QUILLEN STRATIFICATION FOR BLOCK VARIETIES

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Abstract. The classical results on stratifications for cohomology varieties of finite groups and their modules due to Quillen [19, 20] and Avrunin-Scott [3] carry over to the varieties associated with finitely-generated modules over p-blocks of finite groups, introduced in [16].

1 Introduction

Throughout this paper, k is an algebraically closed field of prime characteristic p. By a Theorem of Evens [12] and Venkov [22, 23], the cohomology ring $H^*(G, k)$ of a finite group G is a finitely generated graded commutative $k-$algebra. Thus its maximal ideal spectrum V_G is an affine variety. Quillen showed in [19, 20] that this variety has a stratification indexed by the conjugacy classes of non trivial elementary abelian $p-$subgroups of G. The cohomology variety $V_G(M)$ of a finitely generated $kG-$module M, introduced by Carlson [9, 10], is the maximal ideal spectrum of the quotient of $H^*(G, k)$ by the kernel of the canonical graded algebra homomorphism $H^*(G, k) \to \mathrm{Ext}^*_G(M, M)$ induced by tensoring with M over k. Avrunin and Scott showed in [3] that $V_G(M)$ has a similar stratification, generalising Quillen's results on V_G.

Since the definition of $V_G(M)$ involves the cohomology ring $H^*(G, k)$ - which is an invariant of the principal block of kG - , if M belongs to a non principal block b of kG, the variety $V_G(M)$ is not in general an invariant of M viewed as $kGb-$module. This motivates in [16] the definition of a variety $V_{G,b}(M)$, obtained as the maximal ideal spectrum of the quotient of the *block cohomology* $H^*(G, b)$ by the kernel of the canonical map $H^*(G, b) \to \mathrm{Ext}^*_{kGb}(M, M)$ defined in [15]. It is shown in [16, 4.4] that there is a finite surjective morphism $V_{G,b}(M) \to V_G(M)$; in particular, by a Theorem of Alperin and Evens [2], the dimension of $V_{G,b}(M)$ is equal to the complexity of M. The above morphism $V_{G,b}(M) \to V_G(M)$ is an isomorphism if b is the principal block. Moreover, by [16, 5.5], the variety $V_{G,b}(M)$ is invariant under splendid stable, derived or Morita equivalences.

The main result of Section 2 provides a way to compute the block varieties $V_{G,b}(M)$ in terms of the truncated restriction iM of M, where i is a source idempotent in $(kGb)^P$ for some defect group P of b and where iM is viewed as kP-module. Section 3 contains the technicalities related to the Evens norm map, which we are going to use in Section 4 in order to see that the proof of the stratification of Avrunin and

Scott for $V_G(M)$ in [3] can be adapted to get a stratification for $V_{G,b}(M)$, which coincides with that of $V_G(M)$ if b is the principal block of kG. Section 5 is finally devoted to describing an example which shows that it really matters to work with the truncated restriction $\mathrm{Res}_P(iM)$ and not just $\mathrm{Res}^G_P(M)$. This is because both the block cohomology $H^*(G,b)$ and the variety $V_{G,b}(M)$ are defined with respect to the choice of a source idempotent i, uniquely up to unique isomorphism.

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2 Block varieties and source idempotents

We describe briefly some aspects of the local structure of a block of a finite group in terms of Brauer pairs, introduced by Alperin and Broué in [1], and developed further in work of Broué and Puig [7] and Puig [18] (see Thévenaz [21] for a more detailed account).

Let G be a finite group and b a block of kG; that is, b is a primitive idempotent of $Z(kG)$. Let P be a defect group of b. Then P is a maximal $p-$subgroup of G with the property that $\mathrm{Br}_P(b) \neq 0$, where $Br_P : (kG)^P \to kC_G(P)$ is the Brauer homomorphism [21, §11]. Thus there is a primitive idempotent $i \in (kGb)^P$ satisfying $\mathrm{Br}_P(i) \neq 0$; any such idempotent is called a *source idempotent of the block* b. The algebra $ikGi$, together with the group homomorphism $P \to (ikGi)^\times$ mapping $u \in P$ to ui is a *source algebra of the block* b (cf. Puig [18]). By [7, 1.8], for any subgroup Q of P there is a unique block e_Q of $kC_G(Q)$ satisfying $\mathrm{Br}_Q(i)e_Q = \mathrm{Br}_Q(i)$.

The category $\mathcal{F}_{G,b}$ has as objects the set of subgroups of P; for any two subgroups Q, R, the set of morphisms from Q to R in $\mathcal{F}_{G,b}$ is the set of (necessarily injective) group homomorphisms $\varphi : Q \to R$ for which there is an element $x \in G$ satisfying $\varphi(u) = xux^{-1}$ for all $u \in Q$ and satisfying ${}^x(Q,e_Q) \subseteq (R,e_R)$ (the latter condition is equivalent to $xe_Qx^{-1} = e_{xQx^{-1}}$). The category $\mathcal{F}_{G,b}$ is, up to canonical isomorphism of categories, independent of the choice of the source idempotent i.

The *block cohomology of the block* b *of* kG, introduced in [15], is the graded subalgebra $H^*(G,b)$ of $H^*(P,k)$ consisting of all $\zeta \in H^*(P,k)$ which satisfy the stability conditions $\mathrm{Res}^P_Q(\zeta) = \mathrm{Res}_\varphi(\zeta)$ for any subgroup Q of P and any group homomorphism $\varphi : Q \to P$ belonging to the category $\mathcal{F}_{G,b}$. In other words, $H^*(G,b) = \varprojlim H^*(Q,k)$, where the inverse limit is taken over the category $\mathcal{F}_{G,b}$.

The restriction from G to P induces a homomorphism $H^*(G,k) \to H^*(G,b)$; if b is the principal block of kG, this is an isomorphism by the characterisation of $H^*(G,k)$ in terms of stable elements in [11]. Restriction from P to any subgroup Q of P induces a graded algebra homomorphism $r_Q : H^*(G,b) \to H^*(Q,k)$ whose image is contained in $(H^*(Q,k))^{N_G(Q,e_Q)}$.

The block cohomology algebra $H^*(G,b)$ is defined with respect to a choice of a defect group P and a source idempotent i. Since all pairs consisting of a defect group P of b and a $((kGb)^P)^\times-$conjugacy class γ of source idempotents i in $(kGb)^P$ are transitively permuted by the action of G by conjugation (cf. [18, 1.2]), $H^*(G,b)$ is defined in this way uniquely up to isomorphism, and it is in fact unique up to *unique* isomorphism, because the stabiliser $N_G(P_\gamma)$ of such a pair acts trivially on $H^*(G,b)$.

This justifies the notation $H^*(G,b)$, which makes no mention of the choice of a source idempotent.

There is a canonical injective graded algebra homomorphism

$$\nu : H^*(G,b) \longrightarrow HH^*(kGb)$$

from the block cohomology into the Hochschild cohomology of the block algebra kGb (cf. [15]) and for any finitely generated $kGb-$module M, tensoring by $- \underset{kGb}{\otimes} M$ induces a graded algebra homomorphism

$$\alpha_M : HH^*(kGb) \longrightarrow \mathrm{Ext}^*_{kGb}(M,M) \ .$$

The variety $V_{G,b}(M)$ is defined in [16,4.1] as the maximal ideal spectrum of the quotient $H^*(G,b)$ by the kernel of $\alpha_M \circ \nu$. In particular, $V_{G,b}(M)$ is a subvariety of the maximal ideal spectrum $V_{G,b}$ of $H^*(G,b)$, which is called the *block variety* of the block b. The *cohomology variety* $V_G(M)$, introduced by Carlson [9, 10], is the maximal ideal spectrum of the quotient of $H^*(G,k)$ by the kernel of the homomorphism $H^*(G,k) \to \mathrm{Ext}^*_{kG}(M,M)$ induced by the functor $- \underset{k}{\otimes} M$. The variety $V_G(M)$ is a subvariety of the maximal ideal spectrum V_G of $H^*(G,k)$. There is a finite surjective morphism $V_{G,b}(M) \to V_G(M)$, which is an isomorphism if b is the principal block (cf. [16,4.4]).

Again, this definition of $V_{G,b}(M)$ depends on the choice of the defect group P and the source idempotent i, because both $H^*(G,b)$ and the algebra homomorphism ν depend on this choice. As above, $V_{G,b}(M)$ is defined in this way uniquely up to unique isomorphism. The definition of ν involves the *normalised transfer map* T_{kGi} : $HH^*(kP) \to HH^*(kGb)$, and the welcome consequence of the following Theorem is, that one can compute $V_{G,b}(M)$ without all this technology (which is, though, needed in the proof):

Theorem 2.1. *Let G be a finite group, b a block of kG, P a defect group of b and i a source idempotent of b in $(kGb)^P$. The inclusion $\iota : H^*(G,b) \to H^*(P,k)$ induces a finite surjective morphism $\iota^* : V_P \to V_{G,b}$, and for any finitely generated $kGb-$module M we have*

$$V_{G,b}(M) = \iota^*(V_P(iM)) \ ,$$

where iM is considered as $kP-$module.

Proof. By [16, 4.3], $H^*(P,k)$ is Noetherian as a module over $H^*(G,b)$, which implies that ι^* is finite surjective. The homomorphism ν is, by [15, 5.6(iii)], defined as the unique graded algebra homomorphism which makes the following diagram commutative:

$$\begin{tikzcd} H^*(G,b) \arrow[r, "\nu"] \arrow[d, "\iota"'] & HH^*(kGb) \\ H^*(P,k) \arrow[r, "\delta_P"'] & HH^*(kP) \arrow[u, "T_{kGi}"'] \end{tikzcd}$$

Here δ_P is the algebra homomorphism induced by the "diagonal induction" functor $\mathrm{Ind}_{\Delta P}^{P\times P}$ (cf. [15, 4.5]) and T_{kGi} is the normalised transfer map defined in [15, 3.1], with respect to the $kGb - kP-$bimodule kGi; this makes sense as the relative projective element π_{kGi} is invertible (see [15, 3.1] and [15, 5.6]). By [15, 5.6(iii)] again, the image of ν lies actually in the subalgebra $HH^*_{kGi}(kGb)$ of *$kGi-$stable elements* in $HH^*(kGb)$ (cf. [15, 3.1(iii)]). Similarly, by [15, 5.6(ii)], the image of $\delta_P \circ \iota$ is contained in the subalgebra $HH^*_{ikG}(kP)$ of $ikG-$stable elements in $HH^*(kP)$; here ikG is viewed as $kP - kGb-$bimodule. Since on these subalgebras of stable elements, the normalised transfer T_{ikG} is inverse to the normalised transfer T_{kGi} by [15, 3.6 (iii)], we may reverse the right vertical arrow in the preceding diagram, in order to get a commutative diagram

$$\begin{array}{ccc} H^*(G,b) & \xrightarrow{\nu} & HH^*(kGb) \\ {\scriptstyle\iota}\big\downarrow & & \big\downarrow{\scriptstyle T_{ikG}} \\ H^*(P,k) & \xrightarrow[\delta_P]{} & HH^*(kP) \end{array}$$

Observe that $iM \cong ikG \underset{kGb}{\otimes} M$. Thus applying [16, 5.1] to kP, kGb, ikG instead of A, B, X, respectively, yields a commutative diagram of graded algebra homomorphisms

$$\begin{array}{ccc} HH^*_{kGi}(kGb) & \xrightarrow{\alpha_M} & \mathrm{Ext}^*_{kGb}(M,M) \\ {\scriptstyle T_{ikG}}\big\downarrow & & \big\downarrow{\scriptstyle \beta_M} \\ HH^*_{ikG}(kP) & \xrightarrow[\alpha_{iM}]{} & \mathrm{Ext}^*_{kP}(iM,iM) \end{array}$$

The homomorphism β_M is induced by the functor $ikG \underset{kGb}{\otimes} -$. It follows from the above remarks that by combining the two preceding commutative diagrams we get a commutative diagram of graded algebra homomorphisms

$$\begin{array}{ccc} H^*(G,b) & \xrightarrow{\alpha_M\circ\nu} & \mathrm{Ext}^*_{kGb}(M,M) \\ {\scriptstyle\iota}\big\downarrow & & \big\downarrow{\scriptstyle \beta_M} \\ H^*(P,k) & \longrightarrow & \mathrm{Ext}^*_{kP}(iM,iM) \end{array}$$

In this diagram the bottom horizontal arrow is equal to the composition $\alpha_{iM} \circ \delta_P$, and this is, by [15, 2.9], equal to the homomorphism induced by tensoring with $-\underset{k}{\otimes} iM$. In other words, the top row in this diagram defines $V_{G,b}(M)$, and the bottom row defines $V_P(iM)$. In order to conclude the proof of the Theorem, it suffices to show that β_M is injective. This follows from a general property of source idempotents: the canonical map $kGi \underset{kP}{\otimes} ikG \to kGb$ induced by multiplication in kGb is split as homomorphism of $kGb - kGb-$bimodules. Thus the identity functor is a direct summand of the composition of the "truncated" restriction functor $ikG \underset{kGb}{\otimes} -$ and the corresponding induction functor $kGi \underset{kP}{\otimes} -$. Therefore the composition of the algebra homomorphisms

$$\mathrm{Ext}^*_{kGb}(M,M) \to \mathrm{Ext}^*_{kP}(iM,iM) \to \mathrm{Ext}^*_{kGb}(kGi \underset{kP}{\otimes} iM, kGi \underset{kP}{\otimes} iM)$$

induced by the functors $ikG \underset{kGb}{\otimes} -$ and $kGi \underset{kP}{\otimes} -$ is injective. But then in particular the first of the two homomorphisms is injective, and that is just β_M. This proves 2.1. □

The above result provides a technique to carry over properties of the cohomology varieties at the level of $p-$subgroups to the variety $V_{G,b}(M)$. We note one easy consequence (which could, of course, also be proved without using 2.1):

Corollary 2.2. *Let M, M' be finitely generated $kGb-$modules. We have*

$$V_{G,b}(M \oplus M') = V_{G,b}(M) \cup V_{G,b}(M')$$

Proof. By [5, 5.7.5] we have $V_P(i(M \oplus M')) = V_P(iM) \cup V_P(iM')$, and thus 2.2 follows from 2.1. □

Corollary 2.3. *Let M be a finitely generated indecomposable kGb-module with P as vertex and a source of dimension prime to p. Then $V_{G,b}(M) = V_{G,b}$.*

Proof. By [13, 6.1], some source of M is a direct summand of iM. Thus $V_P(iM) = V_P$ by [5, 5.8.5]. Since $\iota^* : V_P \to V_{G,b}$ is surjective by 2.1, the statement follows. □

Remark 2.4. In the situation of Theorem 2.1, it is not true in general that $V_{G,b}(M)$ coincides with $\iota^*(\mathrm{Res}^G_P(M))$; that is, it really matters to "cut" the module M down by the source idempotent i. This phenomenon occurs if P has more than one conjugacy class of source idempotents, or equivalently, in Puig's terminology, if P has more than one local point on kGb (cf. [18]). We describe an example in Section 5.

3 Norm maps for bisets

Let G be a finite group. If p is odd, we denote by $H^\cdot(G,k)$ the even part of $H^*(G,k)$; if $p = 2$ we set $H^\cdot(G,k) = H^*(G,k)$. Thus $H^\cdot(G,k)$ is commutative since $H^*(G,k)$ is graded commutative (cf. [4, 3.2]). We refer to [5, 4.1] for the definition and general properties of the Evens norm map $n_H^G : H^\cdot(H,k) \to H^\cdot(G,k)$, where H is a subgroup of G. We use this to define for any two finite groups P, Q and any finite P-Q-biset X on which P, Q act regularly on the left and right, respectively, a norm map

$$n_X : H^\cdot(Q,k) \longrightarrow H^\cdot(P,k)$$

as follows. If X is transitive, then X is isomorphic to a biset of the form $P \underset{R}{\times} {}_\varphi Q$ for some subgroup R of P and an injective group homomorphism $\varphi : R \to Q$. In that case, we set $n_X = n_R^P \circ \mathrm{res}_\varphi$, and in general, we set

$$n_X = \prod_Y n_Y \ ,$$

where Y runs over the set of transitive P-Q-subbisets in X (and where the product is taken in $H^*(P,k)$).

Note that the exact sign of n_R^P and hence of n_X depends on the choice of a system of representatives of the right cosets of R in P (cf. [5, 4.1]), and so all statements on norm maps hold modulo keeping track of signs (but since the signs are irrelevant in the Propositions 3.4 and 3.5 below we do not insist on this aspect).

Lemma 3.1. *Let P, Q, R be finite groups, let $\psi : R \to P$ be an injective group homomorphism and let X be a finite P-Q-biset on which P and Q act regularly on the left and on the right, respectively. We have*

$$\mathrm{res}_\psi \circ n_X = n_{{}_\psi X} \ ,$$

where ${}_\psi X$ is the R-Q-biset obtained from restricting X through ψ on the left.

Proof. We may assume that X is transitive as P-Q-biset, and then the result follows from the Mackey formula [5, 4.1.2(v)] for the Evens norm map. □

For the rest of this section, we fix the following notation. Let G be a finite group, b a block of kG, P a defect group of b and choose a source idempotent $i \in (kGb)^P$. For any subgroup Q of P, denote by e_Q the unique block of $kC_G(Q)$ such that $\mathrm{Br}_Q(i)e_Q \neq 0$ (cf. [8, 1.8]).

As before, denote by $\mathcal{F}_{G,b}$ the category whose objects are the subgroups of P and whose morphisms, for any two subgroups Q, R of P, are the group homomorphisms $\varphi : Q \to R$ such that there is $x \in G$ fulfilling $\varphi(u) = xux^{-1}$ for all $u \in Q$ and ${}^x(Q, e_Q) \subseteq (R, e_R)$. In particular, the automorphism group of Q in $\mathcal{F}_{G,b}$ corresponds to $N_G(Q, e_Q)/C_G(Q)$. Since inner automorphisms of Q act trivially on $H^\cdot(Q, k)$, the action of $N_G(Q, e_Q)$ on $H^\cdot(Q, k)$ induces an action of the group $W(Q) = N_G(Q, e_Q)/QC_G(Q)$ on $H^\cdot(Q, k)$.

The following result is due to Broto, Levi and Oliver:

Proposition 3.2. *([6]) With the notation above, there is a finite P-P-biset X with the following properties.*

(i) Every transitive subbiset of X is isomorphic to $P \underset{Q}{\times} {}_\varphi P$ for some subgroup Q of P and some group homomorphism $\varphi : Q \to P$ belonging to the category $\mathcal{F}_{G,b}$.

(ii) $|X|/|P|$ is prime to p.

(iii) For any subgroup Q of P and any group homomorphism $\varphi : Q \to P$ in $\mathcal{F}_{G,b}$, the Q-P-bisets ${}_\varphi X$ and ${}_Q X$ are isomorphic.

The original motivation for constructing such a biset is an observation by Linckelmann and Webb, that its existence implies the existence of a stable summand $\hat{B}(G, b)$ of the classifying space $BP^\wedge_+$ viewed as p-complete spectrum such that the cohomology of $\hat{B}(G, b)$ with coefficients in k is precisely the block cohomology $H^*(G, b)$.

Lemma 3.3. *Let X be a finite P-P-biset fulfilling the conditions in 3.2. Then, for any subgroup Q of P, there is a Q-Q-subbiset of ${}_Q X_Q$ isomorphic to Q.*

Proof. It suffices to show that X has a P-P-subbiset isomorphic to P. By 3.2(i) and 3.2(ii), X has a P-P-subbiset isomorphic to ${}_\varphi P$ for some automorphism φ of P in $\mathcal{F}_{G,b}$. The stability condition 3.2(iii) implies the result. □

Proposition 3.4. *Let X be a finite P-P-biset fulfilling the conditions in 3.2, let Q be a subgroup of P and let Y be the Q-Q-subbiset of ${}_QX_Q$ which is the union of all Q-Q-orbits of length $|Q|$.*

(i) The image of the norm map $n_{X_Q} : H^{\cdot}(Q,k) \to H^{\cdot}(P,k)$ is contained in $H^{\cdot}(G,b)$.

(ii) The set Y is non empty.

(iii) For any $\zeta \in H^{\cdot}(Q,k)^{W(Q)}$ such that $\mathrm{res}^Q_R(\zeta) = 0$ for any proper subgroup R of Q we have $n_{{}_QX_Q}(\zeta) = n_Y(\zeta)$.

(iv) For any $\zeta \in H^{\cdot}(Q,k)^{W(Q)}$ we have $n_Y(\zeta) = \zeta^{|Y|/|Q|}$.

Proof. (i) Let R be a subgroup of P and let $\varphi : R \to P$ be a group homomorphism in $\mathcal{F}_{G,b}$. Using 3.1 and 3.2(iii), we get $\mathrm{res}_\varphi \circ n_{X_Q} = n_{{}_\varphi X_Q} = n_{{}_R X_Q} = \mathrm{res}^P_R \circ n_{X_Q}$.

(ii) follows from 3.3.

(iii) Any Q-Q-orbit of ${}_QX_Q$ outside Y is isomorphic to $Q \underset{R}{\times} {}_\psi Q$ for some proper subgroup R of Q and some group homomorphism $\psi : R \to Q$ in $\mathcal{F}_{G,b}$, from which the statement follows.

(iv) The number of Q-Q-orbits in Y is equal to $|Y|/|Q|$, and any such orbit is isomorphic to ${}_\varphi Q$ for some automorphism φ of Q in $\mathcal{F}_{G,b}$. Moreover, for any $\zeta \in H^{\cdot}(Q,k)^{W(Q)}$ we have then $n_{{}_\varphi Q}(\zeta) = \zeta$, from which the result follows. □

We apply this to translate [4, 5.6.2] to block cohomology.

Proposition 3.5. *Let X be a finite P-P-biset fulfilling the conditions in 3.2. Let E be an elementary abelian subgroup of P and let σ_E be a homogeneous element in $H^{\cdot}(E,k)$ satisfying $\mathrm{res}^E_F(\sigma_E) = 0$ for any proper subgroup F of E. Let Y be the E-E-subbiset of ${}_EX_E$ which is the union of all E-E-orbits of length $|E|$. Write $|Y|/|E| = p^a m$ for some nonnegative intergers a, m, such that $(p,m) = 1$.*

(i) For any $\eta \in H^{\cdot}(E,k)^{W(E)}$ there is $\eta' \in H^{\cdot}(G,b)$ such that $r_E(\eta') = (\sigma_E \cdot \eta)^{p^a}$.

(ii) There is an element $\rho_E \in H^{\cdot}(G,b)$ such that $r_E(\rho_E) = (\sigma_E)^{p^a}$ and such that $r_F(\rho_E) = 0$ whenever F is an elementary abelian subgroup of P such that no G-conjugate of (E,e_E) is containend in (F,e_F).

Proof. (i) We may assume that η is homogeneous. Set $\zeta = n_{X_E}(1 + \sigma_E \cdot \eta)$. By 3.4, we have $\zeta \in H^{\cdot}(G,b)$. Moreover, $r_E(\zeta) = n_{{}_EX_E}(1 + \sigma_E \cdot \eta) = n_Y(1 + \sigma_E \cdot \eta) = (1+\sigma_E \cdot \eta)^{p^a m} = (1+(\sigma_E \cdot \eta)^{p^a})^m = 1 + m(\sigma_E \cdot \eta)^{p^a} + \tau$, where τ is a sum of elements of degree stricly bigger than $\deg((\sigma_A \cdot \eta)^{p^a}) = p^a \cdot \deg(\sigma_E \cdot \eta)$. Define η' to be the homogeneous part of ζ in degree $p^a \cdot \deg(\sigma_E \cdot \eta)$, divided by m.

(ii) Applying (i) to $\eta = 1$ yields a homogeneous element $\rho_E \in H^{\cdot}(G,b)$ such that $r_E(\rho_E) = (\sigma_E)^{p^a}$. By the construction in (i), ρ_E is a scalar multiple of the homogenous part of $n_{X_E}(1+\sigma_E)$ in degree $p^a \cdot \deg(\sigma_E)$. Let F be another elementary abelian subgroup of P. Then $r_F(\rho_E)$ is a scalar multiple of the homogeneous part of $n_{{}_FX_E}(1 + \sigma_E)$ in degree $p^a \cdot \deg(\sigma_E)$. If (E,e_E) has no G-conjugate contained in (F,e_F), then the biset ${}_FX_E$ is a union of transitive bisets of the form $F \underset{H}{\times} {}_\psi E$, where H is a subgroup of F of order smaller than $|E|$, and where $\psi : H \to E$ is an injective group homomorphism. Thus $n_{{}_FX_E}(\sigma_E) = 0$, and so $r_F(\rho_E) = 0$. □

4 The Quillen stratification of $V_{G,b}(M)$

Throughout this section, let G be a finite group, let b be a block of kG and let P be a defect group of b. Choose a source idempotent $i \in (kGb)^P$ and denote, for any subgroup Q of P, by e_Q the unique block of $kC_G(Q)$ satisfying $\mathrm{Br}_Q(i)e_Q = \mathrm{Br}_Q(i)$. For any subgroup Q of P, the graded algebra homomorphism r_Q (which is the inclusion $H^*(G,b) \hookrightarrow H^*(P,k)$ followed by the restriction map $\mathrm{res}^P_Q : H^*(P,k) \to H^*(Q,k)$) induces a morphism of varieties

$$r_Q^* : V_Q \longrightarrow V_{G,b} \ .$$

Recall that since $H^*(P,k)$ is Noetherian over $H^*(G,b)$ by [16, 4.3], the morphism $r_P^* = \iota^* : V_P \to V_{G,b}$ is finite surjective. We will try to follow as closely as possible the lines of the presentation given in Benson [5, 5.6]; there are two major technical adjustments: the extensive use of Puig's notion of local pointed groups [18] (for which we refer again to the account given in Thévenaz [21]) and the application of the Evens norm map with respect to a biset fulfilling the conditions in 3.2.

Definition 4.1. Let M be a finitely generated kGb-module. For any local pointed group Q_δ on kGb, we define the following subvarieties of V_Q:
$V_Q^+ = V_Q - \bigcup_{R<Q} (\mathrm{res}^Q_R)^*(V_R)$, $\quad V_Q^+(iM) = V_Q(iM) \cap V_Q^+$;
$V_{Q_\delta}(M) = V_Q(jM)$, where $j \in \delta$, $\quad V^+_{Q_\delta}(M) = V_{Q_\delta}(M) \cap V_Q^+$.
Furthermore, we define the following subvarieties of $V_{G,b}$:
$V_{G,Q} = r_Q^*(V_Q)$, $\quad V^+_{G,Q} = r^*_Q(V^+_Q)$;
$V_{G,Q}(M) = r^*_Q(V_Q(iM))$, $\quad V^+_{G,Q}(M) = r^*_Q(V^+_Q(iM))$.
$V_{G,Q_\delta}(M) = r^*_Q(V_{Q_\delta}(M))$, $\quad V^+_{G,Q_\delta}(M) = r^*_Q(V^+_{Q_\delta}(M))$.
Finally, we set
$W(Q) = N_G(Q, e_Q)/QC_G(Q)$ and $W(Q_\delta) = N_G(Q_\delta)/QC_G(Q)$.

Theorem 4.2. *Let M be a finitely generated kGb-module.*

(i) The variety $V_{G,b}(M)$ is the disjoint union of the locally closed subvarieties $V^+_{G,E}(M)$, where E runs over a set of subgroups of P such that (E, e_E) runs over a set of representatives of the G-conjugacy classes of those b-Brauer pairs contained in (P, e_P) for which E is elementary abelian and $C_P(E)$ is a defect group of the block e_E.

(ii) Let E be an elementary abelian subgroup of P such that $C_P(E)$ is a defect group of e_E. The group $W(E)$ acts on the variety $V^+_E(iM)$, and r^*_E induces an inseparable isogeny $V^+_E(iM)/W(E) \to V^+_{G,E}(M)$.

(iii) Let E be an elementary abelian subgroup of P such that $C_P(E)$ is a defect group of e_E. Then $V^+_{G,E}(M)$ is the union of the subvarieties $V^+_{G,E_\delta}(M)$, with δ running over the set of local points of E on kGb such that $E_\delta \subseteq P_\gamma$.

If one specialises the above theorem to the principal block of kG, the statements (i) and (ii) in 4.2 are equivalent to the Quillen stratification due to Avrunin and Scott [3]. Since a subgroup E of P can have more than one local point on kGb, statement (iii) gives some additional information on the subvarieties $V^+_{G,E}(M)$.

If one specialises 4.2 to the case where M is indecomposable with P as vertex and a source of dimension prime to p, then, by 2.4, we have $V_{G,b}(M) = V_{G,b}$, and thus 4.2 yields a stratification for the block variety $V_{G,b}$. The following Proposition describes in that case the subvarieties $V^+_{G,b}(M)$ more precisely:

Proposition 4.3. *Let M be a finitely generated indecomposable kGb-module with P as vertex and a source of dimension prime to p.*

(i) For any subgroup Q of P we have $V_Q(iM) = V_Q$ and $V^+_Q(iM) = V^+_Q$.

(ii) For any subgroup Q of P we have $V_{G,Q}(M) = V_{G,Q}$ and $V^+_{G,Q}(M) = V^+_{G,Q}$.

Proof. By [13, 6.1], some indecomposable direct summand of iM as kP-module is a source of M. Thus, for any subgroup Q of P, the restriction of iM to kQ has a direct summand of dimension prime to p, by the assumptions. But then $V_Q(iM) = V_Q$ (cf. [5, 5.8.5]), and the second equality in (i) follows from the first. The two equalities in (ii) follow from applying r^*_Q to the equalities in (i). □

Combining 4.2 and 4.3 yields the obvious analogue for block cohomology of Quillen's stratification in [19, 20]. We break up the proof of 4.2 into a series of Lemmas; we keep the notation introduced above.

Lemma 4.4. *We have $V_{G,b}(M) = \bigcup_E r^*_E(V_E(iM))$, where E runs over the set of elementary abelian subgroups of P.*

Proof. By 2.1, we have $V_{G,b}(M) = \iota^*(V_P(iM))$. Thus the Lemma follows from [5, 5.7.4] applied to P and iM instead of G and M, respectively. □

Lemma 4.5. *For any subgroup Q of P and any idempotent $i' \in (kGb)^Q$ we have $V_Q(i'M) = \bigcup_{R_\epsilon} (\mathrm{res}^Q_R)^*(V_{R_\epsilon}(M))$, where R_ϵ runs over the set of local pointed groups on $i'kGi'$ such that $R \subseteq Q$.*

Proof. Choose a primitive decomposition J of i' in $(kGb)^Q$. That is, $i' = \sum_{j\in J} j$, and the elements of J are pairwise orthogonal primitive idempotents in $(kGb)^Q$. Thus $i'M = \bigoplus_{j\in J} jM$ as direct sum of kQ-modules, and hence $V_Q(i'M) = \bigcup_{j\in J} V_Q(jM)$. Let $j \in J$. Then the conjugacy class of j in $((i'kGi')^Q)^\times$ is a (not necessarily local) point δ of Q on $i'kGi'$. Let R_ϵ be a defect pointed group of Q_δ. By [17, Cor. 1] (see also [21, (23.1)]), this means that there is $l \in \epsilon$ such that $j = \mathrm{Tr}^Q_R(l)$ and such that the different Q-conjugates ulu^{-1} of l are pairwise orthogonal as u runs over a set of representatives of the right R-cosets in Q. Thus $jM \cong \mathrm{Ind}^Q_R(lM)$, and therefore $V_Q(jM) = (\mathrm{res}^Q_R)^*(V_R(lM))$, from which the Lemma follows. □

Lemma 4.6. *For any local pointed group Q_δ on kGb we have*

$$V^+_{Q_\delta}(M) = V_{Q_\delta}(M) - \bigcup_{R_\epsilon} (\mathrm{res}^Q_R)^*(V_{R_\epsilon}(M)) \ ,$$

where R_ϵ runs over the set of local pointed groups on kGb properly contained in Q_δ.

Proof. By [5, 5.7.7] we have $(\mathrm{res}^Q_R)^*(V_R(jM)) = (\mathrm{res}^Q_R)^*(V_R) \cap V_Q(jM)$, where $j \in \delta$ and where R is any subgroup of Q. Thus $V^+_{Q_\delta}(M) = V^+_Q \cap V_Q(jM) = V_Q(jM) - \underset{R<Q}{\cup} ((\mathrm{res}^Q_R)^*(V_R) \cap V_Q(jM)) = V_Q(jM) - \underset{R<Q}{\cup} (\mathrm{res}^Q_R)^*(V_R(jM))$, and now the Lemma follows from 4.5 applied to the varieties $V_R(jM)$ appearing in the last expression. □

Lemma 4.7. *Let Q_δ, R_ϵ be local pointed groups on kGb contained in P_γ. If Q_δ and R_ϵ are G-conjugate, then $V_{G,Q_\delta}(M) = V_{G,R_\epsilon}(M)$ and $V^+_{G,Q_\delta}(M) = V^+_{G,R_\epsilon}(M)$.*

Proof. Let $x \in G$ such that $R_\epsilon = {}^x(Q_\delta)$. Then the group isomorphism $\varphi : Q \to R$ mapping $u \in Q$ to xux^{-1} has the property that $\mathrm{res}^P_Q(\zeta) = \mathrm{res}_\varphi(\zeta)$ for all $\zeta \in H^*(G,b)$, and thus the morphisms $r^*_Q \circ (\mathrm{res}_\varphi)^*$ and r^*_R from V_R to $V_{G,b}$ are equal. Thus $V_{G,Q_\delta}(M) = V_{G,R_\epsilon}(M)$. The second equality is clear. □

Lemma 4.8. *We have $V_{G,b}(M) = \underset{E_\delta}{\cup} V^+_{G,E_\delta}(M)$, where E_δ runs over a set of representatives of the set of G-conjugacy classes of local pointed groups on kGb such that $E_\delta \subseteq P_\gamma$ and such that E is elementary abelian.*

Proof. By 4.4 and 4.6, the variety $V_{G,b}(M)$ is the union of the subvarieties $V^+_{G,E_\delta}(M)$, where E_δ runs over the set of all local pointed groups on kGb contained in P_γ such that E is elementary abelian. The Lemma follows now from 4.7. □

Lemma 4.9. *For any subgroup Q of P we have $V^+_Q(iM) = \underset{\delta}{\cup} V^+_{Q_\delta}(M)$ and $V^+_{G,Q}(M) = \underset{\delta}{\cup} V^+_{G,Q_\delta}(M)$, where δ runs over the set of local points of Q on kGb such that $Q_\delta \subseteq P_\gamma$.*

Proof. By 4.5 we have $V_Q(iM) = \underset{R_\epsilon}{\cup} (\mathrm{res}^Q_R)^*(V_{R_\epsilon}(M))$, where R_ϵ runs over the set of local pointed groups on kGb such that $R_\epsilon \subseteq P_\gamma$ and $R \subseteq Q$. Intersecting with V^+_Q yields the first equality, and applying r^*_Q yields the second equality. □

Proposition 4.10. *We have $V_{G,b}(M) = \underset{E}{\cup} V^+_{G,E}(M)$, and in particular, $V_{G,b} = \underset{E}{\cup} V^+_{G,E}$, where E runs over a set of subgroups of P such that (E, e_E) runs over a set of representatives of the G-conjugacy classes of b-Brauer pairs contained in (P, e_P) for which E is elementary abelian and $C_P(E)$ is a defect group of e_E.*

Proof. Any b-Brauer pair is G-conjugate to a b-Brauer pair of the form (Q, e_Q) for some subgroup Q of P such that $C_P(Q)$ is a defect group of e_Q (see [1, 4.5]). Thus the first equality follows from combining 4.8 and 4.9, and the second equality follows from the first and 4.3. □

Lemma 4.11. *Let Q be a subgroup of P such that $C_P(Q)$ is a defect group of the block e_Q. The action of $N_G(Q, e_Q)$ on $H^*(Q,k)$ induces an action of $W(Q)$ on V_Q and V^+_Q, which preserves the subvariety $V^+_Q(iM)$.*

Proof. Since $QC_G(Q)$ acts trivially on $H^*(Q,k)$, the action of $N_G(Q_\delta)$ induces an action of $W(Q_\delta)$ on V_Q. This action preserves V^+_Q. The action of the group $N_G(Q, e_Q)$

permutes the set of all local points of Q on kGb satisfying $\mathrm{Br}_Q(\delta)e_Q \neq 0$. Since $C_P(Q)$ is a defect group of e_Q, by [14, 3.3(iii)], for any local point δ of Q on kGb such that $\mathrm{Br}_Q(\delta)e_Q \neq 0$ we have $Q_\delta \subseteq P_\gamma$. Thus $N_G(Q, e_Q)$ acts on the set of local points δ of Q on kGb fulfilling $Q_\delta \subseteq P_\gamma$, and now 4.11 follows from 4.9. □

We have set up the machinery in such a way that the rest of the proof of 4.2 follows now exactly that of [5, 5.6.3].

Proof of Theorem 4.2. Let E be an elementary abelian subgroup of P such that $C_P(E)$ is a defect group of e_E. By the argument in [5, 5.6], there is a homogeneous element $\sigma_E \in H^\cdot(E,k)^{W(E)}$ such that V_E^+ consists of all maximal ideals in $H^*(E,k)$ not containing σ_E, and such that $\mathrm{res}^E_F(\sigma_E) = 0$ for any proper subgroup F of E. Thus V_E^+ can be identified to the maximal ideal spectrum of the algebra $H^\cdot(E,k)[\sigma_E^{-1}]$, obtained from localising $H^\cdot(E,k)$ at σ_E. By [5, 5.4.8], the quotient $V_E^+/W(E)$ can be identified to the maximal ideal spectrum of $(H^\cdot(E,k)[\sigma_E^{-1}])^{W(E)}$. Let ρ_E be the element in $H^\cdot(G,b)$ fulfilling 3.5(iv). Then $V_{G,E}^+$ consists of all maximal ideals in $H^\cdot(G,b)$ containing $\ker(r_E)$ and not containing ρ_E. Since r_E maps ρ_E to a power of σ_E, r_E induces an algebra homomorphism

$$H^\cdot(G,b)[\rho_E^{-1}] \longrightarrow (H^\cdot(E,k)[\sigma_E^{-1}])^{W(E)}$$

such that, by 3.5, the image of this homomorphism contains a p^a-th power of every element in $(H^\cdot(E,k)[\sigma_E^{-1}])^{W(E)}$. Upon taking varieties, this is equivalent to saying that r_E^* induces an inseparable isogeny $V_E^+/W(E) \to V_{G,E}^+$. Since $W(E)$ acts on the subvariety $V_E^+(iM)$ by 4.11, passing down to subvarieties proves (ii).

If F is another elementary abelian subgroup of P such that $C_P(F)$ is a defect group of e_F and such that (F, e_F) contains no G-conjugate of (E, e_E), then $\rho_F \in \ker(r_E)$ by 3.5. By the above description of $V_{G,E}^+$, it follows that $V_{G,E}^+$ and $V_{G,F}^+$ are disjoint. Since $V_{G,E}^+(M)$ is a subvariety of $V_{G,E}^+$, this concludes the proof of (i). Finally, statement (iii) is a particular case of 4.9. □

5 An example

Let p be a prime such that $p \geq 5$ and let k be a field of characteristic p containing a primitive 3^{rd} root of unity. For any positive integer n denote by C_n a cyclic group of order n. Let Q be a finite non trivial abelian $p-$group and set $P = Q \times Q$. We consider the group

$$G = (C_3 \times P) \rtimes C_2 \ ,$$

where the non trivial element t of C_2 acts on $C_3 \times P$ by inverting the elements of C_3 and by exchanging the two factors Q of P; that is, $(u,v)^t = (v,u)$ for any $(u,v) \in Q \times Q = P$.

Since the Sylow-p-subgroup P of G is normal, P is the defect group of any block of kG. The blocks of kG correspond bijectively to the $G-$orbits of blocks of $kC_G(P)$. Since $C_G(P) = C_3 \times P$, the algebra $kC_G(P)$ has three $p-$blocks e_0, e, e', corresponding to the linear characters ζ_0, ζ, ζ' of C_3 with values in k, where we choose notation such that e_0 is the principal block and hence ζ_0 is the trivial character of C_3. Then e_0 is

$G-$stable, while the two blocks e, e' get permuted by the action of the involution t. It follows that kG has exactly two blocks, namely the principal block $b_0 = e_0$ and a unique non principal block $b = e + e'$.

The structure of the principal block b_0 of kG is as follows: the canonical map $G \to P \rtimes C_2$ with kernel $C_3 = O_{p'}(G)$ induces an algebra isomorphism $kGb_0 \cong k(P \rtimes C_2)$.

The non principal block b of kG is a nilpotent block (cf. [8]): since t permutes e and e', we have $N_G(P,e) = C_G(P)$. The pairs (P,e) and (P,e') are exactly the two maximal $b-$Brauer pairs. The idempotents e, e' are in fact source idempotents; that is, they remain primitive in $(kGb)^P$. To see this, observe first that e and $e' = e^t$ are orthogonal, and therefore $ete = 0$. Thus $ekGe = kC_G(P)e$. Since $kC_3e \cong k$ we have $kC_G(P)e \cong kP$. This shows not only, that e is primitive in $(kGb)^P$, but also that the source algebra $ekGe$ of b is isomorphic to kP. The same argument works for e'.

Since b is nilpotent, in particular the block cohomology of b is isomorphic to $H^*(P,k)$.

We define an indecomposable $kGb-$module M as follows. Consider $kQ \underset{k}{\otimes} k$ as $kP-$module through the canonical isomorphism $kP \cong kQ \underset{k}{\otimes} kQ$. Extend $kQ \underset{k}{\otimes} k$ to a $kC_G(P)-$module by letting act any element $c \in C_3$ as multiplication with the scalar $\zeta(c)$. In this way, $kQ \underset{k}{\otimes} k$ becomes an indecomposable $kC_G(Q)-$module belonging to the block e. Set

$$M = \mathrm{Ind}_{C_G(P)}^G (kQ \underset{k}{\otimes} k) \ .$$

By Mackey's formula, $\mathrm{Res}_{C_G(P)}^G(M) \cong (kQ \underset{k}{\otimes} k) \oplus {}^t(kQ \underset{k}{\otimes} k) \cong (kQ \underset{k}{\otimes} k) \oplus (k \underset{k}{\otimes} kQ)$. Since t exchanges these two summands, which are both indecomposable as $kC_G(P)-$modules, it follows that M is indeed indecomposable. Moreover, as $kP-$modules, we have

$$eM \cong kQ \underset{k}{\otimes} k \quad \text{and} \quad e'M \cong k \underset{k}{\otimes} kQ \ .$$

Through the Künneth isomorphism $H^*(P,k) \cong H^*(Q,k) \underset{k}{\otimes} H^*(Q,k)$, the ideal $H^+(Q,k) \underset{k}{\otimes} H^*(Q,k)$ is the annihilator of $\mathrm{Ext}^*_{kP}(eM,eM)$, while the annihilator of $\mathrm{Ext}^*_{kP}(e'M,e'M)$ is $H^*(Q,k) \underset{k}{\otimes} H^+(Q,k)$, where $H^+(Q,k)$ denotes the ideal generated by the elements of positive degree in $H^*(Q,k)$. Thus $V_P(eM)$ and $V_P(e'M)$ are different subvarieties of $H^*(P,k) = H^*(G,b)$, or equivalently, $V_P(M) \neq V_P(eM)$.

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