  $G$ , let  $(P, e)$  be a maximal  $(G, b)$ -subpair, and for any subgroup  $Q$  of  $P$ , denote by  $e_Q$  the unique block of  $C_G(Q)$  such that  $(Q, e_Q) \subseteq (P, e)$ .

Given a positive characteristic  $p$ -functor  $W$  and a subgroup  $Q$  of  $P$ , we set  $W_1(Q) = Q$  and  $P_1(Q) = N_P(Q)$ . For any positive integer  $i$ , we define inductively  $W_{i+1}(Q) = W(P_i(Q))$  and  $P_{i+1}(Q) = N_P(W_{i+1}(Q))$ . For all positive integers  $i$  we have  $W_i(Q) \subseteq P_i(Q)$ , and if  $P_i(Q)$  is a proper subgroup of  $P$ , in fact  $P_i(Q)$  is a proper subgroup of  $P_{i+1}(Q)$ . In particular,  $P_i(Q) = P$  for all large enough  $i$ . We will say that  $Q$  is *well-placed in  $P$*  (with respect to  $W$  and  $\mathcal{F}_{(P,e)}(G, b)$ ) if  $P_i(Q)$  is a defect group of the block  $e_{W_i(Q)}$  as block of  $N_G(W_i(Q), e_{W_i(Q)})$  for all positive integer  $i$ . Clearly  $P$  is always well-placed in  $P$ .

The next Lemma states essentially that every subgroup of  $P$  is isomorphic to a well-placed subgroup with respect to  $\mathcal{F}_{(P,e)}(G, b)$  and a positive characteristic  $p$ -functor.

**Lemma 3.1.** *Let  $W$  be a positive characteristic  $p$ -functor. For any subgroup  $Q$  of  $P$ , there is an element  $x \in G$  such that  ${}^x(Q, e_Q) \subset (P, e)$ ,  ${}^x N_P(Q) \subseteq P$  and such that  ${}^x Q$  is well-placed in  $P$ .*

*Proof.* Define sequences of subgroups and blocks as follows. Let  $V_1 := Q$ ,  $v_1 := e_Q$ . Let  $(R_1, r_1)$  be a  $b$ -subpair which is maximal with respect to normalising  $(V_1, v_1)$  and such that  $(N_P(Q), e_{N_P(Q)}) \leq (R_1, r_1)$ . For  $i \geq 1$  let  $V_{i+1} = W(R_{i+1})$  and let  $(V_{i+1}, v_{i+1})$  be the  $b$ -subpair contained in  $(R_{i+1}, r_{i+1})$ . Let  $(R_{i+1}, r_{i+1})$  be a  $b$ -subpair which is maximal with respect to normalising  $(V_{i+1}, v_{i+1})$  and such that  $(R_i, r_i) \leq (R_{i+1}, r_{i+1})$ . Note that if  $(S, f)$  is a maximal  $b$ -subpair containing  $(R_{i+1}, r_{i+1})$ , then  $R_{i+1} = N_S(V_{i+1})$ . On the other hand,  $N_S(R_i) \subset N_S(V_{i+1})$ . Thus, either  $R_i = S$  or  $R_{i+1}$  properly contains  $R_i$ . In other words, there exists an integer  $t$  such that for all  $i \geq t$ ,  $(R_i, r_i) = (R_t, r_t)$  is a maximal  $b$ -Brauer pair,  $(V_i, v_i) = (W(R_t), v_t)$ . Let  $x \in G$  be such that  ${}^x(R_t, r_t) = (P, e)$ . Then  ${}^x(Q, e_Q) \leq (P, e)$ , and since for every  $i \geq 1$ ,  ${}^x R_i \subset {}^g R_t = P$ , it is clear that  ${}^x(Q, e_Q)$  is well placed in  $(P, e)$ . The second assertion is clear since  $N_P(Q) \subset R_1 \subset {}^{x^{-1}}P$ .  $\square$

The next results states roughly speaking, that “if a positive characteristic  $p$ -functor controls fusion locally, it controls fusion globally”. This generalises a result by Alperin and Gorenstein (cf. [9, Ch. X, Theorem 9.3])

**Proposition 3.2.** *Let  $W$  be a positive characteristic  $p$ -functor. Assume that for any non-trivial subgroup  $Q$  of  $P$  and any maximal  $(N_G(Q, e_Q), e_Q)$ -subpair  $(R, f)$ , the group  $N_{N_G(Q, e_Q)}(W(R))$  controls fusion in  $\mathcal{F}_{(R,f)}(N_G(Q, e_Q), e_Q)$ . Then  $N_G(W(P))$  controls fusion in  $\mathcal{F}_{(P,e)}(G, b)$ .*

*Proof.* Set  $H = N_G(W(P))$ . Suppose, if possible that the result is not true. Then by 3.1 above, there exists a non-trivial subgroup  $Q$  of  $P$  such that  $(Q, e_Q)$  is well placed in  $(P, e)$  such that  $N_G(Q, e_Q)$  is not contained in  $C_G(Q)N_H(Q, e_Q)$ .

We introduce the following notation. For  $i \geq 1$ , let  $W_i = W_i(Q)$ ,  $P_i = P_i(Q)$ ,  $e_i = e_{W_i}$ ,  $N_i = N_G(W_i, e_i)$ ,  $M_i = N_G(W_i)$  and  $L_i = N_i \cap N_G(W_{i+1})$ . Let  $f_i$  be the block of  $M_i$  satisfying  $e_i f_i = e_i$ . Let  $s_i$  be the block of  $L_i$  such that  $\text{Br}_{P_i}(s_i) = \text{Br}_{P_i}(e_i)$ .

Set  $\mathcal{F}_i = \mathcal{F}_{(P_i, e_{P_i})}(N_i, e_i)$ , set  $\mathcal{G}_i = \mathcal{F}_{(P_i, e_{P_i})}(L_i, s_i)$ , and set  $\mathcal{H}_i = \mathcal{F}_{(P_i, e_{P_i})}(M_i, f_i)$ .

It is clear from 2.3 that  $\mathcal{G}_i \subset \mathcal{F}_i$ . On the other hand,  $P_i C_{M_{i+1}}(W_i) \subset L_i \subset N_{M_{i+1}}(W_i)$ . Since  $(W_{i+1}, e_{i+1}) \leq (P_i, e_{P_i})$ ,  $\text{Br}_{P_i}(f_i)e_{P_i} = e_{P_i}$  and hence by 2.3 it follows that  $\mathcal{G}_i \subset \mathcal{H}_{i+1}$ . Since, clearly  $\mathcal{H}_{i+1} = \mathcal{F}_{i+1}$ , we get that  $\mathcal{G}_i \subset \mathcal{F}_{i+1}$ .

By the hypothesis of proposition, we have that  $\mathcal{G}_i = \mathcal{F}_i$ , hence, we get that for all  $i \geq 1$ ,  $\mathcal{F}_1 \subset \mathcal{F}_i \subset \mathcal{F}_{i+1}$ .

Let  $i$  be such that  $P_i = P$ , so that  $\mathcal{F}_{i+1} = \mathcal{F}_{(P, e)}(H, c)$ , where  $c$  is the Brauer correspondent of  $b$ . Let  $g$  be an element of  $N_G(Q, e_Q)$ . Then conjugation by  $g$  determines an element, say  $\phi$  of  $\text{End}_{\mathcal{F}_1}(Q)$ . Then  $\phi$  is induced by conjugation with an element  $x \in H$ , hence  $g = zx$  for some  $z \in C_G(Q)$ . Thus,  $N_G(Q, e_Q) \subset C_G(Q)(H \cap N_G(Q, e_Q))$ , contradicting our choice of  $(Q, e_Q)$ .  $\square$

#### 4 ON THE LOCAL STRUCTURE OF CENTRAL $p$ -EXTENSIONS

Let  $G$  be a finite group, let  $b$  be a block of  $G$ , and let  $(P, e)$  be a maximal  $(G, b)$ -subpair. We assume in this section that  $P$  contains a subgroup  $Z$  of  $Z(G)$ . We set  $\bar{G} = G/Z$  and  $\bar{P} = P/Z$ ; for any element or subset  $a$  of  $kG$ , we denote by  $\bar{a}$  its canonical image in  $k\bar{G}$ . It is well-known that the image  $\bar{b}$  of  $b$  in  $k\bar{G}$  is a block of  $\bar{G}$  having  $\bar{P}$  as defect group. The following (equally well-known) Lemma relates the local structures of  $b$  and  $\bar{b}$ .

**Lemma 4.1.** *For every  $(G, b)$ -subpair  $(Q, f)$  there is a unique  $(\bar{G}, \bar{b})$ -subpair of the form  $(\bar{Q}, g)$  such that  $\bar{f}g = \bar{f}$ , and then the canonical map  $G \rightarrow \bar{G}$  induces a surjective group homomorphism  $N_G(Q, f)/C_G(Q) \rightarrow N_{\bar{G}}(\bar{Q}, g)/C_{\bar{G}}(\bar{Q})$  whose kernel is an abelian  $p$ -group. In particular, if  $O_p(N_G(Q, f)/QC_G(Q)) = 1$ , this map induces an isomorphism  $N_G(Q, f)/QC_G(Q) \cong N_{\bar{G}}(\bar{Q}, g)/QC_{\bar{G}}(\bar{Q})$ .*

*Proof.* It is well-known (and easy to check) that the group  $\overline{C_G(Q)}$  is a normal subgroup of  $C_{\bar{G}}(\bar{Q})$  and that  $C_{\bar{G}}(\bar{Q})/\overline{C_G(Q)}$  is an abelian  $p$ -group. Thus any block of  $C_{\bar{G}}(\bar{Q})$  is contained in  $\overline{kC_G(Q)}$ . Hence the sum of the different  $C_{\bar{G}}(\bar{Q})$ -conjugates of  $\bar{f}$  is the unique block  $g$  of  $C_{\bar{G}}(\bar{Q})$  fulfilling  $\bar{f}g = \bar{f}$ , and we have  $N_{\bar{G}}(\bar{Q}, g) = \overline{N_G(Q, f)C_{\bar{G}}(\bar{Q})}$ . The Lemma follows.  $\square$

The above Lemma implies in particular, that the maximal  $(G, b)$ -subpair  $(P, e)$  determines a unique maximal  $(\bar{G}, \bar{b})$ -subpair  $(\bar{P}, f)$  by the condition  $\bar{e}f = \bar{e}$ . With this choice of maximal subpairs, 4.1 translates to the following statement.

**Proposition 4.2.** *The canonical map  $G \rightarrow \bar{G}$  induces a surjective functor  $\mathcal{F}_{(P,e)}(G, b) \rightarrow \mathcal{F}_{(\bar{P},f)}(\bar{G}, \bar{b})$ . In particular,  $b$  is  $SL(2, p)$ -free, if and only if  $\bar{b}$  is  $SL(2, p)$ -free.*

*Proof.* Clear by 4.1.  $\square$

**Proposition 4.3.** *Let  $H$  be a subgroup of  $G$  containing  $N_G(P)$  and denote by  $c$  the unique block of  $H$  such that  $\text{Br}_P(c) = \text{Br}_P(b)$ . Assume that there is a subgroup  $Q$  of  $P$  containing  $Z$  such that  $Q$  is normal in  $H$  and such that  $C_{\bar{G}}(\bar{Q}) \subseteq \bar{H}$ . Then  $\text{Br}_{\bar{P}}(\bar{c}) = \text{Br}_{\bar{P}}(\bar{b})$ , and we have  $\mathcal{F}_{(P,e)}(G, b) = \mathcal{F}_{(P,e)}(H, c)$  if and only if  $\mathcal{F}_{(\bar{P},f)}(\bar{G}, \bar{b}) = \mathcal{F}_{(\bar{P},f)}(\bar{H}, \bar{c})$ .*

*Proof.* The equality  $\text{Br}_{\bar{P}}(\bar{c}) = \text{Br}_{\bar{P}}(\bar{b})$  is clear by [10, Statement 5]. Suppose that  $\mathcal{F}_{\bar{G}, \bar{b}} = \mathcal{F}_{\bar{H}, \bar{c}}$ . Let  $(R, t)$  be a centric radical  $(G, b)$ -subpair. Let  $s$  be the unique block of  $C_{\bar{G}}(\bar{R})$  such that  $\bar{t}s = \bar{t}$ . By Lemma 4.1, we have  $N_G(R, t)/RC_G(R) \cong N_{\bar{G}}(\bar{R}, s)/\bar{R}C_{\bar{G}}(\bar{R}) = N_{\bar{H}}(\bar{R}, s)/\bar{R}C_{\bar{G}}(\bar{R}) \cap \bar{H}$ . Now  $Q$  is normal in  $H$ , and thus the canonical image of  $N_Q(R)$  is normal in  $N_G(R, t)/RC_G(R)$ . Therefore we have  $N_Q(R) \subseteq RC_G(R)$ . As the subpair  $(R, t)$  is centric, we have  $N_Q(R) \subseteq R$ , which forces  $Q \subseteq R$ . Thus  $C_{\bar{G}}(\bar{R}) \subseteq \bar{H}$  by the assumptions, and so  $(\bar{R}, s)$  is also an  $(\bar{H}, \bar{c})$ -subpair and  $(R, t)$  is an  $(H, c)$ -subpair. Therefore  $N_H(R, t)/RC_G(R)$  is a subgroup of  $N_G(R, t)/RC_G(R) \cong N_{\bar{G}}(\bar{R}, s)/\bar{R}C_{\bar{G}}(\bar{R}) = N_{\bar{H}}(\bar{R}, s)/\bar{R}C_{\bar{H}}(\bar{R})$ . But then Lemma 4.1, applied to  $H$  and  $c$  instead of  $G$  and  $b$ , respectively, shows that  $N_H(R, t) = N_G(R, t)$ , which implies the equality  $\mathcal{F}_{(P,e)}(G, b) = \mathcal{F}_{(P,e)}(H, c)$ . The converse is trivial.  $\square$

## 5 ON $SL(2, p)$ -FREE BLOCKS

**Proposition 5.1.** *Let  $G$  be a finite group and let  $b$  be a block of  $G$ . Suppose that  $SL(2, p)$  is involved in  $N_G(Q, f)/C_G(Q)$  for some non-trivial  $(G, b)$ -subpair  $(Q, f)$ . Then  $SL(2, p)$  is involved in  $N_G(Q, e)/C_G(Q)$  for some centric and radical  $(G, b)$ -subpair  $(Q, e)$ .*

*Proof.* Fix a maximal  $(G, b)$ -subpair  $(P, e_P)$ , and for any subgroup  $Q$  of  $P$ , denote by  $(Q, e_Q)$  the unique  $(G, b)$ -subpair contained in  $(P, e_P)$ . Let  $Q$  be a subgroup of  $P$  with  $|Q|$  maximum such that  $SL(2, p)$  is involved in  $N_G(Q, e_Q)/C_G(Q)$ . Replacing  $(Q, e_Q)$  with a  $G$ -conjugate if necessary, we may assume that  $N_P(Q)$  is a defect group of  $kN_G(Q, e_Q)e_Q$ , so that in particular,  $R = QC_P(Q)$  is a defect group of  $kQC_G(Q)e_Q$  and  $(R, e_R)$  is a maximal  $(QC_G(Q), e_Q)$ -pair. Since  $QC_G(Q)$  is normal in  $N_G(Q, e_Q)$ , the Frattini argument gives  $N_G(Q, e_Q) = C_G(Q)[N_G(R, e_R) \cap N_G(Q, e_Q)]$ . But then,  $N_G(Q, e_Q)/C_G(Q) \cong N_G(R, e_R) \cap N_G(Q, e_Q)/N_G(R, e_R) \cap C_G(Q)$ . On the other hand, since  $C_G(R) \subseteq C_G(Q)$ ,  $N_G(R, e_R) \cap N_G(Q, e_Q)/N_G(R, e_R) \cap C_G(Q)$  is isomorphic to a subquotient of  $N_G(R, e_R)/C_G(R)$ . Hence  $SL(2, p)$  is involved in  $N_G(R, e_R)/C_G(R)$ . The choice of  $Q$  now implies that  $R = Q$  whence  $(Q, e_Q)$  is a centric  $(G, b)$ -pair.

Let  $M$  be the inverse image of  $O_p(N_G(Q, e_Q)/QC_G(Q))$  in  $N_G(Q, e_Q)$  and let  $S = M \cap N_P(Q)$ . Then  $S$  is a defect group of  $kMe_Q$ ,  $(S, e_S)$  is a maximal  $(M, e_Q)$ -pair. Since  $N_G(Q, e_Q)$  normalises  $M$ , the Frattini argument again gives that

$N_G(Q, e_Q) = M[N_G(S, e_S) \cap N_G(Q, e_Q)]$ . But  $M = (QC_G(Q))S$  whence  $N_G(Q, e_Q) = C_G(Q)[N_G(S, e_S) \cap N_G(Q, e_Q)]$ . Arguing as before, we conclude that  $S = Q$  and hence that  $M = QC_G(Q)$ . This completes the proof.  $\square$

The main application of 5.1 is the following proposition which shows that the property of being  $SL(2, p)$ -free passes down to corresponding blocks of normalisers of subpairs.

**Proposition 5.2.** *Let  $G$  be a finite group and let  $b$  be an  $SL(2, p)$ -free block of  $G$ . For every  $(G, b)$ -subpair  $(Q, f)$  the block  $f$  of  $N_G(Q, f)$  is  $SL(2, p)$ -free.*

*Proof.* Let  $(R, g)$  be a centric radical  $(N_G(Q, f), f)$ -subpair. Then  $Q \subseteq R$  by 2.2, and hence  $C_G(R) \subseteq N_Q(Q, f)$ . Thus  $(R, g)$  is a  $(G, b)$ -Brauer pair, and hence  $SL(2, p)$  is not a subquotient of  $N_G(R, g)/C_G(R)$  by 5.1. But then  $SL(2, p)$  is obviously not a subquotient of  $N_{N_G(Q, f)}(R, g)/C_{N_G(Q, f)}(R)$ .  $\square$

## 6 PROOF OF THEOREM 1.4 AND THEOREM 1.7

*Proof of Theorem 1.4.* Let  $G$  be a finite group, let  $b$  be a block of  $G$ , let  $(P, e)$  be a maximal  $(G, b)$ -subpair, and for any subgroup  $Q$  of  $P$ , denote by  $e_Q$  the unique block of  $C_G(Q)$  such that  $(Q, e_Q) \subseteq (P, e)$ . Let  $W$  be a Glauberman functor, set  $N = N_G(W(P))$  and denote by  $c$  the unique block of  $N$  such that  $\text{Br}_P(c) = \text{Br}_P(b)$ . Assume that  $p$  is odd.

Suppose that Theorem 1.4 fails for the blocks  $b$  and  $c$  of  $G$  and  $N$ , respectively, and assume that  $|G|$  has minimal order with this property. We are going to derive a contradiction, proceeding in several steps.

**6.1.** *We have  $O_p(G) \neq 1$ .*

*Proof.* If  $O_p(G) = 1$ , then for any nontrivial  $(G, b)$ -Brauer pair  $(Q, f)$ , the group  $N_G(Q, f)$  is a proper subgroup of  $G$ . Since  $f$  is  $SL(2, p)$ -free by 5.2, the induction hypothesis implies that Theorem 1.4 holds for the block  $f$  of  $N_G(Q, f)$ . But then 3.2 implies, that Theorem 1.4 holds for the block  $b$  of  $G$ , contradicting our choice of  $b$ .  $\square$

From now on, we set  $Q = O_p(G)$ . Since  $Q$  is normal in  $G$ , the block  $b$  lies in  $kC_G(Q)$  (cf. [1, (2.9)(1)]). Thus  $b = \text{Tr}_{N_G(Q, e_Q)}^G(e_Q)$ . But then  $\mathcal{F}_{(P, e)}(G, b) = \mathcal{F}_{(P, e)}(N_G(Q, e_Q), e_Q)$ . If  $N_G(Q, e_Q)$  is a proper subgroup of  $G$ , the induction hypothesis implies that Theorem 1.4 holds for the block  $e_Q$  of  $N_G(Q, e_Q)$ , and hence for the block  $b$ , contradicting again our choice of  $b$ . This proves the following.

**6.2.** *We have  $G = N_G(Q, e_Q)$  and  $b = e_Q$ .*

Then  $b$  is a block for any subgroup of  $G$  containing  $C_G(Q)$ . In particular,  $b$  is a block of  $QC_G(Q)$ . Set  $R = QC_P(Q)$ . Then  $(R, e_R)$  is a maximal  $(QC_G(Q), b)$ -subpair (cf. [1, (2.9)(6)]). Note that  $C_G(R) \subseteq C_G(Q)$  and that  $R$  is normal in  $P$ . Since the maximal  $(QC_G(Q), b)$ -subpairs are  $QC_G(Q)$ -conjugate, a Frattini argument shows that

**6.3.** *we have  $G = N_G(R, e_R)C_G(Q)$ .*

If  $R = Q$ , then  $(R, e_R) = (Q, b)$  is  $(G, b)$ -centric. The group  $L$  occurring in the Külshammer-Puig-extension [11, Theorem 1.8]

$$1 \longrightarrow Q \longrightarrow L \longrightarrow G/QC_G(Q) \longrightarrow 1$$

is  $p$ -constrained and does not have  $SL(2, p)$  as subquotient (cf. 2.4). Thus  $W(P)$  is normal in  $L$ . Since  $\mathcal{F}_P(L) = \mathcal{F}_{(P, e)}(G, b)$ , this implies  $\mathcal{F}_{(P, e)}(G, b) = \mathcal{F}_{(P, e)}(N, c)$ , contradicting our choice of  $b$ .

Thus  $Q$  is a proper subgroup of  $R$ . Since  $Q = O_p(G)$ , it follows that  $N_G(R, e_R)$  is a proper subgroup of  $G$ . So Theorem 1.4 applies to the block  $e_R$  of  $N_G(R, e_R)$ , which has still  $P$  as defect group as  $R$  is normal in  $P$ .

We assume now that  $PC_G(Q)$  is a proper subgroup of  $G$ , and derive a contradiction; that is, we are going to show that then  $(G, b)$  cannot be a counterexample to Theorem 1.4. We do this by showing that  $N_G(S, e_S) = N_N(S, e_S)C_G(S)$  for any subgroup  $S$  of  $P$  containing  $Q$  such that  $(S, e_S)$  is  $(G, b)$ -centric. We argue by induction over the order of  $S$ . Up to replacing  $(S, e_S)$  by some  $G$ -conjugate, we may assume that  $N_P(S)$  is a defect group of  $e_S$  as block of  $N_G(S, e_S)$ . The subgroup  $N_G(S, e_S) \cap SC_G(Q)$  is normal in  $N_G(S, e_S)$  and contains  $C_G(S)$ . Thus  $e_S$  is a block of  $N_G(S, e_S) \cap SC_G(Q)$  having as defect group the group  $T = N_P(S) \cap SC_G(Q) = N_P(S) \cap SC_P(Q) = N_{SR}(Q)$ , as  $R = QC_P(Q)$ . Therefore,  $(T, e_T)$  is a maximal  $(N_G(S, e_S) \cap SC_G(Q), e_S)$ -subpair. The Frattini argument yields

$$N_G(S, e_S) = (N_G(S, e_S) \cap N_G(T, e_T)) \cdot (N_G(S, e_S) \cap SC_G(Q)) .$$

Now  $(S, e_S)$  is also a  $(PC_G(Q), b)$ -subpair contained in  $(P, e)$ . As  $PC_G(Q)$  is assumed to be a proper subgroup of  $G$ , it follows that  $N_G(S, e_S) \cap SC_G(Q) = (N_N(S, e_S) \cap SC_G(Q))C_G(S)$ . If  $S$  does not contain  $R$ , then  $S$  is properly contained in  $SR$ , hence properly contained in  $T = N_{SR}(S)$ . By induction, we get  $N_G(T, e_T) = N_N(T, e_T)C_G(T)$ . Together we get  $N_G(S, e_S) \subseteq NC_G(S)$ , hence  $N_G(S, e_S) = N_N(S, e_S)C_G(S)$ .

Thus, we may assume that  $R \subseteq S$ . Then  $C_G(S) \subseteq C_G(R) \subseteq N_G(R, e_R) \cap PC_G(Q)$ . Therefore,  $e_R$  is a block of the group  $N_G(R, e_R) \cap PC_G(Q)$ , having still  $(P, e)$  as maximal subpair. Let  $x \in N_G(S, e_S)$ . Since  $G = N_G(R, e_R)C_G(Q)$ , we can write  $x = nc$  for some  $n \in N_G(R, e_R)$  and some  $c \in C_G(Q)$ . Then  ${}^c(S, e_S) = {}^{n^{-1}}(S, e_S)$ . This implies that  ${}^cS \subseteq N_G(R, e_R) \cap PC_G(Q)$  and that  $(R, e_R) \subseteq {}^c(S, e_S)$ . Thus  ${}^c(S, e_S)$  is a  $(N_G(R, e_R) \cap PC_G(Q), e_R)$ -subpair. Therefore, there is  $y \in N_G(R, e_R) \cap PC_G(Q)$  such that  ${}^{y^c}(S, e_S) \subseteq (P, e)$ . We have  $x = nc = (ny^{-1})(yc)$ . The element  $yc$  belongs to the group  $PC_G(Q)$ , and conjugation by  $yc$  is a morphism in the category  $\mathcal{F}_{(P, e)}(PC_G(Q), b)$  from  $S$  to  ${}^{y^c}S$ . As  $PC_G(Q)$  is assumed to be a proper subgroup of  $G$ , this implies that  $yc \in (N \cap PC_G(Q))C_G(S)$ . The element  $ny^{-1}$  belongs to the group  $N_G(R, e_R)$ , and conjugation by  $ny^{-1}$  is a morphism from  ${}^{y^c}S$  to  ${}^xS = S$  in the category  $\mathcal{F}_{(P, e)}(N_G(R, e_R), e_R)$ . Since  $N_G(R, e_R)$  is a proper subgroup of  $G$ , it follows that  $ny^{-1} \in N_N(R, e_R)C_G({}^{y^c}S)$ . Together, we get  $x = (ny^{-1})(yc) \in NC_G(S)$ , hence  $N_G(S, e_S) = N_N(S, e_S)C_G(S)$ . This contradicts the fact that  $(G, b)$  is a counterexample to the Theorem. Therefore,



**6.4.** *we have  $G = PC_G(Q)$ .*

Set  $Z = Z(P) \cap Q$ ; since  $Q$  is normal in  $G$ , the group  $Z$  is non-trivial. Set  $\bar{G} = G/Z$ , and denote by  $\bar{b}$  the image of  $b$  in  $k\bar{G}$ . Thus  $\bar{b}$  is a block of  $k\bar{G}$  with defect group  $\bar{P} = P/Z$ . By 4.2, the block  $\bar{b}$  is  $SL(2, p)$ -free. Denote by  $H$  the inverse image in  $G$  of  $N_{\bar{G}}(W(\bar{P}))$ . Then  $H$  is the normaliser in  $G$  of a subgroup of  $P$  which contains  $Q$  properly, and so  $H$  is a proper subgroup of  $G$  fulfilling the hypotheses of 4.3. Denote by  $d$  the unique block of  $H$  such that  $\text{Br}_P(d) = \text{Br}_P(b)$ , and denote by  $\bar{d}$  the image of  $d$  in  $k\bar{H}$ . By the induction hypothesis, we have  $\mathcal{F}_{(\bar{P}, f)}(\bar{G}, \bar{b}) = \mathcal{F}_{(\bar{P}, f)}(\bar{H}, \bar{d})$ , where  $f$  is the unique block of  $C_{\bar{G}}(\bar{P})$  such that  $\bar{e}f = f$ . But then 4.3 implies that we have  $\mathcal{F}_{(P, e)}(G, b) = \mathcal{F}_{(P, e)}(H, d)$ . Since  $H$  is a proper subgroup of  $G$ , by induction again, we have  $\mathcal{F}_{(P, e)}(G, b) = \mathcal{F}_{(P, e)}(N, c)$  by 2.3. This contradicts our choice of  $b$  and completes the proof of Theorem 1.4.  $\square$

*Proof of Theorem 1.7.* Since  $b$  is  $SL(2, p)$ -free, we apply Theorem 1.4 with the Glauberman functor  $W$  mapping  $P$  to  $Q = K_\infty(P)$ . Then the subpair  $(Q, e_Q)$  is centric, and its normaliser controls fusion in  $\mathcal{F}_{(P, e)}(G, b)$ . The Theorem follows immediately from 2.8 and 2.7.  $\square$

## 7 PROOF OF THEOREM 1.5 AND 1.6

*Proof of Theorem 1.5.* In this section, we keep the notation of Theorem 1.5; that is, we let  $G$  be a finite group, and let  $H$  be a normal subgroup of  $G$ . Let  $c$  be a  $G$ -stable block of  $H$  and let  $b$  be a block of  $G$ , which covers  $c$ ; that is,  $b$  satisfies  $bc = b$ . Let  $(P, e)$  be a maximal  $(G, b)$ -subpair and set  $Q = P \cap H$ . Then clearly  $Q$  is a defect group of the block  $c$  of  $N$ . Let  $(Q, e_Q)$  be the unique  $(G, b)$ -subpair contained in  $(P, e)$  and let  $f$  be a block of  $C_H(Q)$  covered by the block  $e_Q$  of  $C_G(Q)$ , i.e. such that  $e_Q f \neq 0$ . Then  $(Q, f)$  is a maximal  $(H, c)$ -subpair. If  $x \in N_G(Q, e_Q)$ , then  ${}^x f$  is a block of  $kC_H(Q)$  which is covered by  $e_Q$ , hence  $x = yz$  for some  $y \in C_G(Q)$  and some  $z \in [N_G(Q, e_Q) \cap N_G(Q, f)]$ . In other words, we have

$$N_G(Q, e_Q) = C_G(Q)[N_G(Q, e_Q) \cap N_G(Q, f)].$$

The group in square brackets has a block which induces up to the block  $kN_G(Q, e_Q)e_Q$  and thus contains a defect group of  $kN_G(Q, e_Q)e_Q$ . For some  $y \in C_G(Q)$ , we thus have

$${}^y P \leq N_G(Q, e_Q) \cap N_G(Q, f),$$

hence

$$P \leq N_G({}^{y^{-1}}(Q, e_Q)) \cap N_G({}^{y^{-1}}(Q, f)),$$

Since  ${}^{y^{-1}}(Q) = Q$  and since  $e_Q {}^{y^{-1}} f = {}^{y^{-1}}(e_Q f) \neq 0$ , on replacing  $(Q, f)$  by  ${}^{y^{-1}}(Q, f)$ , we may assume that  $P$  stabilises  $f$ , and this proves the first statement of the Theorem.

For any subgroup  $R$  of  $P$ , we let  $e_R$  be the unique block of  $C_G(R)$  such that  $(R, e_R) \leq (P, e)$ , and for a subgroup  $S$  of  $Q$ , we let  $f_S$  be the unique block of  $C_H(S)$  such that  $(S, f_S) \leq (Q, f)$ . Note that whenever  $R$  is a subgroup of  $P$ , the pair  $(R \cap H, f_{R \cap H})$  is stabilised by  $N_P(R \cap H)$  because this last group stabilises  $R \cap H$  and  $(Q, f)$ .

Let  $\mathcal{F}$  denote the Brauer category  $\mathcal{F}_{(P,e)}(G, b)$  and let  $\mathcal{H}$  denote the Brauer category  $\mathcal{F}_{(Q,f)}(H, c)$ . Let  $\mathcal{C}$  denote the Alperin-Goldschmidt conjugation family for  $\mathcal{F}$ .

Let  $\mathcal{D}$  denote the set of objects  $R$  of  $\mathcal{F}$  such that

- (i)  $N_P(R)$  is a defect group of  $kN_G(R, e_R)e_R$ .
- (ii)  $N_P(R \cap H)$  is a defect group of  $kN_G(R \cap H, e_{R \cap H})e_{R \cap H}$ .
- (iii)  $N_P(R \cap H)$  stabilises  $f_{R \cap H}$ .

**7.1.** *Every object in  $\mathcal{F}$  is isomorphic to an object in  $\mathcal{D}$  and  $\mathcal{C} \cap \mathcal{D}$  is a conjugation family for  $\mathcal{F}$ .*

*Proof.* Consider  $(R, e_R) \leq (P, e)$ . Let  $(S, u)$  be a  $(G, b)$ -Brauer pair such that  $S$  is maximal with respect to normalising  $(R, e_R)$ . Since  $N_G(R, e_R) \leq N_G(R \cap H, e_{R \cap H})$ , we may find a  $(G, b)$ -subpair  $(T, v)$  such that  $T$  is maximal with respect to normalising  $(R \cap H, e_{R \cap H})$  and such that  $S \leq T$ . Note that  $T$  is a defect group of  $kN_G(R \cap H, e_{R \cap H})e_{R \cap H}$  and  $S$  is a defect group of  $N_G(R, e_R)e_R$ .

For some element  $x$  of  $G$ , we have  ${}^x(T, v) \leq (P, e)$ . Thus we have  ${}^x(R \cap H, e_{R \cap H}) \leq {}^x(R, e_R) \leq {}^x(S, u) \leq {}^x(T, v) \leq (P, e)$ .

Clearly,  ${}^xT$  is a defect group of  $kN_G({}^x(R \cap H, e_{R \cap H})){}^xe_{R \cap H}$ , and  ${}^xS$  is a defect group of  $kN_G({}^x(R, e_R)){}^xe_R$ . Also,  ${}^xS = N_P({}^xR)$  and  ${}^xT = N_P({}^x(R \cap H))$ .

Hence, on replacing  $(R, e_R)$  by  ${}^x(R, e_R)$ , we may assume that  $(R, e_R)$  satisfies (i) and (ii) above. Statement (iii) is immediate from (i) and (ii), since  $P$  stabilises  $(Q, f)$ . This proves the first part of the proposition. Since the set of objects  $R$  of  $\mathcal{F}_{(P,e)}(G, b)$  for which  $(R, e_R)$  is a centric and radical  $(G, b)$ -subpair is invariant under  $\mathcal{F}$  isomorphism, this proves also the second part of Statement 7.1.  $\square$

Let  $\mathcal{E}$  be the Alperin-Goldschmidt conjugation family for  $\mathcal{F}_{(Q,f)}(H, c)$ .

**7.2.** *If  $R \in \mathcal{C} \cap \mathcal{D}$ , then  $R \cap H \in \mathcal{E}$ .*

*Proof.* Let  $R \in \mathcal{C} \cap \mathcal{D}$  and let  $\tilde{e}_{R \cap H}$  and  $\tilde{f}_{R \cap H}$  respectively denote the blocks of  $N_G(R \cap H)$  and  $N_H(R \cap H)$  induced from  $e_{R \cap H}$  and  $f_{R \cap H}$ . Since  $N_P(R \cap H)$  is a defect group of  $kN_G(R \cap H, e_{R \cap H})e_{R \cap H}$ ,  $N_P(R \cap H)$  is a defect group of  $kN_G(R \cap H)\tilde{e}_{R \cap H}$ . Since the block  $\tilde{e}_{R \cap H}$  of  $kN_G(R \cap H)$  covers the block  $\tilde{f}_{R \cap H}$  of  $kN_H(R \cap H)$ ,  $N_Q(R \cap H)$  is a defect group of  $kN_H(R \cap H)\tilde{f}_{R \cap H}$ ; hence the defect groups of  $kN_H(R \cap H, f_{R \cap H})f_{R \cap H}$  have order  $|N_Q(R \cap H)|$ . On the other hand, since  $(R, e_R) \in \mathcal{D}$ ,  $N_Q(R \cap H) \subseteq N_H(R \cap H, f_{R \cap H})$ , thus  $N_Q(R \cap H)$  is a defect group of  $kN_H(R \cap H, f_{R \cap H})f_{R \cap H}$ .

Next we show that  $(R \cap H, f_{R \cap H})$  is  $(H, c)$ -centric. For this, by the above remarks, it suffices to show that  $C_Q(R \cap H) = Z(R \cap H)$ . Choose  $p$ -regular  $y \in C_H(R \cap H) \cap N_G(R, e_R)$ . Then  $[R, y] \subseteq R \cap H$ , so that  $[R, y, y] = 1$ , and hence  $[R, y] = 1$  as  $y$  is  $p$ -regular. Hence  $[C_H(R \cap H) \cap N_G(R, e_R)]/C_H(R)$  is a  $p$ -group. On the other hand,  $C_H(R \cap H) \cap N_G(R, e_R)$  is clearly a normal subgroup of  $N_G(R, e_R)$ , and  $O_p(N_G(R, e_R)/RC_G(R)) = 1$ . Hence,  $C_H(R \cap H) \cap N_G(R, e_R) \subseteq RC_G(R)$ . Since

$C_P(R) = Z(R)$ , we get  $C_Q(R \cap H) \cap N_Q(R) \subseteq R$ . Since  $R$  normalises  $C_Q(R \cap H)$ , this means that  $R$  is its own normaliser in the  $p$ -group  $C_Q(R \cap H)R$  whence  $C_Q(R \cap H) \subseteq R$ .

It remains to show that  $O_p(N_H(R \cap H, f_{R \cap H})/(R \cap H)C_H(R \cap H)) = 1$ . So, let  $M$  be the full inverse image of  $O_p(N_H(R \cap H, f_{R \cap H})/(R \cap H)C_H(R \cap H))$  in  $N_H(R \cap H, f_{R \cap H})$ . Since  $N_P(R \cap H)C_H(R \cap H)/(R \cap H)C_H(R \cap H)$  is a Sylow- $p$  subgroup of  $N_H(R \cap H, f_{R \cap H})/(R \cap H)C_H(R \cap H)$ , we have  $M = (M \cap P)C_H(R \cap H)$ . We will show that  $M \cap P \subseteq R \cap H$ .

We have  $N_G(R \cap H, e_{R \cap H}) = C_G(R \cap H)[N_G(R \cap H, f_{R \cap H}) \cap N_G(R \cap H, e_{R \cap H})]$ , and  $C_H(R \cap H)$  is normal in  $N_G(R \cap H, e_{R \cap H})$ ; hence  $C_H(R \cap H)[M \cap N_G(R \cap H, e_{R \cap H})]$  is normal in  $N_G(R \cap H, e_{R \cap H})$ . Since  $C_H(R \cap H) \subset [M \cap N_G(R \cap H, e_{R \cap H})]$ , this means that  $[M \cap N_G(R \cap H, e_{R \cap H})]$  is normal in  $N_G(R \cap H, e_{R \cap H})$  and hence is normal in  $N_G(R, e_R)$ . By the definition of  $M$ , it follows that  $M \cap N_G(R \cap H, e_{R \cap H})/C_H(R \cap H)$  is a  $p$ -group. On the other hand, we have shown before that  $C_H(R \cap H) \cap N_G(R, e_R)/C_H(R)$  is a  $p$ -group. Hence,  $M \cap N_G(R, e_R)/C_H(R)$  is a normal  $p$  subgroup of  $N_G(R, e_R)/C_H(R)$ , and is therefore isomorphic to a normal  $p$ -subgroup of  $N_G(R, e_R)/C_G(R)$ . But then by choice of  $(R, e_R)$  it follows that  $M \cap N_G(R, e_R) \subset RC_G(R)$  whence  $M \cap N_P(R) \subset RC_P(R) \cap H \subset R \cap H$ . Since  $R$  normalises  $M \cap P$ , we see that  $M \cap P \subseteq R \cap H$ . This completes the proof.  $\square$

Now let  $V$  be a normal subgroup of  $Q$  and suppose that  $N_H(V)$  controls fusion in  $\mathcal{H}$  and let  $W$  be as in the statement of the Theorem.

**7.3.**  $N_H(W)$  controls fusion in  $\mathcal{H}$ . Further, if  $S$  is a subgroup of  $Q$  containing  $W$  then  $N_G(S, e_S) \subset N_G(W)$ .

*Proof.* Let  $(S, f_S) \leq (Q, f)$  and let  $x \in N_G(Q, f)$ . Since  $x^{-1}(S, f_S) \leq (Q, f)$ , we have that  $N_H(x^{-1}(S, f_S)) \subset C_H(x^{-1}S)N_H(V)$  whence  $N_H(S, f_S) \subset C_H(S)N_H(xV)$ . Thus  $N_H(xV)$  controls fusion in  $\mathcal{F}_{H,c}$  for all  $x \in N_G(Q, f)$ . It follows by Lemma 2.1 that if  $S \in \mathcal{E}$ , then  $N_H(S, e_S) \subseteq N_H(xV)$  for all  $x \in N_G(Q, f)$ , so that in particular,  $N_H(S, e_S) \subseteq N_H(W)$ . Hence  $N_H(W)$  controls fusion in  $\mathcal{F}_{H,c}$ .

Let  $S$  be a subgroup of  $Q$  containing  $W$  and let  $x \in N_G(S, e_S)$ . By the Frattini argument, we may write  $x = yz$ , where  $y \in N_G(Q, f)$  and  $z \in H$ . Then  $z(S, f_S) = y^{-1}x(S, f_S) \leq (Q, f)$ . Since  $N_H(W)$  controls fusion in  $\mathcal{F}_{H,c}$ , we may write  $z = ct$ , where  $c \in C_H(S) \subset N_H(W)$  and  $t \in N_H(W)$ . Since by definition of  $W$ ,  $y \in N_G(W)$ , we have  $x = yct \in N_G(W)$ .  $\square$

Let  $R \in \mathcal{C} \cap \mathcal{D}$ . Then by 7.2,  $R \cap H \in \mathcal{E}$ . In particular, by Lemma 2.1, we have that  $W \subset R \cap H$  and it follows by 7.3 that  $N_G(R \cap H, f_{R \cap H}) \subset N_G(W)$ . Hence,  $N_G(R, e_R) \subset N_G(R \cap H, e_{R \cap H}) \subset C_G(R \cap H)[N_G(R \cap H, e_{R \cap H}) \cap N_G(R \cap H, f_{R \cap H})] \subset N_G(W)$ . Theorem 1.5 now follows from 7.2 and the fact that  $P \subseteq N_G(W)$ .  $\square$

*Proof of 1.6.* By a standard argument we may assume that  $G$  stabilises the block  $c$ . Then 1.6 is an immediate consequence of 2.3 and Theorems 1.4 and 1.5.  $\square$

**Remark 7.4** The advantage of Theorem 1.6 is that if we wish to produce a single local subgroup controlling fusion in  $\mathcal{F}_{(P,e)}(G, b)$ , it is not really necessary to assume

that  $b$  is  $SL(2, p)$  free. This could be useful in some instances; for example, suppose that  $G = XwrS_n$  for some large integer  $n$  and some non-Abelian finite simple group  $X$ , while  $H$  is the base-group of the wreath product. It is quite possible for automizers of “diagonal-type”  $(G, b)$ -subpairs to involve  $SL(2, p)$  because of the action of the  $S_n$ , while automizers (in  $H$ ) of  $(H, c)$  might not involve  $SL(2, p)$ .

## 8 PROOF OF 1.10

*Proof of 1.10.* It is clear that the pair  $(P, e)$  is a maximal  $(N_G(R), c)$ -subpair. For a subgroup  $Q$  of  $R$ , we let  $(Q, f_Q)$  be the unique  $(N_G(R), c)$  subpair contained in  $(P, e)$ .

In [13], it is shown that if we are considering a group  $G$  such that DPC holds in every section of  $G$ , then in calculating the various quantities  $k_d(B, \lambda)$ , it is only necessary to consider chains of  $(G, b)$ -pairs whose initial objects are pairs  $(Q, e_Q)$  contained in  $(P, e)$  which are  $(G, b)$ -centric and radical. By Lemma 2.2, we have that for any such subpair  $(Q, e_Q)$ ,  $R \leq Q$ , and thus  $N_G(Q, e_Q) \subset C_G(Q)N_G(R) \subset N_G(R)$ . The fact that  $R \leq Q$  also implies that  $f_Q = e_Q$ . It follows that in the subpair version of (W)DPC, the contribution in  $kGb$  from chains beginning with  $(Q, e_Q)$  is the same as the contribution in  $kN_G(R)c$  from chains beginning with  $(Q, e_Q)$ . Similarly, it follows that if DPC holds in every proper section of  $G$ , then checking DPC for  $G$  reduces to checking that there is a defect-preserving bijection between irreducible characters of  $B$  lying over  $\lambda$  and irreducible characters in  $c$  lying over  $\lambda$ .  $\square$

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*Note added in proof:* Since this work was written, a related result of M. Lechuga (Theorem 7.11 in his thesis *Contribution à l'étude locale dans les groupes finis*, Publ. Math. Univ. Paris 7, tome IV, 1994) has been brought to our attention. Lechuga's result concerns the particular Glauberman functor  $ZL$  (defined by L. Puig), is valid for  $p \geq 5$ , and makes use of J. G. Thompson's classification of quadratic pairs. While, as stated, it does not imply the involvement of  $SL(2, p)$  in the relevant automizer, the  $PSL(2, p^n)$  and  $PSU(3, p^m)$  components he mentions arise because of quadratic action, so the presence of a genuine  $SL(2, p)$  is implicit.

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