TRANSFER IN HOCHSCHILD COHOMOLOGY OF BLOCKS OF FINITE GROUPS

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1 Introduction and Notation

1.1 The cohomology of a finite group G with coefficients in a complete discrete valuation ring $\mathcal{O}$ having a residue field k of prime characteristic p is the ring $H^*(G,\mathcal{O}) = Ext^*_{\mathcal{O}G}(\mathcal{O},\mathcal{O})$, where $\mathcal{O}$ is considered as trivial $\mathcal{O}G-$module; that is, with $x \in G$ acting as identity on $\mathcal{O}$. It is apparent from this definition, that $H^*(G,\mathcal{O})$ is an invariant of the *principal block* b_0 of $\mathcal{O}G$ (this is the unique primitive idempotent of the center $Z(\mathcal{O}G)$ of the group algebra $\mathcal{O}G$ of G over $\mathcal{O}$ acting as the identity on the trivial $\mathcal{O}G-$module).

Since in general a $p-$block of G need not have an augmentation, this definition does not generalize to arbitrary blocks. There is, however, a "$p-$local" characterization of $H^*(G,\mathcal{O})$ as the subring of the cohomology ring $H^*(P,\mathcal{O})$, where P is a Sylow-$p-$subgroup of G, consisting of all elements in $H^*(P,\mathcal{O})$ whose restriction to Q is stable under the action of $N_G(Q)$ on $H^*(Q,\mathcal{O})$ for every subgroup Q of P (cf. [9, Ch. XII, 10.1]). This characterization now does generalize to an arbitrary $p-$block b of G (this is a primitive idempotent of $Z(\mathcal{O}G)$) by taking for P a defect group of b and replacing $N_G(Q)$ by $N_G(Q,e_Q)$, where (Q,e_Q) is a $b-$*Brauer pair* contained in (P,e_P) for some fixed choice of a suitable block e_P of $kC_G(P)$ (cf. [1]). We are going to call the ring thus obtained the *cohomology ring of the block* b (see section 5 for a precise definition).

It is well-known that the usual cohomology ring $H^*(G,\mathcal{O})$ embeds into the Hochschild cohomology ring $HH^*(\mathcal{O}G)$ through "diagonal induction" (see 4.5 below). We develop here the machinery to show that the cohomology ring of the block b embeds into the cohomology ring $HH^*(\mathcal{O}Gb)$ of the block algebra $\mathcal{O}Gb$ of b.

In section 2 we define for any two symmetric algebras A, B over a commutative ring R and any bounded complex X of finitely generated left and right projective $A-B-$bimodules a *transfer map* $t_X : HH^*(B) \longrightarrow HH^*(A)$ and study its basic properties. We show in particular, analogously to B. Keller's results in [12] on transfer in cyclic homology, that t_X depends only on the image of X in the appropriate Grothendieck group.

In order to study the multiplicative structure of $HH^*(A)$ we define in section 3 the notion of $X-$*stable elements* in $HH^*(A)$, which are then shown to form a graded subalgebra $HH^*_X(A)$ of $HH^*(A)$; moreover, we show that under some non degeneracy hypothesis on X there is a *normalized transfer* T_X inducing a surjective $R-$algebra homomorphism $HH^*_{X^*}(B) \longrightarrow HH^*_X(A)$, where X^* is the $R-$dual of X.

Section 4 is devoted to showing that the transfer maps between $HH^*(RG)$ and $HH^*(RH)$, where H is a subgroup of the finite group G, obtained from induction and restriction, are compatible with the usual transfer and restriction maps between the cohomology rings $H^*(G,R)$ and $H^*(H,R)$ through the "diagonal" embeddings of the cohomology rings into the Hochschild cohomology rings of G and H, respectively.

In section 5 we give the definition of the cohomology ring of a $p-$block b of a finite group G over the ring $\mathcal{O}$ and apply the results of the previous sections in order to show that indeed this cohomology ring embeds into $HH^*(\mathcal{O}Gb)$ and that the normalized transfer of a suitable $p-$permutation complex always induces the identity on the image of this embedding; this should be thought of as a generalization of the well-known fact, that if H controls $p-$fusion in G then the restriction from G to H

induces an isomorphism $H^*(G,\mathcal{O}) \cong H^*(H,\mathcal{O})$ by the "$p-$local" characterization of the cohomology rings. By Mislin's theorem [15], the converse of this last statement is true, too, and in fact, a major guideline for developing this material is that we expect there to be some generalization of Mislin's theorem to arbitrary $p-$blocks of finite groups; see remark 5.8 below.

1.2 Notation. All algebras and rings are associative with unit element, all modules are supposed to be finitely generated unitary, and, if not stated otherwise, left modules.

1.2.1 If A, B, C are algebras over a commutative ring R, by an $A-B-$bimodule M we mean a bimodule whose left and right $R-$module structure coincide; that is, we may consider M as $A \underset{R}{\otimes} B^0-$module with $a \otimes b$ acting on $m \in M$ as amb, where $a \in A$, $b \in B$, and B^0 is the algebra obtained from endowing B with the opposite product $b.b' = b'b$, $b, b' \in B$. The $R-$dual $M^* = Hom_R(M,R)$ becomes then a $B-A-$bimodule via $(b.m^*.a)(m) = m^*(amb)$ for any $m \in M$, $m^* \in M^*$, $a \in A$, $b \in B$. If furthermore N is an $A-C-$bimodule, $Hom_{A\otimes 1}(M,N)$ is considered as $B-C-$bimodule through $(b.\varphi.c)(m) = \varphi(mb)c$, where $b \in B$, $\varphi \in Hom_{A\otimes 1}(M,N)$, $c \in C$ and $m \in M$.

For a finite group G we consider any $RG-RG-$bimodule M as $R(G\times G)-$module with $(x,y) \in G \times G$ acting on $m \in M$ as xmy^{-1} (and vice versa).

1.2.2 By a complex we always mean a chain complex (that is, its differential has degree -1), and we implicitely consider any cochain complex X over any ring as chain complex through $X_n = X^{-n}$, where n is an integer; note that then $H^n(X) = H_{-n}(X)$ for any integer n.

1.2.3 Let A, B, C be $R-$algebras, X a complex of $A-B-$bimodules with differential δ, Y a complex of $B-C-$bimodules with differential ϵ, and Z a complex of $A-C-$bimodules with differential γ. We define $X \underset{B}{\otimes} Y$ to be the complex of $A-C-$bimodules whose component in degree n is equal to

$$\bigoplus_{i\in\mathbb{Z}} X_i \underset{B}{\otimes} Y_{n-i}$$

and differential induced by the maps

$$\begin{pmatrix} (-1)^i Id_{X_i} \otimes \epsilon_{n-i} \\ \delta_i \otimes Id_{Y_{n-i}} \end{pmatrix} : X_i \underset{B}{\otimes} Y_{n-i} \longrightarrow X_i \underset{B}{\otimes} Y_{n-1-i} \oplus X_{i-1} \underset{B}{\otimes} Y_{n-i}.$$

Similarly, we define $Hom_{A\otimes 1}(X,Z)$ to be the complex of $B-C-$bimodules whose component in degree n is equal to

$$\prod_{i\in\mathbb{Z}} Hom_{A\otimes 1}(X_{-i}, Z_{n-i})$$

with differential induced by the maps

$$((-1)^n Hom_{A\otimes 1}(\delta_{-i+1}, Z_{n-i}), Hom_{A\otimes 1}(X_{-i}, \gamma_{n-i})).$$

1.2.4 For any integer i and any complex X with differential δ we denote by $X[i]$ the complex whose component in degree n is equal to X_{n-i}, with differential given by the maps $(-1)^i\delta_{n-i} : X_{n-i} \longrightarrow X_{n-i-1}$. This sign convention has the property, that if X is a complex of left $A-$modules and i an integer, then the natural isomorphisms $A \underset{A}{\otimes} X_n \cong X_n$ for any integer n induce an isomorphism of complexes $A[i] \underset{A}{\otimes} X \cong X[i]$, where $A[i]$ is the complex whose component in degree i is A and zero in all other degrees, while if X is a complex of right $B-$modules we have to multiply the canonical isomorphisms $X_n \underset{B}{\otimes} B \cong X_n$ by the sign $(-1)^{ni}$ in order to obtain an isomorphism of complexes $X \underset{B}{\otimes} B[i] \cong X[i]$. The above isomorphisms define natural isomorphisms between the functors $A[i] \underset{A}{\otimes} -$, the degree i shift functor $[i]$ and $- \underset{B}{\otimes} B[i]$ on the category of complexes of $A - B-$bimodules.

1.2.5 The *cone of a map of complexes* $f : X \longrightarrow Y$ is the complex $C(f)$ whose component in degree n is equal to $X_{n-1} \oplus Y_n$, with differential induced by the maps

$$X_n \oplus Y_{n+1} \longrightarrow X_{n-1} \oplus Y_n$$

whose restriction to X_n is the map $(-\delta_n, f_n)$ and whose restriction to Y_{n+1} is the map $(0, \epsilon_{n+1})$, where δ and ϵ are the differentials of X and Y, respectively.

The cone comes along with obvious natural maps $Y \longrightarrow C(f)$ and $C(f) \longrightarrow X[1]$.

1.2.6 For an $R-$algebra A we denote by $Mod(A)$ the category of finitely generated $A-$modules, by $C(A)$ the category of complexes of finitely generated $A-$modules and by $K(A)$ its homotopy category. We denote by $C^b(A)$ and $K^b(A)$ the full subcategories of $C(A)$ and $K(A)$, respectively, consisting of bounded complexes of $A-$modules. Recall that an $A-$module U is relatively $R-$projective if it is isomorphic to a direct summand of an $A-$module of the form $A \underset{R}{\otimes} V$ for some $R-$module V; we denote by $\overline{\mathrm{Mod}}(A)$ the *$R-$stable category of* $Mod(A)$; that is, the objects of $\overline{\mathrm{Mod}}(A)$ are the same as the objects of $Mod(A)$, and the morphisms are equivalence classes of $A-$homomorphisms for the relation declaring two $A-$homomorphisms between two $A-$modules to be equivalent if their difference factors through a relatively $R-$projective $A-$module (thus, if R is a field, this is just the usual stable category of $Mod(A)$).

1.2.7 We say that a finite-dimensional algebra A over a field k is *split* if $End_A(S) \cong k$ for every simple $A-$module, or, equivalently, if $A/J(A)$ is a direct sum of matrix algebras over k.

2 Transfer in Hochschild cohomology of symmetric algebras

A transfer associated to a suitable complex of bimodules has been considered by B. Keller [12] for cyclic homology and S. Bouc [4] for Hochschild homology. In the case of induction from a subalgebra, M. Feshbach has developed in [10] a transfer for the cohomology of Hopf algebras. We define here a transfer for Hochschild cohomology of symmetric algebras, using adjunction maps in a way which is similar to M. Broué's definition in [5] of relative traces for symmetric algebras. The properties given in 2.11, 2.12 are analogous to results in [12] and [4] on cyclic and Hochschild homology, respectively.

Let R be a commutative ring. Recall that an $R-$algebra A is *symmetric* if it is projective as $R-$module and isomorphic to its $R-$dual $A^* = Hom_R(A, R)$ as

$A-A-$bimodule. For any projective $A-$module M any choice of such a bimodule isomorphism $A \cong A^*$, or equivalently, of a symmetrizing form $s \in A^*$ on A (cf. 6.3), gives rise to a natural isomorphism of functors (cf. 6.5)

2.1. $Hom_A(M,-) \cong M^* \underset{A}{\otimes} -$

from $Mod(A)$ to $Mod(R)$. Note that then M^* is projective as right $A-$module, since $A^* \cong A$ is so. By naturality, this extends to complexes of bimodules as follows: if B is another $R-$algebra and X a bounded complex of $A-B-$bimodules which are projective as left $A-$modules, there is a natural isomorphism of functors

2.2. $Hom_{A\otimes 1}(X,-) \cong X^* \underset{A}{\otimes} -$

from the category $C(A)$ of complexes of $A-$modules to the category $C(B)$ of complexes of $B-$modules mapping the subcategory $C^b(A)$ of bounded complexes of $A-$modules to the corresponding subcategory $C^b(B)$.

In particular, the functor $X^* \underset{A}{\otimes} -$ is a right adjoint to $X \underset{B}{\otimes} -$ (see [5, 1.12, 2.3] or section 6 below), and we denote by

2.3.

$$\epsilon_X : B \longrightarrow X^* \underset{A}{\otimes} X,$$

$$\eta_X : X \underset{B}{\otimes} X^* \longrightarrow A$$

the chain maps of complexes of $B-B-$bimodules and $A-A-$bimodules, respectively, representing the unit and counit of this adjunction.

The adjunction maps in 2.3 depend on the choice of the symmetrizing form s on A in the following way:

2.4. *if $s' \in A^*$ is another symmetrizing form on A, there is a unique invertible element $u \in Z(A)^\times$ such that $s' = u.s$; that is, $s'(a) = s(ua)$ for any $a \in A$, and then the corresponding adjunction maps ϵ'_X, η'_X satisfy*

$$\epsilon'_X = (Id_{X^*} \otimes u.Id_X) \circ \epsilon_X,$$

$$\eta'_X = u^{-1}.\eta_X.$$

If in addition B is symmetric and if all components of X are projective as right $B-$modules, too, the above discussion applies to X^*, and thus the functors $X \underset{B}{\otimes} -$ and $X^* \underset{A}{\otimes} -$ are both left and right adjoint to each other. In particular, choosing a symmetrizing form on B we obtain chain maps of complexes of bimodules

2.5.

$$\epsilon_{X^*} : A \longrightarrow X \underset{B}{\otimes} X^*,$$

$$\eta_{X^*} : X^* \underset{A}{\otimes} X \longrightarrow B$$

representing the unit and counit of $X \underset{B}{\otimes} -$ being a right adjoint to $X^* \underset{A}{\otimes} -$. It is possible to give explicit descriptions of ϵ_M, η_M (see [5] or 6.6).

Note that the above discussion includes the case where X is an $A-B-$bimodule, considered then as complex concentrated in degree zero.

Example 2.6 Let G be a finite group. The map sending $\lambda \in (RG)^*$ to $\sum_{x \in G} \lambda(x^{-1})x \in RG$ is an isomorphism of $RG-RG-$bimodules; in particular, RG is symmetric and the symmetrizing form $s \in (RG)^*$ corresponding to this bimodule isomorphism is the $R-$linear map sending 1_G to 1_R and any non trivial element of G to 0_R.

Let H be a subgroup of G and $M = (RG)_H$ the $RG-RH-$bimodule obtained from restricting the regular $RG-RG-$bimodule RG to RH on the right. That is, the functor $M \underset{RH}{\otimes} -$ is the usual induction functor Ind_H^G. The $R-$dual M^* of M is, via the above bimodule isomorphism $(RG)^* \cong RG$, naturally isomorphic to the $RH-RG-$bimodule ${}_H(RG)$ obtained from restricting the regular $RG-RG-$bimodule RG to RH on the left, and the functor $M^* \underset{RG}{\otimes} -$ is then naturally isomorphic to the restriction functor Res_H^G. In particular, $M^* \underset{RG}{\otimes} M \cong RG$ as $RH-RH-$bimodules, and, with this identification, the bimodule homomorphisms given by adjunction are as follows:

$\epsilon_M : RH \longrightarrow RG$ is the inclusion,
$\eta_M : RG \underset{RH}{\otimes} RG \longrightarrow RG$ is given by multiplication in RG,
$\epsilon_{M^*} : RG \longrightarrow RG \underset{RH}{\otimes} RG$ maps $a \in RG$ to $\sum_{x \in [G/H]} ax \otimes x^{-1}$, and
$\eta_{M^*} : RG \longrightarrow RH$ is the natural projection mapping $x \in H$ to x and $x \in G-H$ to 0.

Recall that if X is a bounded complex of $A-B-$bimodules, the degree zero component of $X \underset{B}{\otimes} X^*$ is equal to $\underset{n}{\oplus} X_n \underset{B}{\otimes} X_n^*$, where n runs over the set of integers. With some abusive identifications (see 6.9), the adjunction maps have the following properties (the formal proof is left to the reader):

Proposition 2.7. *Let A, B, C be symmetric $R-$algebras, X, X' bounded complexes of $A-B-$bimodules, Y a bounded complex of $B-C-$bimodules, and suppose that all components of X, X', Y are projective as left and right modules.*

(i) We have $\epsilon_{X \oplus X'} = \epsilon_X + \epsilon_{X'}$ and $\eta_{X \oplus X'} = \eta_X + \eta_{X'}$.

(ii) We have $\epsilon_{X \underset{B}{\otimes} Y} = (Id_{Y^*} \otimes \epsilon_X \otimes Id_Y) \circ \epsilon_Y$ and $\eta_{X \underset{B}{\otimes} Y} = \eta_X \circ (Id_X \otimes \eta_Y \otimes Id_{X^*})$.

(iii) We have $\epsilon_X = \sum_n (-1)^n \epsilon_{X_n}$ and $\eta_X = \sum_n (-1)^n \eta_{X_n}$, where n runs over the set of integers.

The *Hochschild cohomology* of an $R-$algebra A whixh is projective as $R-$module this is the ring

$$HH^*(A) = Ext^*_{A \otimes A^0}(A).$$

More explicitely, we denote by $\mathcal{P}_A$ a projective resolution of A; that is, $\mathcal{P}_A$ is a right bounded complex of projective $A-A-$bimodules endowed with a quasi-isomorphism $\mu_A : \mathcal{P}_A \longrightarrow A$, where A is viewed as complex concentrated in degree zero. For explicit calculations it may sometimes be helpful to use the standard projective resolution of A as $A-A-$bimodule, which in degree $n \geq 0$ is equal to $A^{\otimes(n+2)}$ (the $(n+2)-$fold tensorproduct of A by itself over R) with differential

$$\delta_n^A : A^{\otimes(n+2)} \longrightarrow A^{\otimes(n+1)}$$

mapping $a_0 \otimes a_1 \otimes ... \otimes a_{n+1}$ to $\sum_{0 \leq k \leq n} (-1)^k a_0 \otimes .. \otimes a_k a_{k+1} \otimes .. \otimes a_{n+1}$ (where $n > 0$ and $a_j \in A$ for $0 \leq j \leq n+1$), together with the map from the degree zero component $A \underset{R}{\otimes} A$ to A given by multiplication in A.

By definition, $HH^n(A)$ is the cohomology in degree n of the cochain complex $Hom_{A\otimes A^0}(\mathcal{P}_A, A)$. By standard results of homological algebra, we have natural isomorphisms

2.8.

$$HH^n(A) = H^n(Hom_{A\otimes A^0}(\mathcal{P}_A, A)) \cong H^n(Hom_{A\otimes A^0}(\mathcal{P}_A, \mathcal{P}_A))$$

$$\cong Hom_{K(A\otimes A^0)}(\mathcal{P}_A, \mathcal{P}_A[n]).$$

It is this latter form of the Hochschild cohomology that we are going to use most of the time, since it is convenient for dealing with the multiplicative structure in $HH^*(A)$, which is simply induced by composing chain maps.

Observe that $HH^0(A) \cong Z(A)$ lies in the center of $HH^*(A)$ and that $HH^n(A)$ is a module over $Z(A)$ which is annihilated by the projective ideal $Z^{pr}(A)$ of $Z(A)$ for $n > 0$ (cf. [5]). We define the *stable Hochschild cohomology* $\overline{HH}^*(A)$ to be the quotient of $HH^*(A)$ by the ideal generated by $Z^{pr}(A)$; thus $\overline{HH}^n(A) = HH^n(A)$ for $n > 0$ and $\overline{HH}^0(A) = HH^0(A)/Z^{pr}(A)HH^0(A) \cong \overline{Z}(A)$.

Recall another standard fact on homological algebra: if A, B are symmetric $R-$algebras and X is a bounded complex of $A-B-$bimodules which are projective as left and right modules, the (total) complex $X^* \underset{A}{\otimes} \mathcal{P}_A \underset{A}{\otimes} X$ is a projective resolution of the complex $X^* \underset{A}{\otimes} X$; that is, it is right bounded (as $\mathcal{P}_A$ is so) consisting of projective $B-B-$bimodules and quasi-isomorphic to $X^* \underset{A}{\otimes} X$ through the chain map $Id_{X^*} \otimes \mu_A \otimes Id_X$, where $\mu_A : \mathcal{P}_A \longrightarrow A$ is the given quasi-isomorphism as above. Therefore the adjunction map $\epsilon_X : B \longrightarrow X^* \underset{A}{\otimes} X$ lifts to a chain map

$$\mathcal{P}_B \longrightarrow X^* \underset{A}{\otimes} \mathcal{P}_A \underset{A}{\otimes} X$$

that we are going to denote by ϵ_X again; this chain map is in general not unique, but unique up to homotopy. Similarly, η_X lifts to a chain map

$$X \underset{B}{\otimes} \mathcal{P}_B \underset{B}{\otimes} X^* \longrightarrow \mathcal{P}_A$$

still denoted by η_X, which is again unique up to homotopy. With this notation we define now a transfer as follows:

Definition 2.9 Let A, B be symmetric $R-$algebras, fix symmetrizing forms $s \in A^*$, $t \in B^*$, and let X be a bounded complex of $A-B-$bimodules which are projective as left and right modules. The *transfer associated with* X is the unique linear graded map

$$t_X : HH^*(B) \longrightarrow HH^*(A)$$

sending, for any $n \geq 0$, the homotopy class $[\zeta]$ of a chain map $\zeta : \mathcal{P}_B \longrightarrow \mathcal{P}_B[n]$ to the homotopy class $[\eta_X[n] \circ (Id_X \otimes \zeta \otimes Id_{X^*}) \circ \epsilon_{X^*}]$ of the composition of chain maps

2.9.1

$$\mathcal{P}_A \stackrel{\epsilon_{X^*}}{\longrightarrow} X \underset{B}{\otimes} \mathcal{P}_B \underset{B}{\otimes} X^* \stackrel{Id_X \otimes \zeta \otimes Id_{X^*}}{\longrightarrow} X \underset{B}{\otimes} \mathcal{P}_B[n] \underset{B}{\otimes} X^* \stackrel{\eta_X[n]}{\longrightarrow} \mathcal{P}_A[n].$$

The degree zero component $t_X^0 : HH^0(B) \longrightarrow HH^0(A)$ of t_X induces a linear map $Z(B) \longrightarrow Z(A)$ through the natural isomorphisms $HH^0(A) \cong Z(A)$ and $HH^0(B) \cong Z(B)$ that we are going to denote abusively again by t_X^0, if no confusion arises.

If M is an $A-B-$bimodule which is projective as left and right module, we denote by t_M the transfer associated with the complex equal to M in degree zero and zero in all other degrees.

Remark 2.10 The above definition depends on the choice of the symmetrizing forms s on A and t on B in the following way: if $s' \in A^*$ and $t' \in B^*$ are some other symmetrizing forms on A and B, there are unique invertible elements $u \in Z(A)^\times$ and $v \in Z(B)^\times$ such that $s' = u.s$ and $t' = v.t$. It follows from 2.4 that the corresponding transfer map t'_X associated with the choice of the symmetrizing forms s' and t' instead of s and t satisfies

$$t'_X([\tau]) = u^{-1} t_X(v[\tau])$$

for any $[\tau] \in HH^*(B)$.

In the case of group algebras we assume always that the bimodule isomorphisms are the standard ones as in 2.6.

The following proposition is an immediate consequence of 2.7:

Proposition 2.11. *Let A, B, C be symmetric $R-$algebras, X, X' bounded complexes of $A-B-$bimodules and Y a bounded complex of $B-C-$bimodules. Assume that all components of X, X', Y are projective as left and right modules. For any choice of symmetrizing forms on A, B, C we have:*

(i) $t_{X \oplus X'} = t_X + t_{X'}$,

(ii) $t_{X \underset{B}{\otimes} Y} = t_X \circ t_Y$,

(iii) $t_X = \sum_{n \in \mathbb{Z}} (-1)^n t_{X_n}$.

(iv) $t_{X[i]} = (-1)^i t_X$ *for any integer* i.

We list some further easy properties of the transfer, including in particular the statement analogous to [12, 2.4] saying that the transfer map t_X depends only on the image of X in the Grothendieck group of the triangulated subcategory of $K^b(A \otimes B^0)$ of bounded complexes whose components are projective as left and right modules:

Proposition 2.12. *Let A, B be symmetric $R-$algebras with symmetrizing forms $s \in A^*$, $t \in B^*$, X, Y bounded complexes of $A-B-$bimodules which are projective as left and right modules, and let $f : X \longrightarrow Y$ be a chain map. Denote by $C(f)$ the mapping cone of f.*

(i) The components of $C(f)$ are projective as left and right modules, and we have $t_{C(f)} = t_Y - t_X$.

(ii) If X is acyclic we have $t_X = 0$.

(iii) If f is a quasi-isomorphism we have $t_X = t_Y$.

(iv) If $0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$ is an exact sequence of $A-B-$bimodules which are projective as left and right modules, we have $t_M = t_{M'} + t_{M''}$.

(v) If M is a projective $A-B-$bimodule then $t_M(HH^n(B)) = 0$ for all $n > 0$ and $t_M(HH^0(B)) \subset Z^{pr}(A)HH^0(A)$.

Proof. (i) The component in degree i of $C(f)$ is equal to $X_{i-1} \oplus Y_i$ and hence (i) follows from 2.11 (iii).

(ii) If X is acyclic, the complex $X \underset{B}{\otimes} \mathcal{P}_B \underset{B}{\otimes} X^*$ is acyclic, right bounded and its components are projective (as bimodules), hence this complex splits and is therefore homotopic to zero, which implies the statement.

(iii) If f is a quasi-isomorphism the cone $C(f)$ is acyclic, and thus (iii) follows from (i) and (ii).

(iv) is a particular case of (ii).

Statement (v) follows from the fact that $M \underset{B}{\otimes} \mathcal{P}_B \underset{A}{\otimes} M^*$ is a projective resolution of the projective bimodule $M \underset{B}{\otimes} M^*$, hence a split complex which is homotopy equivalent to $M \underset{B}{\otimes} M^*$ viewed as complex concentrated in degree zero.

Remark 2.13 By [20, 2.5] the Hochschild cohomology ring is invariant under derived equivalences. For symmetric algebras this can be seen as follows: with the notation of 2.12, if A and B are derived equivalent, then J. Rickard proved in [20] that there is a bounded complex X of $A-B-$bimodules whose components are projective as left and right modules such that we have homotopy equivalences $X \underset{B}{\otimes} X^* \simeq A$ and $X^* \underset{A}{\otimes} X \simeq B$ as complexes of bimodules. Then t_X and t_{X^*} are mutually inverse ring isomorphisms between $HH^*(A)$ and $HH^*(B)$, since all occurring adjunction maps are homotopy equivalences.

Similarly, if M is an $A-B-$bimodule inducing a stable equivalence of Morita type between A and B (a concept due to Broué [5]; see [13] for some properties of this notion), then t_M and t_{M^*} induce mutually inverse ring isomorphisms $\overline{HH}^*(A) \cong \overline{HH}^*(B)$ by 2.12 (v).

3 Stable elements

We provide here an abstract setting for the notion of "stable elements" in Hochschild cohomology of symmetric algebras and study its connections with transfer maps; this is intended to be a tool for dealing with the multiplicative structure of Hochschild cohomology rings. In section 5 we establish a link to the "classical" definition of stable elements in group cohomology as given in [9, Ch. XII, section 10].

Let R be a commutative ring. We keep the notation of the previous section for adjunction maps.

Definition 3.1 Let A, B be symmetric $R-$algebras with symmetrizing forms $s \in A^*$, $t \in B^*$, and X a bounded complex of $A-B-$bimodules which are projective as left and right modules.

(i) We denote by π_X the image $\eta_X \circ \epsilon_{X^*}(1_A)$ in $Z(A)$ of 1_A under the composition of bimodule homomorphisms $A \underset{\epsilon_{X^*}}{\longrightarrow} X \underset{B}{\otimes} X^* \underset{\eta_X}{\longrightarrow} A$ and call π_X the *relatively $X-$projective element of* $Z(A)$ (with respect to the choice of the symmetrizing forms s and t).

(ii) If π_X is invertible in $Z(A)$ we denote by

$$T_X : HH^*(B) \longrightarrow HH^*(A)$$

the graded linear map defined by $T_X([\tau]) = \pi_X^{-1} t_X([\tau])$ for any $[\tau] \in HH^*(B)$, and call T_X the *normalized transfer associated with* X.

(iii) An element $[\zeta] \in HH^*(A)$ is called $X-$*stable* if there is $[\tau] \in HH^*(B)$ such that for any non negative integer n the following diagram is homotopy commutative

3.1.1

$$\begin{array}{ccc} \mathcal{P}_A \underset{A}{\otimes} X & \xrightarrow{\simeq} & X \underset{B}{\otimes} \mathcal{P}_B \\ {\scriptstyle \zeta_n \otimes Id_X} \downarrow & & \downarrow {\scriptstyle Id_X \otimes \tau_n} \\ \mathcal{P}_A[n] \underset{A}{\otimes} X & \xrightarrow{\simeq} & X \underset{B}{\otimes} \mathcal{P}_B[n] \end{array}$$

where ζ_n and τ_n are the components in degree n of ζ and τ, respectively, and where the horizontal arrows are given by the natural homotopy equivalences $\mathcal{P}_A \underset{A}{\otimes} X \simeq X \underset{B}{\otimes} \mathcal{P}_B$ lifting the natural isomorphism $A \underset{A}{\otimes} X \cong X \underset{B}{\otimes} B$. We denote by $HH^*_X(A)$ the set of $X-$stable elements in $HH^*(A)$.

(iv) We say that an element $z \in Z(A)$ is $X-$*stable*, if the image of z in $HH^0(A)$ under the natural isomorphism $Z(A) \cong HH^0(A)$ lies in $HH^0_X(A)$, and denote by $Z_X(A)$ the set of $X-$stable elements in $Z(A)$.

Remark 3.2 The element π_X defined in 3.1(i) depends on the choice of the symmetrizing forms $s \in A^*$ and $t \in B^*$. In fact, denoting again abusively by $t^0_X : Z(B) \longrightarrow Z(A)$ the linear map induced by the degree zero component of t_X via the natural isomorphisms $Z(A) \cong HH^0(A)$ and $Z(B) \cong HH^0(B)$, we have

3.2.1

$$\pi_X = t^0_X(1_B).$$

Thus if π'_X is the relatively $X-$projective element with respect to another choice of symmetrizing forms $s' \in A^*$ and $t' \in B^*$, there are unique invertible elements $u \in Z(A)^\times$, $v \in Z(B)^\times$ such that $s' = u.s$, $t' = v.t$ (cf. 2.4), and then by 2.10, we have

3.2.2

$$\pi'_X = u^{-1} t^0_X(v).$$

Using 2.6 it is possible to construct examples where $\pi_X = 1_A$ and $\pi'_X = 0_A$ (see 3.9 below), so not even the property of π_X being invertible is independent of the choice of the symmetrizing forms. The previous formula shows in particular

3.2.3 if π_X is invertible, then setting $s' = (\pi_X)^{-1}.s$ and $t' = t$ we have $\pi'_X = 1_A$.

Observe that at the levels of complexes, the composition of chain maps

3.2.4

$$\mathcal{P}_A \underset{\epsilon_{X^*}}{\longrightarrow} X \underset{B}{\otimes} \mathcal{P}_B \underset{B}{\otimes} X^* \underset{\eta_X}{\longrightarrow} \mathcal{P}_A$$

is homotopic to either endomorphism of $\mathcal{P}_A$ induced by left or right multiplication with the element π_X on $\mathcal{P}_A$, since $\eta_X \circ \epsilon_{X^*}$ "lifts" the endomorphism of A given by left or right multiplication with the central element π_X (cf. 3.1.1).

The property of an element $[\zeta] \in HH^*(A)$ to be $X-$stable does not depend on the choice of the symmetrizing forms on A and B. Note finally that an element $z \in Z(A)$ is $X-$stable if and only if there is an element $y \in Z(B)$ such that the two chain endomorphisms of X induced by left multiplication with z and by right multiplication with y on X are homotopic.

Lemma 3.3. *Let A, B be symmetric $R-$algebras with symmetrizing forms $s \in A^*$, $t \in B^*$, X a bounded complex of $A - B-$bimodules which are projective as left and right modules, and let $[\zeta] \in HH^n(A)$ and $[\tau] \in HH^n(B)$, where n is a nonnegative integer.*

The diagram

3.3.1

$$\begin{array}{ccc} \mathcal{P}_A \underset{A}{\otimes} X & \xrightarrow{\simeq} & X \underset{B}{\otimes} \mathcal{P}_B \\ {\scriptstyle \zeta\otimes Id_X}\downarrow & & \downarrow{\scriptstyle Id_X\otimes\tau} \\ \mathcal{P}_A[n] \underset{A}{\otimes} X & \underset{\simeq}{\longrightarrow} & X \underset{B}{\otimes} \mathcal{P}_B[n] \end{array}$$

is homotopy commutative if and only if any of the following diagrams is homotopy commutative:

3.3.2

$$\begin{array}{ccc} \mathcal{P}_A & \xrightarrow{\epsilon_{X^*}} & X \underset{B}{\otimes} \mathcal{P}_B \underset{B}{\otimes} X^* \\ {\scriptstyle \zeta}\downarrow & & \downarrow{\scriptstyle Id_X\otimes\tau\otimes Id_{X^*}} \\ \mathcal{P}_A[n] & \underset{\epsilon_{X^*}[n]}{\longrightarrow} & X \underset{B}{\otimes} \mathcal{P}_B[n] \underset{B}{\otimes} X^* \end{array}$$

3.3.3

$$\begin{array}{ccc} X^* \underset{A}{\otimes} \mathcal{P}_A \underset{A}{\otimes} X & \xrightarrow{\eta_X} & \mathcal{P}_B \\ {\scriptstyle Id_{X^*}\otimes\zeta\otimes Id_X}\downarrow & & \downarrow{\scriptstyle \tau} \\ X^* \underset{A}{\otimes} \mathcal{P}_A[n] \underset{A}{\otimes} X & \underset{\eta_X[n]}{\longrightarrow} & \mathcal{P}_B[n] \end{array}$$

3.3.4

$$\begin{array}{ccc} \mathcal{P}_B \underset{B}{\otimes} X^* & \xrightarrow{\simeq} & X^* \underset{A}{\otimes} \mathcal{P}_A \\ {\scriptstyle \tau\otimes Id_{X^*}}\downarrow & & \downarrow{\scriptstyle Id_{X^*}\otimes\zeta} \\ \mathcal{P}_B[n] \underset{B}{\otimes} X^* & \underset{\simeq}{\longrightarrow} & X^* \underset{A}{\otimes} \mathcal{P}_A[n] \end{array}$$

3.3.5

$$\begin{array}{ccc} \mathcal{P}_B & \xrightarrow{\epsilon_X} & X^* \underset{A}{\otimes} \mathcal{P}_A \underset{A}{\otimes} X \\ {\scriptstyle \tau}\downarrow & & \downarrow{\scriptstyle Id_{X^*}\otimes\zeta\otimes Id_X} \\ \mathcal{P}_B[n] & \underset{\epsilon_X[n]}{\longrightarrow} & X^* \underset{A}{\otimes} \mathcal{P}_A[n] \underset{A}{\otimes} X \end{array}$$

3.3.6

$$\begin{array}{ccc} X \underset{B}{\otimes} \mathcal{P}_B \underset{B}{\otimes} X^* & \xrightarrow{\eta_X} & \mathcal{P}_A \\ {\scriptstyle Id_X\otimes\tau\otimes Id_{X^*}}\downarrow & & \downarrow{\scriptstyle \zeta} \\ X \underset{B}{\otimes} \mathcal{P}_B[n] \underset{B}{\otimes} X^* & \underset{\eta_X[n]}{\longrightarrow} & \mathcal{P}_A[n] \end{array}$$

Proof. If 3.3.1 is homotopy commutative, tensoring this diagram by $-\underset{B}{\otimes} X^*$ and composing it with the commutative diagram

$$\begin{array}{ccc} \mathcal{P}_A \underset{A}{\otimes} A & \xrightarrow{Id_{\mathcal{P}_A}\otimes\epsilon_{X^*}} & \mathcal{P}_A \underset{A}{\otimes} X \underset{B}{\otimes} X^* \\ {\scriptstyle \zeta\otimes Id_{\mathcal{P}_A}}\downarrow & & \downarrow{\scriptstyle \zeta\otimes Id_X\otimes Id_{X^*}} \\ \mathcal{P}_A[n] \underset{A}{\otimes} A & \underset{Id_{\mathcal{P}_A}[n]\otimes\epsilon_{X^*}}{\longrightarrow} & \mathcal{P}_A[n] \underset{A}{\otimes} X \underset{B}{\otimes} X^* \end{array}$$

shows that 3.3.2 is homotopy commutative. By applying this argument in the appropriate way it follows that the homotopy commutativity of the diagrams 3.3.1, 3.3.2 and 3.3.3 is equivalent. Since the horizontal arrows in 3.3.1 are homotopy equivalences we may reverse them, and then, again applying appropriate variations of the above argument shows that 3.3.1 is homotopy commutative if and only if any of the diagrams 3.3.4, 3.3.5, 3.3.6 is so.

Lemma 3.4. *Let A, B be symmetric $R-$algebras with symmetrizing forms $s \in A^*$, $t \in B^*$ and X a bounded complex of $A-B-$bimodules which are projective as left and right modules. Let $[\zeta] \in HH^*_X(A)$ and $[\tau] \in HH^*(B)$ such that the diagram 3.1.1 is homotopy commutative for all nonnegative integers n.*

(i) We have $[\tau] \in HH^*_{X^*}(B)$.

(ii) We have $t_X([\tau]) = \pi_X[\zeta]$.

(iii) For any $[\sigma] \in HH^*(B)$ we have $t_X([\tau][\sigma]) = [\zeta]t_X([\sigma])$ and $t_X([\sigma][\tau]) = t_X([\sigma])[\zeta]$.

(iv) We have $t_{X^*\underset{A}{\otimes}X}([\tau]) = \pi_{X^*\underset{A}{\otimes}X}[\tau]$.

Proof. Statement (i) follows from the fact that 3.3.1 is homotopy commutative if and only if 3.3.4 is so. Using 3.3.2 we get $t_X([\tau]) = [\eta_X \circ \epsilon_{X^*} \circ \zeta]$, and since $\eta_X \circ \epsilon_{X^*}$ is homotopic to multiplication by π_X on $\mathcal{P}_A$ we obtain (ii). Similarly, composing the diagrams 3.3.2 and 3.3.6 with the diagram defining $t_X([\tau])$ in each component yields the two equalities in (iii). For (iv) apply t_{X^*} to the equality (ii) and use then the analogue of (iii) with X^* instead of X, together with the fact that $\pi_{X^* \underset{A}{\otimes} X} = t^0_{X^* \underset{A}{\otimes} X}(1_B) = t^0_{X^*}(\pi_X)$ by 3.2.1 and 2.11(ii).

Proposition 3.5. *Let A, B be symmetric $R-$algebras with symmetrizing forms $s \in A^*$, $t \in B^*$, and X a bounded complex of $A-B-$bimodules whose components are projective as left and right modules.*

(i) The set $HH^*_X(A)$ of $X-$stable elements in $HH^*(A)$ is a graded subalgebra of $HH^*(A)$; in particular, $Z_X(A)$ is a subalgebra of $Z(A)$.

(ii) The space $Im(t_X)$ is a graded $HH^*_X(A)-HH^*_X(A)-$subbimodule of $HH^*(A)$.

(iii) We have $t_X(HH^*_{X^*}(B)) = \pi_X HH^*_X(A)$.

(iv) For every direct summand X' of the complex X we have $HH^*_X(A) \subset HH^*_{X'}(A)$.

Proof. Clearly $HH^*_X(A)$ is a graded $R-$submodule of $HH^*(A)$, and the composition of diagrams of the form 3.1.1 shows that indeed $HH^*_X(A)$ is a ring, which shows (i). Statement (ii) follows from 3.4(iii), and (iii) is a consequence of 3.4(i) and 3.4(ii). Moreover, if 3.3.1 is homotopy commutative, decomposing the complex X into a direct sum of complexes yields a decomposition of the diagram 3.3.1 into a direct sum of homotopy commutative diagrams, which implies (iv).

Theorem 3.6. *Let A, B be symmetric $R-$algebras with symmetrizing forms $s \in A^*$, $t \in B^*$, and X be a bounded complex of $A-B-$bimodules which are projective as left and right modules.*

(i) If π_X is invertible, the normalized transfer T_X induces a graded surjective $R-$algebra homomorphism

$$R_X : HH^*_{X^*}(B) \longrightarrow HH^*_X(A).$$

(ii) If π_X is invertible and $s' \in A^*$, $t' \in B^*$ are symmetrizing forms such that the corresponding relatively $X-$projective element π'_X is again invertible, then, denoting by R'_X the algebra homomorphism induced by the normalized transfer T'_X associated with the choice of s', t' instead of s, t, we have $R'_X = R_X$.

(iii) If both π_X and π_{X^*} are invertible then R_X and R_{X^*} are mutually inverse $R-$algebra isomorphisms

$$HH^*_{X^*}(B) \cong HH^*_X(A).$$

Proof. (i) By 3.4 (ii) and 3.4 (iii), the map R_X is indeed an $R-$algebra homomorphism, and by 3.5 (iii) it is surjective.

(ii) Let $u \in Z(A)^\times$ and $v \in Z(B)^\times$ be the unique elements such that $s' = u.s$ and $t' = v.t$. Applying 3.4 to $[\tau] \in HH^*_{X^*}(B)$ yields $t^0_X(v)t_X([\tau]) = \pi_X t_X(v[\tau])$. Multiplying this equation by u^{-1} shows then that $\pi'_X t_X([\tau]) = \pi_X t'_X([\tau])$ by 2.10. Since both π_X and π'_X are invertible it follows that $R_X([\tau]) = R'_X([\tau])$.

Statement (iii) is an easy verification, using again 3.4(i) and 3.4(ii).

Proposition 3.7. *Let A, B, C be symmetric $R-$algebras, X a bounded complex of $A-B-$bimodules, and Y a bounded complex of $B-C-$bimodules. Suppose that all components of both X and Y are projective as left and right modules. Let n be a nonnegative integer, $[\zeta] \in HH^n(A)$ and $[\gamma] \in HH^n(C)$ making the following diagram homotopy commutative:*

3.7.1

$$\begin{array}{ccc}
\mathcal{P}_A & \xrightarrow{\epsilon_{Y^* \underset{B}{\otimes} X^*}} & X \underset{B}{\otimes} Y \underset{C}{\otimes} \mathcal{P}_C \underset{C}{\otimes} Y^* \underset{B}{\otimes} X^* \\
\zeta \big\downarrow & & \big\downarrow Id_{X \underset{B}{\otimes} Y} \otimes \gamma \otimes Id_{Y^* \underset{B}{\otimes} X^*} \\
\mathcal{P}_A[n] & \xrightarrow[\epsilon_{Y^* \underset{B}{\otimes} X^*}[n]]{} & X \underset{B}{\otimes} Y \underset{C}{\otimes} \mathcal{P}_C[n] \underset{C}{\otimes} Y^* \underset{B}{\otimes} X^*
\end{array}$$

Suppose there is $z \in Z(A)$ such that left multiplication by z and right multiplication by π_Y on X induce homotopic endomorphisms of X.
Then the following diagram is homotopy commutative:

3.7.2

$$\begin{array}{ccc}
\mathcal{P}_A & \xrightarrow{\epsilon_{X^*}} & X \underset{B}{\otimes} \mathcal{P}_B \underset{B}{\otimes} X^* \\
z.\zeta \big\downarrow & & \big\downarrow Id_X \otimes t_Y(\gamma) \otimes Id_{X^*} \\
\mathcal{P}_A[n] & \xrightarrow[\epsilon_{X^*}[n]]{} & X \underset{B}{\otimes} \mathcal{P}_B[n] \underset{B}{\otimes} X^*
\end{array}$$

In particular, $z[\zeta]$ is $X-$stable and $t_Y([\gamma])$ is X^*-stable.

Proof.
By 2.7(ii) we have a homotopy commutative diagram

3.7.3

$$\begin{array}{ccc}
\mathcal{P}_A & \xrightarrow{\epsilon_{X^*}} & X \underset{B}{\otimes} \mathcal{P}_B \underset{B}{\otimes} X^* \\
Id_{\mathcal{P}_A} \big\downarrow & & \big\downarrow Id_X \otimes \epsilon_{Y^*} \otimes Id_{X^*} \\
\mathcal{P}_A & \xrightarrow[\epsilon_{Y^* \underset{B}{\otimes} X^*}]{} & X \underset{B}{\otimes} Y \underset{C}{\otimes} \mathcal{P}_C \underset{C}{\otimes} Y^* \underset{B}{\otimes} X^*
\end{array}$$

(with the usual identification $(X \underset{B}{\otimes} Y)^* \cong Y^* \underset{B}{\otimes} X^*$, see 6.7). Moreover, by 2.7(ii) we have $(Id_X \otimes \eta_Y \otimes Id_{X^*}) \circ \epsilon_{Y^* \underset{B}{\otimes} X^*} = (Id_X \otimes \eta_Y \circ \epsilon_{Y^*} \otimes Id_{X^*}) \circ \epsilon_{X^*}$, and as $\eta_Y \circ \epsilon_{Y^*}$ is homotopic to right (or left) multiplication by π_Y on $\mathcal{P}_B$, it follows from the hypothesis on z, that we have a homotopy commutative diagram

3.7.4

$$\begin{array}{ccc}
\mathcal{P}_A & \xrightarrow{\epsilon_{Y^* \underset{B}{\otimes} X^*}} & X \underset{B}{\otimes} Y \underset{C}{\otimes} \mathcal{P}_C \underset{C}{\otimes} Y^* \underset{B}{\otimes} X^* \\
z.Id_{\mathcal{P}_A} \big\downarrow & & \big\downarrow Id_X \otimes \eta_Y \otimes Id_{X^*} \\
\mathcal{P}_A & \xrightarrow[\epsilon_{X^*}]{} & X \underset{B}{\otimes} \mathcal{P}_B \underset{B}{\otimes} X^*
\end{array}$$

Now shift the diagram 3.7.4 by the degree n and match it (vertically) together with the diagrams 3.7.1 and 3.7.3; this yields the diagram 3.7.2 as required.

Corollary 3.8. *Let A, B, C , X and Y be as in 3.7. If π_Y is invertible in $Z(B)$ we have $HH^*_{X\underset{B}{\otimes}Y}(A) \subset HH^*_X(A)$ and $T_Y(HH^*_{Y^*\underset{B}{\otimes}X^*}(C)) \subset HH^*_{X^*}(B)$.*

Proof. Recall that the property of being stable with respect to some complex is independent of the choice of symmetrizing forms; thus, as π_Y is invertible we may in fact assume that $\pi_Y = 1_B$ by 3.2.3 and 3.6(ii). Then $z = 1_A$ fulfills the hypothesis in 3.7, and thus 3.8 follows from 3.7.

Example 3.9 Let G be a finite group, H a subgroup of G and $M = (RG)_H$. With respect to the canonical symmetrizing forms on RG and RH we have $\pi_M = [G : H].1_{RG}$ and $\pi_{M^*} = 1_{RH}$ as follows from the explicit description of the adjunction maps in 2.6. If there is $z \in Z(G)$ such that z lies not in H, then $s' = z.s$ is again a symmetrizing form on RG, and now the relatively projective elements with respect to s' and the canonical form on RH are $\pi'_M = [G : H].z$ and $\pi'_{M^*} = 0_{RH}$ by 3.2.2.

4 Transfer and group cohomology

Let R be a commutative ring. The cohomology of a finite group G with coefficients in R is the ring (cf. [2])

4.1.
$$H^*(G,R) = Ext^*_{RG}(R,R),$$

where R is considered as $RG-$module with the trivial action of the elements of G on R. Again, it is well-known that there is a natural isomorphism

4.2.
$$H^n(G,R) \cong Hom_{K(RG)}(\mathcal{P}_R, \mathcal{P}_R[n])$$

for any nonnegative integer n, where $\mathcal{P}_R$ is a projective resolution of the trivial $RG-$module R.

If we denote by ΔG the diagonal subgroup $\Delta G = \{(x,x)\}_{x\in G}$ of $G \times G$, there is a unique isomorphism of $RG - RG-$bimodules

4.3.
$$\mathrm{Ind}_{\Delta G}^{G\times G}(R) \cong RG$$

mapping $(x,y)\otimes 1_R$ to xy^{-1}, where x, $y \in G$ (with our convention 1.2.1 of considering the $R(G\times G)-$module $\mathrm{Ind}_{\Delta G}^{G\times G}(R)$ as $RG - RG-$bimodule).

Thus the complex $\mathrm{Ind}_{\Delta G}^{G\times G}(\mathcal{P}_R)$ is a projective resolution of RG; in particular, the isomorphism in 4.3 lifts to a homotopy equivalence of complexes of $RG - RG-$bimodules

4.4.
$$\mathrm{Ind}_{\Delta G}^{G\times G}(\mathcal{P}_R) \simeq \mathcal{P}_{RG},$$

which is unique up to homotopy, where here $\mathcal{P}_{RG}$ is a projective resolution of RG as $RG-RG-$bimodule as above. The following is well-known (and we leave the proof to the reader):

Proposition 4.5. *Let G be a finite group and $\mathcal{P}_R$ a projective resolution of the trivial $RG-$module R. The map sending $\zeta \in Hom_{C(RG)}(\mathcal{P}_R, \mathcal{P}_R[n])$ to $\mathrm{Ind}_{\Delta G}^{G\times G}(\zeta)$, where n is a nonnegative integer, induces an injective $R-$algebra homomorphism*

$$\delta_G : H^*(G,R) \longrightarrow HH^*(RG).$$

Recall that for any subgroup H of G the usual transfer map (see e.g. [2]) t_H^G : $H^*(H,R) \longrightarrow H^*(G,R)$ can be defined as follows: since the restriction to H of a projective resolution $\mathcal{P}_R$ of the trivial $RG-$module is a projective resolution of the trivial $RH-$module, any element of $H^n(H,R)$ can be represented by a chain map $\tau : Res_H^G(\mathcal{P}_R) \longrightarrow Res_H^G(\mathcal{P}_R[n])$, from which we obtain a chain map $Tr_H^G(\tau)$: $\mathcal{P}_R \longrightarrow \mathcal{P}_R[n]$ defined by $Tr_H^G(\tau)(a) = \sum_{x\in[G/H]} x\tau(x^{-1}a)$, where $a \in \mathcal{P}_R$, and then $t_H^G([\tau]) = [Tr_H^G(\tau)]$. Moreover, if $\varphi : H \longrightarrow G$ is any injective group homomorphism, we denote by $res_\varphi : H^*(G,R) \longrightarrow H^*(H,R)$ the graded linear map induced by restriction through φ.

The usual transfer and restriction for group cohomology are without any surprise compatible with the corresponding transfers in Hochschild cohomology through the algebra homomorphisms in 4.5 (recall that we consider RG with the canonical symmetrizing form mapping 1_G to 1_R and any non trivial element of G to 0_R):

Proposition 4.6. *Let G be a finite group and H a subgroup of G. The following diagram is commutative:*

$$\begin{array}{ccc} H^*(H,R) & \xrightarrow{t_H^G} & H^*(G,R) \\ {\scriptstyle \delta_H}\downarrow & & \downarrow{\scriptstyle \delta_G} \\ HH^*(RH) & \xrightarrow[t_{(RG)_H}]{} & HH^*(RG) \end{array}$$

Proof. Let $\mathcal{P}_R$ be a projective resolution of the trivial $RG-$module R. Let n be a nonnegative integer and $\tau \in Hom_{C(RH)}(\mathcal{P}_R, \mathcal{P}_R[n])$. Then Tr_H^G followed by $\mathrm{Ind}_{\Delta G}^{G\times G}$ maps τ to the chain map

4.6.1

$$\mathrm{Ind}_{\Delta G}^{G\times G}(\mathcal{P}_R) \underset{\mathrm{Ind}_{\Delta G}^{G\times G}(Tr_H^G(\tau))}{\longrightarrow} \mathrm{Ind}_{\Delta G}^{G\times G}(\mathcal{P}_R[n])$$

and, the other way round, $Ind_{\Delta H}^{H\times H}$ followed by the diagram defining the transfer $t_{(RG)_H}$ maps τ to the chain map

4.6.2

$$\mathrm{Ind}_{\Delta G}^{G\times G}(\mathcal{P}_R) \longrightarrow Ind_{\Delta H}^{G\times G}(\mathcal{P}_R) \underset{Ind_{\Delta H}^{G\times G}(\tau)}{\longrightarrow} Ind_{\Delta H}^{G\times G}(\mathcal{P}_R[n]) \longrightarrow \mathrm{Ind}_{\Delta G}^{G\times G}(\mathcal{P}_R[n]),$$

where the first and last map are induced by the appropriate adjunction maps. Using the explicit description in 2.6 of these adjunction maps shows that the two chain maps defined in 4.6.1 and 4.6.2 coincide, which completes the proof.

Proposition 4.7. *Let G, H be finite groups and $\varphi : H \longrightarrow G$ an injective group homomorphism. The following diagram is commutative:*

$$\begin{array}{ccc} H^*(G,R) & \xrightarrow{res_\varphi} & H^*(H,R) \\ \delta_G \downarrow & & \downarrow \delta_H \\ HH^*(RG) & \xrightarrow[t_{\varphi(RG)}]{} & HH^*(RH) \end{array}$$

Proof. We may assume that H is a subgroup of G and that φ is the inclusion. Then 4.7 follows from the explicit description of the adjunction maps corresponding to the transfer associated with the bimodule ${}_H(RG)$ in 2.6.

Proposition 4.8. *Let G be a finite group and H a subgroup of G.*

(i) We have a natural isomorphism of functors $Res^{G\times G}_{G\times H}\mathrm{Ind}^{G\times G}_{\Delta G} \cong Ind^{G\times H}_{\Delta H} Res^{\Delta G}_{\Delta H}$ from $Mod(R\Delta G)$ to $Mod(R(G\times H))$.

(ii) For any nonnegative integer n and any chain map $\zeta \in Hom_{C(RG)}(\mathcal{P}_R, \mathcal{P}_R[n])$ we have a commutative diagram of complexes of $RG-RH-$bimodules

$$\begin{array}{ccc} \mathrm{Ind}^{G\times G}_{\Delta G}(\mathcal{P}_R) \underset{RG}{\otimes} (RG)_H & \xrightarrow{\cong} & RG \underset{RH}{\otimes} Ind^{H\times H}_{\Delta H}(\mathcal{P}_R) \\ \delta_G(\zeta)\otimes Id_{RG} \downarrow & & \downarrow Id_{RG}\otimes\delta_H(\zeta) \\ \mathrm{Ind}^{G\times G}_{\Delta G}(\mathcal{P}_R[n]) \underset{RG}{\otimes} (RG)_H & \xrightarrow[\cong]{} & RG \underset{RH}{\otimes} Ind^{H\times H}_{\Delta H}(\mathcal{P}_R[n]) \end{array}$$

(iii) We have $Im(\delta_G) \subset HH^*_{(RG)_H}(RG)$.

Proof. Statement (i) is a particular case of the Mackey formula. Through our convention 1.2.1 of considering $RG-RH-$bimodules as $R(G\times H)-$modules the left column in the diagram of (ii) is obtained from applying the functor $Res^{G\times G}_{G\times H}Ind^{G\times G}_{\Delta G}$ to the chain map ζ, and the right column is obtained from applying the functor $Ind^{G\times H}_{\Delta H}Res^{\Delta G}_{\Delta H}$ to ζ, thus (ii) follows from (i).

The horizontal isomorphisms in the commutative diagram in (ii) yield precisely the homotopy equivalences of the diagram 3.1.1 applied to $(RG)_H$ instead of X, which shows (iii).

5 The cohomology ring of a $p-$block of a finite group

In this section we fix a prime p and a complete discrete valuation ring $\mathcal{O}$ having a residue field $k = \mathcal{O}/J(\mathcal{O})$ of characteristic p. We assume that either $\mathcal{O}$ has characteristic zero or $\mathcal{O} = k$. We adopt the following abuse of notation: if n, $m \in \mathbb{Z}$ such that $n \neq 0$ and $v_p(m) \geq v_p(n)$, where v_p is the $p-$adic valuation, we set $\frac{m}{n}1_\mathcal{O} = (m'1_\mathcal{O})(n'1_\mathcal{O})^{-1}$, where m', $n' \in \mathbb{Z}$ such that $(n',p) = 1$ and $\frac{m}{n} = \frac{m'}{n'}$.

Recall that a *block of a finite group G* is a primitive idempotent b in $Z(\mathcal{O}G)$, the algebra $\mathcal{O}Gb$ is called *block algebra of b*, and a *defect group of b* is a minimal subgroup

P of G such that the map $\mathcal{O}Gb \underset{\mathcal{O}P}{\otimes} \mathcal{O}Gb \longrightarrow \mathcal{O}Gb$ given by multiplication in $\mathcal{O}Gb$ splits as homomorphism of $\mathcal{O}Gb - \mathcal{O}Gb-$bimodules. We refer to [1], [17] , [18] for recalls on pointed groups, fusion in block algebras and Brauer pairs. See [14, section 6] for a short review (and further references) on fusion which is pretty much adapted to our needs in this section. A detailed account of the material on block theory we use here can be found in Thévenaz' book [23].

As mentioned in the introduction, the cohomology ring $H^*(G, \mathcal{O})$ of G has three algebraic characterizations: a "global" characterization as the ring $Ext^*_{\mathcal{O}G}(\mathcal{O}, \mathcal{O})$, a "local" characterization as subring of stable elements in the cohomology ring $H^*(P, \mathcal{O})$ of a Sylow-$p-$subgroup P of G, and a characterization as a subring of "relatively $\Delta G-$projective elements" of the Hochschild cohomology ring $HH^*(\mathcal{O}G)$ by 4.5. The two latter characterizations have obvious generalizations to any $p-$block b of a finite group G, replacing P by a defect group of b and the Hochschild cohomology of $\mathcal{O}G$ by that of $\mathcal{O}Gb$. These two notions still coincide as we are going to show in this section, while it is not clear at this stage, whether there is an analogous "global" description of the cohomology ring of the block b, as there need not be an augmentation (not even the source algebras of b need have an augmentation).

Definition 5.1 Let G be a finite group, b a block of G, P_γ a defect pointed group of $G_{\{b\}}$. Let $i \in \gamma$ and denote for any subgroup Q of P by e_Q the unique block of $kC_G(Q)$ satisfying $Br_Q(i)e_Q \neq 0$. The *cohomology ring of the block b of G associated with P_γ* is the subring

$$H^*(G, b, P_\gamma)$$

of $H^*(P, \mathcal{O})$ which consists of all $[\zeta] \in H^*(P, \mathcal{O})$ satisfying $res_\varphi([\zeta]) = res^P_Q([\zeta])$ for any subgroup Q of P and any automorphism φ of Q induced by conjugation with an element of $N_G(Q, e_Q)$.

Since all defect pointed groups of $G_{\{b\}}$ are conjugate, the above ring is, up to isomorphism, independent of the choice of P_γ, and we will write $H^*(G, b)$ instead of $H^*(G, b, P_\gamma)$ if the choice of P_γ is obvious from the context.

Remarks 5.2

5.2.1 It follows from Alperin's fusion lemma for Brauer pairs (cf. [23, (48.3)]) that if an element $[\zeta] \in H^*(P, \mathcal{O})$ belongs to $H^*(G, b, P_\gamma)$ then in fact $res_\varphi([\zeta]) = res^P_Q([\zeta])$ for any injective group homomorphism $\varphi : Q \longrightarrow P$ induced by conjugation with an element $x \in G$ satisfying ${}^x(Q, e_Q) \subset (P, e_P)$, or equivalently, $\tilde{\varphi} \in E_G((Q, e_Q), (P, e_P))$ (cf. [18]). Alperin's fusion lemma implies also that in order to check whether an element $[\zeta]$ of $H^*(P, \mathcal{O})$ belongs to $H^*(G, b, P_\gamma)$ it suffices to check that $res_\varphi([\zeta]) = res^P_Q([\zeta])$ for any $Q \subset P$ such that (Q, e_Q) is self-centralizing (that is, e_Q has $Z(Q)$ as defect group) and for any automorphism φ of Q induced by conjugation with an element of $N_G(Q, e_Q)$.

5.2.2 If b_0 is the principal block of G then for any subgroup Q of P the block e_Q is the principal block of $kC_G(Q)$, hence $N_G(Q, e_Q) = N_G(Q)$, and therefore, by [9, Ch. XII, 10.1], we have

$$H^*(G, b, P_\gamma) \cong H^*(G, \mathcal{O}),$$

the usual cohomology ring of G with coefficients in $\mathcal{O}$.

5.2.3 If the defect group P of b is abelian, then the inertial quotient $E = N_G(P_\gamma)/C_G(P)$ controls fusion (cf. [1] or [23, (49.6)]) and hence we have

$$H^*(G, b, P_\gamma) = H^*(P, \mathcal{O})^E \cong H^*(P \rtimes E, \mathcal{O}).$$

5.2.4 The cohomology ring $H^*(G, b, P_\gamma)$ is in fact an invariant of the source algebra $i\mathcal{O}Gi$ of b, where $i \in \gamma$. This is because if (Q, e_Q) is self-centralizing, there is a unique local point δ of Q on $i\mathcal{O}Gi$ (cf. [23, (41.1)]) and by a theorem of Puig [18, 3.1], the group $E_G(Q, e_Q) = N_G(Q, e_Q)/QC_G(Q)$ viewed as subgroup of the outer automorphism group of Q coincides with the group of $i\mathcal{O}Gi-$fusion $F_{i\mathcal{O}Gi}(Q_\delta)$ (see [18] or the appendix of [14] for a brief account on these ideas). One might therefore as well write $H^*(i\mathcal{O}Gi) = H^*(G, b, P_\gamma)$ and call this the *cohomology ring of* $i\mathcal{O}Gi$. More generally, we can define the *cohomology ring of an interior* $P-$*algebra* A *over* $\mathcal{O}$ to be the subring of elements $[\zeta]$ in $H^*(P, \mathcal{O})$ satisfying $res^P_Q([\zeta]) = res_\varphi([\zeta])$ whenever Q is a subgroup of P and $\varphi : Q \longrightarrow P$ an injective group homomorphism such that the class of φ modulo inner automorphisms of P belongs to the set of $A-$fusion $F_A(Q, P) = \underset{\gamma,\delta}{\cup} F_A(Q_\delta, P_\gamma)$, where γ and δ run over the sets of local points of P and Q on A, respectively.

The next lemma provides the technicalities in order to establish a connection between stable elements in group cohomology in the sense of [9, Ch. XII, section 10] and the stability notion developed in section 3.

Lemma 5.3. *Let P be a finite group, Q a subgroup of P and $\varphi : Q \longrightarrow P$ an injective group homomorphism. Let n be a nonnegative integer and $[\zeta] \in H^n(P, \mathcal{O})$. The following statements are equivalent.*

(i) We have $res_\varphi([\zeta]) = res^P_Q([\zeta])$.

(ii) The diagram

$$\begin{array}{ccc}
\mathcal{P}_{\mathcal{O}Q} & \xrightarrow{\epsilon_\varphi} & {}_\varphi(\mathcal{P}_{\mathcal{O}P})_\varphi \\
{\scriptstyle \delta_Q(\zeta)}\downarrow & & \downarrow{\scriptstyle \delta_P(\zeta)} \\
\mathcal{P}_{\mathcal{O}Q}[n] & \xrightarrow[\epsilon_\varphi[n]]{} & {}_\varphi(\mathcal{P}_{\mathcal{O}P}[n])_\varphi
\end{array}$$

is homotopy commutative, where ϵ_φ is a chain map lifting the homomorphism of $\mathcal{O}Q - \mathcal{O}Q-$bimodules $\mathcal{O}Q \longrightarrow {}_\varphi(\mathcal{O}P)_\varphi$ sending $u \in Q$ to $\varphi(u)$.

(iii) The diagram

$$\begin{array}{ccc}
\mathcal{P}_{\mathcal{O}P} & \xrightarrow{\epsilon_{\mathcal{O}P \underset{\mathcal{O}Q}{\otimes} \varphi(\mathcal{O}P)}} & (\mathcal{O}P)_\varphi \underset{\mathcal{O}Q}{\otimes} \mathcal{P}_{\mathcal{O}P} \underset{\mathcal{O}Q}{\otimes} {}_\varphi(\mathcal{O}P) \\
{\scriptstyle \delta_P(\zeta)}\downarrow & & \downarrow{\scriptstyle Id_{(\mathcal{O}P)_\varphi} \otimes \delta_P(\zeta) \otimes Id_{\varphi(\mathcal{O}P)}} \\
\mathcal{P}_{\mathcal{O}P}[n] & \xrightarrow[\epsilon_{\mathcal{O}P \underset{\mathcal{O}Q}{\otimes} \varphi(\mathcal{O}P)}[n]]{} & (\mathcal{O}P)_\varphi \underset{\mathcal{O}Q}{\otimes} \mathcal{P}_{\mathcal{O}P}[n] \underset{\mathcal{O}Q}{\otimes} {}_\varphi(\mathcal{O}P)
\end{array}$$

is homotopy commutative.

Moreover, if the above statements hold then $t_{\mathcal{O}P \underset{\mathcal{O}Q}{\otimes} {}_\varphi(\mathcal{O}P)}(\delta_P([\zeta])) = [P:Q]\delta_P([\zeta])$*; in particular,* $\pi_{\mathcal{O}P \underset{\mathcal{O}Q}{\otimes} {}_\varphi(\mathcal{O}P)} = [P:Q]1_{\mathcal{O}P}$.

Proof. Denote by $\mathcal{P}_\mathcal{O}$ a projective resolution of the trivial $\mathcal{O}P-$module $\mathcal{O}$. The equality $res_\varphi([\zeta]) = res^P_Q([\zeta])$ holds if and only if the diagram of complexes of $\mathcal{O}Q-$modules

5.3.1

$$\begin{array}{ccc} res^P_Q(\mathcal{P}_\mathcal{O}) & \xrightarrow{\simeq} & res_\varphi(\mathcal{P}_\mathcal{O}) \\ {\scriptstyle res^P_Q(\zeta)}\downarrow & & \downarrow{\scriptstyle res_\varphi(\zeta)} \\ res^P_Q(\mathcal{P}_\mathcal{O}[n]) & \xrightarrow[\simeq]{} & res_\varphi(\mathcal{P}_\mathcal{O}[n]) \end{array}$$

is homotopy commutative, where the horizontal maps lift the identity on $\mathcal{O}$ viewed as homomorphism of $\mathcal{O}Q-$modules (this makes sense, since the complexes $res^P_Q(\mathcal{P}_\mathcal{O})$ and $res_\varphi(\mathcal{P}_\mathcal{O})$ are both projective resolutions of the trivial $\mathcal{O}Q-$module).

Apply now the induction functor $Ind^{Q\times Q}_{\Delta Q}$ to this diagram (where we consider an $\mathcal{O}Q-$module as $\mathcal{O}\Delta Q-$module through the obvious isomorphism $\Delta Q \cong Q$); as for any $\mathcal{O}P-$module U there is a natural isomorphism $Ind^{Q\times Q}_{\Delta Q}({}_\varphi U) \cong {}_\varphi(Ind^{\varphi(Q)\times\varphi(Q)}_{\Delta\varphi(Q)}(U))_\varphi$ of $\mathcal{O}Q-\mathcal{O}Q-$bimodules mapping $(y,y')\otimes u$ to $(\varphi(y),\varphi(y'))\otimes u$, where $x,x'\in Q$ and $u\in U$, we obtain a diagram

5.3.2

$$\begin{array}{ccc} Ind^{Q\times Q}_{\Delta Q}(\mathcal{P}_\mathcal{O}) & \longrightarrow & {}_\varphi(Ind^{\varphi(Q)\times\varphi(Q)}_{\Delta\varphi(Q)}(\mathcal{P}_\mathcal{O}))_\varphi \\ {\scriptstyle \delta_Q(\zeta)}\downarrow & & \downarrow{\scriptstyle \delta_{\varphi(Q)}(\zeta)} \\ Ind^{Q\times Q}_{\Delta Q}(\mathcal{P}_\mathcal{O}[n]) & \longrightarrow & {}_\varphi(Ind^{\varphi(Q)\times\varphi(Q)}_{\Delta\varphi(Q)}(\mathcal{P}_\mathcal{O}[n]))_\varphi \end{array}$$

(where we write $\mathcal{P}_\mathcal{O}$ instead of $res^P_Q(\mathcal{P}_\mathcal{O})$ and $res^P_{\varphi(Q)}(\mathcal{P}_\mathcal{O})$). Since the identity functor on the category of $\mathcal{O}\Delta Q-$modules is isomorphic to a direct summand of the functor $Res^{Q\times Q}_{\Delta Q} Ind^{Q\times Q}_{\Delta Q}$ it follows that the diagram 5.3.2 is homotopy commutative if and only 5.3.1 is so.

Now the functor taking an $\mathcal{O}P-$module U to the $\mathcal{O}Q-\mathcal{O}Q-$bimodule ${}_\varphi(Ind^{\varphi(Q)\times\varphi(Q)}_{\Delta\varphi(Q)}(U))_\varphi$ is isomorphic to a direct summand of the functor sending U to ${}_\varphi(Ind^{P\times P}_{\Delta P}(U))_\varphi$ via the natural transformation sending $(\varphi(y),\varphi(y'))\otimes u$ to $(\varphi(y),\varphi(y'))\otimes u$, where $y,y'\in Q$ and $u\in U$ (this is in fact a particular case of Mackey's formula). Composing 5.3.2 with this natural transformation yields therefore a diagram

5.3.3

$$\begin{array}{ccc} Ind^{Q\times Q}_{\Delta Q}(\mathcal{P}_\mathcal{O}) & \longrightarrow & {}_\varphi(Ind^{P\times P}_{\Delta P}(\mathcal{P}_\mathcal{O}))_\varphi \\ {\scriptstyle \delta_Q(\zeta)}\downarrow & & \downarrow{\scriptstyle \delta_P(\zeta)} \\ Ind^{Q\times Q}_{\Delta Q}(\mathcal{P}_\mathcal{O}[n]) & \longrightarrow & {}_\varphi(Ind^{P\times P}_{\Delta P}(\mathcal{P}_\mathcal{O}[n]))_\varphi \end{array}$$

which is homotopy commutative if and only if 5.3.2 is so, since the horizontal arrows identify the left column to a direct summand of the right column, up to homotopy. Moreover, through the identification ${}_\varphi(\mathcal{O}P) \underset{\mathcal{O}P}{\otimes} Ind^{P\times P}_{\Delta P}(\mathcal{P}_\mathcal{O}) \underset{\mathcal{O}P}{\otimes} (\mathcal{O}P)_\varphi$ $\cong {}_\varphi(Ind^{P\times P}_{\Delta P}(\mathcal{P}_\mathcal{O}))_\varphi$ the horizontal maps are precisely the adjunction map $\epsilon_{(\mathcal{O}P)_\varphi}$

and its shift by the degree n, since they lift the bimodule homomorphism $\mathcal{O}Q \cong Ind_{\Delta Q}^{Q\times Q}(\mathcal{O}) \longrightarrow {}_\varphi(Ind_{\Delta P}^{P\times P}(\mathcal{O}))_\varphi \cong {}_\varphi(\mathcal{O}P)_\varphi$ sending $u \in Q$ to $\varphi(u)$. This shows the equivalence of (i) and (ii).

Observe in particular that this shows also that the diagram in (ii) is homotopy commutative if we take for φ the inclusion homomorphism $Q \subset P$.

By 3.3.2 and 3.3.5, the diagram in statement (ii) is homotopy commutative if and only if the diagram

5.3.4

$$\begin{array}{ccc}
\mathcal{P}_{\mathcal{O}P} & \longrightarrow & (\mathcal{O}P)_\varphi \underset{\mathcal{O}Q}{\otimes} \mathcal{P}_{\mathcal{O}Q} \underset{\mathcal{O}Q}{\otimes} {}_\varphi(\mathcal{O}P) \\
{\scriptstyle \delta_P(\zeta)}\downarrow & & \downarrow{\scriptstyle Id\otimes\delta_Q(\zeta)\otimes Id} \\
\mathcal{P}_{\mathcal{O}P}[n] & \longrightarrow & (\mathcal{O}P)_\varphi \underset{\mathcal{O}Q}{\otimes} \mathcal{P}_{\mathcal{O}Q}[n] \underset{\mathcal{O}Q}{\otimes} {}_\varphi(\mathcal{O}P)
\end{array}$$

is homotopy commutative.

Consider now the diagram in (ii) with the inclusion $Q \subset P$ instead of the homomorphism φ, tensor this diagram with $(\mathcal{O}P)_\varphi \underset{\mathcal{O}Q}{\otimes} - \underset{\mathcal{O}Q}{\otimes} {}_\varphi(\mathcal{O}P)$ and compose it with the diagram 5.3.4 above; this yields the diagram of statement (iii) and hence the equivalence of (ii) and (iii).

The last statement is a consequence of 2.11(ii), 4.6 and 4.7 together with the well-known fact, that on ordinary group cohomology, restriction to a subgroup followed by the transfer is just multiplication by the index of the subgroup.

Proposition 5.4. *Let G be a finite group, b a block of G, P_γ a defect pointed group of $G_{\{b\}}$ and let $i \in \gamma$. Consider $i\mathcal{O}Gi$ as $\mathcal{O}P-\mathcal{O}P-$bimodule and let $[\zeta] \in H^*(G, b, P_\gamma)$.*

(i) We have $t_{i\mathcal{O}Gi}(\delta_P([\zeta])) = \frac{rk_\mathcal{O}(i\mathcal{O}Gi)}{|P|}\delta_P([\zeta])$; in particular, $\pi_{i\mathcal{O}Gi} = \frac{rk_\mathcal{O}(i\mathcal{O}Gi)}{|P|}1_{\mathcal{O}P}$.

(ii) For any nonnegative integer n, the following diagram is homotopy commutative:

$$\begin{array}{ccc}
\mathcal{P}_{\mathcal{O}P} & \xrightarrow{\epsilon_{i\mathcal{O}Gi}} & i\mathcal{O}Gi \underset{\mathcal{O}P}{\otimes} \mathcal{P}_{\mathcal{O}P} \underset{\mathcal{O}P}{\otimes} i\mathcal{O}Gi \\
{\scriptstyle \delta_P(\zeta_n)}\downarrow & & \downarrow{\scriptstyle Id\otimes\delta_P(\zeta_n)\otimes Id} \\
\mathcal{P}_{\mathcal{O}P}[n] & \xrightarrow[\epsilon_{i\mathcal{O}Gi}[n]]{} & i\mathcal{O}Gi \underset{\mathcal{O}P}{\otimes} \mathcal{P}_{\mathcal{O}P}[n] \underset{\mathcal{O}P}{\otimes} i\mathcal{O}Gi
\end{array}$$

where ζ_n is the degree n component of ζ; in particular, $\delta_P([\zeta])$ is $i\mathcal{O}Gi-$stable.

Proof. By [14, 6.6], any indecomposable direct summand of $i\mathcal{O}Gi$ is isomorphic to $\mathcal{O}P \underset{\mathcal{O}Q}{\otimes} {}_\varphi(\mathcal{O}P)$ for some subgroup Q of P and some injective group homomorphism $\varphi : Q \longrightarrow P$ such that $\tilde{\varphi} \in E_G((Q, e_Q), (P, e_P))$ where the notation is as in 5.1. Thus we have $res_\varphi([\zeta]) = res_Q^P([\zeta])$. Since $rk_\mathcal{O}(\mathcal{O}P \underset{\mathcal{O}Q}{\otimes} {}_\varphi(\mathcal{O}P)) = [P : Q]|P|$, statement (i) follows from the last statement in 5.3 and 2.11(ii), and statement (ii) is a consequence of the equivalence of 5.3(i) and 5.3(ii) together with the additivity 2.7(i) of adjunction maps.

It is possible to make the adjunction maps associated with $\mathcal{O}Gi$ and its dual $i\mathcal{O}G$ explicit (we need this for the calculus of the relative projective elements):

Lemma 5.5. *Let G be a finite group, b a block of G, P_γ a defect pointed group of $G_{\{b\}}$ and let $i \in \gamma$. The isomorphism $(\mathcal{O}G)^* \cong \mathcal{O}G$ mapping the canonical symmetrizing form s to $1_{\mathcal{O}G}$ induces an isomorphism of $\mathcal{O}P - \mathcal{O}Gb-$bimodules $(\mathcal{O}Gi)^* \cong i\mathcal{O}G$ and multiplication in $\mathcal{O}Gb$ induces an isomorphism $i\mathcal{O}G \underset{\mathcal{O}Gb}{\otimes} \mathcal{O}Gi \cong i\mathcal{O}Gi$. With the identifications through these isomorphisms, the adjunction maps associated with $\mathcal{O}Gi$ and $i\mathcal{O}G$ are given as follows:*

$\epsilon_{\mathcal{O}Gi} : \mathcal{O}P \longrightarrow i\mathcal{O}Gi$ maps $u \in P$ to ui;

$\eta_{\mathcal{O}Gi} : \mathcal{O}Gi \underset{\mathcal{O}P}{\otimes} i\mathcal{O}G \longrightarrow \mathcal{O}Gb$ is induced by multiplication in $\mathcal{O}Gb$;

$\epsilon_{i\mathcal{O}G} : \mathcal{O}Gb \longrightarrow \mathcal{O}Gi \underset{\mathcal{O}P}{\otimes} i\mathcal{O}G$ maps $a \in \mathcal{O}Gb$ to $\sum_{x\in[G/P]} axi \otimes ix^{-1}$;

$\eta_{i\mathcal{O}G} : i\mathcal{O}Gi \longrightarrow \mathcal{O}P$ maps $c \in i\mathcal{O}Gi$ to $\sum_{u\in P} s(cu^{-1})u$.

Moreover, we have $\pi_{\mathcal{O}Gi} = Tr_P^G(i)$ and $\pi_{i\mathcal{O}G} = s(i)1_{\mathcal{O}P} = \frac{rk_{\mathcal{O}}(i\mathcal{O}G)}{|G|}1_{\mathcal{O}P}$.

Proof. The fact that the first two maps are as stated follows, for instance, from the explicit description of adjunction maps in 6.6, and then the remaining two maps are obtained by the duality 6.8 (or again from the explicit descriptions in 6.6).

It follows that $\pi_{\mathcal{O}Gi} = Tr_P^G(i)$ and $\pi_{i\mathcal{O}G} = \sum_{u\in P} s(iu^{-1})u$. In order to compute the latter expression we may assume that $\mathcal{O}$ has characteristic zero; indeed this expression is invariant under extensions, so we may assume that k is perfect, and then it suffices to observe that k occurs as residue field of some complete discrete valuation ring (cf. [21]). Then $|G|s$ is equal to the regular character of $\mathcal{O}G$; thus $|G|s(iu^{-1})$ is the value of the character of the projective $\mathcal{O}P-$module $i\mathcal{O}G$ at the element u^{-1}, thus equal to $rk_{\mathcal{O}}(i\mathcal{O}G)$ if $u = 1$ and zero otherwise. The lemma follows.

The next theorem is now the announced embedding of $H^*(G, b, P_\gamma)$ into $HH^*(\mathcal{O}Gb)$:

Theorem 5.6. *Let G be a finite group, b a block of G, P_γ a defect pointed group of $G_{\{b\}}$ and let $i \in \gamma$. Assume that $(i\mathcal{O}Gi)(P)$ is a split $k-$algebra. Consider $\mathcal{O}Gi$ and $i\mathcal{O}G$ as $\mathcal{O}Gb - \mathcal{O}P-$bimodule and $\mathcal{O}P - \mathcal{O}Gb-$bimodule, respectively.*

(i) We have $\pi_{\mathcal{O}Gi} = Tr_P^G(i) \in Z(\mathcal{O}Gb)^\times$ and $\pi_{i\mathcal{O}G} = \frac{rk_{\mathcal{O}}(i\mathcal{O}G)}{|G|}1_{\mathcal{O}P} \in \mathcal{O}^\times 1_{\mathcal{O}P}$.

(ii) If $[\zeta] \in H^*(G, b, P_\gamma)$ then $\delta_P([\zeta])$ is $i\mathcal{O}G-$stable in $HH^*(\mathcal{O}P)$.

(iii) The map $T_{\mathcal{O}Gi} \circ \delta_P$ induces an injective $\mathcal{O}-$algebra homomorphism

$$H^*(G, b, P_\gamma) \longrightarrow HH^*_{\mathcal{O}Gi}(\mathcal{O}Gb).$$

Proof. (i) It follows from the previous lemma that the relative projective elements are as stated in (i). Since $(i\mathcal{O}Gi)(P)$ is split we may apply [16, Prop. 1] (see also [22, 9.3]), which shows that $\pi_{\mathcal{O}Gi}$ is invertible. In order to show that $\pi_{i\mathcal{O}G}$ is invertible, we may assume that $\mathcal{O} = k$. Then $ker(Br_P) \subset ker(s)$, where s is the canonical symmetrizing form on kG, and therefore $s(i) = s(Br_P(i))$. Now $Br_P(i)$ is a primitive idempotent in $kC_G(P)$, and applying the last statement of 5.5 to $Br_P(i)$ instead of i shows that $s(Br_P(i)) = \frac{dim_k(Br_P(i)kC_G(P))}{|C_G(P)|}$. This last expression is non zero in k, since $Br_P(i)kC_G(P)$ is, up to isomorphism, the unique projective indecomposable module of the nilpotent block e_P of $kC_G(P)$ (cf. [7]), which is split by the assumptions.

(ii) Since $i\mathcal{O}Gi \cong i\mathcal{O}G \underset{\mathcal{O}Gb}{\otimes} \mathcal{O}Gi$ and $\pi_{\mathcal{O}Gi}$ is invertible, we can apply 3.8 to $A = C = \mathcal{O}P$, $B = \mathcal{O}Gb$, $X = i\mathcal{O}G$ and $Y = \mathcal{O}Gi$. Then 5.4(ii) implies that $\delta_P(H^*(G, b, P_\gamma)) \subset HH^*_{i\mathcal{O}Gi}(\mathcal{O}P) \subset HH^*_{i\mathcal{O}G}(\mathcal{O}P)$.

(iii) follows from (ii) together with 3.6.

The last theorem of this section is a generalization of the well-known fact that if a subgroup of a finite group controls $p-$fusion, then the restriction to this subgroup induces an isomorphism on $mod - p$ cohomology.

Theorem 5.7. *Let G, H be finite groups, b, c blocks of G, H, respectively, having a common defect group P, let γ and δ be local points of P on $\mathcal{O}Gb$ and $\mathcal{O}Hc$, respectively, and let $i \in \gamma$ and $j \in \delta$. For any subgroup Q of P, denote by e_Q and f_Q the unique blocks of $kC_G(Q)$ and $kC_H(Q)$, respectively, satisfying $Br^G_Q(i)e_Q \neq 0$ and $Br^H_Q(j)f_Q \neq 0$.*

Let X be a bounded complex of $\mathcal{O}Gb-\mathcal{O}Hc-$bimodules whose components are direct sums of direct summands of the bimodules $\mathcal{O}Gi \underset{\mathcal{O}Q}{\otimes} j\mathcal{O}H$, with Q running over the set of subgroups of P and set $Y = iXj$, considered as complex of $\mathcal{O}P - \mathcal{O}P-$bimodules.

(i) We have $\pi_Y = \pi_{Y^*} = \sum_{n\in\mathbb{Z}} (-1)^n \frac{rk_{\mathcal{O}}(Y_n)}{|P|} 1_{\mathcal{O}P}$.

(ii) We have $t_Y(\delta_P(H^*(P, \mathcal{O}))) \subset \delta_P(H^*(P, \mathcal{O}))$.

If moreover $E_G((Q, e_Q), (P, e_P)) \subset E_H((Q, f_Q), (P, f_P))$ for any subgroup Q of P, then the following hold:

(iii) We have $H^*(H, c, P_\delta) \subset H^*(G, b, P_\gamma)$ and $\delta_P(H^*(H, c, P_\delta)) \subset HH^*_Y(\mathcal{O}P) \cap HH^*_{Y^*}(\mathcal{O}P)$.

(iv) We have $t_Y(\delta_P([\zeta])) = \pi_Y\delta_P([\zeta])$ for any $[\zeta] \in H^*(H, c, P_\delta)$; in particular, if π_Y is invertible then T_Y restricts to the identity on $H^*(H, c, P_\delta)$ through δ_P.

Proof. By the hypotheses on the components of X, for any integer n, any indecomposable direct summand W of Y_n or Y^*_n is of the form $\mathcal{O}P \underset{\mathcal{O}Q}{\otimes} {}_\varphi(\mathcal{O}P)$ for some subgroup Q of P and some injective group homomorphism $\varphi : Q \longrightarrow P$, thus (i) follows from the last statement in 5.3 and 2.11(iii). Moreover, by 4.6 and 4.7 we have $t_W(\delta_P([\zeta])) = \delta_P(t^P_Q res_\varphi[\zeta])$ for any $[\zeta] \in H^*(P, \mathcal{O})$ which implies (ii).

If $E_G((Q, e_Q), (P, e_P)) \subset E_H((Q, f_Q), (P, f_P))$ for any subgroup Q of P, then $H^*(H, c, P_\delta) \subset H^*(G, b, P_\gamma)$ by the definition 5.1. Also, with W and φ as before, we have now $\tilde{\varphi} \in E_H((Q, f_Q), (P, f_P))$. This shows (iv), using again the last statement of 5.3. By matching together the diagrams that we get from 5.3(iii) we obtain (iii).

Remark 5.8 With the notation and hypotheses of 5.7, if $E_G((Q, e_Q), (P, e_P)) = E_H((Q, f_Q), (P, f_P))$ for any subgroup Q in P then $H^*(G, b, P_\gamma) = H^*(H, c, P_\delta)$. One might wonder if the converse of this statement holds. If the blocks b and c are the principal blocks of G and H, respectively, this is true by Mislin's theorem [15]. It is also true if the defect group P of b and c is abelian, since this case reduces to Mislin's theorem [15] by 5.2.3. We expect the general case to hold via the following outline of an idea of proof: the $\mathcal{O}P-\mathcal{O}P-$bimodule $i\mathcal{O}Gi$ is a permutation $\mathcal{O}(P\times P)-$module, hence defines up to isomorphism a unique $P-P-$biset on which P acts regularly on each side. Such a biset can be interpreted as an endomorphism of the classifying space BP viewed as object of the appropriate category of spectra, and the image of this endomorphism is then defined to be the "classifying space" $B(i\mathcal{O}Gi)$ of the source algebra $i\mathcal{O}Gi$. Using results and methods from Benson and Feshbach [3], if b is the principal block of G, we obtain the usual classifying space of G. Next, we expect the cohomology of $B(i\mathcal{O}Gi)$ to be the cohomology ring of b (possibly after some suitable $p-$completion), and then we have all ingredients to try to imitate Mislin's proof in this more general situation.

6 Appendix: Symmetric algebras and adjunction maps

We collect here, without proofs, some general abstract nonsense on symmetric algebras and adjunction maps that we use in this paper. Much of this material can be found in [5].

We fix a commutative ring R. Let A, B, C be $R-$algebras, let M, M' be $A-B-$bimodules and N a $B-C-$bimodule.

6.1. *There is a natural isomorphism of bifunctors*

$$Hom_{A\otimes 1}(M \underset{B}{\otimes} -, -) \cong Hom_{B\otimes 1}(-, Hom_{A\otimes 1}(M, -))$$

given by the isomorphisms

$$\begin{cases} Hom_{A\otimes 1}(M \underset{B}{\otimes} V, U) & \to Hom_{B\otimes 1}(V, Hom_{A\otimes 1}(M, U)) \\ \varphi & \mapsto (v \mapsto (m \mapsto \varphi(m\otimes v))) \end{cases}$$

for any $A-$module U and $B-$module V.

In other words, the functor $Hom_{A\otimes 1}(M, -)$ has $M \underset{B}{\otimes} -$ as left adjoint. The unit and counit of this adjunction are represented by the bimodule homomorphisms

$$\begin{cases} B & \to Hom_{A\otimes 1}(M, M) \\ b & \mapsto (m \mapsto mb) \end{cases}$$

and

$$\begin{cases} M \underset{B}{\otimes} Hom_{A\otimes 1}(M, A) & \to A \\ m\otimes\beta & \mapsto \beta(m) \end{cases}$$

6.2. *If M is projective as left $A-$module, there is a natural isomorphism of functors*

$$Hom_{A\otimes 1}(M,A)\underset{A}{\otimes} - \cong Hom_{A\otimes 1}(M,-)$$

given by the isomorphisms

$$\begin{cases} Hom_{A\otimes 1}(M,A)\underset{A}{\otimes} U & \to Hom_{A\otimes 1}(M,U) \\ \varphi\otimes u & \mapsto (m\mapsto \varphi(m)u) \end{cases}$$

for any $A-$module U.

Applying this statement to $U = M$ shows that we have in particular an isomorphism of $B-B-$bimodules

$$Hom_{A\otimes 1}(M,A)\underset{A}{\otimes} M \cong Hom_{A\otimes 1}(M,M).$$

6.3 We assume from now on that A is symmetric; that is, A is projective as $R-$module and there is an isomorphism of $A-A-$bimodules $\Phi : A \longrightarrow A^* = Hom_R(A,R)$.

Then $s = \Phi(1_A)$ is a *symmetrizing form on* A; that is, $s(aa') = s(a'a)$ for all $a, a' \in A$, and the map sending $a \in A$ to $a.s \in A^*$ defined by $a.s(a') = s(aa')$ for all $a' \in A$ is an isomorphism of $A-A-$bimodules (and this is then in fact equal to Φ, so fixing a bimodule isomorphism $A \cong A^*$ is equivalent to choosing a symmetrizing form on A). If s' is another symmetrizing form on A there is a unique invertible element $z \in Z(A)^\times$ such that $s' = z.s$, since every automorphism of A as $A-A-$bimodule is given by multiplication with an invertible element of $Z(A)$. Since A is projective as $R-$module we have a natural isomorphism $A^* \otimes A^* \cong (A\otimes A)^*$. Consider then the composition of $A-A-$bimodule homomorphisms

6.3.1

$$A \cong A^* \underset{\mu^*}{\longrightarrow} (A\otimes A)^* \cong A\otimes A$$

where μ^* is the $R-$dual of the bimodule homomorphism $\mu : A\otimes A \longrightarrow A$ given by multiplication in A; write then the image of 1_A in $A\otimes A$ under the map 6.3.1 as sum

6.3.2

$$\sum_{x\in\mathfrak{X}} x\otimes x'$$

where $\mathfrak{X}$ is a finite subset of A and $x' \in A$ for any $x \in \mathfrak{X}$. Then we have

6.3.3

$$\sum_{x\in\mathfrak{X}} s(x'a)x = a$$

for all $a \in A$. The inverse map of Φ maps $\varphi \in A^*$ to $\sum_{x\in\mathfrak{X}} \varphi(x')x$.

If A is $R-$free, then $\mathfrak{X}$ can be chosen to be an $R-$basis of A and the set $\mathfrak{X}' = \{x'\}_{x\in\mathfrak{X}}$ is the dual basis with respect to s (i.e, we have $s(xx') = 1$ and $s(yx') = 0$ for all $x, y \in \mathfrak{X}$ such that $x \neq y$.

6.4. *There is a natural equivalence of functors*

$$Hom_{A\otimes 1}(-,A) \cong Hom_R(-,R)$$

from $Mod(A)$ to $Mod(A^0)$ given for any $A-$module U by the isomorphism

$$\begin{cases} Hom_{A\otimes 1}(U,A) & \to Hom_R(U,R) = U^* \\ \varphi & \mapsto s\circ\varphi \end{cases}$$

whose inverse sends $\tau \in U^*$ to the map sending $u \in U$ to $\sum_{x\in\mathfrak{X}} \tau(x'u)x$.

In particular, we have an isomorphism of $B-A-$modules

$$\begin{cases} Hom_{A\otimes 1}(M,A) & \cong M^* \\ \varphi & \mapsto s\circ\varphi \end{cases}$$

6.5 If moreover M is projective as left $A-$module we can combine 6.2 and 6.4 to obtain a natural equivalence of functors

$$Hom_{A\otimes 1}(M,-) \cong M^* \underset{A}{\otimes} -$$

from $Mod(A)$ to $Mod(B)$. Note that in particular $Hom_{A\otimes 1}(M,M) \cong M^* \underset{A}{\otimes} M$ as $B-B-$bimodules, and that the image of Id_M in $M^*\underset{A}{\otimes} M$ under this isomorphism is of the form $\sum_{m\in\mathcal{M}} s\circ\varphi_m \otimes m$, where $\mathcal{M}$ is a finite subset of M and $\varphi_m \in Hom_{A\otimes 1}(M,A)$ for any $m \in M$ satisfying $\sum_{m\in\mathcal{M}} \varphi_m(m')m = m'$ for all $m' \in M$.

Furthermore, the above equivalence of functors means that $M^* \underset{A}{\otimes} -$ is a right adjoint to $M \underset{B}{\otimes} -$. Since $M^{**} \cong M$ it follows that if B is symmetric, too, and M projective as right $B-$module, $M \underset{B}{\otimes} -$ is also a right adjoint to $M^* \underset{A}{\otimes} -$. Using the above isomorphisms we can make the corresponding adjunction maps explicit:

6.6 We assume from now on that both A and B are symmetric with symmetrizing forms $s \in A^*$ and $t \in B^*$, and that M is projective as left $A-$module and as right $B-$module. Denote by $\mathfrak{X}$, $\mathfrak{X}'$ finite subsets of A as in 6.3 and by $\mathfrak{Y}$, $\mathfrak{Y}'$ corresponding finite subsets of B with respect to the symmetrizing form t on B.

6.6.1. *There is a natural isomorphism of bifunctors*

$$Hom_{A\otimes 1}(M \underset{B}{\otimes} -, -) \cong Hom_{B\otimes 1}(-, M^* \underset{A}{\otimes} -)$$

whose unit and counit are represented by the bimodule homomorphisms

$$\epsilon_M : B \longrightarrow M^* \underset{A}{\otimes} M$$

mapping $b \in B$ to $\sum_{m\in\mathcal{M}} (s\circ\varphi_m)\otimes mb$, where $\mathcal{M}$ is a finite subset of M and $\varphi_m \in Hom_{A\otimes 1}(M,A)$ for each $m \in \mathcal{M}$ satisfying $\sum_{m\in\mathcal{M}} \varphi_m(m')m = m'$ for all $m' \in M$, and

$$\eta_M : M \underset{B}{\otimes} M^* \longrightarrow A$$

mapping $m \otimes m^*$ to $\sum_{x\in\mathfrak{X}} m^*(x'm)x$.

6.6.2. *There is a natural isomorphism of bifunctors*

$$Hom_{B\otimes 1}(M^* \underset{A}{\otimes} -, -) \cong Hom_{A\otimes 1}(-, M \underset{B}{\otimes} -)$$

whose unit and counit are represented by the bimodule homomorphisms

$$\epsilon_{M^*} : A \longrightarrow M \underset{B}{\otimes} M^*$$

mapping $a \in A$ *to* $\sum_{n\in\mathcal{N}} an \otimes (t \circ \psi_n)$*, where* $\mathcal{N}$ *is a finite subset of* M *and* $\psi_n \in Hom_{1\otimes B}(M, B)$ *for any* $n \in \mathcal{N}$ *satisfying* $\sum_{n\in\mathcal{N}} n\psi_n(m') = m'$ *for all* $m' \in M$*, and*

$$\eta_{M^*} : M^* \underset{A}{\otimes} M \longrightarrow B$$

mapping $m^* \otimes m$ *to* $\sum_{y\in\mathfrak{Y}} m^*(my')y$*.*

6.7 If N is projective as left $B-$module there is an isomorphism of $C-A-$bimodules

$$(M \underset{B}{\otimes} N)^* \cong N^* \underset{B}{\otimes} M^*$$

given by the sequence of isomorphisms $(M \underset{B}{\otimes} N)^* = Hom_R(M \underset{B}{\otimes} N, R) \cong Hom_{B\otimes 1}(N, Hom_R(M, R)) = Hom_{B\otimes 1}(N, M^*) \cong N^* \underset{B}{\otimes} M^*$, where we use the adjunction 6.1 with R instead of A and 6.5 in the last isomorphism. Note that we did use that B is symmetric, while we did not use that A is so.

6.8 The adjunction maps in 6.6 are dual to each other in the sense that the following diagrams are commutative:

$$\begin{array}{ccc} A & \xrightarrow{\epsilon_{M^*}} & M \underset{B}{\otimes} M^* \\ \downarrow & & \downarrow \\ A^* & \xrightarrow[(\eta_M)^*]{} & (M \underset{B}{\otimes} M^*)^* \end{array}$$

$$\begin{array}{ccc} M^* \underset{A}{\otimes} M & \xrightarrow{\eta_{M^*}} & B \\ \downarrow & & \downarrow \\ (M^* \underset{A}{\otimes} M)^* & \xrightarrow[(\epsilon_M)^*]{} & B^* \end{array}$$

where the vertical isomorphisms are given by the symmetrizing forms on A, B and appropriate versions of 6.7.

6.9 Suppose now that all A, B, C are symmetric and that M, N are projective as left and right modules. We have commutative diagrams

$$\begin{array}{ccc}
C & \xrightarrow{\epsilon_{M \underset{B}{\otimes} N}} & (M \underset{B}{\otimes} N)^* \underset{A}{\otimes} (M \underset{B}{\otimes} N) \\
\downarrow{\scriptstyle Id_C} & & \downarrow{\scriptstyle \cong} \\
C \xrightarrow[\epsilon_N]{} N^* \underset{B}{\otimes} N \cong N^* \underset{B}{\otimes} B \underset{B}{\otimes} N & \xrightarrow[Id_{N^*} \otimes \epsilon_M \otimes Id_N]{} & N^* \underset{B}{\otimes} (M^* \underset{A}{\otimes} M) \underset{B}{\otimes} N \ ,
\end{array}$$

where the vertical isomorphism on the right is obtained from 6.7, and

$$\begin{array}{ccc}
(M \underset{B}{\otimes} N) \underset{C}{\otimes} (M \underset{B}{\otimes} N)^* & \xrightarrow{\eta_{M \underset{B}{\otimes} N}} & A \\
\downarrow{\scriptstyle \cong} & & \downarrow{\scriptstyle Id_A} \\
M \underset{B}{\otimes} (N \underset{C}{\otimes} N^*) \underset{B}{\otimes} M & \xrightarrow[Id_M \otimes \eta_N \otimes Id_{M^*}]{} M \underset{B}{\otimes} B \underset{B}{\otimes} M^* \cong M \underset{B}{\otimes} M^* \xrightarrow[\eta_M]{} & A \ ,
\end{array}$$

where the vertical isomorphism on the left is again obtained from 6.7.

6.10 The above discussion extends to complexes: if X is a bounded complex of $A - B-$bimodules whose components are projective as left and right modules, the adjunction in 6.1 extends to an adjunction

$$Hom_{C(A)}(X \underset{B}{\otimes} -, -) \cong Hom_{C(B)}(-, Hom_{A\otimes 1}(X, -))$$

as follows: if U is a complex of $A-$modules and V a complex of $B-$modules, for any integers $i, n \in \mathbb{Z}$, we match together the adjunction maps

$$Hom_{A\otimes 1}(X_i \underset{B}{\otimes} V_{n-i}, U_n) \cong Hom_{B\otimes 1}(V_{n-i}, Hom_{A\otimes 1}(X_i, U_n))$$

multiplied by the sign $(-1)^{i(n-i)}$ (this sign reflects the fact, that in the isomorphism just above, the left side belongs to the degree n component of the considered complexes, while the right side to degree $n - i$).

We obtain again a natural isomorphism of functors

$$Hom_{A\otimes 1}(X, -) \cong X^* \underset{A}{\otimes} -$$

and hence $X \underset{B}{\otimes} -$ and $X^* \underset{A}{\otimes} -$ are adjoint functors between $C(A)$ and $C(B)$. The signs in the above adjunction show that 2.7(iii) holds. Also, for any further bounded complexes Y of $A - C-$bimodules and Z of $B - C-$bimodules whose components are projective as left and right modules, we have again isomorphisms

$Hom_{A\otimes 1}(X, Y) \cong X^* \underset{A}{\otimes} Y$ as complexes of $B - C-$bimodules and

$(X \underset{B}{\otimes} Z)^* \cong Z^* \underset{B}{\otimes} X^*$ as complexes of $C - A-$bimodules.

References

1. J.L. Alperin, M. Broué, *Local Methods in Block Theory*, Ann. of Math. **110** (1979), 143–157.
2. D. J. Benson, *Representations and cohomology*, Cambridge University Press I, II, 1991.
3. D. J. Benson, M. Feshbach, *Stable splittings of classifying spaces of finite groups*, Topology **31** (1992), 157–176.
4. S. Bouc, *Bimodules, trace généralisée et transferts en homologie de Hochschild*, preprint (1997).
5. M. Broué, *On representations of symmetric algebras: an introduction*, Notes by M. Stricker, Mathematik Department ETH Zürich (1991).
6. M. Broué, *Equivalences of Blocks of Group Algebras, in: Finite Dimensional Algebras and Related Topics, V. Dlab, L. L. Scott (eds.)*, Kluwer Academic Publishers, Dordrecht, Netherlands, 1994.
7. M.Broué, L. Puig, *A Frobenius Theorem for Blocks*, Invent. Math. **56** (1980), 117–128.
8. M.Broué, L. Puig, *Characters and Local Structure in G-Algebras*, J. Algebra **63** (1980), 306–317.
9. H. Cartan, S. Eilenberg, *Homological Algebra*, Princeton University Press, 1956.
10. M. Feshbach, *Transfer for Hopf Algebras*, J. Algebra **180** (1996), 287–305.
11. D. Happel, *Triangulated categories in the representation theory of finite dimensional algebras*, London Math. Soc. Lecture Notes 119, 1988.
12. B. Keller, *Invariance and localization for cyclic homology of DG-algebras*, J. Pure and Applied Algebra **123** (1998), 223–273.
13. M. Linckelmann, *Stable equivalences of Morita type for self-injective algebras and p-groups* **223** (1996), 87–100.
14. M. Linckelmann, *On splendid derived and stable equivalences between blocks of finite groups*, preprint (1997).
15. G. Mislin, *Group homomorphisms inducing isomorphisms in mod-p cohomology*, Comment. Math. Helvet. **65** (1990), 454–461.
16. C. Picaronny, L. Puig, *Quelques remarques sur un théorème de Knörr*, J. Algebra **109** (1987), 69–73.
17. L. Puig, *Pointed groups and construction of characters*, Math. Z. **176** (1981), 265–292.
18. L. Puig, *Local fusion in block source algebras*, J. Algebra **104** (1986), 358–369.
19. L. Puig, *Pointed groups and construction of modules*, J. Algebra **116** (1988), 7–129.
20. J. Rickard, *Derived equivalences as derived functors*, J. London Math. Soc. **43** (1991), 37–48.
21. J. P. Serre, *Corps Locaux*, Hermann, Paris, 1968.
22. J. Thévenaz, *Duality in G-algebras*, Math. Z. **200** (1988), 317–332.
23. J. Thévenaz, *G-Algebras and Modular Representation Theory*, Oxford Science Publications, Clarendon Press, Oxford, 1995.
24. C. A. Weibel, *Introduction to homological algebra*, Cambridge University Press, 1994.

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