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# Quantum-like model of subjective expected utility

August 24, 2017

## 1 Introduction

Recently quantum probability (QP) started to be widely used to model the processes of decision making (DM) in cognitive psychology, behavioral economics and finance, see, e.g., monographs Khrennikov (2004, 2010), Busemeyer and Bruza (2012), Bagarello (2012), Haven and Khrennikov (2013), Asano et al. (2015), reviews Pothos and Busemeyer (2013), Khrennikov (2015a), Haven and Sozzo (2016), the recent handbook edited by Haven and Khrennikov (2017a), and the first textbook for students (Haven et al. 2017).<sup>1</sup> One may consider the appeal to QP to model DM instead of the usage of the conventional probabilistic measures too exotic. Yet, we recall that as early as the 1970s, Tversky, Kahneman and other researchers in psychology and economics following the seminal paradoxes by Allais (1953), Ellsberg (1961) have been demonstrating cases where classical probability (CP) prescription and actual human thinking persistently diverge, seeking to explain these deviations from the normative DM frameworks (Kahneman and Tversky 1972; Tversky and Kahneman 1974, Shafir 1994, Kahneman, 2003; Kahneman and Thaler, 2006).



The main inquiry of the experimental studies was often focused on human evaluation and revision of probabilities in uncertain DM situations. Following questions naturally emerged:

*Do people obey the rules of classical probability theory, and if not, in which circumstances? Are there any other laws that can be applied to formalize human judgments and preferences?*

After demonstrating first comprehensive evidence on deviation of human preference formation from the postulates of von Neumann and Morgenstern

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<sup>1</sup>The first steps in this direction were done in 1990th, see, e.g., Khrennikov (1999).



(1944) theories (VNM), Ellsberg (1961, p.646) **conceived** that: there would be simply no way to infer meaningful probabilities for those events from their [participants] choices, and theories which purported to describe their uncertainty in terms of probabilities would be quite inapplicable in that area (unless quite different operations for measuring probability were devised).

A wide array of elegant generalizations of classical probabilistic formulations of rational decision theories (VNM)<sup>2</sup> were devised following the emerging empirical evidence. Generalized utility theories focused particularly on eliminating the linearity in probabilistic measures, and seeking to relax the assumption of agents' possessing firm and state-independent probabilistic estimates (Kahneman and Tversky, 1979; Machina, 1982, 1989; 2005, Gilboa and Schmeidler, 1989, 1994; Schmeidler, 1989; Tversky and Kahneman, 1992; Klibanoff et al., 2005, and others.)

As articulated by Machina (1989), the main appeal of the devised generalizations of VNM formulations was to reach three goals; the empirical (fit to the experimental data), the theoretical (allow to use the formulation in the most general settings, from trading on the financial markets, to insurance and gambling) and finally, the normative status (logical implications such as rationality of the assumptions in VNM).

Another stream of case by case explanations, based on heuristics and biases pioneered<sup>3</sup> by Kahneman and Tversky, 1972; Tversky and Kahneman 1974, also gained wide recognition in behavioral economics, that mainly emerged due to expensive collection of empirical evidence on human preference formation and judgment. Yet, the convincing explanations of each such bias and its effect on DM formation were placed under critique largely due to their lack of theoretical coherence and normative appeal (e.g., Wolford, 1991; Gigenzenger, 1996)<sup>4</sup>.

Today, we still find ourselves at the theoretical cross-roads, with considerable divisions across conflicting, entrenched theoretical positions that revolve around the following dilemmas:

- Should we continue to rely on CP as the basis for descriptive and normative predictions in decision making (and perhaps ascribe inconsistencies

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<sup>2</sup>We allude to von Neumann and Morgenstern's (1944) expected utility formulation under objective risk, as well as Savage's (1954) subjective expected utility over consequences in uncertain states of the world.

<sup>3</sup>See also an earlier excellent survey of the behavioral factors that ought to falsify the postulates of modern decision theories under uncertainty and risk by Simon (1959).

<sup>4</sup>We remark that Prospect Theory by Kahneman and Tversky (1979), and the advanced version, Cumulative Prospect Theory by Tversky and Kahneman (1992) can be considered as great accomplishments in the pursuit for a generalized and structured DM framework, encompassing some of the effects of human heuristics and biases.



to methodological idiosyncrasies)?

- Should we abandon probability theory completely and instead pursue explanations based on heuristics and biases, as proposed by Tversky and Kahneman?

Yet, a need for a DM framework with theoretical foundations that can be utilized in economics, finance and other domains persists. Hence, by generalizing or replacing classical VNM framework, one is compelled to maintain the theoretical foundations of the alternative decision theory. Probabilistic and statistical methods are undeniably the cornerstones of modern scientific methodology in all spheres of social science. Thus, although the heuristic approach to decision making cannot be discarded completely and serves as an important tool to research the nature of human reasoning, it appears that it is more natural to approach novel probabilistic models to formalize preference formation. Hence, in the present contribution we proceed with the slogan:

*QP instead of heuristics and biases!*

Application of the laws of QP, instead of CP, can resolve some paradoxes of classical DM theory, see section 2. The number of different ‘paradoxes’ generated by the classical DM theory is startling. The authors of a recent review (Ert and Erev 2015) distinguished 35 basic paradoxes. The history of decision theory, can be characterized by advancement of the theoretical frameworks via creation and resolution of paradoxes through modifications of the theory. As an example, von Neumann-Morgenstern (VNM) expected utility theory was generalized to Prospect decision theory after numerous empirical studies (cf. Tversky and Kahenman, 1992; Shafir 1994). However, any modification suffered from new paradoxes.



It seems that the use of QP can resolve all such paradoxes (including Allais (1953), Ellsberg (1961) and Machina (2009) paradoxes), at least this is claimed in the recent paper of Asano et al. (2017). In this paper the authors develop a quantum-like model of selection of lotteries under uncertainty based on QP realization of subjective expected utility (SEU) approach. In particular, this model reproduces VNM expected utility theory (in this case probabilities can be interpreted objectively) and the Prospect theory (including its representation with cumulative probability weighting function, Tversky and Kahenman, 1992). Moreover, the quantum-like model of lottery selection recreates one special form of the probability weighting function used in the Prospect theory. We recall that in the prospect theory, the probability weighting function is the important concept to explain the violation of independence axiom in VNM theory. Actually, from phenomenological



discussions, various weighting functions have been proposed (Prelec, 1998; Rieger and Wang, 2006; Tversky and Kahneman, 1992; Gonzales and Wu, 1999; Wu and Gonzales, 1996).

$$w_{\lambda,\delta}(x) = \frac{\delta x^\lambda}{\delta x^\lambda + (1-x)^\lambda}, \quad (1)$$

which was discussed in (Gonzalez and Wu, 1999). The parameters  $\lambda$  and  $\delta$  control the curvature and elevation of the function, respectively. Such a phenomenological function with  $\lambda = 1/2$  corresponds to the subjective probabilities derived from the usage of quantum probability framework, see Asano et al. (2017) for a detailed discussion. Thus, quantum theory provides an argument in favor of one special type of the weighting function of the formulation of the Prospect theory.

The quantum-like SEU-model Asano et al. (2017) does not only reproduce the output of the Prospect theory (for the aforementioned special choice of the weighting function), but (depending on the belief-state of an agent) can lead to new decision rules, including the existence of new parameters (besides the subjective probabilities). These parameters are given by relative phases expressing correlations between different outcomes of lotteries  $A$  and  $B$  (within a single lottery or between two lotteries). The presence of phases induces the effects of constructive and destructive interference. An agent, say Alice, can make a decision  $A \succeq B$  even if  $\langle A \rangle_\sigma \geq \langle B \rangle_\sigma$ , where  $\langle A \rangle_\sigma, \langle B \rangle_\sigma$  are the subjective expected utilities of the lotteries. The decision depends crucially on the sign of the factor of the form  $\cos \theta$  representing the interference effect, where  $\theta$  is a combination of the relative phases, see sections ??, 6.

In the quantum-like SEU-model by Asano et al. (2017), a classical VNM utility function is used to construct a *comparison operator*  $D$ . It encodes operationally the process of comparison of two lotteries. Hermitian operators (whereby their eigenvectors signify the possible monetary outcomes), represent the random outcomes of the lotteries. The belief state of an agent is mathematically realized as a quantum state that can be either a pure, or mixed state.

In this paper we advance the representation scheme of the model proposed in (Asano et al., 2017), by representing Alice's beliefs about the lotteries' outcomes by two orthonormal bases in *the same Hilbert space*. In Asano et al. (2017) each lottery was represented in its own space and the pair of lotteries was represented in the direct sum of these two belief-spaces (cf. Pothos and Busemeyer, 2009; Broekaert et al., 2017), *the Cartesian product of the belief-state spaces for individual lotteries*. The use of a *single belief*

*space* gives the possibility (absent in the model by Asano et al. (2017)) to represent complementary (incompatible) systems of events corresponding to different lotteries. At the same time this new approach reduces the dimension of the state space. It also describes the subtle features of the DM process that were not present in the previous model by Asano et al. (2017).



In the present model with the common belief-state space, the DM-process is split into four sub-processes, see section 7. The previous model Asano et al. (2017) (with the direct sum of belief-state spaces) handled only one of these processes (Process 1, see section 7). The new counterparts of the DM-process model describe mathematically Alice's reflections in respect to the selection of lotteries. These reflections are modeled with the aid of *quantum transition probabilities*. For complementary lotteries, these probabilities are nontrivial and their presence generates complex reflections of a decision maker, cf. (Asano et al., 2017). The transition probabilities are involved in the creation of more complex subjective probabilities for lotteries' outcomes than the probabilities of Process 1 only, see Asano et al. (2017). The devised model is quite complicated from the viewpoint of mathematical computations. We reproduce the detailed model derivation in the special appendix (appendix 1).

The structure of the model is very rich. To demonstrate at least some of its distinguishing features, we analyze in very detail the example of lotteries with two outcomes, see section 8 and appendix 2. This simple example shows that, in fact, a quantum-like agent uses the probability amplitudes (an not the squares of their absolute values) as weights for averaging of utility function. (So, roughly speaking an agent works with square roots of quantum probabilities.) Of course, the straightforward probability interpretation of this construction is impossible (since amplitudes need not be positive real numbers). For the probabilistic interpretation, one has to proceed with four counterparts of the process of decision making considered in section 7. At the same time modeling based on amplitudes is attractive by its simplicity. One can proceed in this direction by using signed and complex "probabilities" which are widely used in quantum mechanics and recently strated to be applied to decision making, see de Barros et al. (2016, 2017).



The example presented in section 8 also reflect the very special feature of the quantum probability update: generation of nonzero (subjective) probability from zero prior probability (Basieva et al., 2017), cf. with *Cromwell's rule* for the classical Bayesian update.

## 2 From the von Neumann-Morgenstern expected utility theory to quantum(-like) modeling of subjective expected utility

In their book von Neumann and Morgenstern (1944) introduced an expected utility function over lotteries, or gambles. The type of uncertainty which was embedded in their expected utility<sup>5</sup> approach was objective uncertainty (i.e. an uncertainty which is formalizable by using objective probabilities). A key theorem in the VNM theory establishes the so called expected utility representation, which in essence requires that preferences over lotteries satisfy a specified number of axioms<sup>6</sup>. As economic history has shown, some of their axioms were not as ‘natural’ as expected and Allais’ paradox (see Allais, 1953) showed a violation of the so called substitution axiom<sup>7</sup>. Whilst VNM developed an axiomatic choice framework along objective uncertainty, it surely is the case that real life decisions can revolve around subjective choice situations. The purpose of Savage’s theory is to consider choice under subjective uncertainty (see Savage, 1954). In words, the Savage model can be summarized as follows (Kreps (1988) (p. 195-196): “ Savage models ...hold that you should assess probabilities for the subjectively uncertain events, probabilities that add up to one, and then choose whichever gamble gives the highest subjective expected utility.”

Savage (1954) formulated the famous *Sure Thing Principle* which is an essential axiom (amongst seven other axioms) which allow for the existence of an equivalence relation between a preference over acts<sup>8</sup> and an ordering of expected utilities.



We remark that in purely probabilistic terms, this principle is equivalent to the validity of the law of total probability (see Khrennikov, 2010). Hence, a violation of this law for our quantum-like model of DM (will be equivalent to a violation of Savage’s Sure Thing Principle (see Busemeyer et al., 2006). We note that the well known Ellsberg paradox (see Ellsberg, 1961) specifically refers to a violation of the sure-thing principle<sup>9</sup>.

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<sup>5</sup>We note that in economics there has been a long standing discussion between expected value and expected utility. This debate relates to the so called St. Petersburg paradox (see Blavatsky , 2005 and Samuelson, 1977).

<sup>6</sup>In essence, those preferences need to satisfy the ‘substitution’ and ‘continuity’ axioms.

<sup>7</sup>This axiom says that for three lotteries,  $a, b$  and  $c$  the preference of lottery  $a$  over lottery  $b$  will imply that the weighted average of lotteries  $a$  and  $c$  is preferred over the weighted average over lotteries  $b$  and  $c$ .

<sup>8</sup>An act, as per Kreps (1988), p. 128, is a function from a set of states to a set of prizes.

<sup>9</sup>The sure thing principle can also be found back in von Neumann-Morgenstern’s theory but only if one considers probabilities without finite support (see Krep, 1988, p. 59 and



In the expected utility representation of the Savage approach (as well as in the VNM approach), the utility function will be bounded and unique (up to an affine transformation). The linearity<sup>10</sup> of the preference function is tightly connected to the substitution axiom which forms part of the VNM theory. Violation of this axiom, was shown to occur via the Allais paradox which we already mentioned above. More paradoxes exist. The Machina paradox (see Machina, 2009) will challenge a whole variety of expected utility approaches, such as the max-min expected utility (see Gilboa and Schmeidler, 1989) and the Choquet expected utility (see Gilboa and Schmeidler, 1994). See also Haven and Sozzo (2016) for more of a discussion on why non-classical probability can be an answer in the presence of such paradox (see also Machina, 1983, 1987 and Erev et al., 2016).

All these DM-theories are mathematically formalized with the aid of *classical probability* (CP), the axiomatics of Kolmogorov (1933). Those axiomatics are based on the set-representation of events and the measure-representation of probabilities. Let us make the following point. The constraints posed on a DM-model by the CP-calculus can have fundamental consequences. The most important set of CP-constraints is related to the set-representation of events. In fact, this is the special representation of *classical Boolean logic*. Thus, all probabilistic utility models (not only the expected utility ones) are based on the implicit assumption that all agents use the special calculus of propositions known as the *Boolean algebra*. This assumption precedes, e.g. the axioms about the rationality of agents. To even formulate such axioms, one has to appeal to Boolean logic. We also remark that *expected* utility theory uses mathematical expectation which corresponds to the CP-model.

It would be interesting to investigate what consequences will emerge if we consider a relaxation of some of the axioms of CP for DM under uncertainty. However, this general project has a very high complexity: one can create a huge variety of novel ‘non-Kolmogorovean models’ and to analyze all possible consequences for DM is really impossible. In particular, in mathematical applications we can find a variety of generalized averaging procedures leading to nonclassical notions of mathematical expectation. Therefore, it would be natural to try to go beyond CP (and Boolean logic) on the basis of some concrete and well developed non-CP model which has already demonstrated its applicability to the solution of non-trivial problems in not only natural science but also in economics, psychology and other areas of social science.


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following).

<sup>10</sup>Mark Machina did note in Machina (1982) (p. 295) that ....“from the fact that differentiable functions are ‘locally linear’, and that for preference functionals over probability distributions, linearity is equivalent to expected utility maximization.”

Such a non-CP model is now well known: this is the probabilistic counterpart of the mathematical apparatus of quantum mechanics - QP. It is essential we make a remark about possible interpretations of quantum mechanics. The variety of interpretations of quantum mechanics is huge and we have no possibility to present even the most important ones (see Khrennikov, 2010).

Let us point to the recently developed subjective probability interpretation, known as *Quantum Bayesianism* (QBism) (see e.g. Fuchs and Schack, 2011). By this interpretation, QP is the machinery used by agents to update probabilities for outcomes of experiments. *QBism stresses the private agent perspective in quantum theory*. The only shortcoming of QBism (from our perspective) is that it handles only decisions about outcomes of observations done for quantum physical systems.

This shortcoming of QBism was discussed in the papers by Khrennikov (2016) and Haven and Khrennikov (2017b), where we extended the applications of QBism to areas outside of quantum physics. Thus, QP is treated as formalizing the DM-process by an agent who follows the rules of quantum logic. The latter relaxes some basis rules of classical logic. Here, in particular, an agent can violate the law of distributivity between conjunction and disjunction. The QBism interpretation of the quantum formalism is very supporting for its applications to decision making, since it justifies the use of subjective probability. 

In this paper we shall present a concrete QP-based model of the DM-process under uncertainty which is generated by a complex information environment, including internal representations of lotteries by a decision maker. Here lotteries should not be reduced to mechanical devices such as roulettes. These are generators of events with complex inter-relation, inside each of the lotteries as well as between lotteries.

### 3 Quantum-like model of selection of lotteries

There are two lots, say  $A = (x_i, P_i)$  and  $B = (y_i, Q_i)$ , where  $(x_i)$  are outcomes and  $(P_i)$  and  $(Q_i)$  are probabilities of these outcomes. All of the outcomes are different from each other. Which lot do you select?

An agent, say Alice, can simulate the experience that she draws the lot  $A$  (or  $B$ ) and gets the outcome  $x_i$  (or  $y_i$ ). Let us represent such an event by  $(A, x_i)$  (or  $(B, x_i)$ ). As usual, Alice assigns the utilities of  $(A, x_i)$  and  $(B, y_i)$  by  $u(x_i)$ . (In the present model the utility of an event depends on only

outcome.) Here,  $u(x)$  is a utility function of outcome  $x$ . By using the utility function the agent evaluates various comparisons for making the preference  $A \succeq B$  or  $B \succeq A$ .

The first mathematically consistent theory of decision making was VNM expected utility theory based on VNM axioms (Completeness, Transitivity, Independence, Continuity). VNM axioms are given for the relation of utilities like  $u \succeq v$  and the operation using probability like  $pu + (1 - p)v$ . This motivates an agent to operate with the expected utilities,  $E_A = \sum u(x_i)P_i$  and  $E_B = \sum u(y_i)Q_i$ , and to use their difference as the criterion for making the preference.

However, the VNM decision theory is not free of paradoxes, section 2. This problem is fundamentally coupled to the interpretation of probability used in the VNM theory. VNM used the frequency (statistical) interpretation of probability. Therefore it is natural to test models of decision making based on other interpretations. The most powerful alternative to the frequency probability is the *subjective probability*.

The subjective probability is not the frequency of an event obtained on the basis a large number of trial experiments, rather it is the *weight of awareness* assigned for unmeasured event. Why do people prefer to appeal to subjective probability? It **can be difficult for them to simulate a large number of trial experiments in their heads**. Thus they use the subjectively assigned weights. **Our model** is based on the assumption that agents proceed with subjective probabilities and that a “natural” operation exists, which is different from the form of  $pu + (1 - p)v$ . In our model of decision making we describe it by using the framework of quantum theory. We emphasize that the quantum formalism operates with a state before measurement. In quantum-like models of cognition and decision making a quantum state is treated as the belief state of an agent. Such state represents subjective recognition for uncertain (unmeasured) events. And the measurement is regarded as an acquisition of subjective experience.

Consider the space of belief states of an agent. In accordance with the quantum-like modeling of cognition belief-states are represented by normalized vectors of a complex Hilbert space  $H$ . These are so-called pure states. More generally, belief-states are represented by density operators encoding classical probabilistic mixtures of pure states.

Lotteries  $A$  and  $B$  are mathematically realized as two orthonormal bases in  $H : (|i_a\rangle)$  and  $(|i_b\rangle)$ . Here we use the Dirac notation  $|i\rangle$  for the  $i$ th vector of a basis. In a Hilbert space, each vector  $|u\rangle$  determines the continuous linear functional on  $H$  which is denoted as  $\langle u|$ . Then action of the functional  $\langle v|$  on the vector  $|x\rangle$  is represented by the formal multiplication of these two symbols:  $\langle v||u\rangle \equiv \langle v|u\rangle$ , the scalar product of these two vectors. This algebra

will be heavily used in this paper, see especially appendix 1. We can also represent lotteries by Hermitian operators

$$A = \sum_i x_i |i_a\rangle \langle i_a| \quad B = \sum_j y_j |j_b\rangle \langle j_b|. \quad (2)$$

We also remark that any pure state  $|u\rangle$  determines the orthogonal projector; in the Dirac notations,  $\sigma_u = |u\rangle \langle u|$ . This is the density operator representation of a pure state.

Any vector  $|i_a\rangle$  represents the event  $(A, x_i)$  - “selecting of the  $A$ -lottery which generates the outcome  $x_i$ . The same can be said about vectors of the  $B$ -basis. These events are not real, but imaginable. Alice plays with potential outcomes of the lotteries and compares them.<sup>11</sup>

We now consider the notion of the quantum transition (conditional) probability. For our applications, it is sufficient to consider transitions between the states  $(|i_a\rangle)$  and  $(|j_b\rangle)$ . We have

$$\langle m_b | i_a \rangle = \sqrt{p(m_b|i_a)} e^{i\theta_{i_a \rightarrow m_b}}, \quad (3)$$

where  $p(m_b|i_a) = p(i_a \rightarrow m_b)$  is the probability of transition from the state  $i_a$  to the state  $m_b$ . Thus

$$p(m_b|i_a) = |\langle m_b | i_a \rangle|^2 \quad (4)$$

This is the Born rule of quantum theory. Symmetry of a scalar product implies that

$$p(i_a \rightarrow m_b) = p(m_b \rightarrow i_a), \text{ i.e., } p(m_b|i_a) = p(i_a|m_b).$$

We also remark that the corresponding transition phases are related as  $\theta_{i_a \rightarrow m_b} = -\theta_{m_b \rightarrow i_a}$ .

## 4 Probabilities and phases

Here we shall discuss the meaning of coefficients in expansion of a quantum state  $|\psi\rangle$  with respect to an orthonormal basis. For simplicity, we consider the two dimensional state space (qubit space). Here we represent some dichotomous observable by Hermitian operator  $A$  with the eigenvalues  $(x_1, x_2)$

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<sup>11</sup>This is a kind of counterfactual reasoning. From this viewpoint, we treat the quantum formalism as a mathematical device for counterfactual reasoning. Of course, we well aware that this not the only possible representation for such a reasoning; in future other models of counterfactual reasoning can be in the use.

and eigenvectors  $|1\rangle, |2\rangle$ . Any state  $|\psi\rangle$  can be expanded with respect to this basis:

$$|\psi\rangle = c_1|1\rangle + c_2|2\rangle, \quad (5)$$

where  $c_1, c_2$  are complex numbers and

$$|c_1|^2 + |c_2|^2 = 1. \quad (6)$$

By using the quantum terminology the state  $|\psi\rangle$  is *superposition* of the (eigen)states  $|1\rangle, |2\rangle$ . We remark that the use of the linear space representation is very common in a variety of cognitive and psychological models, see, e.g., [?]. Thus one might think that the only uncommon feature of the model is the use of complex numbers. However, since each complex number  $z$  can be represented as  $z = u + iv$ , where  $u, v$  are real numbers, any complex linear model of dimension  $n$  can be treated as the real model of dimension  $2n$ .

*The main distinguishing feature of the quantum model is that the coefficients have a probabilistic meaning given by the famous Born's rule.* For the state  $|\psi\rangle$  of the form (5), the numbers

$$p_j = |c_j|^2 \quad (7)$$

are interpreted as the probabilities of the outcomes  $x_j$  of the observable  $A$  having the basis of eigenvectors  $|1\rangle, |2\rangle$ . This is the fundamental rule of quantum theory and its validity has been tested in huge variety of experiments. At the same time to be honest with economists who will use this rule, we have to stress that this is an axiom of quantum theory and, in fact, the conventional theory does not provide any justification of this rule. It is used,

**because it works well.**<sup>12</sup>

Thus the absolute values of the coefficients in the expansion (5) have the clear meaning: these are square roots of probabilities,  $|c_j| = \sqrt{p_j}$ . However, any complex number has also the phase:  $c_j = |c_j|e^{i\theta_j}, j = 1, 2$ . The interpretation of phases is more complicated. Why do we need phases at all? Why is it not sufficient to work with states with real coefficients? From the viewpoint of the Born rule, it seems that it would be sufficient to proceed with superpositions of the form:

$$|\psi\rangle = \sqrt{p_1}|1\rangle + \sqrt{p_2}|2\rangle. \quad (8)$$

<sup>12</sup>Some non-conventional approaches to quantum theory can provide derivations of this rule. However, in such approaches quantum mechanics is considered not as the fundamental theory (as it should be in accordance with the Copenhagen interpretation), but as a theory emergent from some more fundamental theory. In particular, one of the authors derived quantum theory and Born's rule from theory of classical random fields, see, e.g., [?]. However, in this random field model Born's rule is not fundamental, but just an approximation. Hence, one can expect deviations from it.



One of the possibilities to provide a consistent interpretation to the phases is to consider the dynamical model of states generation. This model is basic in quantum computing [1] and it is widely used outside of physics in theory of decision making [2, 3]. The crucial point is that to have the law of conservation of probability, see (6), we have to consider the *unitary dynamics*. And a unitary dynamics can generate nontrivial phases starting with superpositions of the form (8). The dynamics of the quantum state is described by the Schrödinger equation:


$$i\frac{\partial|\psi\rangle}{\partial t}(t) = \mathcal{H}|\psi\rangle(t), \quad |\psi\rangle(0) = |\psi_0\rangle, \quad (9)$$

where  $\mathcal{H}$  is the generator of quantum dynamics, a Hermitian positively definite operator. It has the dimension of frequency, i.e., 1/time.<sup>13</sup> Therefore  $\mathcal{H}$  can be called the *oscillation operator*. To understand better its meaning, let us consider its eigenvalues  $\omega_1, \omega_2$  and corresponding eigenvectors  $|e_1\rangle, |e_2\rangle$ . In this basis the Schrödinger equation is the system of two equations ( $j = 1, 2$ ):

$$i\frac{dz_j}{dt}(t) = \omega_j z_j(t), \quad (10)$$


Its solution has the form

$$z_j(t) = e^{-i\omega_j t} z_{0j}. \quad (11)$$

These are two oscillatory processes. **They** combination gives the complete state-oscillations: 

$$|\psi\rangle(t) = e^{-i\omega_1 t} z_{01}|e_1\rangle + e^{-i\omega_2 t} z_{02}|e_2\rangle. \quad (12)$$

Thus even if, for the initial state, the coefficients  $z_{0j} \in \mathbf{R}$ , the dynamics generates nontrivial phases and complex coefficients.

Following the model of dynamical decision making [4, 5], agent's state evolves driven by the Schrödinger equation until the moment of decision making  $T = T_{\text{dm}}$ . The simplest problem of decision making can be represented  as **measurement** of some observable, say dichotomous, represented by a Hermitian operator  $A$  with eigenvectors  $|1\rangle, |2\rangle$ . By expanding the state  $|\psi\rangle(t)$  with respect to this basis we get the representation:

$$|\psi\rangle(t) = c_1(t)|1\rangle + c_2(t)|2\rangle, \quad (13)$$

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<sup>13</sup>In physics the left-hand side of the equation contains also the Planck constant having the dimension of action = time × energy. Therefore the generator has the dimension of energy. It is called Hamiltonian and has the meaning of the energy observable. In applications outside of physics we treat it formally as dynamics' generator. In financial modeling [6]  $\mathcal{H}$  was interpreted as a kind of financial energy; in social modeling [7] it was interpreted as a kind of social energy. However, such interpretations suffer of the absence of measurement methodology.

where  $c_j(t) = e^{-i\gamma_j t} \sqrt{p_j(t)}$ . Now at the instant of the self-measurement of the mental observable  $A$  an agent uses the state

$$|\psi\rangle = e^{-i\theta_1} \sqrt{p_1} |1\rangle + e^{-i\theta_2} \sqrt{p_2} |2\rangle, \quad (14)$$

where  $\theta_j = \gamma_j T$  and  $p_j = p_j(T)$ . If the operator  $A$  coincides with  $\mathcal{H}$ , then  $\theta_j = \omega_j T$  (but this is the very special situation).


For this state, an agent makes the  $A$ -observation and she obtains the output  $a_j$  with the probability  $p_j$ . This is *the objective probability model of decision making*: for a large ensemble of agents the probability-frequency of the output  $x_j$  equals to  $p_j$  []. Another model is based on *the subjective interpretation of probability* []. An agent assigns subjective probabilities of the outputs by extracting them from the state (14), then she computes odds  $O(1/2) = \frac{p_1}{p_2}$  and she makes her choice depending on the value of odds.

In this paper we shall study a more complex problem of comparison of two lotteries which cannot be reduced to quantum-like modeling of a single observable. The process of comparison involves two in general incompatible observables  $A$  and  $B$ . We shall proceed with the subjective interpretation of probabilities. However, the main feature of the quantum-like process of decision making, namely, reduction of this process to elementary oscillations, will be crucial even in the coming model of comparison of lotteries, see section 7. 

## 5 Belief-state

The state of Alice's beliefs about the lottery  $A$  can be represented as superposition

$$|\Psi_A\rangle = \sum_i \sqrt{P_i} e^{i\theta_{ai}} |i_a\rangle;$$

The probability of realization of the event  $(A, x_i)$  is given by the Born rule and equals to  $P_i = |\langle i_a | \Psi_A \rangle|^2$ . In the same way the state of **believes** about the lottery  $B$  can be represented as superposition 

$$|\Psi_B\rangle = \sum_i \sqrt{Q_i} e^{i\theta_{bi}} |i_b\rangle;$$

To point that an index serves to describe the lottery  $A$  (lottery  $B$ ), we shall label it by its own index, say  $i_a$  or  $j_b$ . And we omit these labels,  $a, b$ , when the meaning of **indexes** be clear or their coupling to  $A$  and  $B$  would not be important.



Now, Alice also superpose her belief-states about the lotteries and her total belief-state is created via superposition of her believes about the  $A$ -lottery and the  $B$ -lottery:

$$|\Psi\rangle = |\Psi_A\rangle + |\Psi_B\rangle. \quad \text{🗨️} \quad (15)$$

However, since in general the states representing Alice's believes about the lotteries are not orthogonal<sup>14</sup>, i.e., in general  $\langle\Psi_A|\Psi_B\rangle \neq 0$ , the vector  $|\Psi\rangle$  is not normalized and the state of combined believes is obtained via normalization:  $|\Phi\rangle = |\Psi\rangle/\| |\Psi\rangle\|$ . Since the normalization factor is positive, it does not play any role in the process of comparison of lotteries, see section 6. Therefore we can proceed with the vector  $|\Psi\rangle$ .

In further calculations it is useful to use the operator representation of  $|\Psi\rangle$  :

$$\sigma \equiv \sigma_\Psi = |\Psi\rangle\langle\Psi| = \frac{1}{2}(\sigma_A + \sigma_B + \sigma_{B\rightarrow A} + \sigma_{A\rightarrow B}),$$

where

$$\begin{aligned} \sigma_A &= |\Psi_A\rangle\langle\Psi_A| = \sum_{ij} \sqrt{P_i P_j} e^{i(\theta_{ai} - \theta_{aj})} |i_a\rangle\langle j_a| \\ \sigma_B &= |\Psi_B\rangle\langle\Psi_B| = \sum_{ij} \sqrt{Q_i Q_j} e^{i(\theta_{bi} - \theta_{bj})} |i_b\rangle\langle j_b| \\ \sigma_{B\rightarrow A} &= |\Psi_A\rangle\langle\Psi_B| = \sum_{ij} \sqrt{P_i Q_j} e^{i(\theta_{ai} - \theta_{bj})} |i_a\rangle\langle j_b| \\ \sigma_{A\rightarrow B} &= |\Psi_B\rangle\langle\Psi_A| = \sum_{ij} \sqrt{P_i Q_j} e^{-i(\theta_{ai} - \theta_{bj})} |j_b\rangle\langle i_a|. \end{aligned}$$

We remark that, since  $|\Psi\rangle$  is not normalized,  $\text{tr}\sigma \neq 1$ . (But we repeat that this is not important in the process of lotteries selection described in section 6.)

## 6 Comparison operator

In the classical expected utility theory Alice calculates the averages of the utility function. In the quantum-like model Asano et. al (2017) the utility function determines the *comparison operator*.

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<sup>14</sup>Nonorthogonality of the states  $|\Psi_A\rangle$  and  $|\Psi_B\rangle$  means that the believes about two lotteries are not complementary. There is an ‘‘overlap’’ between them. The presence of such overlap plays the important role in the process of decision making, see section 8.



Let us introduce the operators  $E_{i_a \rightarrow j_b} = |j_b\rangle\langle i_a|$  and  $E_{j_b \rightarrow i_a} = |i_a\rangle\langle j_b|$ . We have, e.g.,  $E_{i_a \rightarrow j_b}|i_a\rangle = |j_b\rangle$ . This operator describes the process of transition from preferring the state  $|i_a\rangle$  to preferring the state  $|j_b\rangle$ . The operator  $E_{j_b \rightarrow i_a} = |i_a\rangle\langle j_b|$  describes transition in the opposite direction. We stress that these are transitions between the belief-states of Alice. **So, they happen in her mind.** We remark that  $E_{j_b \rightarrow i_a} = E_{i_a \rightarrow j_b}^*$ , i.e., elementary transitions in opposite directions are represented by adjoint operators.

Now we introduce the two comparison operators:

$$D_{B \rightarrow A} = \sum_{nm} (u(x_n) - u(y_m)) e^{i\gamma_{m_b \rightarrow n_a}} E_{m_b \rightarrow n_a} = \sum_{nm} (u(x_n) - u(y_m)) e^{i\gamma_{m_b \rightarrow n_a}} |n_a\rangle\langle m_b|.$$

The operator  $D_{B \rightarrow A}$  represents the utility of selection of the lottery  $A$  relatively to the utility of selection of the lottery  $B$ . We can say that by transition from the potential outcome  $(B, y_m)$  to the potential outcome  $(A, x_n)$  Alice **earns utility  $u(x_n)$  and at the same time she loses utility  $u(y_m)$** . If  $u(x) = x$  and  $x$  has the meaning of cash amounts (say USD), then by such a transition Alice (potentially) earns  $x_n - y_m$  USD. In the same way we interpret the transition operator:

$$D_{A \rightarrow B} = \sum_{nm} (u(y_m) - u(x_n)) e^{i\gamma_{n_a \rightarrow m_b}} E_{n_a \rightarrow m_b} = \sum_{nm} (u(y_m) - u(x_n)) e^{i\gamma_{n_a \rightarrow m_b}} |m_b\rangle\langle n_a|.$$

This operator represents the utility of selection of the lottery  $B$  relatively to the utility of selection of the lottery  $A$ . And finally we define the **comparison operator**:

$$D = D_{B \rightarrow A} - D_{A \rightarrow B}.$$

The operator  $D$  compares these two relative utilities. This operator has the form:

$$\begin{aligned} D &= \sum_{nm} (u(x_n) - u(y_m)) e^{i\gamma_{m_b \rightarrow n_a}} |n_a\rangle\langle m_b| - \sum_{nm} (u(y_m) - u(x_n)) e^{i\gamma_{n_a \rightarrow m_b}} |m_b\rangle\langle n_a| \\ &= \sum_{nm} \delta_{nm} (e^{i\gamma_{m_b \rightarrow n_a}} |n_a\rangle\langle m_b| + e^{i\gamma_{n_a \rightarrow m_b}} |m_b\rangle\langle n_a|), \end{aligned} \tag{16}$$

where

$$\delta_{nm} = u(x_n) - u(y_m).$$

Since all quantum observables are represented by Hermitian operators, the phases should be related as follows:

$$\gamma_{n_a \rightarrow m_b} = -\gamma_{m_b \rightarrow n_a}. \tag{17}$$

At the same time the operators  $D_{B \rightarrow A}$  and  $D_{A \rightarrow B}$  not Hermitian. We have that  $D_{A \rightarrow B}^* = -D_{B \rightarrow A}$  and  $D = D_{B \rightarrow A} + D_{B \rightarrow A}^*$ . Hence, they **can not** be treated as observables. Thus the comparison operator  $D$  gives us the integral judgment. Only heuristically we can treat the  $D$ -based judgment as the result of comparison of two relative utilities represented by the operators. The quantum analog of (subjective) expected utility theory is based on the natural decision rule:

**Decision rule.** *If the average of the comparison operator  $D$  is non-negative, i.e.,  $\langle D \rangle_\psi = \text{tr} D \sigma_\psi \geq 0$ , then  $A \succeq B$ .*

We remark that the comparison operator  $D$  has no direct relation to comparison of these two concrete lotteries  $A$  and  $B$ . It was created on the basis of previous experience of decision making and it was memorized in Alice's **brain**. It is natural to assume that for each class  $C$  of decision problems Alice has its own comparison operator  $D(C)$ , e.g., the financial operator, the private life operator, and so on. Of course, these operators are not fixed for ever and they can be modified on the basis of the new (negative) experience. However, we do not consider this problem of learning in this paper.

The appendix contains calculations of the quantum averages. The structures of the comparison operator  $D$  and the belief-state  $\sigma$  induce nontrivial decomposition of the process of decision making into a few sub-processes. We shall analyze these subprocesses separately, see section 7.

From calculations in the appendix we get:

$$\frac{1}{2} \text{tr} D \sigma_A = \sum_{ij;m} \delta_{jm} \sqrt{p(m_b|i_a) P_i P_j} \cos \Theta_{ij;m}^A, \quad (18)$$

where  $\Theta_{ij;m}^A = \theta_{i_a \rightarrow m_b} - \gamma_{i_a \rightarrow m_b} + \theta_{a_i} - \theta_{a_j}$ .

$$\frac{1}{2} \text{tr} D \sigma_B = \sum_{ij;n} \delta_{nj} \sqrt{p(n_a|i_b) Q_i Q_j} \cos \Theta_{ij;n}^B, \quad (19)$$

where  $\Theta_{ij;n}^B = \theta_{i_b \rightarrow n_a} - \gamma_{i_b \rightarrow n_a} + \theta_{b_i} - \theta_{b_j}$ .

Calculations in appendix shows that it is natural to consider the following combinations of traces for comparison operators and “transition states”, see (46), (47):


$$\Delta_1 = \frac{1}{2} \text{tr} (D_{B \rightarrow A} \sigma_{A \rightarrow B} - D_{A \rightarrow B} \sigma_{B \rightarrow A}) = \sum_{ij} \delta_{ij} \sqrt{P_i Q_j} \cos \Theta_{ij}. \quad (20)$$

where  $\Theta_{ij} = \theta_{b_j} - \theta_{a_i} + \gamma_{j_b \rightarrow i_a}$ ;

$$\Delta_2 = \frac{1}{2} \text{tr} (D_{B \rightarrow A} \sigma_{B \rightarrow A} - D_{A \rightarrow B} \sigma_{A \rightarrow B}) = \sum_{ij;nm} \delta_{nm} \sqrt{p(m_b|i_a) p(n_a|j_b) P_i Q_j} \cos \Gamma_{ij,nm}, \quad (21)$$

where  $\Gamma_{ij,nm} = \theta_{j_b \rightarrow n_a} - \theta_{i_a \rightarrow m_b} + \gamma_{n_a \rightarrow m_b} + \theta_{b_j} - \theta_{a_i}$ .

## 7 Analysis of the basic counterparts of the process of comparison of lotteries

As we have seen, the average of the comparison operator is naturally decomposed into four counterparts representing special subprocesses of the process of decision making. We start with the simplest (mathematically) expression. 

**Process 1:** Its output is represented by the quantity  $\Delta_1$ , see (20). To simplify considerations, let us assume that all phases  $\Theta_{ij}$  in the sum are equal, i.e.,  $\Theta_{ij} \equiv \Theta$ . Thus

$$\Delta_1 = \left[ \sum_i u(x_i) \sqrt{P_i} \sum_j \sqrt{Q_j} - \sum_j u(y_j) \sqrt{Q_j} \sum_i \sqrt{P_i} \right] \cos \Theta.$$

Following Asano et al. (2017), consider the normalized difference


$$\frac{\Delta_1}{\sum_{ij} \sqrt{P_i Q_j}} = \left[ \sum_i u(x_i) \tilde{P}_i - \sum_j u(y_j) \tilde{Q}_j \right] \cos \Theta, \quad (22)$$

where the quantities

$$\tilde{P}_i = \frac{\sqrt{P_i}}{\sum_i \sqrt{P_i}}, \quad \tilde{Q}_j = \frac{\sqrt{Q_j}}{\sum_j \sqrt{Q_j}} \quad (23)$$

can be interpreted as subjective probabilities.<sup>15</sup> Alice assigns these probabilities to outcomes of the lotteries in the process of comparison of the relative utility of the  $B \rightarrow A$  transition (based on her beliefs encoded in the  $\sigma_{B \rightarrow A}$  counterpart of her belief state) with the relative utility of the  $A \rightarrow B$  transition (based on her beliefs encoded in the  $\sigma_{A \rightarrow B}$  counterpart of her belief state). We remark that the quantities

$$\langle u \rangle_{\tilde{P}} = \sum_i u(x_i) \tilde{P}_i, \text{ and } \langle u \rangle_{\tilde{Q}} = \sum_j u(y_j) \tilde{Q}_j$$

are *expected utilities* for the lotteries with respect to these subjective probabilities. Thus, for this part of the process of comparison of two lotteries, Alice assigns subjective probabilities ( $\tilde{P}_i$ ) and ( $\tilde{Q}_j$ ) given by the **square root transformation of the original probabilities** ( $P_i$ ) and ( $Q_j$ ). (The latter can be 


<sup>15</sup>We remark that  $\sum_i \tilde{P}_i = 1$  and  $\sum_j \tilde{Q}_j = 1$ .

treated as objective statistical probabilities.) Then she calculates subjective expected utilities and compare them. The final step of comparison is taking into account the sign of the factor  $\cos \Theta$ . This is really nontrivial quantum counterpart of the decision process.

We remark that the square root transformation of probabilities can be directly coupled to selection of weighting functions in *the prospect theory*, see (1) in introduction.

**Process 2:** Now we analyze the counterpart of the decision making process based on the comparison average given by the quantity

$$\frac{1}{2} \text{tr} D \sigma_A = \sum_{ij;m} \delta_{jm} \sqrt{p(m|i) P_i P_j} \cos \Theta_{ij;m},$$

see (18). In this subprocess Alice uses only the part of her belief-state (given by  $\Psi_A$ ) representing her believes about the  $A$ -lottery. She compares these   $A$ -believes with possible transitions to the  $B$ -states. Such transitions are expressed through the transition probabilities  $p(m|i) = p(i_a \rightarrow m_b)$  and the phases  $\theta_{i_a \rightarrow m_b}$  and  $\gamma_{i_a \rightarrow m_b}$ . These phases represent correlations between the events  $(A, x_i)$  and  $(B, x_m)$ . These quantities are of the subjective nature. Alice tries to treat two lotteries separately, but she has a variety of correlations between them coming from the analysis of the situation and the previous experience. The transition probabilities  $p(m|i)$  are also subjective quantities.

To simplify analysis, we again assume that all transition phases  $\Theta_{ij;m}$  are equal, i.e.,  $\Theta_A \equiv \Theta_{ij;m}$ . Besides the subjective probabilities  $\tilde{P}_i$ , we consider subjective probabilities of transition from the lottery  $A$  to the lottery  $B$  given by

$$\tilde{Q}_{m;A} = \frac{\sum_i \sqrt{p(m|i) P_i}}{\sum_{i,m} \sqrt{p(m|i) P_i}}. \quad (24)$$

(We have  $\sum_m \tilde{Q}_{m;A} = 1$ .)

If Alice were acting on the basis of the classical (Kolmogorov) probability and if she were not using the square root weighting of probabilities, then the quantity  $\tilde{Q}_{m;A}$  would be equal to the original probability  $Q_m = \sum_i p(m|i) P_i$ . Consider now the normalized quantity

$$\begin{aligned} \frac{\text{tr} D \sigma_A}{2 \sum_{ij;m} \sqrt{p(m|i) P_i P_j}} &= \left[ \sum_j u(x_j) \tilde{P}_j - \sum_m u(y_m) \tilde{Q}_{m;A} \right] \cos \Theta_A \\ &= [\langle u \rangle_{\tilde{P}} - \langle u \rangle_{\tilde{Q}_A}] \cos \Theta_A, \end{aligned}$$

where the quantities in the brackets are expected utilities with respect to the corresponding subjective probability distributions. Thus in this decision

subprocess Alice assigns subjective probabilities to the  $A$ -events (by using the square root transformation). Then she assigns subjective probabilities  $p(m|i)$  for transitions  $i_a \rightarrow m_b$ . They can be interpreted in the following way. Alice assumed that the event  $(A, x_i)$  would happen (with the probability  $\tilde{P}_i$ , but with the probability  $p(m|i)$  she changes her mind and assumes that the event  $(B, y_m)$  would happen. The output of such mental fluctuations is assignment of subjective probability to the events  $(B, y_m)$  conditioned on the believes about the  $A$ -states, the probability  $\tilde{Q}_{m;A}$ . Then Alice compute the difference between expected utilities. If  $\cos\Theta_A \geq 0$ , then she uses this difference to order the lotteries as  $A \succeq B$ . However, if  $\cos\Theta_A \leq 0$ , then  $B \succeq A$ . Thus the ordering is opposite to the expected utility ordering. We recall that this is not the final ordering of the lotteries, but just ordering generated by the subprocess under consideration.

**Process 3:**

The counterpart of the decision process based on comparison corresponding Alice's beliefs about the  $B$ -lottery is analyzed similarly - again under the assumption of coincidence of all phases: in (19)  $\Theta_B \equiv \Theta_{ij;n}$ . Consider subjective probabilities of transition from the lottery  $B$  to the lottery  $A$  given by

$$\tilde{P}_{n;B} = \frac{\sum_i \sqrt{p(n|i)Q_i}}{\sum_{i,n} \sqrt{p(n|i)Q_i}}. \quad (25)$$

(We have  $\sum_m \tilde{P}_{n;B} = 1$ .) Then we have

$$\begin{aligned} \frac{\text{tr}D\sigma_B}{2 \sum_{i,n} \sqrt{p(n|i)Q_i}Q_j} &= \left[ \sum_n u(x_n)\tilde{P}_{n;B} - \sum_j u(y_j)\tilde{Q}_j \right] \cos \Theta_B \\ &= [\langle u \rangle_{\tilde{P}_B} - \langle u \rangle_{\tilde{P}}] \cos \Theta_B. \end{aligned} \quad (26)$$

Thus the output of this counterpart of the decision process is based on comparison of subjective expected utilities and relative phases. One of the expected utilities is based on the subjective account of probability of the transition (in Alice's mind) from the believes about the  $B$ -lottery to the  $A$ -lottery and another is so to say straightforward subjective probability based on the square root transform of the initial probabilities ( $Q_j$ ).

**Process 4:**

Finally, we analyze the most complicated counterpart of the process of decision making corresponding to the comparison term  $\Delta_2$ . This term compares the utilities of transitions  $B \rightarrow A$  and  $A \rightarrow B$  when Alice appeals to the counterparts of her state representing believes about these transitions. This process is characterized by ambiguity and intensive fluctu-



ations of Alice's mind in both directions; the intensity of these fluctuations is given by the transition probabilities  $p(n_a|j_b) = p(j_b \rightarrow n_a)$  and  $p(m_b|i_a) = p(i_a \rightarrow m_b)$ . Correlations between outcomes of lotteries are encoded in the phases  $\theta_{j_b \rightarrow n_a}, \theta_{i_a \rightarrow m_b} = -\theta_{m_b \rightarrow i_a}$ . Since the general form of dependence of  $\Delta_2$  on the phases is very complex, we again assume that

$$\Delta_2 = \sum_{ij;nm} \delta_{nm} \sqrt{p(m_b|i_a)p(n_a|j_b)P_iQ_j} \cos \Gamma, \quad (27)$$

where  $\Gamma = \theta_{j_b \rightarrow n_a} - \theta_{i_a \rightarrow m_b} + \gamma_{n_a \rightarrow m_b} + \theta_{bj} - \theta_{ai}$ . Now, as in the previous processes, we can represent this comparison term as difference of two expected utilities with respect to the subjective probabilities ( $\tilde{P}_{i;B}$ ) and ( $\tilde{Q}_{j;A}$ ):

$$\begin{aligned} & \frac{\Delta_2}{\sum_{ij;nm} \sqrt{p(m_b|i_a)p(n_a|j_b)P_iQ_j}} \cos \Gamma \\ &= \left[ \sum_n u(x_n) \frac{\sum_i \sqrt{p(n|j)Q_j}}{\sum_{jn} \sqrt{p(n|j)Q_j}} - \sum_m u(y_m) \frac{\sum_i \sqrt{p(m|i)P_i}}{\sum_{im} \sqrt{p(m|i)P_i}} \right] \cos \Gamma = \\ & \quad [\langle u \rangle_{\tilde{Q}_A} - \langle u \rangle_{\tilde{P}_B}] \cos \Gamma. \end{aligned}$$

To finalize this the most complex counterpart of the process of comparison of the lotteries, Alice takes into account the signs of difference between subjective expected utilities and and of the interference cos-term.

**Complete process of lottery selection.** Under the simplification assumption about phases, we can write the average of the comparison operator as

$$\begin{aligned} \langle D \rangle_{\Psi} &= c_1[\langle u \rangle_{\tilde{P}} - \langle u \rangle_{\tilde{Q}}] \cos \Theta + c_2[\langle u \rangle_{\tilde{P}} - \langle u \rangle_{\tilde{Q}_A}] \cos \Theta_A \\ & \quad + c_3[\langle u \rangle_{\tilde{P}_B} - \langle u \rangle_{\tilde{P}}] \cos \Theta_B + c_4[\langle u \rangle_{\tilde{Q}_A} - \langle u \rangle_{\tilde{P}_B}] \cos \Gamma. \end{aligned} \quad (28)$$

where the weights  $c_j > 0, j = 1, 2, 3, 4$ , can be found in the above considerations for the subprocesses 1-4.

In the accordance with the decision rule, if  $\langle D \rangle_{\Psi} \geq 0$ , Alice selects the A-lottery.

## 8 Example: lotteries with two outcomes

Consider two lotteries,  $A$  and  $B$ , having two outcomes  $(x_1, x_2)$  and  $(y_1, y_2)$  and the utilities,  $u_i = u(x_i)$  and  $v_j = u(y_j)$ . The probabilities  $P_i, Q_j$  will be specified later. We start with calculation of the matrix of the comparison operator  $D$  in the basis  $|1_a\rangle, |2_a\rangle$ . By definition we have

$$D = (u_1 - v_1)[|1_a\rangle\langle 1_b| + |1_b\rangle\langle 1_a|] + (u_1 - v_2)[|1_a\rangle\langle 2_b| + |2_b\rangle\langle 1_a|]$$

$$+(u_2 - v_1)[|2_a\rangle\langle 1_b| + |1_b\rangle\langle 2_a|] + (u_2 - v_2)[|2_a\rangle\langle 2_b| + |2_b\rangle\langle 2_a|].$$

Consider in the qubit space two bases

$$|1_a\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad |2_a\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \quad (29)$$

and

$$|1_b\rangle = (|1_a\rangle + |2_a\rangle)/\sqrt{2} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad |2_b\rangle = (|1_a\rangle - |2_a\rangle)/\sqrt{2} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix}. \quad (30)$$



**We no find** the matrix of  $D$  in the basis  $|1_a\rangle, |2_a\rangle$ . We have  $\langle 1_a|1_a\rangle\langle 1_b|1_a\rangle = 1/\sqrt{2}$ ,  $\langle 1_a|1_a\rangle\langle 1_b|2_a\rangle = 1/\sqrt{2}$ ,  $\langle 2_a|1_a\rangle\langle 1_b|1_a\rangle = 0$ ,  $\langle 2_a|1_a\rangle\langle 1_b|2_a\rangle = 0$ .

$$D_{11,b \rightarrow a} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}; D_{11,a \rightarrow b} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}; D_{11} = \frac{1}{\sqrt{2}} \begin{pmatrix} 2 & 1 \\ 1 & 0 \end{pmatrix} \quad (31)$$

In the same way we obtain

$$D_{21,b \rightarrow a} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix}; D_{21,a \rightarrow b} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 \\ -1 & 0 \end{pmatrix}; D_{21} = \frac{1}{\sqrt{2}} \begin{pmatrix} 2 & -1 \\ -1 & 0 \end{pmatrix} \quad (32)$$

$$D_{12,b \rightarrow a} = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}; D_{12,a \rightarrow b} = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}; D_{12} = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 \\ 1 & 2 \end{pmatrix} \quad (33)$$

$$D_{22,b \rightarrow a} = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 0 \\ 1 & -1 \end{pmatrix}; D_{22,a \rightarrow b} = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 \\ 0 & -1 \end{pmatrix}; D_{22} = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 \\ 1 & -2 \end{pmatrix} \quad (34)$$

Hence,

$$D = \frac{1}{\sqrt{2}} \left[ (u_1 - v_1) \begin{pmatrix} 2 & 1 \\ 1 & 0 \end{pmatrix} + (u_1 - v_2) \begin{pmatrix} 2 & -1 \\ -1 & 0 \end{pmatrix} + (u_2 - v_1) \begin{pmatrix} 0 & 1 \\ 1 & 2 \end{pmatrix} + (u_2 - v_2) \begin{pmatrix} 0 & 1 \\ 1 & -2 \end{pmatrix} \right]$$

$$D = \frac{1}{\sqrt{2}} \begin{pmatrix} 4u_1 - 2v_1 - 2v_2 & 2u_2 - 2v_1 \\ 2u_2 - 2v_1 & 2v_2 - 2v_1 \end{pmatrix}$$

## 8.1 Starting with the uniform probability distributions

Let now  $P_1 = P_2 = Q_1 = Q_2 = 1/2$ , i.e.,

$$|\psi_A\rangle = (|1_a\rangle + |2_a\rangle)/\sqrt{2} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad |\psi_B\rangle = (|1_b\rangle + |2_b\rangle)/\sqrt{2} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}. \quad (35)$$

$$|\psi\rangle = \frac{|\psi_A\rangle + |\psi_B\rangle}{\| |\psi_A\rangle + |\psi_B\rangle \|} = \frac{1}{\sqrt{(1+\sqrt{2})^2 + 1}} \begin{pmatrix} 1 + \sqrt{2} \\ 1 \end{pmatrix}. \quad (36)$$



By we obtain the following inequality determining selection of lotteries:  $\langle \psi | D | \psi \rangle$

$$\begin{aligned} &= C2 \left( (1+\sqrt{2})^2 u_1 + (1+\sqrt{2}+1)u_2 - ((1+\sqrt{2})^2 + 2(1+\sqrt{2})+1)v_1 - ((1+\sqrt{2})^2 - 1)v_2 \right) = \\ &= C' \left( [(1+\sqrt{2})u_1 + u_2] - [(1+\sqrt{2})v_1 + v_2] \right) \geq 0, \end{aligned} \quad (37)$$

where  $C, C'$  are positive factors.

The main point is that the weights of utilities for both lotteries coincide with the coefficients (up to normalization) in the expansions of the state  $|\psi\rangle$  (which combines Alice's **believes** about the lotteries) with respect to the corresponding bases. We can rewrite the expectation of the comparison operator as

$$\langle \psi | D | \psi \rangle = \tilde{C} \left( [u_1 \tilde{P}_1 + u_2 \tilde{P}_2] - [v_1 \tilde{Q}_1 + v_2 \tilde{Q}_2] \right) \geq 0, \quad (38)$$

where  $C'$  is a positive factor and the subjective probabilities are given by the expressions:

$$\begin{aligned} \tilde{P}_1 &= \tilde{Q}_1 = \frac{(1+\sqrt{2})}{2+\sqrt{2}} = \frac{1}{\sqrt{2}}, \\ \tilde{P}_2 &= \tilde{Q}_2 = \frac{1}{2+\sqrt{2}}. \end{aligned}$$



Thus by using the state expansion with respect to the  $A$ -basis, see (36), and by *treating amplitudes (the coefficients with respect to the  $A$ -basis) as subjective probabilities*, Alice calculates SEU for the  $A$ -lottery. We stress that she uses the subjective probabilities encoded in the state  $|\psi\rangle$  combining believes about both lotteries and not probabilities (objective or subjective) encoded in the state  $|\psi_A\rangle$  representing solely believes about the  $A$ -lottery. Now we remark that the complete belief state  $|\psi\rangle$  can be represented as well in the form:

$$|\psi\rangle = \frac{1+\sqrt{2}}{\sqrt{(1+\sqrt{2})^2 + 1}} |1_b\rangle + \frac{1}{\sqrt{(1+\sqrt{2})^2 + 1}} |2_b\rangle. \quad (39)$$

Alice uses this representation to calculate SEU for the  $B$ -lottery.


How can one explain, e.g., the increase of probability of the outcome  $x_1$  comparing with the outcome  $x_2$ ? In the state  $|\psi_A\rangle$  both these outcomes are



 equally possible. Now Alice started to combine (by using the rules of quantum **logics**)<sup>16</sup> the believes about the two lotteries and she **finds** that some believes about the  $B$ -lottery encoded in the state  $|\psi_B\rangle$  can be interpreted as the believes in favor of the outcomes  $x_1$  and  $x_2$ , but the additional weight assigned to  $x_1$  is higher than the weight assigned additionally to  $x_2$ . (In our example the latter is zero.) This is a kind of *constructive and destructive interference* for probabilities assigned to believes in favor of the outcomes  $x_1$  and  $x_2$ , respectively. We recall that 

$$|\psi_B\rangle = (|1_b\rangle + |2_b\rangle)/\sqrt{2} = \frac{1}{2}[(|1_a\rangle + |2_a\rangle) + (|1_a\rangle - |2_a\rangle)]. \quad (40)$$

The believes in favor of  $x_1$  which are present in the states  $|1_b\rangle, |2_b\rangle$  interfere constructively and believes in favor of  $x_2$  interfere destructively.

This process of comparison of the lotteries can be decomposed in the four processes, see section 7, describing reflections of Alice in the process of decision making. And this is the right way to represent mathematically the process of comparison of lotteries. The above simple and heuristically clear picture with the straightforward identification of subjective probabilities with the coefficients in the expansions of the belief-state with respect to lotteries' bases does not work in the general case. The reason is simple: the coefficients need not be positive, see appendix 2, where we consider the case of lotteries with arbitrary objective probabilities  $P_1, P_2$  and  $Q_1, Q_2$ . 

## 8.2 Starting with deterministic lotteries and violation of Cromwell's rule

Now we consider more interesting features of our model. Consider the degenerate case  $P_1 = Q_1 = 1$  and  $P_2 = Q_2 = 0$ . Thus  $|\psi_A\rangle = |1_a\rangle$  and  $|\psi_B\rangle = |1_b\rangle$ . Here

$$|\psi\rangle = \frac{|1_a\rangle + |1_b\rangle}{\| |1_a\rangle + |1_b\rangle \|}.$$

Surprisingly this state coincides with the state (36) corresponding to the lotteries with probabilities  $P_1 = P_2 = Q_1 = Q_2 = 1/2$ . How can it happen?

The state  $|\psi_A\rangle = |1_a\rangle$  corresponds to the clear picture of the  $A$ -lottery (in Alice's brain): the probability of the outcome  $x_1$  equals to one. In the same way the state  $|\psi_B\rangle = |1_b\rangle$  corresponds to the clear picture of the  $B$ -lottery: the probability of the outcome  $y_1$  equals to one. In these two (separate) pictures the possibilities of the outcomes  $x_2$  and  $y_2$  are simply ignored.

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<sup>16</sup>These rules are formally represented by linear algebra in complex Hilbert space. So, Alice expands  $|\psi_B\rangle$  with respect to the  $A$ -basis.

Then Alice starts to combine these two pictures in one, by using the rules of quantum logics. By analyzing believes corresponding to the outcome  $x_1$  she finds that some of these believes can be interpreted as believes in favor of the outcome  $y_2$  of the  $B$ -lottery as well. (Since the state  $|\psi_A\rangle = |1_a\rangle$  can be represented as  $|\psi_A\rangle = \frac{1}{\sqrt{2}}(|1_b\rangle + |2_b\rangle)$ .) So, she cannot more believe that the outcome  $y_2$  would never possible and she assigns to it nonzero subjective probability; in this way the utility  $v_2 = u(y_2)$  comes into the game. In the same way by analyzing her believes about the outcome  $y_1$  (encoded in the state  $|\psi_B\rangle = |1_b\rangle$ ) she finds that some of these believes can be interpreted as believes in favor of the outcome  $x_2$  of the  $A$ -lottery as well. (Since the state  $|\psi_B\rangle = |1_b\rangle$  can be represented as  $|\psi_B\rangle = \frac{1}{\sqrt{2}}(|1_a\rangle + |2_a\rangle)$ .)

In fact, such assignment of nonzero probabilities by starting from zero priors is one of the most intriguing features of the quantum update of probabilities and quantum Bayesian inference. As we know well, the classical Bayesian update of probabilities cannot generate nonzero probability from zero prior probability. This feature of Bayesian inference is sometimes referred as *the Cromwell rule* []. As was shown in ??, in quantum Bayesian inference the the Cromwell rule can be violated. Moreover, this work presents design of a psychological experiment and the corresponding experimental statistical data demonstrating violation of the Cromwell rule and, hence, inapplicability of classical Bayesian inference.

## 9 Classical expected utility model from quantum-like model

The quantum-like model developed in this paper has very rich structure and it is natural to expect that the classical VNM model of lottery selection can be reproduced as its counterpart. Before to proceed in this direction we shall consider one special class of orthonormal bases in Hilbert state spaces.

### 9.1 Mutually unbiased bases

The bases (29), (30) considered in the above example of comparison of lotteries are very special. These are so-called *mutually unbiased bases* (MUB). We recall that in quantum theory of information two orthonormal bases ( $e_i$ ) and ( $f_j$ ) are called mutually unbiased if

$$|\langle e_i | f_j \rangle|^2 = \frac{1}{d}, \quad (41)$$

where  $d$  is the dimension of the Hilbert state space. MUBs play very important role in the variety of problems of quantum information. In terms of transition probabilities the equality (41) has the following meaning: all transition probabilities from the states ( $e_i$ ) to the states ( $f_j$ ) are equal. In the process of comparison of lotteries Alice considers possible transitions from a state  $|i_a\rangle$  to all states of the  $B$ -basis and vice versa, see section 7, processes 2-4. If the lotteries are represented by MUBs, then by being in the state  $|i_a\rangle$  Alice assigns equal probability for transition to any state  $|j_b\rangle$  and vice versa. On one hand, such a comparison process is more complicated than a process for mutually biased bases. Uniformity of transition probabilities induces more uncertainty. However, the use of MUBs has also the important advantage: Alice need not analyze the structure of transition probabilities, she uses the uniform distribution for transitions. This strategy can be profitable in the absence of information about probabilities of transitions. Therefore MUBs might be really preferred by agents processing information by using the rules of quantum information theory.

## 9.2 Derivation of VNM-model of lottery selection

In this section we consider special pairs of MUBs such that

$$\langle e_i | f_j \rangle = \frac{1}{\sqrt{d}}, \quad (42)$$

i.e., transition amplitudes do not have phases. The basis (29), (30) is precisely of this type. So, let ( $|i_a\rangle$ ) and ( $|j_b\rangle$ ) be MUBs of this type. We remark that for two lotteries  $A = \{(x_i, P_i)\}$  and  $B = \{(y_i, Q_i)\}$  we can always assume (by selecting some additional zero probabilities) that the sets of their outcomes coincide:

$$\{x_i\} = \{y_i\} \quad (43)$$

Consider now the comparison operator  $D$  corresponding to these bases, see (16), and having no phases:

$$D = \sum_{nm} (u(x_n) - u(y_m)) \left[ |n_a\rangle\langle m_b| + |m_b\rangle\langle n_a| \right], \quad (44)$$

where the bases satisfy (42). We have

$$\begin{aligned} \langle i_a | D | i_a \rangle &= \sum_{nm} (u(x_n) - u(y_m)) \left[ \langle i_a | n_a \rangle \langle m_b | i_a \rangle + \langle i_a | m_b \rangle \langle n_a | i_a \rangle \right] \\ &= 2 \sum_m (u(x_i) - u(y_m)). \end{aligned}$$

For the  $B$ -basis, we have  $\langle i_b | D | i_b \rangle = 2 \sum_n (u(x_n) - u(y_i))$ .

Now suppose that Alice combines her images about the lotteries in the classical manner, i.e., she does not form superposition of the form (15), but she forms the mixtures of the states  $\sigma_A$  and  $\sigma_B$ , i.e.,

$$\sigma = \frac{1}{2}(\sigma_A + \sigma_B) \quad (45)$$

By using previously calculated averages for the basis vectors we obtain

$$\begin{aligned} \text{tr} D \sigma_A &= 2 \sum_i P_i \sum_m (u(x_i) - u(y_m)) = 2[d \sum_i u(x_i) P_i - \sum_m u(y_m)] \\ &= 2[d \langle u \rangle_P - \sum_m u(y_m)]. \end{aligned}$$

In the same way

$$\begin{aligned} \text{tr} D \sigma_B &= 2 \sum_i Q_i \sum_n (u(x_n) - u(y_i)) = 2[\sum_n (u(x_n) - d \sum_i Q_i u(y_i))] \\ &= 2[\sum_n (u(x_n) - d \langle u \rangle_Q)]. \end{aligned}$$

Thus

$$\frac{1}{d} \text{tr} D \sigma = \langle u \rangle_P - \langle u \rangle_Q.$$

Hence, if Alice combines uncertainties about lotteries in the classical manner, by using the mixed states, instead of superposition (so, by such a combination she does not increase uncertainty), then she, in fact, acts by comparing two (objective) expected utilities. Such a process of the lottery selection is described by VNM-model.

## Appendix 1: calculations of quantum averages

We start with calculation of average  $\text{tr} D_{B \rightarrow A} \sigma_A$ .

$$\begin{aligned} \text{tr} D_{B \rightarrow A} \sigma_A &= \text{tr} \left( \sum_{nm} \delta_{nm} e^{i\gamma_{m_b \rightarrow n_a}} |n_a\rangle \langle m_b| \right) \left( \sum_{ij} \sqrt{P_i P_j} e^{i(\theta_{a_i} - \theta_{a_j})} |i_a\rangle \langle j_a| \right) \\ &= \sum_{ij; nm} \delta_{nm} \sqrt{P_i P_j} e^{i(\theta_{a_i} - \theta_{a_j} + \gamma_{m_b \rightarrow n_a})} \langle j_a | n_a \rangle \langle m_b | i_a \rangle \end{aligned}$$

By using the definition of quantum transition (conditional) probabilities, see (3), (4), we obtain:

$$\text{tr}D_{B \rightarrow A}\sigma_A = \sum_{ij;m} \delta_{jm} \sqrt{p(m_b|i_a)P_iP_j} e^{i(\theta_{i_a \rightarrow m_b} + \gamma_{m_b \rightarrow j_a} + \theta_{a_i} - \theta_{a_j})}$$

In the same way we obtain

$$\begin{aligned} \text{tr}D_{A \rightarrow B}\sigma_A &= -\text{tr} \left( \sum_{nm} \delta_{nm} e^{i\gamma_{n_a \rightarrow m_b}} |m_b\rangle \langle n_a| \right) \left( \sum_{ij} \sqrt{P_iP_j} e^{i(\theta_{a_i} - \theta_{a_j})} |i_a\rangle \langle j_a| \right) \\ &= - \sum_{ij;nm} \delta_{nm} \sqrt{P_iP_j} e^{i(\theta_{a_i} - \theta_{a_j} + \gamma_{n_a \rightarrow m_b})} \langle n_a|i_a\rangle \langle j_a|m_b\rangle \\ &= - \sum_{ij;m} \delta_{im} \sqrt{p(j_a|m_b)P_iP_j} e^{i(\theta_{m_b \rightarrow i_a} + \gamma_{i_a \rightarrow m_b} + \theta_{a_i} - \theta_{a_j})} \end{aligned}$$

This gives us expression (18) for  $\text{tr}D\sigma_A$ . The calculations leading to expression (19) for  $\text{tr}D\sigma_B$  just repeat the previous ones.

Now we find traces for comparison operators and “transition states”:

$$\begin{aligned} \text{tr}D_{B \rightarrow A}\sigma_{B \rightarrow A} &= \text{tr} \left( \sum_{nm} \delta_{nm} e^{i\gamma_{m_b \rightarrow n_a}} |n_a\rangle \langle m_b| \right) \left( \sum_{ij} \sqrt{P_iQ_j} e^{i(\theta_{a_i} - \theta_{b_j})} |i_a\rangle \langle j_b| \right) \\ &= \sum_{ij;nm} \delta_{nm} \sqrt{P_iQ_j} e^{i(\theta_{a_i} - \theta_{b_j} + \gamma_{n_a \rightarrow m_b})} \langle j_b|n_a\rangle \langle m_b|i_a\rangle \\ &= \sum_{ij;nm} \delta_{nm} \sqrt{p(n_a|j_b)p(m_b|i_a)P_iQ_j} e^{i(\theta_{i_a \rightarrow m_b} - \theta_{j_b \rightarrow n_a} + \gamma_{n_a \rightarrow m_b} + \theta_{a_i} - \theta_{b_j})}. \end{aligned}$$

$$\begin{aligned} \text{tr}D_{A \rightarrow B}\sigma_{B \rightarrow A} &= -\text{tr} \left( \sum_{nm} \delta_{nm} e^{i\gamma_{n_a \rightarrow m_b}} |m_b\rangle \langle n_a| \right) \left( \sum_{ij} \sqrt{P_iQ_j} e^{i(\theta_{a_i} - \theta_{b_j})} |i_a\rangle \langle j_b| \right) \\ &= - \sum_{ij;nm} \delta_{nm} \sqrt{P_iQ_j} e^{i(\theta_{a_i} - \theta_{b_j} + \gamma_{n_a \rightarrow m_b})} \langle j_b|m_b\rangle \langle n_a|i_a\rangle \\ &= - \sum_{ij} \delta_{ij} \sqrt{P_iQ_j} e^{i(\theta_{a_i} - \theta_{b_j} + \gamma_{i_a \rightarrow j_b})}. \end{aligned}$$

In the same way

$$\text{tr}D_{A \rightarrow B}\sigma_{A \rightarrow B} = - \sum_{ij;nm} \delta_{nm} \sqrt{p(m_b|i_a)p(n_a|j_b)P_iQ_j} e^{i(\theta_{j_b \rightarrow n_a} - \theta_{i_a \rightarrow m_b} + \gamma_{m_b \rightarrow n_a} + \theta_{b_j} - \theta_{a_i})}$$

and

$$\begin{aligned} \text{tr} D_{B \rightarrow A} \sigma_{A \rightarrow B} &= \text{tr} \left( \sum_{nm} \delta_{nm} e^{i\gamma_{m_b \rightarrow n_a}} |n_a\rangle \langle m_b| \right) \left( \sum_{ij} \sqrt{P_i Q_j} e^{i(\theta_{b_j} - \theta_{a_i})} |j_b\rangle \langle i_a| \right) = \\ &= \sum_{ij} \delta_{ij} \sqrt{P_i Q_j} e^{i(\theta_{b_j} - \theta_{a_i} + \gamma_{j_b \rightarrow i_a})}. \end{aligned}$$

Thus

$$\Delta_1 = \frac{1}{2} \text{tr} (D_{B \rightarrow A} \sigma_{A \rightarrow B} - D_{A \rightarrow B} \sigma_{B \rightarrow A}) = \sum_{ij} \delta_{ij} \sqrt{P_i Q_j} \cos(\theta_{b_j} - \theta_{a_i} + \gamma_{j_b \rightarrow i_a}). \quad (46)$$

Then

$$\begin{aligned} \Delta_2 &= \frac{1}{2} \text{tr} (D_{B \rightarrow A} \sigma_{B \rightarrow A} - D_{A \rightarrow B} \sigma_{A \rightarrow B}) = \\ &= \frac{1}{2} \sum_{ij; nm} \delta_{nm} \sqrt{p(n_a | j_b) p(m_b | i_a) P_i Q_j} e^{i(\theta_{i_a \rightarrow m_b} - \theta_{j_b \rightarrow n_a} + \gamma_{n_a \rightarrow m_b} + \theta_{a_i} - \theta_{b_j})} \\ &\quad - \frac{1}{2} \sum_{ij; nm} \delta_{nm} \sqrt{p(m_b | i_a) p(n_a | j_b) P_i Q_j} e^{i(\theta_{j_b \rightarrow n_a} - \theta_{i_a \rightarrow m_b} - \gamma_{n_a \rightarrow m_b} + \theta_{b_j} - \theta_{a_i})}. \end{aligned}$$

Finally, we get

$$\Delta_2 = \sum_{ij; nm} \delta_{nm} \sqrt{p(m_b | i_a) p(n_a | j_b) P_i Q_j} \cos(\theta_{j_b \rightarrow n_a} - \theta_{i_a \rightarrow m_b} + \gamma_{n_a \rightarrow m_b} + \theta_{b_j} - \theta_{a_i}). \quad (47)$$

## Appendix 2: lotteries with two outcomes and arbitrary probabilities

We consider the same bases for the lotteries as in section 8, see (29), (30), but now the probabilities  $P_1, P_2$  and  $Q_1, Q_2$  are arbitrary.

$$|\psi_A\rangle = \sqrt{P_1} |1_a\rangle + \sqrt{P_2} |2_a\rangle, \quad |\psi_B\rangle = \sqrt{Q_1} |1_b\rangle + \sqrt{Q_2} |2_b\rangle. \quad (48)$$

The complete belief-state can be written as

$$|\psi\rangle = \frac{c_1 |1_a\rangle + c_2 |2_a\rangle}{\|c_1 |1_a\rangle + c_2 |2_a\rangle\|}, \quad (49)$$

where

$$c_1 = \sqrt{P_1} + \frac{\sqrt{Q_1} + \sqrt{Q_2}}{\sqrt{2}}, \quad c_2 = \sqrt{P_2} + \frac{\sqrt{Q_1} - \sqrt{Q_2}}{\sqrt{2}}. \quad (50)$$

Hence, the decision inequality has the form (with some factor  $C > 0$ ) :

$$\langle \psi | D | \psi \rangle = C \left( [2c_1^2 u_1 + 2c_1 c_2 u_2] - [(c_1 + c_2)^2 v_1 + (c_1^2 - c_2^2) v_2] \right) \geq 0. \quad (51)$$

It can be rewritten as

$$\langle \psi | D | \psi \rangle \sim 2c_1 [c_1 u_1 + c_2 u_2] - (c_1 + c_2) [(c_1 + c_2) v_1 + (c_1 - c_2) v_2] \geq 0. \quad (52)$$

Thus Alice assigns to the outcomes of the lotteries the weights  $c_1, c_2$  and  $d_1 = c_1 + c_2, d_2 = c_1 - c_2$ . As we have seen in (49), the weights  $c_1, c_2$  correspond to the coefficients in the expansion of the complete state  $|\psi\rangle$  with respect to the  $A$ -basis. It is easy to see that

$$|\psi\rangle = \frac{d_1 |1_b\rangle + d_2 |2_b\rangle}{\|d_1 |1_b\rangle + d_2 |2_b\rangle\|}. \quad (53)$$

The comparison inequality (52) can be written as comparison of two subjective utilities with respect to the probabilities:

$$\tilde{P}_1 = \frac{c_1}{c_1 + c_2} = \frac{\sqrt{P_1} + \frac{\sqrt{Q_1} + \sqrt{Q_2}}{\sqrt{2}}}{\sqrt{P_1} + \sqrt{P_2} + \sqrt{2Q_1}}, \quad (54)$$

$$\tilde{P}_2 = \frac{c_2}{c_1 + c_2} = \frac{\sqrt{P_2} + \frac{\sqrt{Q_1} - \sqrt{Q_2}}{\sqrt{2}}}{\sqrt{P_1} + \sqrt{P_2} + \sqrt{2Q_1}} \quad (55)$$

$$\tilde{Q}_1 = \frac{c_1 + c_2}{2c_1} = \frac{\sqrt{P_1} + \sqrt{P_2} + \sqrt{2Q_1}}{2[\sqrt{P_1} + \frac{\sqrt{Q_1} + \sqrt{Q_2}}{\sqrt{2}}]} \quad (56)$$

$$\tilde{Q}_2 = \frac{c_1 - c_2}{2c_1} = \frac{\sqrt{P_1} - \sqrt{P_2} + \sqrt{2Q_2}}{2[\sqrt{P_1} + \frac{\sqrt{Q_1} + \sqrt{Q_2}}{\sqrt{2}}]}. \quad (57)$$

Thus, for some  $\tilde{C} > 0$ ,

$$\langle \psi | D | \psi \rangle = \tilde{C} \left( (c_1 + c_2) [u_1 \tilde{P}_1 + u_2 \tilde{P}_2] - 2c_1 [v_1 \tilde{Q}_1 + v_2 \tilde{Q}_2] \right) \geq 0. \quad (58)$$

However, as was emphasized in section 8, generally the quantities  $\tilde{P}_1, \tilde{P}_2$  and  $\tilde{Q}_1, \tilde{Q}_2$  cannot be interpreted as probabilities since  $\tilde{P}_2$  and  $\tilde{Q}_2$  can become negative (and, hence,  $\tilde{P}_1$  and  $\tilde{Q}_1$  can become larger than 1). Therefore to keep to the probabilistic reasoning, we have to split the process of comparison of the lotteries in the four components (see section 7); each of this component can be interpreted as comparison of two subjective utilities.

At the same time appearance of negative and even complex “probabilities” is rather common in quantum theory, starting with works of Dirac [1] and Wigner [2], see also [3]. One can proceed formally with such signed or complex distributions and develop sufficiently advanced mathematical formalism, including analogs of the central limit theorem and the law of large numbers [4]. Recently signed “probabilities” were actively used to model the process of decision making [5]. One can consider the calculus of signed (or even complex) distributions as an alternative to the quantum probability calculus.

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