Quantum like modeling of decision making: quantifying uncertainty with the aid of Heisenberg-Robertson inequality

March 14, 2018

Abstract

This paper contributes to quantum-like modeling of decision making (DM) under uncertainty through application of Heisenberg's uncertainty principle (in the form of the Robertson inequality). In this paper we apply this instrument to quantify uncertainty in DM performed by quantum-like agents. As an example, we apply the Heisenberg uncertainty principle to the determination of mutual interrelation of uncertainties for "incompatible questions" used to be asked in political opinion pools. We also consider the problem of representation of decision problems, e.g., in the form of questions, by Hermitian operators, commuting and noncommuting, corresponding to compatible and incompatible questions respectively. Our construction unifies the two different situations (compatible versus incompatible mental observables), by means of a single Hilbert space and of a deformation parameter which can be tuned to describe these opposite cases. One of the main foundational consequences of this paper for cognitive psychology is formalization of the mutual uncertainty about incompatible questions with the aid of Heisenberg's uncertainty principle implying the mental state dependence of (in)compatibility of questions.

Keywords: Compatible and incompatible questions; decision making; Heisenberg uncertainty principle; mental state; order effect

I Introduction

During the recent years the quantum-like approach to modeling of cognition and decision making (DM) under uncertainty has been increasingly applied to behavioral results surprising or problematic from classical perspectives.[1] One of the main distinguishing features of this approach is the possibility to treat mutually incompatible ("complementary") DM problems, e.g., questions, inside the common model based on quantum probability. Experts in "classical DM-theory" were well aware about the existence of such problems, e.g., in the form of the disjunction, conjunction, and order effects (see, e.g., Tversky & Shafir, 1992). The attempts to represent incompatible problems in the classical probabilistic framework led to a number of paradoxes and theoretical proposals augmenting classical probabilistic inference with additional assumptions (e.g., Costello & Watts, 2014; Tentori et al., 2013). The best known are the Allais (1953), Ellsberg (1961) and Machina (1982) paradoxes, but in their review Erev et al. (2016) count 35 basic paradoxes of classical DM-theory.

Quantum-like modeling of DM, or more generally, cognition is based on the quantum methodology and formalism, but not on quantum biophysics (cf., e.g., works of Hameroff (1994) and Penrose (1989) about reduction of

[1] As some representative works, we can mention the following : Aerts et al., 2016; Asano et al., 2012, 2015, 2017; Bagarello, 2012, 2015; Bagarello et al., 2017; Boyer-Kassem, Duchene and Guerci, 2016; Busemeyer et al., 2006, 2011; Busemeyer and Bruza, 2012; de Barros, 2012; de Barros and Oas, 2014; Dzhafarov and Kujala, 2014, 2016, 2017; Dzhafarov et al. 2017; Haven and Khrennikov, 2013, 2017; Haven and Sozzo, 2015; Khrennikov, 2003, 2004a, 2004b, 2010, 2016; Khrennikov and Basieva, 2014; Khrennikov and Haven, 2007; Khrennikova and Haven, 2016; Plotnitsky, 2014; Pothos and Busemeyer, 2009, 2013; Pothos et al., 2011; and Trueblood and Busemeyer, 2012; Wang and Busemeyer, 2013, Zhang and Dzhafarov 2015, and references therein.

cognition to quantum physical processes in the brain). In the quantum-like framework the brain is a black box, such that its information processing can be described by the formalism of quantum theory. “Mental observables”, e.g., in the form of questions, are represented by Hermitian operators (and in more general framework by so-called positive operator valued measures, Asano et al., 2015). The mental state (or the belief state) of an agent is represented like a quantum state, i.e., a normalized vector of the state space (or, more generally, a density operator representing the classical statistical mixture of pure states).

Therefore we can apply the *Heisenberg uncertainty principle* to characterize interrelation of uncertainties of two incompatible questions (or tasks) A and B. In the general form the Heisenberg uncertainty principle is expressed in the form of the *Robertson inequality:*

$$\sigma_A(\psi)\sigma_B(\psi) \geq |\langle [A, B]/2\rangle_\psi|, \tag{1.1}$$

where $[A, B] = AB - BA$ is the commutator of the operators, $\langle [A, B]/2\rangle_\psi$ is the mean value of the commutator with respect to the state ψ, $\sigma_A(\psi), \sigma_B(\psi)$ are the standard deviations of the observables A and B with respect to the state ψ.

The operators representing the position and momentum observables satisfy a very special commutation relation (the canonical commutation relation): $[q, p] = i\mathbb{1}$, where $\mathbb{1}$ is the unit operator. By using this relation and the Robertson inequality we obtain the original Heisenberg inequality:

$$\sigma_q(\psi)\sigma_p(\psi) \geq 1/2. \tag{1.2}$$

We emphasize that the latter imposes a state-independent constraint onto the product of standard deviations, since the right-hand side of (1.2) does not depend on the state ψ. This is very important property of the Heisenberg inequality. In general we do not have a state independent estimate of the form $\sigma_q(\psi)\sigma_p(\psi) \geq c$, where $c > 0$ does not depend on ψ (cf. with (1.2)). The lower bound for the *interrelation between the standard deviations is state dependent.*

Thus, even for noncommuting mental observables A and B, the right-hand side of the Robertson inequality (1.1) can be equal to zero. In this case the observables A and B are similar to classical observables. In particular, if $[A, B]\psi = 0$ for some mental state, we assume an equivalence with classical probability description in the form of random variables, see see section III. In quantum foundations this issue was studied in very detail by Ozawa (2006, 2011, 2016) and we shall apply his approach to DM and cognition, section III. In that section we shall refer to the condition of *spectral commutativity.* The latter is equivalent to condition $[A, B]\psi = 0$ in the case of dichotomous observables (which we focus on in this paper). However, for general observables $[A, B]\psi = 0$ does not imply spectral commutativity and hence does not imply the possibility of using the classical probability model.[2] (Note that the condition $[A^n, B^m]\psi = 0$ for any n, m is equivalent to the spectral commutativity of A, B.)

The more general situation, $\langle [A, B] \rangle_\psi = 0$, is more complicated from the interpretational viewpoint (section III). We note that in the finite-dimensional space (used for representation of beliefs) it is impossible to construct Hermitian operators satisfying the canonical commutation relation. Moreover, any Hermitian operator has eigenvectors and, for states consistent with them, variance equals zero and (1.1) degenerates to $0 \geq 0$.

The state dependence of the uncertainty relations for mental observables was emphasized by Khrennikov and Haven (2007). The role of the principle of complementarity in cognitive science was analyzed by Khrennikov (1999) and Wang & Busemeyer (2013).

Section II contains the basic mathematical construction of that unifies the two different situations (compatible versus incompatible mental observables) by means of a single Hilbert space and a deformation parameter θ that can be tuned to describe these opposite cases (cf. the work of Buse-

[2]Here we speak about the *noncontextual* classical probability model. Contextual classical measure-theoretic models can serve even for representation of incompatible observables (see Khrennikov, 2010, and Dzhafarov and Kujala, 2014, 2016, 2017; Dzhafarov et al. 2017).

meyer & Pothos (2013), where these cases were treated separately and in different state spaces). The one-parametric families of operators can be used for quantum-like modeling in cognitive psychology and psychophysics - by treating θ as the formal parameter (representing the degree of deformation of compatibility) and selecting it to match experimental data. In section V we use this approach to construct the Hermitian operator representation of questions demonstrating the order effect: we match the deformation parameter θ with the degree of noncommutativity in the sequential joint probability distributions obtained on the basis of the experimental data taken from Moore (2002). These operators can be used in the quantum-like model of Busemeyer and Pothos (2013). Section III presents the most important (for psychological applications) message of this paper: the state dependence of incompatibility of questions. Thus to be or not to be compatible depends not only on questions, but also on the mental state. This statement is very natural from the cognitive viewpoint and our contribution is to put it into the formal mathematical framework.

II Operator representation of incompatible and compatible questions ("mental observables")

We work in finite-dimensional (complex) Hilbert spaces. Such space $\mathcal{H}$ can be represented (by fixing an orthonormal basis) as the space of vectors $\psi = (\psi_1, ..., \psi_n)$ with complex coordinates, endowed with the scalar product given as $\langle \psi, \phi \rangle = \sum_i \psi_i \bar{\phi}_i$. *Mental states* are represented by normalized vectors of $\mathcal{H}$, and mental observables, e.g., in the form of questions, are represented by Hermitian operators.

Consider some decision maker, call her Alice. Following Busemeyer and Pothos (2013), we consider the following pair of aspects of Alice's life represented in the form of questions (mental observables):

- Q_1 : "Are you happy or not?"
- Q_2 : "Are you employed or not?"

We represent each aspect of Alice's life in its own Hilbert state space. The *happiness status* is modeled as a two-state system living in the two-dimensional Hilbert space $\mathcal{H}_H = \mathbb{C}^2$. We introduce an orthonormal basis $\mathcal{F}_H = \{h_+, h_-\}$ of $\mathcal{H}_H$, and a Hermitian operator H, the *happiness operator*, having $h_\pm$ as eigenstates: $Hh_\pm = \pm h_\pm$. Of course, we have $\langle h_j, h_k \rangle = \delta_{jk},\ j,k = \pm$. The interpretation of eigenstates of the happiness operator is clear: if Alice's state is $\Psi = h_+$, then she is definitively happy. But she is unhappy if $\Psi = h_-$. The crucial point is that the state of happiness is not always explicitly determined; Alice can be in the state of superposition of happiness and unhappiness. Such a mental state is represented by a linear combination $\Psi = \alpha_+ h_+ + \alpha_- h_-$, with $|\alpha_+|^2 + |\alpha_-|^2 = 1$. In this case, $|\alpha_+|^2$ is the probability that Alice is happy, while $|\alpha_-|^2$ is the probability that she is not. We see that the answer to Q_1 is sure (i.e., determined with probability one) only if Ψ is an eigenstate of H.

In Busemeyer and Pothos (2013), the questions Q_1 and Q_2, can be thought to be compatible or incompatible. It depends on the mental context. Here compatibility is understood as the absence of the mutual disturbance of these questions. The psychological basis of the mutual disturbance is clear. Suppose Alice is unemployed and she is asked first the question Q_2. By replying to it she modifies her original mental state. In this new ("post-measurement") state her answer to the next question Q_1 can be different from her possible answer to Q_1 as the first question, i.e., in the initial mental state.[3]

In Busemeyer and Pothos (2013), *the issue of incompatible questions was handled in a two-dimensional Hilbert state space, but the issue of compatible questions was handled in a four-dimensional space.* The mathematics behind such modeling is explained in appendix 1.

[3] We note that this picture for how prior questions can activate thoughts which impact on our perspective for later question is well known in social psychology, e.g., Schwarz (2007).

In this paper we propose a unified framework which can be used in both situations: for compatible as well as incompatible questions. Also, rather than measuring the *degree of compatibility* of the questions by means of the angle between, say, h_+ and e_+ (see Busemeyer & Pothos, 2013), we propose below a suitable deformation of the original operators, which is able to include in the same settings the two different situations, by means of some relevant commutator.

We adopt as our Hilbert state space the tensor product of the states spaces $\mathcal{H}_E$ and $\mathcal{H}_H$ representing the states of (un)happiness and (un)employment, $\mathcal{H} = \mathcal{H}_E \otimes \mathcal{H}_H = \mathbb{C}^4$. An orthonormal basis for $\mathcal{H}$ can be constructed out of $\mathcal{F}_E$ and $\mathcal{F}_H$: $\mathcal{F}_\varphi = \{\varphi_{\alpha\beta} = h_\alpha \otimes e_\beta,\ \alpha, \beta = \pm\}$. These are common eigenstates of the operators $\tilde{E} := E \otimes \mathbb{1}_H$ and $\tilde{H} = \mathbb{1}_E \otimes H$, where $\mathbb{1}_E$ and $\mathbb{1}_H$ are the identity operators on $\mathcal{H}_E$ and $\mathcal{H}_H$. (For operators D and C acting in $\mathcal{H}_E$ and $\mathcal{H}_H$, respectfully, their tensor product acts as $D \otimes C(u \otimes v) = Du \otimes Cv$.) We have

$$\tilde{E}\varphi_{\alpha,\beta} = \mu_\alpha \varphi_{\alpha,\beta}; \qquad \tilde{H}\varphi_{\alpha,\beta} = \lambda_\beta \varphi_{\alpha,\beta} \tag{2.1}$$

where $\mu_\alpha, \lambda_\beta = \pm 1$. Of course, the operators $\tilde{H}$ and $\tilde{E}$ commute: $[\tilde{E}, \tilde{H}] = 0$. (This is because they admit a common set of eigenstates, $\mathcal{F}_\varphi$.) When this is so, the Heisenberg uncertainty principle becomes $\sigma_{\tilde{E}}\sigma_{\tilde{H}} \geq 0$, which is not particularly useful.

If, in particular, Ψ is an eigenstate of $\tilde{E}$, then $\sigma_{\tilde{E}} = 0$. Analogously, if Ψ is an eigenstate of $\tilde{H}$, then $\sigma_{\tilde{H}} = 0$. Of course, if Ψ is an common eigenstate of $\tilde{E}$ and $\tilde{H}$, which is possible, since $[\tilde{E}, \tilde{H}] = 0$, then $\sigma_{\tilde{E}} = \sigma_{\tilde{H}} = 0$. The meaning of these results is that, since $\tilde{E}$ and $\tilde{H}$ commute, we are able to know, in principle, what are the eigenvalues of these two operators simultaneously and we are able to answer, with no uncertainty, to Q_1 and Q_2. Notice, however, that this is true if Alice's state Ψ is a common eigenstate of $\tilde{E}$ and $\tilde{H}$. Suppose now that Alice's mental state does not coincide with any common eigenstate of $\tilde{E}$ or $\tilde{H}$, but it is a superposition of the four eigenstates. In this case Q_1 and Q_2 are still compatible, but the state of Alice does not allow us to have any certain answers to our questions.

The key point in our analysis, now, is quite elementary: we deform continuously $\tilde{E}$ and $\tilde{H}$, producing two new operators which, in general, do not commute anymore, and we use this non commutativity as a measure of compatibility of the questions. It is well known that one can use unitary (or just invertible) operators to modify a given operator preserving its eigenvalues and producing eigenvectors related to those of the original operator. Our idea is to deform $\tilde{H}$ and $\tilde{E}$ using this general scheme. However, since the main aim of our procedure is to produce operators which do not commute, we cannot use a single invertible operator S to deform both $\tilde{E}$ and $\tilde{H}$ as shown before, since $S\tilde{E}S^{-1}$ and $S\tilde{H}S^{-1}$ commute in the same way as $\tilde{E}$ and $\tilde{H}$ do. The natural way out is to use two different invertible operators, one for $\tilde{H}$ and a different one for $\tilde{E}$. Of course, the choice of these operators is quite arbitrary, but must be constrained by the requirement that commutativity is changed continuously.

This can be achieved by using unitary matrices constructed out of one-parameter rotations in some plane,

$$R_\theta = \begin{pmatrix} \cos\theta & -\sin\theta & 0 & 0 \\ \sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & \cos\theta & -\sin\theta \\ 0 & 0 & \sin\theta & \cos\theta \end{pmatrix}, \qquad V_\theta = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta & 0 \\ 0 & \sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \tag{2.2}$$

where θ is a parameter which we take in $[0, 2\pi]$.

Specifically, we use R_θ, to deform $\tilde{H}$ by means of two copies of the same rotation acting respectively on $\{\phi_{11}, \phi_{12}\}$ and on $\{\phi_{21}, \phi_{22}\}$. This choice, obviously nonunique, is motivated by its simplicity, and by the fact that it satisfies the requirements listed below. Rotation operator for $\tilde{E}$, V_θ, is just one rotation (in the subspace with the basis $\{\phi_{12}, \phi_{21}\}$). This suffices for our analysis of modulating incompatibility by means of the single parameter θ as will be demonstrated further. Moreover, adjustments of θ together with the choices of the ψ-function allow us to reproduce experimental results (see Section V). Thus our construction provides the possibility to generate rich families of generally incompatible operators by using a single rotation parameter θ.[4]

It is clear that $R_\theta^\dagger = R_\theta^{-1} = R_{-\theta}$, $V_\theta^\dagger = V_\theta^{-1} = V_{-\theta}$ and that $R_\theta \to \mathbb{1}$ and $V_\theta \to \mathbb{1}$ when $\theta \to 0$. Here $\mathbb{1}$ is the identity matrix in $\mathcal{H}$. We define two new Hermitian operators

$$H_\theta = R_\theta \tilde{H} R_\theta^{-1}, \; E_\theta = V_\theta \tilde{E} V_\theta^{-1}$$

with eigenstates $\mathcal{F}_\varphi^H = \{\varphi_{\alpha,\beta}^H = R_\theta \varphi_{\alpha,\beta}, \, \alpha, \beta = \pm\}$ and $\mathcal{F}_\varphi^E = \{\varphi_{\alpha,\beta}^E = V_\theta \varphi_{\alpha,\beta}, \, \alpha, \beta = \pm\}$. The eigenvalues are ± 1, each with multiplicity two. Also, $\mathcal{F}_\varphi^H$ and $\mathcal{F}_\varphi^E$ are related by the unitary operator $T_\theta = R_\theta V_\theta^{-1}$: $\varphi_{\alpha,\beta}^H = T_\theta \varphi_{\alpha,\beta}^E$, for all α and β. Moreover, when $\theta \to 0$, both $\mathcal{F}_\varphi^H$ and $\mathcal{F}_\varphi^E$ converge to the original set $\mathcal{F}_\varphi$, and $H_\theta \to \tilde{H}$ and $E_\theta \to \tilde{E}$. We can also check that

$$[E_\theta, H_\theta] = i U_\theta, \tag{2.3}$$

where the matrix of U_θ can be found in appendix 2, see (6.1). From the latter, we see that $U_\theta = U_\theta^\dagger$ and that $U_\theta \to 0$ when $\theta \to 0$. This reflects the fact that, at this limit, the two operators E_θ and H_θ converge to the commuting matrices $\tilde{E}$ and $\tilde{H}$. Therefore it is natural to call θ the *compatibility parameter*, and the pair (R_θ, V_θ) the *compatibility operators*.

Now, the Robertson inequality implies that

$$\sigma_{E_\theta} \, \sigma_{H_\theta} \geq \frac{1}{2} \, | \, \langle U_\theta \rangle \, |, \tag{2.4}$$

[4]Deformation quantization procedure developed in this paper can be considered as ad hoc. However, even in quantum physics quantization procedures (starting with Schrödinger's quantization) have no intrinsic conceptual justification. They are used because they work well. For example, justification of the Schrödinger quantization via the correspondence, $h \to 0$, with classical phase space mechanics is mainly of a mathematical value (the real physical Planck constant is a constant). In cognitive science and psychology in general the situation is more complex, since there we have no analog of classical phase space mechanics. Mathematically our construction follows the scheme of quantum physical deformation quantizations. There is a parameter of a small value and by sending it to zero we approach a commutative model, a kind of classical formalism.

where $\langle X \rangle = \langle \Psi, X\Psi \rangle$ and $\sigma_X = \sqrt{\langle X^2 \rangle - \langle X \rangle^2}$, for any Hermitian operator X.

Let Alice's mental state be described by a generic normalized vector $\Psi = (c_{++}, c_{+-}, c_{-+}, c_{--})$, with $\sum_{\alpha,\beta=\pm} |c_{\alpha,\beta}|^2 = 1$. Since $E_\theta^2 = H_\theta^2 = \mathbb{1}$ for all values of θ, we have $\langle E_\theta^2 \rangle = \langle H_\theta^2 \rangle = 1$. Moreover, simple computations give

$$\langle E_\theta \rangle = |c_{++}|^2 - |c_{--}|^2 + \big(|c_{+-}|^2 - |c_{-+}|^2\big)\cos(2\theta) + (c_{+-}\,\overline{c_{-+}} + c_{-+}\,\overline{c_{+-}})\sin(2\theta),$$

while

$$\begin{aligned}\langle H_\theta \rangle &= \big(|c_{++}|^2 + |c_{-+}|^2 - |c_{+-}|^2 - |c_{--}|^2\big)\cos(2\theta) + \\ &+ (c_{++}\,\overline{c_{+-}} + c_{+-}\,\overline{c_{++}} + c_{--}\,\overline{c_{-+}} + c_{-+}\,\overline{c_{--}})\sin(2\theta),\end{aligned}$$

whence

$$(\sigma_{E_\theta})^2 = 1 - \langle E_\theta \rangle^2\,, \quad (\sigma_{H_\theta})^2 = 1 - \langle H_\theta \rangle^2\,.$$

Let us now see what happens for a few particular choices of Ψ.

We remark that for all mental states represented by vectors with real coordinates, RHS of the Robertson inequality equals to zero. This is a simple consequence of the fact that the matrix elements of the operators E_θ and H_θ are real numbers.

To obtain nontrivial interrelation between RHS and LHS of the Robertson inequality, we consider the following states having one imaginary component. These are just illustrative examples:

$$\Psi_a = \frac{1}{2}(i,1,1,1);\ \Psi_b = \frac{1}{3}(i,\sqrt{2},\sqrt{3},\sqrt{3}),\ \Psi_c = \left(i\cos(\theta), \frac{\sin(\theta)}{\sqrt{2}}, \frac{\sin(\theta)}{\sqrt{2}}, 0\right). \tag{2.5}$$

In the latter case the state also depends on the deformation parameter θ. The results of numerical simulation are presented at Figures 1, 2.

We remark that the product of standard deviations (LHS of the Robertson inequality) fluctuates as a function of the deformation parameter θ; the average of commutator (RHS of the Robertson inequality) behaves similarly (see Fig. 1 and 2). In complete accordance with quantum theory the graph

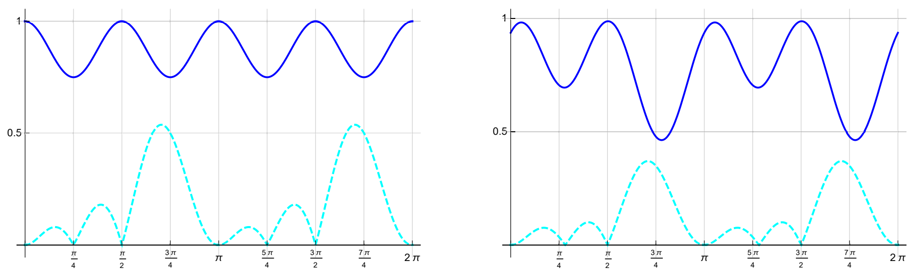

Figure 1: Dashed line: RHS of the Robertson inequality (the commutator value divided by 2) and the solid line: LHS (the product of standard deviations) as functions of θ, for $\Psi = \Psi_a$ (left) and $\Psi = \Psi_b$ (right).

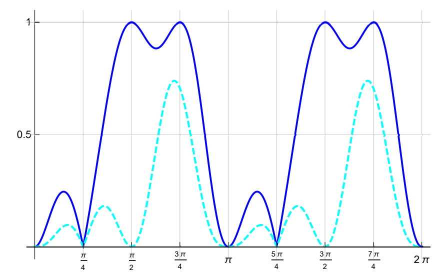

Figure 2: Dashed line: RHS of the Robertson inequality (the commutator value divided by 2); the solid line: LHS (the product of standard deviations), as functions of θ and $\Psi = \Psi_c$.

of RHS is majorized by the graph of LHS. In particular, the product of uncertainties can vanish only for the values of θ for which the average of the commutator vanishes as well, see Fig. 2. Of course, inverse is not true. Nevertheless, all points, where the average of the commutator vanishes, are of special interest (see section III).

From our analysis, it is clear that, for all those values of θ for which $U_\theta = 0$, we go back to the case of compatible questions: in this case (consider, in particular, $\theta = 0$), we recover the orthonormal basis considered in Busemeyer and Pothos (2013), and E_θ and H_θ become those considered in that paper.

So everything discussed in Busemeyer and Pothos (2013) can be restated in our framework. The interesting feature of our settings is that we don't need to change anything to describe the questions, even when they are assumed to be incompatible, and θ parametrizes the *degree of compatibility*.

III Mental state dependent (in)compatibility

For cognitive systems, the state dependence of compatibility is evident. It is clear that, for some mental states of Alice, the questions about happiness and employment are compatible and, for other mental states, they are incompatible. We formalize this issue in the framework of the uncertainty relation based on the Robertson inequality.

Consider first the weakest version of the commutativity between E_θ and H_θ in (2.3). It is possible that, even if $[E_\theta, H_\theta] \neq 0$, still $C_{\theta,\varphi} := \langle\varphi, [E_\theta, H_\theta]\varphi\rangle = 0$, for some specific non zero vector $\varphi \in \mathcal{H}$. In our particular situation, because of the expression in (2.3) for U_θ, $C_{\theta,\varphi} = 0$ if the following equality is satisfied:

$$u_{1,2}\,(\overline{x}\,y - \overline{y}\,x + \overline{z}\,w - \overline{w}\,z) + u_{1,3}\,(\overline{x}\,z - \overline{z}\,x + \overline{w}\,y - \overline{y}\,w) + u_{2,3}\,(\overline{y}\,z - \overline{z}\,y) = 0, \tag{3.1}$$

where x, y, z and w are the components of the normalized column vector φ, $|x|^2 + |y|^2 + |z|^2 + |w|^2 = 1$, while $u_{k,l}$ are related to U_θ as follows:

$$u_{1,2} = -4i\cos(\theta)\sin^3(\theta), \qquad u_{1,3} = i\sin^2(2\theta), \qquad u_{2,3} = -i\sin(4\theta).$$

A first class of solutions of (3.1) is easily found: it is enough to choose x, y, z and w all reals. Then $C_{\theta,\varphi} = 0$ for all values of θ. This suggests that, in order to have nontrivial incompatibility effects of E_θ and H_θ, the state of the system cannot have all components real. Of course, finding the general solution of (3.1) is rather difficult, if not impossible. However, it is not hard to find other particular solutions. For instance, if we assume that $|x| = |y| = |z| = |w| = \frac{1}{4}$, and if we call φ_x, φ_y,... the phases of x, y,..., then it is possible to deduce that these can be chosen for instance as in

$$\varphi_w - \varphi_x = \arccos\left(\cot(2\theta)\right), \qquad \varphi_y = \varphi_w, \quad \varphi_z = 2\varphi_w - \varphi_x,$$

or

$$\varphi_y - \varphi_x = \arctan\left(\frac{\cos(2\theta)}{2\sin^2(\theta)}\right), \qquad \varphi_z = 2\varphi_y - \varphi_x + \frac{\pi}{2}, \quad \varphi_w = 3\varphi_y - 2\varphi_x - \frac{\pi}{2}.$$

Other possibilities are also allowed, but we will not discuss them here.

Now, following Ozawa (2006, 2011, 2016), we classify various variants of (non-)commutativity of Hermitian operators representing quantum observables. Consider two Hermitian operators A and B. We shall also consider their spectral families $E^A(x), E^B(y)$. In the case of discrete spectra (as for finite-dimensional Hilbert spaces) these are just families of projectors on the eigen subspaces of these operators. We restrict our consideration to the latter case.

A, B **commuting:** $[A, B] = 0$. There exists an orthonormal basis consisting of common eigenvectors, i.e., any state ψ can be represented as superposition of common eigenvectors.

A, B **non-commuting:** $[A, B] \neq 0$. An orthonormal basis consisting of common eigenvectors does not exist. (However, common eigenvectors may exist.)

A, B **nowhere-commuting: for any state** ψ, $[A, B]\psi \neq 0$. There exists no common eigenvector.

A, B **spectrally commuting in** ψ : $[E^A(x), E^B(y)]\psi = 0$. Such ψ is a superposition of common eigenvectors of A, B.

A, B **weakly commuting in** ψ : $\langle\psi, [A, B]\psi\rangle = 0$.

Spectral commutativity provides the most natural generalization of compatibility. Of course, spectrally commuting (in a state ψ) observables have the joint probability distribution:

$$p(x, y) = \langle\psi, E^A(x)E^B(y)\psi\rangle = \langle\psi, E^B(y)E^A(x)\psi\rangle.$$

One can say that, for such a mental state ψ, Alice's answers to these questions are based on hidden variables that are "objectively present in her brain". At the same time if these operators do not commute, there exist mental states for which it is impossible to model answers to these questions with the aid of a common classical probability space.

In particular, spectral commutativity implies that, for this concrete state, there is no order effect, nor a disjunction effect. We discuss the latter statement in more detail. (One can say that it immediately follows from the existence of the joint probability distribution, and hence the validity of the formula of total probability. However, the validity of this formula is not evident, since it involves conditional probabilities, and in quantum theory they are not defined by Bayes' formula. So, we have to analyze the process of the update of probabilities starting with a specific state ψ.)

IV Maximizing incompatibility

In sections II, III we investigated the problem of minimization of incompatibility and approaching the weak compatibility, when the right hand side of the Robertson inequality, (1.1), equals to zero. We can also consider the opposite question: *When incompatibility achieves its maximum?*

Maximum expectation value of U_θ equals to its maximum eigenvalue and is reached at the corresponding eigenstate. If we denote

$$sq = \sqrt{(5 - 4\cos(2\theta) + \cos(4\theta))\sin^2(2\theta)},$$

the four eigenvalues of U are $\lambda_{1,2} = \frac{i}{2}\left(\sqrt{2}sq \pm \sin(4\theta)\right)$, $\lambda_{3,4} = -\lambda_{1,2}$. The first two corresponding eigenstates (not normalized) are

$$e_{1,2} = \begin{pmatrix} \pm i \\ \frac{-1}{8}\csc(\theta)\left(\csc(\theta) \mp i\sec(\theta)\right)\left(\sqrt{2}sq \pm i\sin(4\theta)\right) \\ \frac{1}{8}\csc(\theta)\left(\csc(\theta) \mp i\sec(\theta)\right)\left(\mp i\sqrt{2}sq + \sin(4\theta)\right) \\ 1 \end{pmatrix}.$$

In Fig. 3 we show dependence of the right-hand-side of inequality (1.1) as a function of θ. Note that the left-hand-side of (1.1) equals 1 for any θ, because expectation values of H_θ and E_θ are zero at eigenstates of U_θ. Taking into account that squares of H_θ and E_θ are just the identity matrix (independent of θ value), it can be seen that at eigenstates of U_θ both standard deviations σ_{H_θ} and σ_{E_θ} as well as their product are equal to one.

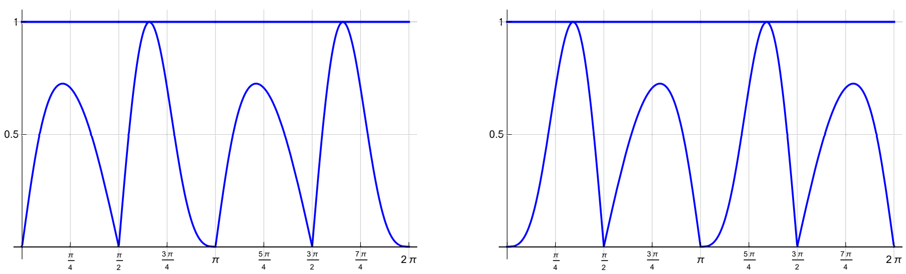

Figure 3: Standard deviations σ_{H_θ}, σ_{E_θ} (upper horizontal lines) and expectation values $|\langle U_\theta/2\rangle_{e_1}|$ (left panel), $|\langle U_\theta/2\rangle_{e_2}|$ (right panel), as functions of θ.

V An application to decision making: modeling the order effect in social opinion polls

Now we apply our construction of the one parametric families of generally noncommuting Hermitian operators to a problem of the failure of commutativity in DM which was discussed, using quantum mechanical language, in Busemeyer and Pothos (2013), Wang and Busemeyer (2013), see Moore (2002) for experimental data. Our aim is to construct a quantum-like representation of questions exhibiting the order effect, i.e., to find a value of the deformation parameter θ and the coordinates of a mental state Ψ matching the aforementioned experimental data.

The problem is quite simple: let us suppose we ask a set of people, $\mathcal{P}$, two questions Q_C and Q_G in two different orders, $O_{C\to G}$ or $O_{G\to C}$. Here Q_C and Q_G stand respectively for *is Clinton honest?* and *is Gore honest?* [5] Order

[5]These questions should be updated, changing the names of the actors, of course. We

$O_{C\to G}$ means that Q_C is asked first, and Q_G after, while $O_{G\to C}$ describes the reversed temporal order. The results of this experiment show that, if we adopt $O_{C\to G}$, then 50% of $\mathcal{P}$ answer yes to Q_C, and 68% of $\mathcal{P}$ answer yes to Q_G. On the other hand, if we consider the reverse order, $O_{G\to C}$, the above percentages change respectively to 57% and 60%. It is clear that the order is important, and the explanation proposed in the literature is that *the first question activates thoughts*, (Busemeyer & Pothos, 2013). Here we want to show how our general settings can be efficiently used in the description.

We shall use the formalism developed in section II. To couple the operators to questions about Clinton and Gore, in this section we shall use the symbols C, G and C_θ, G_θ. Note that these operators can be represented as differences of projectors onto the eigensubspaces corresponding to their eigenvalues ± 1. Thus $C_\theta = P_+^{\theta,C} - P_-^{\theta,C}$, and $G_\theta = P_+^{\theta,G} - P_-^{\theta,G}$.

Consider "joint sequential probabilities" for transitions $Q_C \to Q_G$ and $Q_G \to Q_C$, i.e., the probabilities $p_{Q_C Q_G}(\alpha, \beta)$ that a respondent first gives the answer α to the question Q_C and then the answer β to the question Q_G and the probabilities $p_{Q_G Q_C}(\beta, \alpha)$ for these questions asked in inverse order. The social opinion poll (Moore, 2002, see also Wang & Busemeyer, 2013), give us the following numbers:

$$p_{Q_C Q_G}(++) = 0.4899; p_{Q_C Q_G}(+-) = 0.0447;$$

$$p_{Q_C Q_G}(-+) = 0.1767; p_{Q_C Q_G}(--) = 0.2886;$$

$$p_{Q_G Q_C}(++) = 0.5625; p_{Q_G Q_C}(+-) = 0.1991;$$

$$p_{Q_G Q_C}(-+) = 0.0255; p_{Q_G Q_c}(--) = 0.2130.$$

Our aim is to find representation of the questions Q_C and Q_G by the operators $C_\theta = P_+^{\theta,C} - P_-^{\theta,C}$ and $G_\theta = P_+^{\theta,G} - P_-^{\theta,G}$ for some values of the deformation parameter θ and states Ψ which generate the joint probabilities. We have the following matching conditions (for $\alpha, \beta = \pm$) :

$$\|P_\beta^{\theta,G} P_\alpha^{\theta,C} \Psi\|^2 = p_{Q_C Q_G}(\alpha, \beta), \; \|P_\alpha^{\theta,C} P_\beta^{\theta,G} \Psi\|^2 = p_{Q_G Q_C}(\beta, \alpha). \qquad (5.1)$$

merely follow a literature example here.

Note that these matching conditions are consequences of the projection postulate. [6]

Since the sum of the probabilities for each of the experiments $Q_C \to Q_G$ and $Q_G \to Q_C$ equals one, we have the system of 6 equations. Each state in the four-dimensional complex Hilbert space can be encoded by $3+3$ real parameters, $\Psi = xe_1 + ye_2 + ze_3 + we_4$, where $e_1, ..., e_4$ is the basis in the state space constructed in section II. (But in the present considerations the tensor product structure of the basis is not important.) Here $|x|^2 + ... + |w|^2 = 1$.Thus we have 3 independent real variables $a = |x|, b = |y|, c = |z|$. There are also 3 independent phase variables, ϕ_1, ϕ_2, ϕ_3, given by $x = e^{i\phi_1}u_1, y = e^{i\phi_2}u_2, z = e^{i\phi_3}u_3$. By taking into account the deformation parameter θ, we get the system of 6 equations with 7 variables. This system was solved numerically using the Mathematica software package for minimizing with respect to θ and Ψ the sum of squares of differences between experimental and theoretical probabilities. The optimal parameters are $\theta = 0.427;\ a = 0.538; b = 0.424; c = 0.342;\quad \phi_1 = 0.217; \phi_2 = 0.593; \phi_3 = -0.188$. Thus the state vector can be chosen as

$$\Psi = ae^{i\phi_1}e_1 + be^{i*\phi_2}e_4 + ce^{i\phi_3}e_3 + \sqrt{1-a^2-b^2-c^2}e_4. \qquad (5.2)$$

Matching between quantum and experimental probabilities is very good, $\epsilon = 0.5 \times 10^{-6}$, where ϵ is the averaged sum of squares of differences between (independent) experimental and theoretical probabilities.

Note that, for the optimal θ and Ψ, operators C_θ and G_θ do not commute. Moreover, they are not even weakly commuting (see section III), i.e., the average of their commutator is nonzero: $\langle [C_\theta, G_\theta]\Psi, \Psi\rangle = -0.07i$. For these

[6] In general different assumptions on the joint sequential probability of the observables E_θ and H_θ would be possible - in a way consistent with the quantum formalism. In quantum mechanics, there are in many ways to measure the same observable with different state changes, described by quantum instruments (Ozawa, 2006), that leads to different joint sequential probability distributions. Here, we assume one particular type of instrument determined by the projection postulate, which has been a standard postulate in the conventional quantum measurement theory.

quantum observables and quantum state, the right hand side of the Robertson equality equals to 0.035. By computing the variances from the experimental data we obtain that $\sigma^2_{C_\theta}(\Psi) = 0.9952$ and $\sigma^2_{G_\theta}(\Psi) = 0.7263$, i.e, the Robertson inequality is satisfied: $\sigma_{C_\theta}(\Psi)\sigma_{G_\theta}(\Psi) = 0.80501 > 0.035$.

VI Conclusions

We have proposed a strategy to discuss, within the framework of quantum formalisms, questions which may be compatible or not. Our proposal provides a unifying framework to consider both these situations. We have also shown that this framework can be used in DM, in the presence of failure of commutativity.

We quantified uncertainty in beliefs of people with the aid of the Heisenberg uncertainty relation (in the general form of the Robertson inequality).[7] The general theoretical framework was illustrated by concrete statistical data collected in the political opinion pool (Moore, 2002) and exhibiting *the order effect* (Pothos & Busemeyer, 2013; Wang & Busemeyer, 2013). By using our formalism of generating noncommutative operators based on the deformation parameter technique we reconstructed the operators (representing questions about Clinton and Gore) and the mental state of the representative participant of the poll. Consistency between theoretical model and the experimental data is very high, of the magnitude $\epsilon \sim 10^{-6}$.

[7]The Heinsenberg uncertainty relations played the decisive role in the elaboration of the principle of complementarity by N. Bohr, and in coupling the noncmmutativity of Hermitian operators (representing quantum observables) with incompatibility of these observables. It is strange that this fundamental feature of quantum theory has not yet been utilized in quantum-like modeling of cognition and decision making. One of the reasons for this situation was the absence of a formalism of deformation quantization applicable to mental observables. Such formalism was developed in this paper, and it provides the possibility to couple uncertainty with noncommutativity through the Robertson inequality. We suggest that exploring of the Heinsenberg uncertainty relations started in our paper is an important step in establishing foundations of quantum-like cognition.

We hope that this theoretical work will have implications for specific predictions in decision making, regarding the emergence of quantum-like effects. Notably, in quantum cognition research so far it is typically either assumed that two observables are compatible or assumed that they are incompatible. In the latter case we predict quantum-like effects, such as results corresponding to fallacies according to classical probability theory (conjunction fallacy, order effects, etc.). The novel prediction made in this paper is that the degree of incompatibility may be state-dependent, so that for some mental states we would predict quantum-like effects, but for the same individual and for the same questions, other states would eliminate such effects. The issue of mental state preparation in psychological experiments is still a methodological challenge, as we cannot prepare mental states with the same precision as physical states. However, it is generally accepted that the more background information provided for a task and the more elaborate supporting manipulations (e.g., questions which establish that participants had understood the presented information), the more we can be confident that a specific mental state has been achieved.

Appendix 1.

Hermitian operators are commuting if and only if there exists an orthonormal basis which consists of their common eigenvectors. Suppose that one uses the two-dimensional Hilbert space to represent two compatible questions by Hermitian operators H and E with spectrum $\{-1, +1\}$. Then the eigenvectors h_-, h_+ of H coincide with the eigenvectors e_-, e_+ of E, up to (a single) unitary transformation. Suppose that, for example, $h_- = e_-$ and $h_+ = e_+$. This means that the answers to the questions are perfectly correlated: Alice is (un)happy if and only if she is (un)employed. Now let $h_- = e_+$ and $h_+ = e_-$. This means that the answers are perfectly anti-correlated. Alice is happy only being unemployed and vice versa. (This situation is not unusual: some people may prefer to live relaxed lives by using social funds; they may be unhappy to go each day to work.) Thus representation of compatible questions in the two-dimensional space covers only the very special situation of perfect correlations, or questions which are "unitarily equivalent": $E = UHU^{-1}$, where U is a unitary operator. One cannot be satisfied by such a model, since precise (anti-)correlations are not typical for psychology. Therefore Busemeyer and Pothos (2013) proposed to use four-dimensional representation for compatible questions. Here one can realize all possible joint probability distributions and correlations.

Appendix 2.

The matrix representation of the operator U_θ in the commutation relation (2.3):

$$U_\theta = \begin{pmatrix} 0 & -4i\cos\theta\sin^3\theta & i\sin^2 2\theta & 0 \\ 4i\cos\theta\sin^3\theta & 0 & -i\sin 4\theta & -i\sin^2 2\theta \\ -i\sin^2 2\theta & i\sin 4\theta & 0 & -4i\cos\theta\sin^3\theta \\ 0 & i\sin^2 2\theta & 4i\cos\theta\sin^3\theta & 0 \end{pmatrix}. \tag{6.1}$$

References:

Aerts, D., Broekaert, J., Gabora, L. & Sozzo, S. (2016). Generalizing prototype theory: A formal quantum framework. *Front. Psych.*, https://doi.org/10.3389/fpsyg.2016.00418.

Allais, M. (1953). Le comportement de l'homme rationnel devant le risque: critique des postulats et axiomes de l'école américaine. *Econometrica* **21**, 503-546.

Asano, M., Basieva, I., Khrennikov, A., Ohya, M., & Tanaka, Y. (2012). Quantum-like generalization of the Bayesian updating scheme for objective and subjective mental uncertainties. *J. Math. Psych.*, **56**, 166-175.

Asano, M., Khrennikov, A., Ohya, M., Tanaka, Y., Yamato, I., (2015). *Quantum adaptivity in biology: from genetics to cognition.* Heidelberg-Berlin-New York: Springer.

Asano, M, Basieva, I., Khrennikov, A., Ohya, M., Tanaka, Y. (2017). A quantum-like model of selection behavior. *J. Math. Psych.*, **78**, 2-12.

Bagarello, F. (2012). *Quantum dynamics for classical systems: with applications of the number operator.* New York: J. Wiley.

Bagarello, F. (2015). An operator view on alliances in politics. *SIAM J. Appl.. Math.*, 75(2), 564-584.

Bagarello, F., Di Salvo, R., Gargano, F., Oliveri, F. (2017). (H, ρ)-induced dynamics and the quantum game of life. *Applied Math. Mod.* , **43**, 15-32.

Boyer-Kassem, T., Duchene, S., Guerci, E. (2016). Testing quantum-like models of judgment for question order effect. *Math. Soc. Sci.*, 80, 33-46.

Busemeyer, J. B., Wang, Z., Townsend, J. T. (2006). Quantum dynamics of human decision making. *J. Math. Psych.*, 50, 220-241.

Busemeyer, J. R. , Pothos EM, Franco R and Trueblood J. (2011). A quantum theoretical explanation for probability judgment errors. *Psych. Rev.*, 118, 193-218.

Busemeyer, J. R., Bruza, P. D. (2012). *Quantum models of cognition and decision.* Cambridge: Cambridge Univ. Press.

Dzhafarov, E. N., & Kujala, J. V. (2014). Contextuality is about identity of random variables. *Physica Scripta*, T163, 14009.

Dzhafarov, E. N., & Kujala, J. V. (2016). Context-content systems of random variables: The Contextuality-by-Default theory. *J. Math. Psych.*, 74, 11.

Dzhafarov, E. N., & Kujala, J. V. (2017). Probabilistic foundations of contextuality. *Fortschritte Der Physik - Progress of Physics*, 65, 1600040 (1-11).

Dzhafarov, E. N., Cervantes, V. H., & Kujala, J. V. (2017). Contextuality in canonical systems of random variables. *Phil. Trans. Royal Soc.* A, 375, 20160389.

Costello, F., and Watts, P. (2014). Surprisingly rational: probability theory plus noise explains biases in judgments. *Psych. Rev.* , 121 (3), 463-480.

de Barros, A. J. (2012). Quantum-like model of behavioral response computation using neural oscillators. *Biosystems*, 110, 171-182.

de Barros, A.J., Oas, G. (2014). Negative probabilities and counterfactual reasoning in quantum cognition. *Physica Scripta*, T163, 014008.

Dzhafarov E. N., Kujala J. V. (2014). On selective influences, marginal selectivity, and Bell/CHSH inequalities. *Topics in Cognitive Science*, 6, 121-128.

Erev, I., Ert, E., Plonsky, O., Cohen, D., Cohen, O. (2016). From anomalies to forecasts: toward a descriptive model of decisions under risk, under ambiguity, and from experience. Preprint.

Ellsberg, D. (1961). Risk, ambiguity, and the Savage axioms. *Q. J. Econ.*, 75 (4), 643-669.

Gigerenzer, G. and Todd, P. M. (1999). *Simple heuristics that make us smart.* New York and Oxford: Oxford University Press.

Hameroff, S. (1994). Quantum coherence in microtubules. A neural basis for emergent consciousness? *J. of Cons. Stud.*, 1, 91-118.

Haven, E., Khrennikov, A. (2013). *Quantum social science.* Cambridge: Cambridge Univ. Press.

Haven, E., Khrennikov, A. (2016). Statistical and subjective interpretations of probability in quantum-like models of cognition and decision making. *J. Math. Psych.*, 74, 82-91.

Kahneman, D., Tversky, A. (1973). On the Psychology of Prediction. *Psych. Rev.*, 80 (4), 23751.

Khrennikov, A. (1999). Classical and quantum mechanics on information spaces with applications to cognitive, psychological, social and anomalous phenomena. *Found. Physics*, 29, 1065-1098.

Khrennikov, A. (2003). Quantum-like formalism for cognitive measurements. *Biosystems*, 70, 211-233.

Khrennikov, A. (2004a) On quantum-like probabilistic structure of mental information.*Open Systems and Information Dynamics*, **11** (3), 267-275.

Khrennikov, A. (2004b). *Information dynamics in cognitive, psychological, social, and anomalous phenomena.* Ser.: Fundamental Theories of Physics. Dordreht: Kluwer.

Khrennikov, A. (2010). *Ubiquitous quantum structure: from psychology* *to finance.* Berlin-Heidelberg-New York: Springer.

Khrennikov, A. (2016). Quantum Bayesianism as the basis of general theory of decision-making. *Phil. Trans. Royal Soc.* A, 374, 20150245.

Khrennikov, A., Basieva, I. (2014). Possibility to agree on disagree from quantum information and decision making. *J. Math. Psych.*, 62–63, 1–15.

Khrennikov, A. Yu., Haven, A. Yu. (2007). Does probability interference exist in social science? *AIP Conf. Proc.*, 889, 299-309.

Khrennikova, P., Haven E. (2016). Instability of political preferences and the role of mass media: a dynamical representation in a quantum framework. *Phil. Trans. R. Soc.* A, 374, 20150106.

Kolmogoroff, A. N. (1933). *Grundbegriffe der Wahrscheinlichkeitsrechnung.* Berlin: Springer Verlag. English translation: Kolmogorov, A. N. (1956). *Foundations of theory of probability.* New York: Chelsea Publishing Company.

Machina, M. (1982). Expected utility analysis without the independence axiom. *Econometrica*, 50(2), 277-323.

Moore, D. W. (2002). Measuring new types of question-order effects. *Public Opinion Quarterly*, 66, 8091.

Merzbacher, E. (1970) . *Quantum Mechanics.* New York: Wiley.

Nilsson, H., Juslin, P., and Winman, A. (2016). Heuristics can produce surprisingly rational probability estimates: Comment on Costello and Watts (2014). *Psych. Rev.*, 123, 103111.

Oaksford, M. and Chater, N. (2009). Prcis of Bayesian rationality: the probabilistic approach to human reasoning. *Behavioral and Brain Sciences*, 32, 69-120.

Ozawa, M. (2006). Quantum perfect correlations. *Annals Phys.*, 321, 744769.

Ozawa, M. (2011). Quantum Reality and Measurement: A Quantum Logical Approach. *Found. Phys.*, 41, 592607.

Ozawa, M. (2016). Probabilistic Interpretation of Quantum Theory. *New Generation Computing*, 34, 125-152.

Penrose, R. (1989). *The Emperor's new mind.* New-York: Oxford Univ. Press.

Plotnitsky, A. (2014). Are quantum-mechanical-like models possible, or necessary, outside quantum physics? *Physica Scripta*, T163, 014011.

Pothos, E. M., Busemeyer, J. R. (2009). A quantum probability explanation for violation of rational decision theory. *Proc. Royal. Soc. B*, 276, 2171-2178.

Pothos, E. M., Perry, G., Corr, P.J., Matthew, M.R., Busemeyer, J. R. (2011). Understanding cooperation in the Prisoner's Dilemma game. *Pers. Indiv. Differ.*, 51, 210-215.

Pothos E.M. , Busemeyer J.R. (2013). Can quantum probability provide a new direction for cognitive modeling? *Behavioral Brain Sc.*, 36, 255-327.

Schwarz, N. (2007). Attitude construction: Evaluation in context. *Social Cognition*, 25, 638-656.

Tentori, K., Crupi, V., and Russo, S. (2013). On the determinants of the conjunction fallacy: Probability versus inductive confirmation. *J. Exp. Psych.: General*, 142, 235255.

Trueblood J.S. , Busemeyer J. R. (2012). A quantum probability model of causal reasoning *Frontiers in Cognition*, 2, 289.

Tversky A., Kahenman D. (1983). Extensional versus intuitive reasoning: the conjunction fallacy in probability judgement. *Psych. Rev.*, 90(4), 293-315.

Tversky A., Shafir E. (1992). The disjunction effect in choice under uncertainty. ıPsych. Sc., 3(5), 305-309.

von Neumann, J., Morgenstern, O. (1944). *Theory of games and economic behavior.* Princeton: Princeton Univ. Press.

Wang, Zh., Busemeyer, J.R. (2013). A quantum question order model supported by empirical tests of an a priori and precise prediction. *Topics in Cognitive Sciences*, 5, 689-710.

Wang, Z., Solloway, T., Shiffrin, R. M., and Busemeyer, J. R. (2014). Context effects produced by question orders reveal quantum nature of human judgments. *PNAS*, 111, 9431-9436.

Yates, J. F., and Carlson, B. W. (1986). Conjunction errors: Evidence for multiple judgment procedures, including signed summation. Organizational *Behavior and Human Decision Processes*, 37, 230253.

Zhang, R., and Dzhafarov, E. N. (2015). Noncontextuality with marginal selectivity in reconstructing mental architectures. *Front. Psych.*, https://doi.org/10.3389/fpsyg.2015.00735.